A Meshless Method for the Reissner-Mindlin Plate Equations based on a Stabilized Mixed Weak Form

Jack S. Hale[†*], Pedro M. Baiz[†]

[†]Imperial College London, UK
Department of Aeronautics
SW7 2AZ
j.hale09@imperial.ac.uk
p.m.baiz@imperial.ac.uk

Keywords: Reissner-Mindlin, plates, meshfree, maximum-entropy, mixed methods.

ABSTRACT

Plate theories such as the Reissner-Mindlin theory have seen wide use throughout engineering practice to simulate the mechanical response of structures that are far larger in their planar dimensions than through their thickness [1]. Meshless methods such as the Element Free Galerkin method [2] have been applied to the solution of the Reissner-Mindlin plate equations. Similarly to Finite Element methods, meshless methods must be carefully designed to overcome the well-known shear locking problem.

A relatively recent development in meshless methods are Maximum-Entropy basis functions which provide a 'weak' Kronecker-delta property when constructed on convex node sets, amongst other advantageous properties [3]. Unlike the more commonly used Moving Least-Squares approximation scheme this allows for the direct imposition of Dirichlet boundary conditions.

We begin by recalling the normalised weak form of the Reissner-Mindlin plate problem defined on the middle surface of the plate Ω_0 [1]: Find the transverse displacements and rotations $(z_3, \boldsymbol{\theta}) \in \mathcal{V}_3 \times \mathcal{R}$ such that for all $(y_3, \boldsymbol{\eta}) \in \mathcal{V}_3 \times \mathcal{R}$

$$\int_{\Omega_0} L\epsilon(\boldsymbol{\theta}) : \epsilon(\boldsymbol{\eta}) \; d\Omega + \lambda \bar{t}^{-2} \int_{\Omega_0} (\nabla z_3 - \boldsymbol{\theta}) \cdot (\nabla y_3 - \boldsymbol{\eta}) \; d\Omega = \int_{\Omega_0} g y_3 \; d\Omega \tag{1a}$$

where ν is Poisson's ratio, E is Young's modulus, $\bar{t}$ is the plate thickness, $D = E/12(1-\nu^2)$ is the bending modulus, $\lambda = E\kappa/2(1+\nu)$ is the shear modulus and $\kappa = 5/6$ is a shear correction factor. The operators L and ϵ are defined as:

$$\epsilon(\boldsymbol{v}) = \frac{1}{2}\left((\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^T\right) \qquad L[\epsilon] \equiv D\left[(1-\nu)\epsilon + \nu \operatorname{tr}(\epsilon) I\right] \tag{1b}$$

Many successful treatments of locking in the finite element literature are constructed through the application of a mixed variational formulation. In mixed formulations the shear stress vector $\boldsymbol{\gamma} = (\gamma_{xz}, \gamma_{yz})$ is treated as an independent variable:

$$\gamma = \lambda \bar{t}^{-2} (\nabla z_3 - \boldsymbol{\theta}) \tag{2}$$

giving the equivalent mixed problem as [1]: Find the transverse deflection, rotations and transverse shear stresses $(z_3, \theta, \gamma) \in (\mathcal{V}_3, \mathcal{R}, \mathcal{S})$ such that for all $(y_3, \eta, \psi) \in (\mathcal{V}_3, \mathcal{R}, \mathcal{S})$:

$$\int_{\Omega_0} L\epsilon(\boldsymbol{\theta}) : \epsilon(\boldsymbol{\eta}) \; d\Omega + \int_{\Omega_0} \boldsymbol{\gamma} \cdot (\nabla y_3 - \eta) \; d\Omega = \int_{\Omega_0} g y_3 \; d\Omega \tag{3a}$$

$$\int_{\Omega_0} (\nabla z_3 - \theta) \cdot \psi \; d\Omega - \frac{\bar{t}^2}{\lambda} \int_{\Omega_0} \boldsymbol{\gamma} \cdot \boldsymbol{\psi} \; d\Omega = 0 \tag{3b}$$

In this work, we derive a shear-locking free meshless method using Maximum-Entropy basis functions by considering a stabilised mixed weak form. This stabilisation is in the form of an augmented Lagrangian discussed in [4]. We split the energy from the shear bilinear form into two separate parts; one $a_s^{(\mathcal{R},\mathcal{V}_3)}$ calculated as usual from the displacement function spaces and a second $a_s^{\mathcal{S}}$ from an independently interpolated shear strain function space [4]:

$$a_s = \alpha a_s^{(\mathcal{R},\mathcal{V}_3)} + (\bar{t}^{-2} - \alpha) a_s^{\mathcal{S}} \tag{4}$$

where $0 < \alpha < \bar{t}^{-2}$ is a scalar parameter with units of inverse length squared which is independent of the plate thickness $\bar{t}$. When $\alpha = 0$ we recover the standard mixed formulation, and when $\alpha = \bar{t}^{-2}$ we recover the standard displacement formulation. The effect of this stabilisation, or 'blending', is to ensure the coercivity of the problem on the spaces $\mathcal{V}_{3h} \times \mathcal{R}_h$ and $\mathcal{V}_{3h} \times \mathcal{V}_{3h}$ which was lost in our original mixed problem, allowing the use of incompressible elasticity or Stokes type elements for the Reissner-Mindlin plate problem.

Both the stabilised and standard mixed formulations are known to be well-behaved in the limiting case as $\bar{t} \to 0$, however at the cost of introducing extra degrees of freedom to representing the transverse shear stresses into the problem. Clearly the elimination of these extra unknowns *apriori* to the solution of the linear system of equations is a desirable outcome. To this end, our goal is to construct a discrete operator $\Pi_h : (\mathcal{V}_{3h}, \mathcal{R}_h) \to \mathcal{S}_h$:

$$\begin{aligned} &\int_{\Omega_0} L\epsilon(\boldsymbol{\theta}_h) : \epsilon(\boldsymbol{\eta}_h)\, d\Omega + \alpha\lambda \int_{\Omega_0} (\nabla z_{3h} - \boldsymbol{\theta}_h) \cdot (\nabla y_{3h} - \boldsymbol{\eta}_h)\, d\Omega \\ &+ (\bar{t}^{-2} - \alpha)\lambda \int_{\Omega_0} \Pi_h(\nabla z_{3h} - \boldsymbol{\theta}_h) \cdot (\nabla y_{3h} - \boldsymbol{\eta}_h)\, d\Omega = \int_{\Omega_0} g y_{3h}\, d\Omega \end{aligned} \tag{5}$$

eliminating the shear strain unknowns from the mixed formulation. We construct this operator using the technique detailed in [5]; we discretise the shear strain field using Lagrangian Finite Elements (for simplicity) and then using the elements attached to each node we calculate the volume averaged nodal shear strain. This results in a system of equations in the original displacement unknowns only.

We show the performance of the method using common numerical examples and discuss the method's applicability to the more complicated asymptotic behaviours of Naghdi shells.

References

[1] D. Chapelle, & K.J. Bathe, The Finite Element Analysis of Shells - Fundamentals. 2nd ed. *Springer*, 2010.

[2] T. Belytschko, et al., Element-free Galerkin methods. *Int. J. Numer. Meth. Engng*, 37(2) (1994), 229–256.

[3] M. Arroyo, & M. Ortiz, Local maximum-entropy approximation schemes: a seamless bridge between finite elements and meshfree methods. *Int. J. Numer. Meth. Engng*, 65 (2006), 2167–2202.

[4] D. Boffi, & C. Lovadina, Analysis of new augmented Lagrangian formulations for mixed finite element schemes. *Numerische Mathematik* 75(4) (1997): 405–419.

[5] A. Ortiz, M. A. Puso, & N. Sukumar, Maximum-entropy meshfree method for compressible and near-incompressible elasticity. *Computer Methods in Applied Mechanics and Engineering* 199(25-28) (2010). 1859–1871.