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Meshless methods for the Reissner-Mindlin plate problem based on mixed variational forms

J. S. Hale

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Universidad de Chile

Overview

- Meshless numerical methods
  - Similarities with finite element methods
  - Differences with finite element methods
- Reissner-Mindlin plate problem
  - Physics of the problem
  - Scaling
  - The Kirchhoff limit
- Shear-locking
  - Numerical demonstration in 1D
  - Why does it happen?
  - What are the potential solutions?
- Mixed variational form
  - Projection Operator
  - Stability
  - Results

Meshless numerical methods

What's the difference with Finite Element Methods?

Well, of course, there is no mesh. But really, at least from a mathematical perspective, there is very little difference between a mesh-based and a mesh-less numerical method.

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Theorem (Partition of Unity, Babuška and Melenk 1993)

$$\sum_i \phi_i = 1 \qquad (1)$$

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Meshless numerical methods

So, meshless methods just use different techniques to construct the Partition of Unity, or basis, to approximate the functions within the domain.

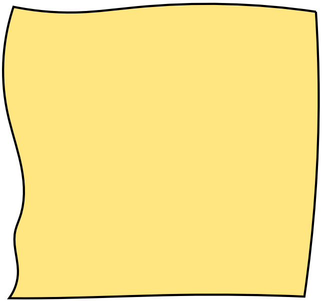

Figure : Domain

Meshless numerical methods

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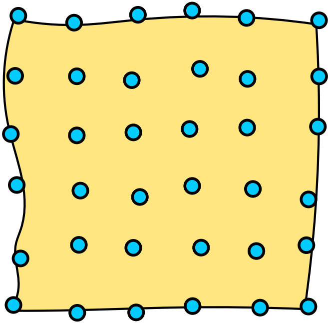

Figure : Seed with nodes

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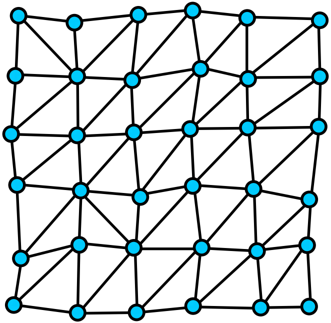

Figure : Mesh

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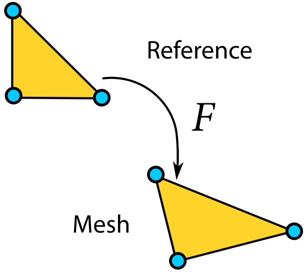

Figure : Construct basis

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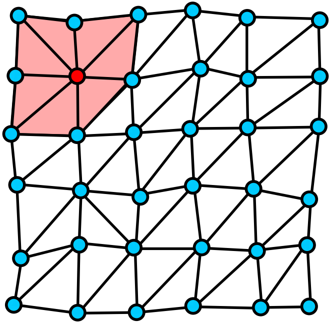

Figure : Support *defined by mesh*

Meshless numerical methods

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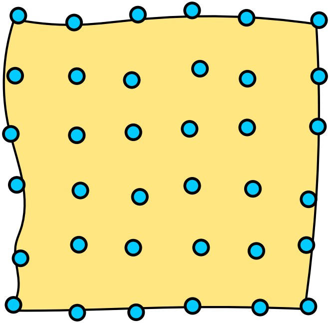

Figure : Meshless; support no longer defined by mesh

Meshless numerical methods

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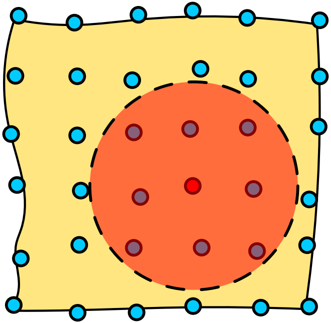

Figure : Give a node a support area

Meshless numerical methods

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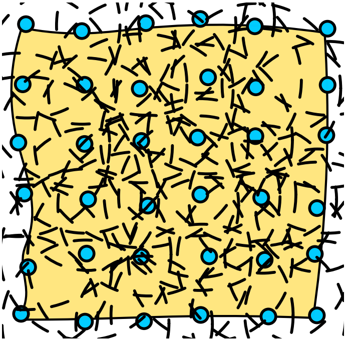

Figure : Give every node a support area

Construct a meshless PU

Typically by minimisation or convex optimisation process.

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Moving Least-Squares (Shepard, Lancaster and Salkauskas)

Quadratic Weighted Least-Squares Minimisation

$$\min_{a} \frac{1}{2} \sum_{i=1}^{N} w_i [p^T a - u_i]^2$$

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$$\min_a \frac{1}{2} \sum_{i=1}^{N} w_i [p^T a - u_i]^2$$

Maximum-Entropy (Sukumar, M. Ortiz and Arroyo)

Entropy functional maximisation

$$\min_\phi \sum_{i=1}^{N} \phi_i \ln \left(\frac{\phi_i}{w_i} \right)$$

FE

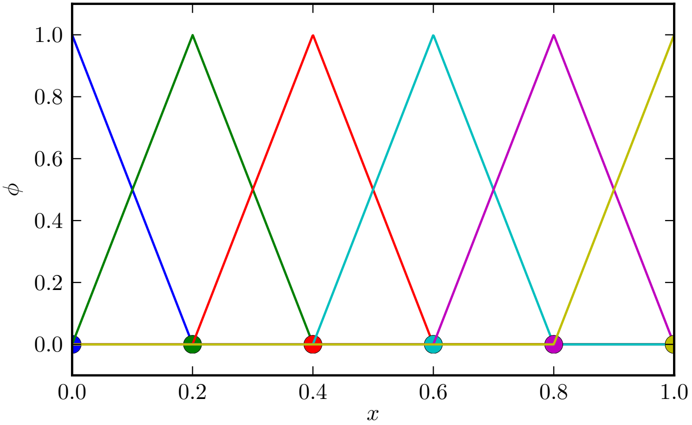

Figure : Finite Element (P_1) basis functions on unit interval

MLS

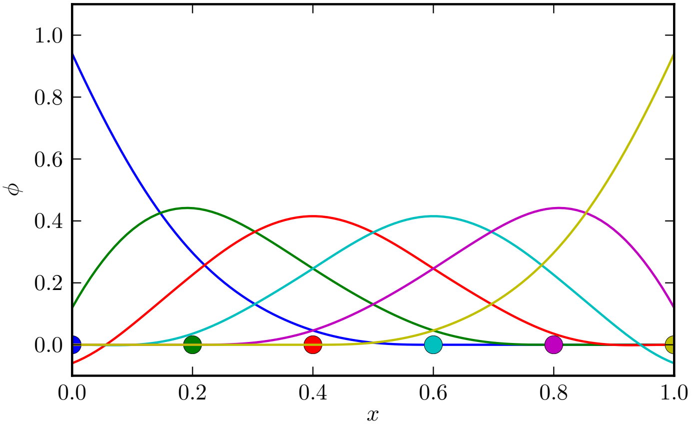

Figure : Moving Least-Squares basis functions on unit interval

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MaxEnt

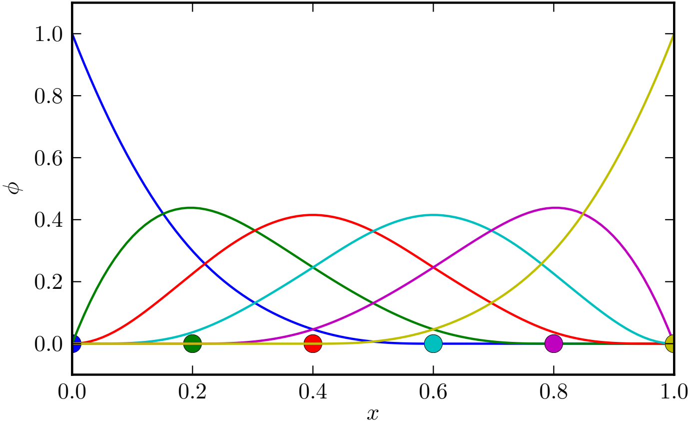

Figure : Maximum-Entropy basis functions on unit interval

Summary

Question

What are the key differences between a mesh-based and a mesh-less PU?

Summary

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What are the key differences between a mesh-based and a mesh-less PU?

Property	Mesh-based	Mesh-less
Space	local element + map	global
Connectivity	mesh	support
Support	local, lower-bandwidth	local, higher-bandwidth
Integration	polynomial	rational
Continuity	C^0 easy, C^1 hard	up to C^∞
Character	interpolant	approximant
Kronecker delta	yes	sometimes

Solving the problem

Step 1: *Begin with the weak (variational) form of your problem*

Poisson Problem (Weak Form)

Find $u \in V$ such that for all $v \in V$ where $V \equiv H_0^1(\Omega)$:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega fv \, dx$$

Solving the problem

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$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega fv \, dx$$

Step 2: *Construct a suitable Partition of Unity* $V_h \subset V$

Poisson Problem (Discrete Form)

Find $u_h \in V_h$ such that for all $v \in V_h$:

$$\int_\Omega \nabla u_h \cdot \nabla v \, dx = \int_\Omega fv \, dx$$

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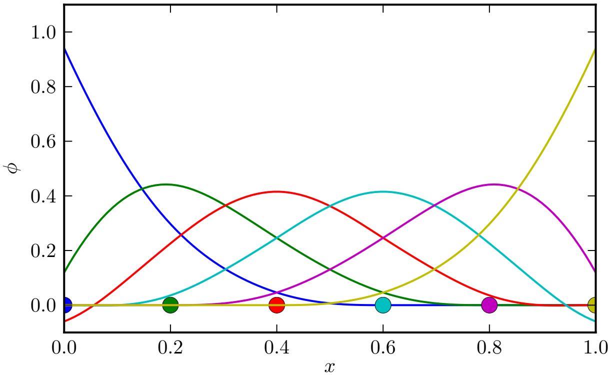
Solving the problem

Problem 1: *Typically for a meshless PU we do not have* $V_h \subset V \equiv H^1_0(\Omega)$ *so we cannot enforce Dirichlet (essential) boundary conditions in a straightforward manner as with finite elements*



Solving the problem

Step 3a: *Modify the weak form of the problem to enforce boundary conditions*

Solving the problem

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Poisson Problem (Discrete Form + Constraint)

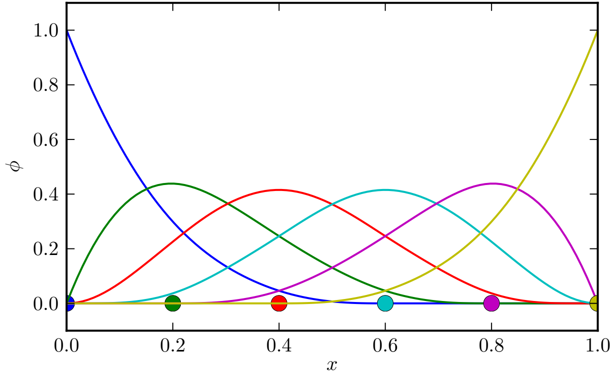
Find $(u_h, \lambda_h) \in V_h \times W_h$ such that for all $(v, \gamma) \in V_h \times W_h$:

$$\int_\Omega \nabla u_h \cdot \nabla v \; dx + \int_\Gamma \lambda_h v \; ds = \int_\Omega fv \; dx$$
$$\int_\Gamma u_h \gamma \; ds = 0$$

Problem 2: *More unknowns, non positive-definite matrix, possible stability problems*

Solving the problem

Step 3b: *Use Maximum-Entropy basis functions* $V_h \subset V \equiv H^1_0(\Omega)$



Solving the problem

Step 4: *Substitute in trial and test basis functions*

Solving the problem

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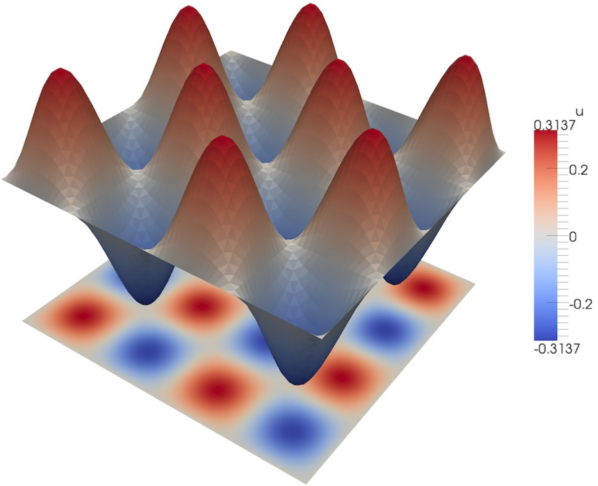
Poisson Problem (Linear System Form)

$$Au = b$$

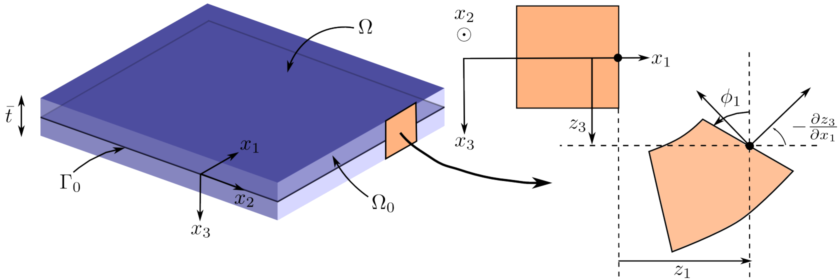
where

$$A_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, dx$$
$$b_i = \int_\Omega \phi_i \, f \, dx$$

Solving the problem



The Reissner-Mindlin Problem



The Reissner-Mindlin Problem

Displacement Weak Form

Find $(z_3, \theta) \in (\mathcal{V}_3 \times \mathcal{R})$ such that for all $(y_3, \eta) \in (\mathcal{V}_3 \times \mathcal{R})$:

$$\int_{\Omega_0} L\epsilon(\theta) : \epsilon(\eta) \; d\Omega + \lambda \bar{t}^{-2} \int_{\Omega_0} (\nabla z_3 - \theta) \cdot (\nabla y_3 - \eta) \; d\Omega = \int_{\Omega_0} g y_3 \; d\Omega \tag{2}$$

or:

$$a_b(\theta; \eta) + \lambda \bar{t}^{-2} a_s(\theta, z_3; \eta, y_3) = f(y_3) \tag{3}$$

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or:

$$a_b(\theta; \eta) + \lambda \bar{t}^{-2} a_s(\theta, z_3; \eta, y_3) = f(y_3) \quad (3)$$

Locking Problem

Whilst this problem is always stable, it is poorly behaved in the thin-plate limit $\bar{t} \rightarrow 0$

Cantilever Beam Problem

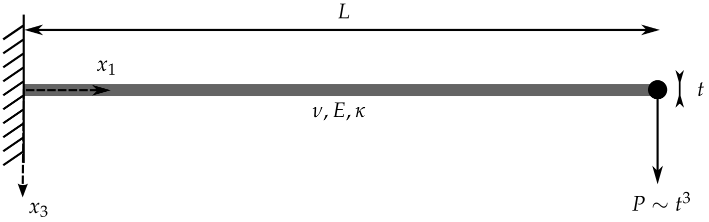

Figure : Cantilever Beam with Point Load

Analytical Solution

Scaling with:

$$\epsilon = \frac{1}{L}\sqrt{\frac{EI}{Gbt\kappa}}, \quad \tilde{p} = L^2/EI$$

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Kirchhoff Theory ($t = 0$, thin)

$$z_3(x_1 = L) = \frac{PL^3}{3EI} = \frac{\tilde{p}L}{3}$$

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Analytical Solution

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$$\epsilon = \frac{1}{L}\sqrt{\frac{EI}{Gbt\kappa}}, \quad \tilde{p} = L^2/EI$$

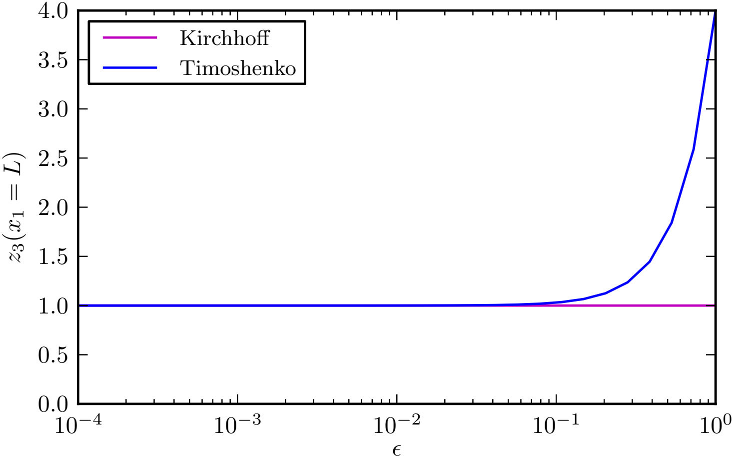
Kirchhoff Theory ($t = 0$, thin)

$$z_3(x_1 = L) = \frac{PL^3}{3EI} = \frac{\tilde{p}L}{3}$$

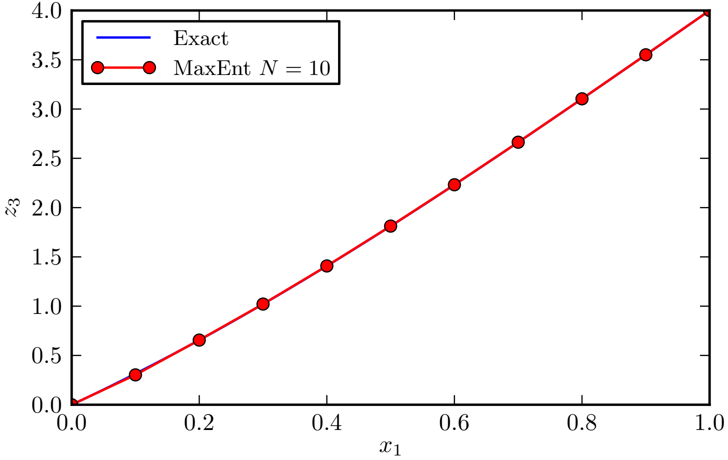
Timoshenko Theory ($t \geq 0$, thin through moderately thick)

$$z_3(x_1 = L) = \frac{PL^3}{3EI}\left(1 + \frac{3EI}{Gbt\kappa L^2}\right) = \frac{\tilde{p}L}{3}\left(1 + 3\epsilon^2\right)$$

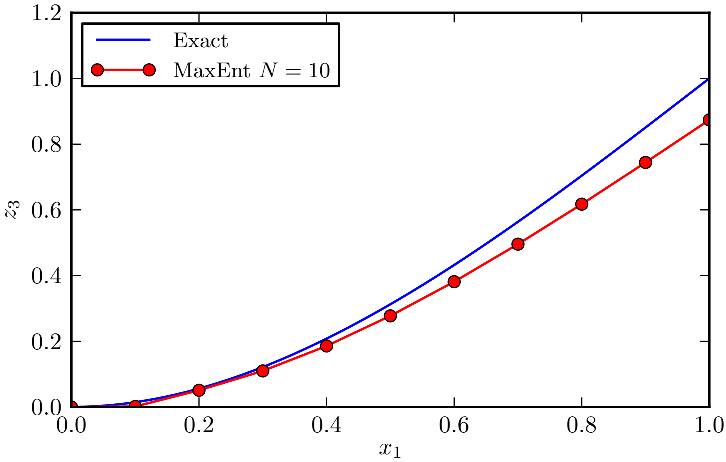
Analytical Solution



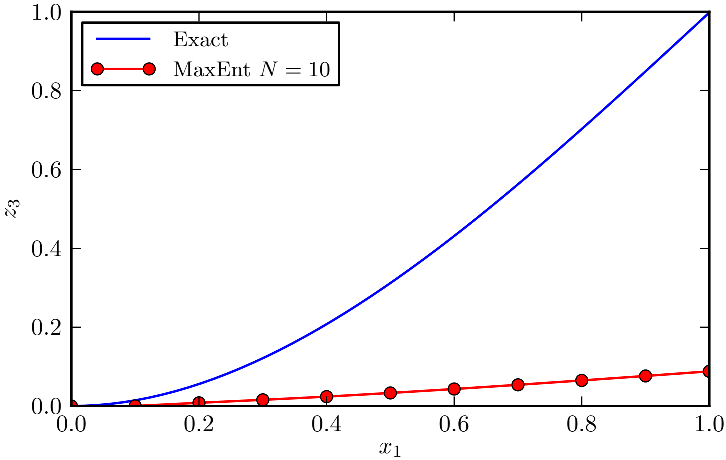
Thick $\epsilon = 1$



Thin $\epsilon = 0.01$



Very thin $\epsilon = 0.001$



Summary

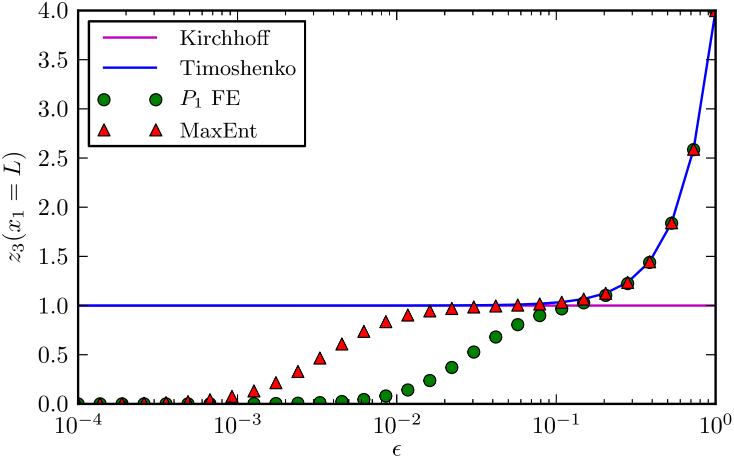

Kirchhoff
Timoshenko
P_1 FE
MaxEnt
$z_3(x_1 = L)$
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
ϵ

Table : The effect of h-refinement on the error $z_{3h}(L)/z_3(L)$ at the tip of the cantilever beam using P_1 finite elements

N	dofs	$\epsilon = 1$	$\epsilon = 0.1$	$\epsilon = 0.01$	$\epsilon = 0.001$	$\epsilon = 0.0001$
1	4	0.92308	0.10714	0.00120	0.00001	0.00000
10	22	0.99917	0.92308	0.10714	0.00120	0.00001
100	202	0.99999	0.99917	0.92308	0.10714	0.00120
1000	2002	1.00000	0.99999	0.99917	0.92308	0.10714
10000	20002	1.00000	1.00000	0.99999	0.99942	0.92292

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Conclusion

To obtain uniform convergence with respect to ϵ using standard methods we must use a huge number of elements

Shear Locking

The Problem

Inability of the basis functions to represent the limiting Kirchhoff mode

$$\nabla z_3 - \eta = 0 \tag{4}$$

Shear Locking

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A solution?

Move to a mixed weak form

Mixed Weak Form

Treat the shear stresses as an *independent* variational quantity:

$$\gamma = \lambda \bar{t}^{-2}(\nabla z_3 - \theta) \in \mathcal{S} \tag{5}$$

Mixed Weak Form

Find $(z_3, \theta, \gamma) \in (\mathcal{V}_3 \times \mathcal{R} \times \mathcal{S})$ such that for all $(y_3, \eta, \psi) \in (\mathcal{V}_3 \times \mathcal{R} \times \mathcal{S})$:

$$a_b(\theta; \eta) + (\gamma; \nabla y_3 - \eta)_{L^2} = f(y_3) \tag{6a}$$

$$(\nabla z_3 - \theta; \psi)_{L^2} - \frac{\bar{t}^2}{\lambda}(\gamma; \psi)_{L^2} = 0 \tag{6b}$$

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Stability Problem

Whilst this problem is well-posed in the thin-plate limit, ensuring stability is no longer straightforward

Stabilised Mixed Weak Form

Displacement Formulation

Locking as $\overline{t} \to 0$

Mixed Formulation

Not necessarily stable

Stabilised Mixed Weak Form

Displacement Formulation

Locking as $\bar{t} \to 0$

Mixed Formulation

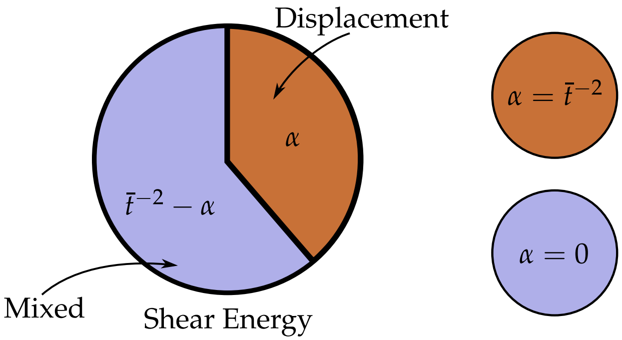
Not necessarily stable

Solution

Combine the displacement and mixed formulation to retain the advantageous properties of both

Stabilised Mixed Weak Form

Split the discrete shear term with a parameter $0 < \alpha < \bar{t}^{-2}$ that is *independent of the plate thickness*:



$$a_s = \alpha a^{\text{displacement}} + (\bar{t}^{-2} - \alpha) a^{\text{mixed}} \tag{7}$$

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Stabilised Mixed Weak Form

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$$(\nabla z_3 - \theta; \psi)_{L^2} - \frac{\bar{t}^2}{\lambda}(\gamma; \psi)_{L^2} = 0 \tag{8b}$$

Stabilised Mixed Weak Form (Brezzi and Arnold 1993, Boffi and Lovadina 1997)

$$a_b(\theta; \eta) + \lambda \alpha a_s(\theta, z_3; \eta, y_3) + (\gamma, \nabla y_3 - \eta)_{L^2} = f(y_3) \tag{9a}$$

$$(\nabla z_3 - \theta, \psi)_{L^2} - \frac{\bar{t}^2}{\lambda(1 - \alpha \bar{t}^2)}(\gamma; \psi)_{L^2} = 0 \tag{9b}$$

Stabilised Mixed Weak Form

Mixed Weak Form

Find $(z_3, \theta, \gamma) \in (\mathcal{V}_3 \times \mathcal{R} \times \mathcal{S})$ such that for all $(y_3, \eta, \psi) \in (\mathcal{V}_3 \times \mathcal{R} \times \mathcal{S})$:

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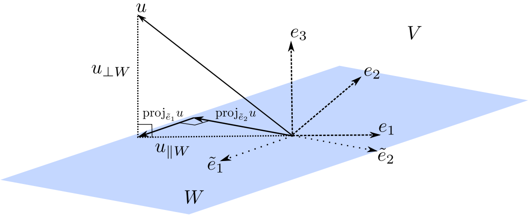
Eliminating the Stress Unknowns

- Find a (cheap) way of eliminating the extra unknowns associated with the shear-stress variables

$$\gamma_h = \frac{\lambda(1 - \alpha \bar{t}^2)}{\bar{t}^2} \Pi_h(\nabla z_{3h} - \theta_h, \psi_h) \tag{10}$$

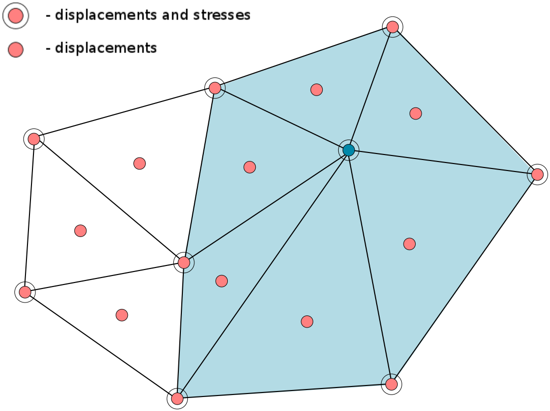
Figure : The Projection Π_h represents a softening of the energy associated with the shear term



Eliminating the Stress Unknowns

- We use a version of a technique proposed by A. Ortiz, Puso and Sukumar for the Incompressible-Elasticity/Stokes' flow problem which they call the "Volume-Averaged Nodal Pressure" technique.
- A more general name might be the "Local Patch Projection" technique.

Eliminating the Stress Unknowns



Eliminating the Stress Unknowns

For one component of shear (for simplicity):

$$(z_{3,x} - \theta_1, \psi_{13})_{L^2} - \frac{\bar{t}^2}{\lambda(1 - \alpha \bar{t}^2)} (\gamma_{13}; \psi_{13})_{L^2} = 0 \tag{11}$$

Substitute in meshfree and FE basis, perform row-sum (mass-lumping) and rearrange to give nodal shear unknown for a node a. Integration is performed over local domain Ω_a:

$$\gamma_{13a} = \sum_{i=1}^{N} \frac{\int_{\Omega_a} N_a \left\{ -\phi_i \quad \phi_{i,x} \right\} \, d\Omega}{\int_{\Omega_a} N_a \, d\Omega} \begin{Bmatrix} \phi_i \\ z_{3i} \end{Bmatrix} \tag{12}$$

Choosing α

The *dimensionally consistent* choice for α is length^{-2}. In the FE literature typically this paramemeter has been chosen as either h^{-1} or h^{-2} where h is the local mesh size.

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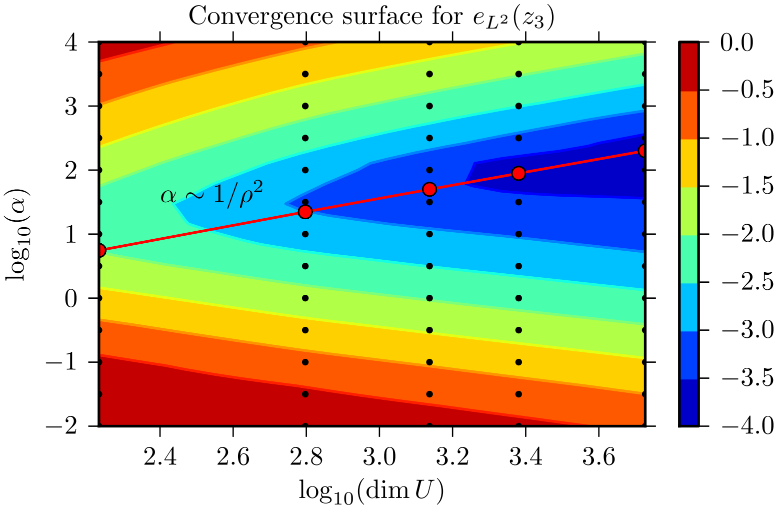
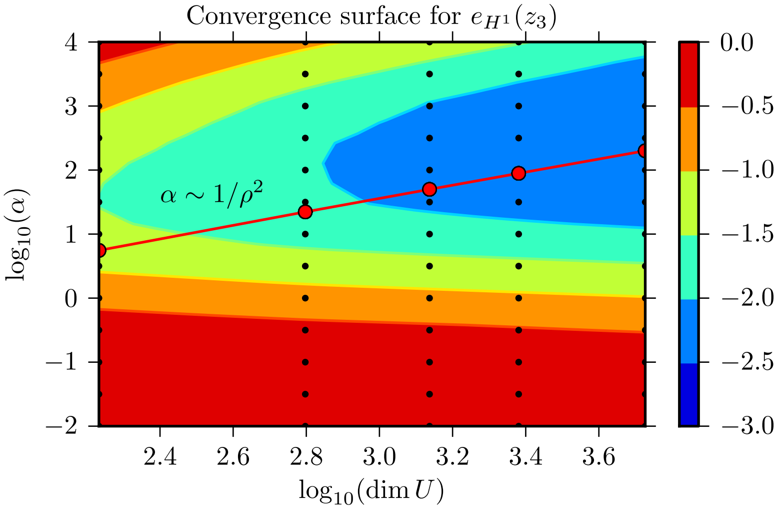
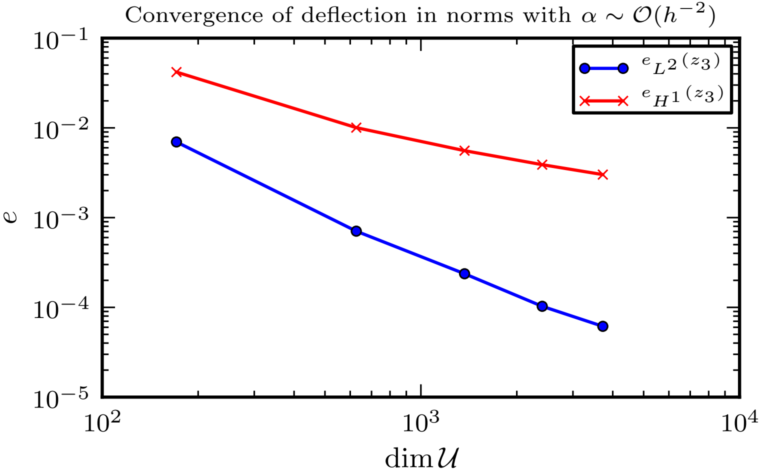
Meshless methods

A sensible place to start would be ρ^{-2} where ρ is the local support size.

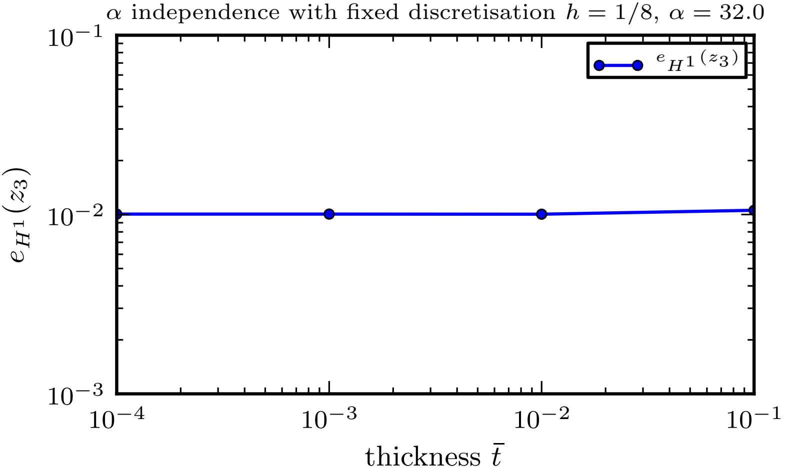

Convergence surface for $e_{L^2}(z_3)$
$\alpha \sim 1/\rho^2$
$\log_{10}(\alpha)$
$\log_{10}(\dim U)$
4
3
2
1
0
−1
−2
2.4
2.6
2.8
3.0
3.2
3.4
3.6
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
−3.5
−4.0


Convergence surface for $e_{H^1}(z_3)$
$\alpha \sim 1/\rho^2$
$\log_{10}(\alpha)$
$\log_{10}(\dim U)$
4
3
2
1
0
−1
−2
2.4
2.6
2.8
3.0
3.2
3.4
3.6
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0

Results - Convergence



Results - α independence



Results - Surface Plots

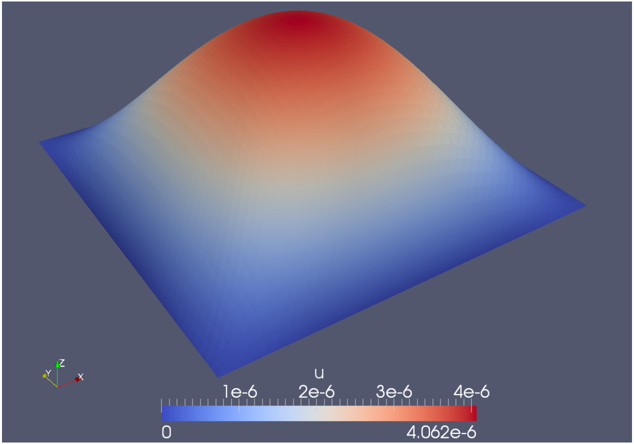


Figure : Displacement z_{3h} of SSSS plate on 12×12 node field + 'bubbles', $t = 10^{-4}, \alpha = 120$

Results - Surface Plots

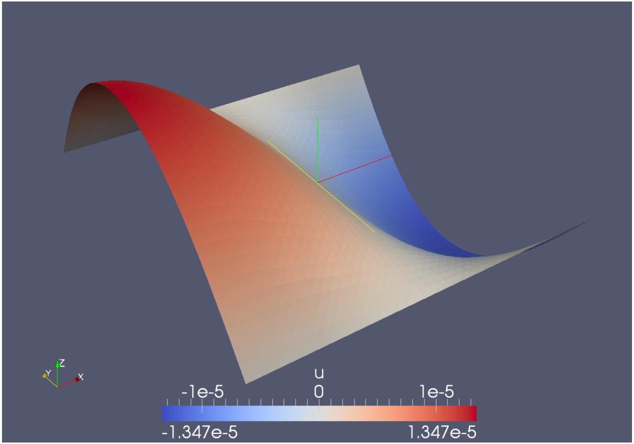


Figure : Rotation component θ_1 of SSSS plate on 12×12 node field + 'bubbles', $t = 10^{-4}, \alpha = 120$

Summary

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