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Rapid testing of stabilised finite element formulations for the Reissner-Mindlin plate problem using the FEniCS project

J. S. Hale*, P. M. Baiz

18th June 2012

Outline

- Main research goal
- Plate theories
- Numerical demonstration of locking using FEniCS
- Mixed formulation
- Stabilisation
- A few results
- Summary

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Our Research

A meshless method for the Reissner-Mindlin plate problem that:

- is based on a sound variational principle
- is free from shear-locking
- avoids problems of previous approaches
- can be extended to the more complicated shell problem

"A locking-free meshfree method for the simulation of shear-deformable plates based on a mixed variational formulation", Accepted in *Computer Methods in Applied Mechanics and Engineering*

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Finite Elements

Of course, there are many successful approaches to the Reissner-Mindlin problem in the Finite Element literature.

How can FE approaches inform the design of a new meshless method?

Unifying themes with FEniCS

- Mixed variational form (robust, general)*
- Stabilisation (bubbles, parameters)*
- Reduction and projection operators to eliminate extra unknowns (‘tricks’ become rigorous)**

*Easy with FEniCS, **Doable, but could be easier

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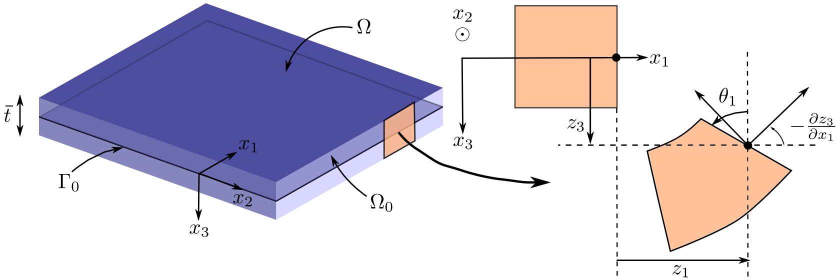
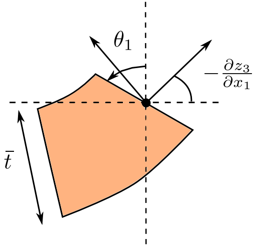
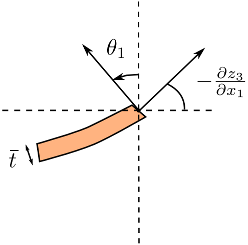
The Reissner-Mindlin Plate Problem



The Kirchhoff Limit



$$\bar{t} \to 0$$



Reissner-Mindlin

Kirchhoff

$$\boldsymbol{\gamma} = \lambda \bar{t}^{-2}(\nabla z_3 - \boldsymbol{\theta}) = 0$$

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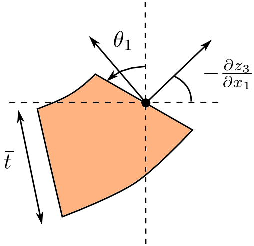

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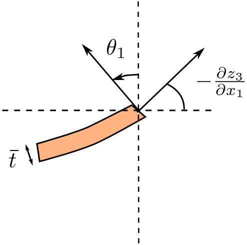

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Reissner-Mindlin Equations

Discrete Displacement Weak Form

Find $(z_{3h}, \boldsymbol{\theta}_h) \in (\mathcal{V}_{3h} \times \mathcal{R}_h)$ such that for all $(y_3, \boldsymbol{\eta}) \in (\mathcal{V}_{3h} \times \mathcal{R}_h)$:

$$\int_{\Omega_0} L\epsilon(\boldsymbol{\theta}_h) : \epsilon(\boldsymbol{\eta}) \; d\Omega + \lambda \bar{t}^{-2} \int_{\Omega_0} (\nabla z_3 - \boldsymbol{\theta}_h) \cdot (\nabla y_3 - \boldsymbol{\eta}) \; d\Omega = \int_{\Omega_0} g y_3 \; d\Omega$$

or:

$$a_b(\boldsymbol{\theta}_h; \eta) + \lambda \bar{t}^{-2} a_s(\boldsymbol{\theta}_h, z_3; \boldsymbol{\eta}, y_3) = f(y_3)$$

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```
...
degree = 1
V_3 = FunctionSpace(mesh, "Lagrange", degree)
R = VectorFunctionSpace(mesh, "Lagrange",
    degree, dim=2)
U = MixedFunctionSpace([V_3, R])

z_3, theta = TrialFunctions(U)
y_3, eta = TestFunctions(U)
...
```

Find $(z_{3h}, \boldsymbol{\theta}_h) \in (\mathcal{V}_{3h} \times \mathcal{R}_h)$ such that for all $(y_3, \boldsymbol{\eta}) \in (\mathcal{V}_{3h} \times \mathcal{R}_h)$:

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$$\epsilon_{ij}(\mathbf{u}) := 1/2(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$$

$$L(\epsilon_{ij}) := D((1-\nu)\epsilon + \nu \operatorname{tr} \epsilon I)$$

$$a_b(\boldsymbol{\theta}_h, \boldsymbol{\eta}) := \int_{\Omega_0} L\epsilon(\boldsymbol{\theta}_h) : \epsilon(\boldsymbol{\eta}) \; d\Omega$$

```
...
e = lambda theta: 0.5*(grad(theta) +
    grad(theta).T)
L = lambda e: D*((1 - nu)*e +
    nu*tr(e)*Identity(2))

a_b = inner(L(e(theta)), e(eta))*dx
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$$a_s(\boldsymbol{\theta}_h, z_3; \boldsymbol{\eta}, y_3) = \int_{\Omega_0} (\nabla z_3 - \boldsymbol{\theta}_h) \cdot (\nabla y_3 - \boldsymbol{\eta}) \; d\Omega$$

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a_s = inner(grad(z_3) - theta, grad(y_3) -
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$$a_b(\boldsymbol{\theta}_h; \eta) + \lambda \bar{t}^{-2} a_s(\boldsymbol{\theta}_h, z_{3h}; \boldsymbol{\eta}, y_3) = f(y_3)$$

```
...
a = a_b + lmbda*t**-2*a_s
f = force*y_3*dx
u_h = solve(a == f, bcs=[bc1, bc2])
z_3h, theta_h = u_h.split()
```

Done!

$$a_b(\boldsymbol{\theta}_h; \eta) + \lambda \bar{t}^{-2} a_s(\boldsymbol{\theta}_h, z_{3h}; \boldsymbol{\eta}, y_3) = f(y_3)$$

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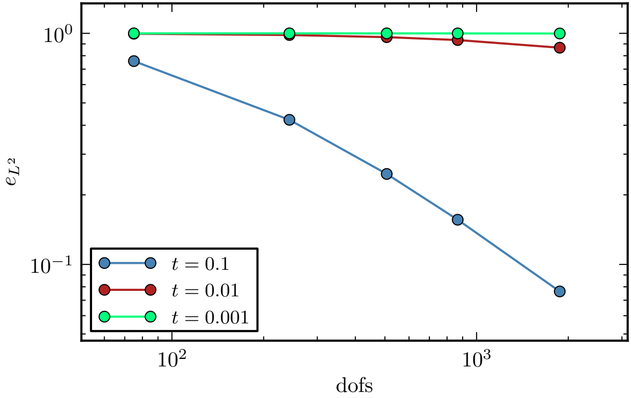

Figure: Locking; Fix discretisation, decrease $\bar{t}$

Locking

Locking

Inability of the basis functions to represent the limiting Kirchhoff mode.

$$\mathcal{V}_b = \{(y_3, \eta) \in (\mathcal{V}_3 \times \mathcal{R}) \mid \nabla y_3 - \eta = 0\}$$

$$\mathcal{V}_b \cap \mathcal{V}_h = \{0\}$$

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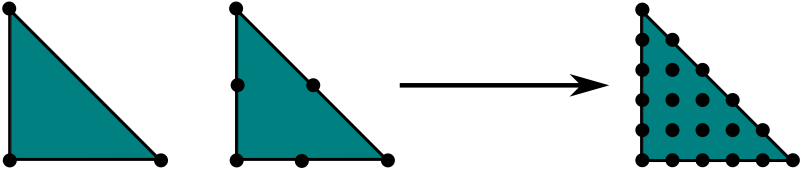

```
for degree in range(0,6):
    V_3 = FunctionSpace(mesh, "Lagrange",
        degree)
    R = VectorFunctionSpace(mesh, "Lagrange",
        degree, dim=2)
    ...
```

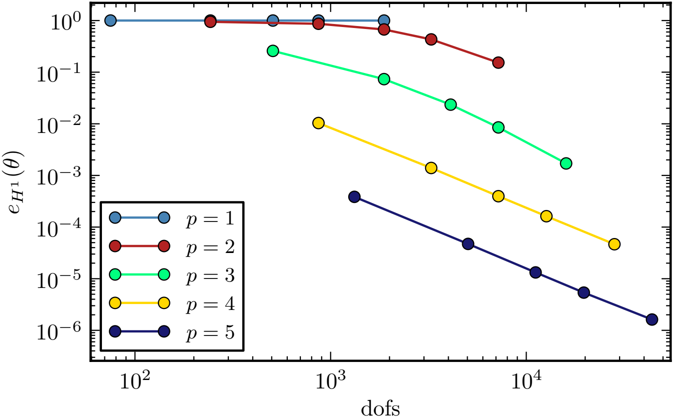

Figure: Fix $\bar{t}$, increase polynomial order p

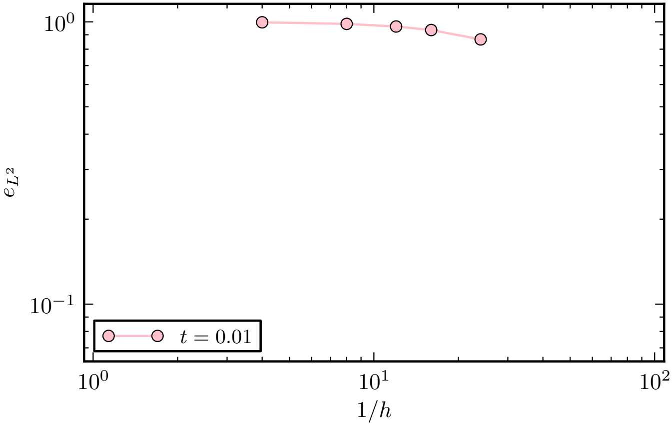

Figure: Fix $\bar{t}$, refine mesh by decreasing h

Error estimate

$$\|u - u_h\| \leq C(\Omega_0, \kappa, E, \nu)\frac{h^p}{\bar{t}}|u|$$

Conclusion

We can never fully eliminate locking with these approaches. It would be better to remove the dependence on $\bar{t}$ entirely.

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Mixed Form

Treat the shear stress as an *independent* variational quantity:

$$\boldsymbol{\gamma}_h = \lambda \bar{t}^{-2} (\nabla z_{3h} - \boldsymbol{\theta}_h) \in \mathcal{S}_h$$

Discrete Mixed Weak Form

Find $(z_{3h}, \boldsymbol{\theta}_h, \boldsymbol{\gamma}_h) \in (\mathcal{V}_{3h}, \mathcal{R}_h, \mathcal{S}_h)$ such that for all $(y_{3h}, \boldsymbol{\eta}, \boldsymbol{\psi}) \in (\mathcal{V}_{3h}, \mathcal{R}_h, \mathcal{S}_h)$:

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Stability

Theorem (Brezzi 1974)

The classical saddle point problem ($\bar{t} = 0$) is stable, if and only if, the following conditions hold:

1. *($\mathcal{Z}$-Ellipticity of a) There exists a constant $\alpha \geq 0$ such that:*

$$a(v, v) \geq \alpha \|v\|_{\mathcal{X}}^2 \quad \forall v \in \mathcal{Z}$$

where $\mathcal{Z}$ is the kernel of the bilinear form b:

$$\mathcal{Z} := \{v \in \mathcal{X} \mid b(v, q) = 0 \quad \forall q \in \mathcal{M}\}$$

2. *(inf-sup condition on b) The bilinear form b satisfies an inf-sup condition:*

$$\inf_{q \in \mathcal{M}} \sup_{v \in \mathcal{X}} \frac{b(v, q)}{\|v\|_{\mathcal{X}} \|q\|_{\mathcal{M}}} = \beta > 0$$

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Displacement Formulation

Locking as $\bar{t} \to 0$

Mixed Formulation

Not necessarily stable

Solution

Combine the displacement and mixed formulation to retain the advantageous properties of both

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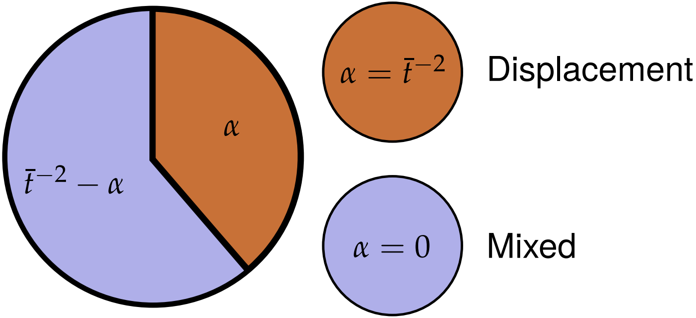

Stabilised Mixed Weak Form

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Stabilised Mixed Weak Form (Brezzi and Arnold 1993, Boffi and Lovadina 1997)

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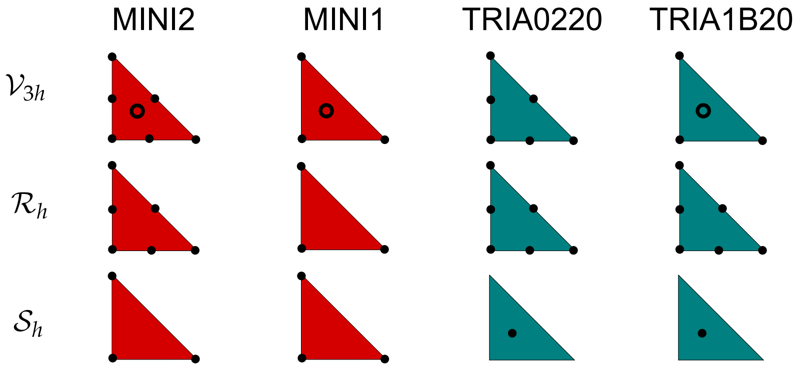
$$\mathcal{Z} := \{v \in \mathcal{X} \mid b(v, q) = 0 \quad \forall q \in \mathcal{M}\}$$

2. *(inf-sup condition on b) The bilinear form b satisfies an inf-sup condition:*

$$\inf_{q \in \mathcal{M}} \sup_{v \in \mathcal{X}} \frac{b(v, q)}{\|v\|_{\mathcal{X}} \|q\|_{\mathcal{M}}} = \beta > 0$$

...

```
...
S = VectorFunctionSpace(mesh, "DG", 0, dim=2)
...
gamma = TrialFunction(S)
psi = TestFunction(S)
...
a = a_b + alpha*lmbda*a_s + inner(gamma,
    grad(y_3) - eta) + inner(grad(z_3) -
    theta, psi) - t**2/(lmbda*(1.0 -
    alpha*t**2))*inner(gamma, psi)
...
```


MINI2
MINI1
TRIA0220
TRIA1B20
$\mathcal{V}_{3h}$
$\mathcal{R}_h$
$\mathcal{S}_h$

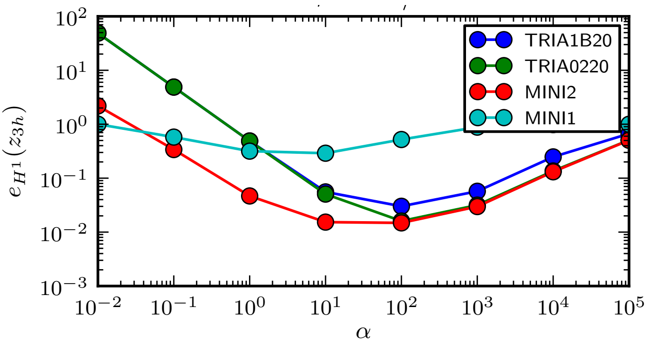


Figure: Various elements, Fix h, vary α.

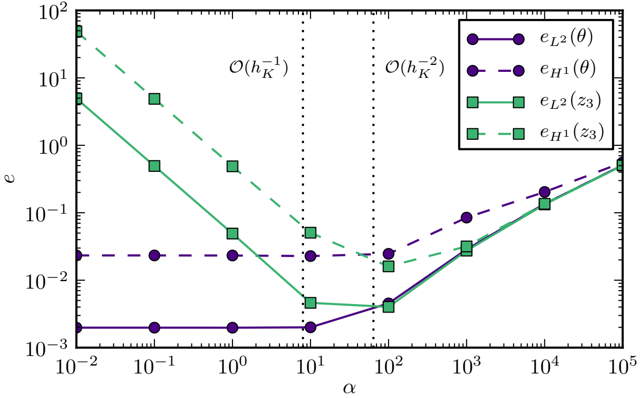


Figure: TRIA0220 element. Fix h, vary α.

```
h = CellSize(mesh)
alpha = h**(-2.0)*constant
```

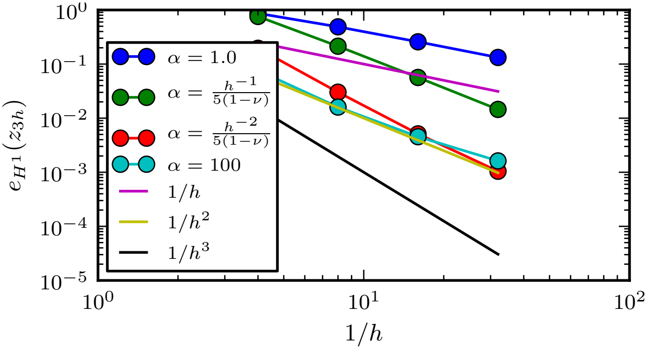


Figure: Trying different α recipes; convergence can be *improved* (Lovadina)

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Conclusions

- The FEniCS project allows for rapid testing of different Finite Element strategies.
- $\sim$ 400 lines of code; Displacement, Mixed, Projections, Errors, Command Line Interface, Output Results etc.
- The stabilisation parameter α should be chosen based on some local discretisation dependent length measure.
- Convergence rates can even be improved by a 'good' choice of α
- Based on these results, we have designed a novel meshfree method based on a stabilised weak form with a Local Patch Projection technique to eliminate the shear-stress unknowns *a priori*

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Thanks for listening.