Symmetries and infinite dimensional Lie algebras

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Abstract

In this lecture course I present the idea of symmetries (of physical systems, mathematical systems, systems in nature, ...) and try to show how it is related to groups, Lie algebras and all such things. In particular, it is shown how the existence of symmetries will help to understand and analyze such systems better. In a first part I deal with finite-dimensional systems, finite groups and Lie groups. In the second part I consider infinite dimensional systems, like they appear in the context of partial differential equations, e.g. for the Korteweg – de Vries equation, and in Conformal Field Theory. In this situation infinite dimensional Lie algebras play a crucial role. Beside others I introduce the Witt algebra and its central extensions the Virasoro algebra. It is shown that the need to introduce central extension is very typical for infinite dimensional situations. In this case one is often forced to regularize an action and this forces us to consider central extensions. It is exemplified for the semi-infinite wedge representation, also known as fermionic Fock space representation, and for the Sugawara construction for affine Lie algebra representations. I close with some remarks on Krichever - Novikov type algebras.

Contents

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1 Introduction

Most people who observe an object, a picture, or a situation which has a certain kind of symmetries feel esthetically pleased. But there is more than this esthetic aspect. Objects with symmetries make feel people more comfortable compared to objects without any ordering structure which helps to understand the object better. We might guess that both aspects are related.

It is undeniable that symmetries make life simpler. For example, to describe to somebody the set in three-dimensional space given by the points lying on the surface S of a ball B, it is enough to tell him what is the center point $c \in \mathbb{R}^3$ of the ball, the radius r, and finally telling him that the object is *rotational symmetric*. Meaning, if $x \in S$ and R is any rotation with center c, then Rx also lies on S. Of course, he needs to understand what a rotation is. In this case the situation is even as optimal as it could be. It is enough just to give one point y on S, i.e. one point y with $\|y - c\| = r$ and all other points will be given by rotating this point. From the point of view of pure symmetry the situation for the full ball is the same. In contrast to the sphere one point is not enough to "generate" all points.

Such kind of symmetries are quite often described with the help of groups. Depending on the situation we have finite groups, infinite groups, discrete groups, continuous groups, Lie groups, ... There are more general symmetries of relevance, which have to be described by more general objects. Examples of such objects are Hopf algebras, quantum groups, and group like categorical objects. We will not consider them here. In fact in the main part of the lecture we will not even discuss groups so much, but will talk more about Lie algebras. To every finite-dimensional Lie group (see the definition below - it is a special case of a continuous group - all rotations around a fixed center in $\mathbb{R}^3$ gives an example) one can assign a finite-dimensional Lie algebra (also defined below). The Lie algebra is in a certain sense the infinitesimal object. For example for the group of rotation in $\mathbb{R}^3$ around the origin, denoted by SO(3), the Lie algebra is the vector space of 3×3 skew-symmetric matrices.

In a Lie group the operation can be very non-linear, in the Lie algebra the operation is linearized. Every action of a Lie group on a set induces a linearized action of the Lie algebra. Very often the latter is more easy to treat with.

We will go even one step further. We will consider infinite-dimensional systems. As symmetries infinite dimensional Lie algebras show up. This will be the main part of the lecture. But nevertheless I will start by discussing some examples for the group and finite situation.

Let me close the introduction by making some remark on the style of this write-up. The presented lecture was not intended to be complete in any respect. It should make appetite to learn more in the field. At certain parts it was quite informal. I was trying to keep this idea in this write-up. In particular, I decided to stay short and not to add additional material to it, beyond that what I prepared for the lecture. Anyway writing one single book would not suffice, for a complete treatment of all the aspects touched here. Also references to literature are given rather erratic.

Finally, let me thank the organisers of the summer school for doing this great job to collect every summer quite a number of extremely interested and motivated university and high school students. Moreover, I like to thank the participants, for their lively feed back.

2 Groups

I assume that most of the readers know the definition of a group. Nevertheless for completeness let me just repeat it.

Definition 1. A group $(G, \cdot)$ consists of a set G and a map (called the product)

$$\cdot : G \times G \to G, \quad (x, y) \mapsto x \cdot y \tag{1}$$

such that

1. the product $\cdot$ is associative, i.e. $\forall x, y, z \in G : (x \cdot y) \cdot z = x \cdot (y \cdot z)$,
2. there exists an element $e \in G$, such that $\forall x \in G : \quad e \cdot x = x \cdot e = x$,
3. for every element $x \in G$ there exists an element in G, denoted by x^{-1} such that $x \cdot x^{-1} = x^{-1} \cdot x = e$.

The element e is called the neutral element. One easily shows that there is only one such element. The element x^{-1} is called the inverse element to x. Again it is uniquely given for x.

Those readers for which this is new, should spend some minutes in reflecting that the set of rotations in 3-dimensional space $\mathbb{R}^3$ with a fixed point c as center with product given by the composition of the rotations is indeed a group. In this description one usually denotes it by $(\mathrm{Rot}(c), \cdot)$. Maybe, you are more used to another description: After choosing a orthonormal basis in $\mathbb{R}^3$ with respect to the standard scalar product, the rotations around the point $0 \in \mathbb{R}^3$ can be exactly described as the set of orthogonal matrices with determinant equal to one. The composition of the rotations corresponds to matrix multiplication. In this description the group is denoted by $\mathrm{SO}(3)$.

2.1 Platonic solids.

Let us consider convex bounded subset of $\mathbb{R}^3$. If we translate the situation by a shift we do not change much. Also an overall rescaling is of no importance. Hence, if we understand the situation with respect to the origin 0 we have the complete picture.

In a certain sense the ball $B := \{x \in \mathbb{R}^3 \mid \|x\| \le 1\}$ is the most symmetric object of the type we are looking for. To make this more precise, let us define for any subset X of $\mathbb{R}^3$

$$\mathrm{Sym}(X) := \{A \in \mathrm{SO}(3) \mid \forall x \in X : Ax \in X\}. \tag{2}$$

In this definition we do neither include reflections nor translations. Obviously, $\mathrm{Sym}(B) = \mathrm{SO}(3)$. Moreover, if we require that X is bounded and convex, then B is the only set which has as symmetry group the full $\mathrm{SO}(3)$.

There are other such sets X which look still very symmetric. These are the five platonic solids (polyhedrons). They are regular convex polyhedron. Regular means that the faces are congruent regular polygons, with the same number of faces meeting at each vertex. In fact the polyhedrons of Figure 1 are the only examples: the tetrahedron $Tetra$, the octahedron $Octo$, the cube $Cube$, the icosahedron $Icosa$ and the dodecahedron $Dodeca$.

The symmetry groups of these polyhedrons do not coincide with the full $\mathrm{SO}(3)$. They are related to very interesting subgroups of it. In fact, for every subset X of $\mathbb{R}^3$ the symmetry group will always be a subgroup of $\mathrm{SO}(3)$. In the extreme

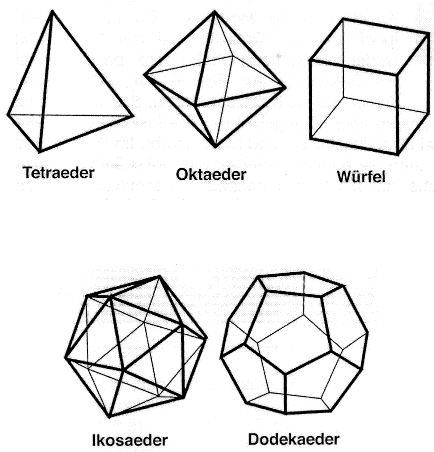

Figure 1: The five platonic solids.

case the subgroup will be the trivial subgroup. Meaning that X does not have any symmetry at all. The larger the subgroup is the more symmetric X is.

The symmetry groups of the platonic solids are quite interesting

$$\begin{aligned} \mathrm{Sym}(Tetra) &= T \cong A_4, && \#T = 12, \\ \mathrm{Sym}(Octo) = \mathrm{Sym}(Cube) &= O \cong S_4, && \#O = 24 \\ \mathrm{Sym}(Icosa) = \mathrm{Sym}(Dodeca) &= I \cong A_5, && \#I = 60 \end{aligned} \tag{3}$$

Here S_n is the symmetric group, consisting of the permutations of n elements, and A_n is the alternating group of n elements, i.e. the subgroup of S_n consisting of even permutations.

That both the octahedron and the cube and both the icosahedron and the dodecahedron respectively, have the same symmetry group has to do with the fact, that these pairs are dual to each others, meaning if one takes the middle points of the faces of the polyhedron as vertices of another polyhedron one obtains just the other element of the pair. Note that the tetrahedron is self-dual.

2.2 Crystals

Whereas above we considered bounded subsets of $\mathbb{R}^3$ we want to have a look on mathematical crystals. Mathematical crystals are discrete subsets of $\mathbb{R}^3$ (called a crystal lattice) which are invariant under a three-dimensional lattice of translations. Meaning there is a fundamental cell containing all informations and this fundamental cell is replicated in all three spacial directions, see Figure 2 for the two-dimensional situation.

Of course, real crystals are always cut in a finite region in the space. Let us look on possible symmetries of a mathematical crystal. The lattice is by the very

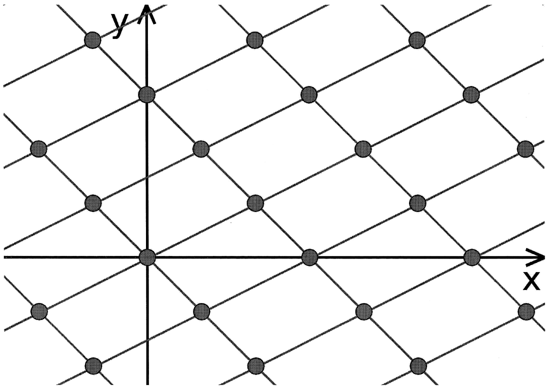

Figure 2: A two-dimensional crystal lattice.

definition a symmetry of the (mathematical) crystal. Which kind of lattices appear gives a coarse classification. The lattice could be an orthogonal lattice, where the basis lengths in all 3 spacial directions are the same, it could be a hexagonal lattice, etc. We will not discuss the complete classification here. Additionally the crystal could have other symmetries, like rotational symmetries of certain order around certain axes, and also reflections might be allowed. All these other symmetries have to be compatible with the translation symmetry coming from the lattice. A quite elementary geometric reasoning shows that only rotational axes of order 2, 3, 4 and 6 are possible.

If one has a look on the platonic solids and their symmetries again, one sees that both the icosahedron (and of course its dual the dodecahedron) have 5-fold rotational symmetry which is excluded in crystals. This corresponds to the fact that such polyhedron cannot be observed in nature as crystals. Of course, real crystals are always cut, but the possible arrangements of faces to the outside of the crystal is governed by the internal structure. There are certain crystals in nature who look nearly as they would be icosahedrons, but a closer inspection shows that they are always distorted.

Hence, it came as a big surprise that in 1984 Shechtman, Blech, Gratius, and Cahn were able to produce an Mn-Al alloy which had a ten-fold rotational symmetry in the refraction picture, see Figure 3, meaning a five-fold symmetry in the habitus [19]. In fact, the alloy exhibits an icosahedral symmetry. At the beginning some people did not believe them, but very recently in 2011 Shechtman got the Nobel prize in Chemistry for his discovery. Of course the question is: was the mathematics wrong. Fortunately the answer is no. What was wrong was the understanding of physicist who only thought that there are two structures of solid materials possible: the amorphous structure or the crystal structure in the above introduced sense. The work of Shechtman et al. showed that there are other structures possible. These new ones are nowadays called quasi-crystals. See e.g. the book of Senechal [18] for more mathematical background on quasi-crystals.

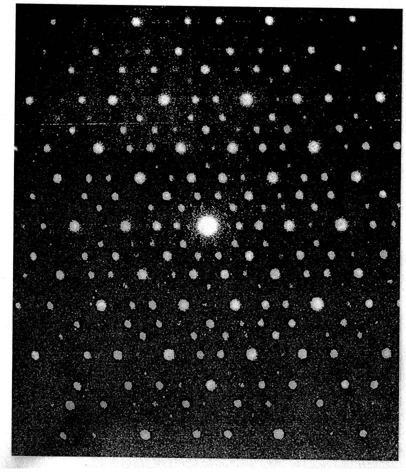

Figure 3: Refraction picture of a Mn-Al quasi-crystal [19].

3 Lie groups

The groups discussed in the previous section where either finite or infinite but discrete (the lattice of translations). We will discuss next continuous groups, this means groups which depend on continuous parameters. More precisely, we restrict our attention to the following extremely important subclass.

Definition 2. A *Lie group* is a group $(G,\cdot)$ for which G is also a differentiable manifold, such that the product and the inversion

$$(x,y) \mapsto x\cdot y\,, \qquad x \mapsto x^{-1} \tag{4}$$

are smooth maps, meaning that they are infinitely often differentiable.

Example 3. The standard, non-trivial example is the following. Let $\mathrm{GL}(n,\mathbb{R})$ be the set of invertible $n\times n$ real-valued matrices. If we take all square matrices then this set can be identified with the vector space $\mathbb{R}^{n\cdot n}$. Hence trivially the space of all square matrices is a differentiable manifold. Parameters are given by the entries a_{ij} in the matrix $A=(a_{ij})$. The subset $\mathrm{GL}(n,\mathbb{R})$ is given as complement of the closed subset consisting of matrices A with $\det(A)=0$. Hence it is an open subset of all matrices and as such a differentiable manifold. As group structure we take the matrix multiplication. Let $C=A\cdot B$ then

$$c_{ik}=\sum_{j=1}^{n} a_{ij}\cdot b_{jk}.$$

Consequently the parameters of the product are differentiable functions of the parameters of the individual factors. For an invertible matrix A its inverse will be given by

$$A^{-1}=1/(\det(A))\cdot {}^{\mathrm{ad}}.$$

Here, ${}^{\mathrm{ad}}A$ is the adjugate matrix of A. Its elements are as subdeterminants differentiable functions of the entries of A, the same is of course true for the determinant. As the determinant is different from zero the quotient is also differentiable with respect to the entries of A. Altogether we showed that $\mathrm{GL}(n,\mathbb{R})$ is a (real) Lie group. The same is true for complex-valued matrices $\mathrm{GL}(n,\mathbb{C})$, which is now also a complex Lie group.

This allows us to construct a lot of more examples. One can show that every closed subgroup of a Lie group is again a Lie group. Next we will deal again with the group SO(3), the (matrix) rotation group. The most conceptional definition is the definition that its elements are invariant with respect to the standard scalar product and do not change the orientation. This is equivalent to ${}^{t}A \cdot A = I$, the identity matrix, and $\det A = 1$. These conditions obviously define a closed submanifold and a subgroup of $\mathrm{GL}(3,\mathbb{R})$. Hence SO(3) (and more generally any $\mathrm{SO}(n)$) is a Lie group. It is of dimension 3 (as real differentiable manifold).

3.1 Examples of a symmetry and its consequences

The following example is taken from the book of Schottenloher [16] which contains much more examples – but unfortunately it is only available in German.

We are in $\mathbb{R}^3$ and let F be a *central force*. This means that

$$F : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}^3, \quad q \mapsto F(q) = \varphi(\|q\|)\frac{q}{\|q\|}, \quad \text{with} \quad \varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}. \tag{5}$$

In other words, the direction of the force F at a point q is always pointing to or from the origin $(0,0,0)$, and its strength depends only on the distance to the origin. The gravitational force or the electrostatic force are examples of such central forces. In both cases the force is given by

$$F(q) = -k\frac{q}{\|q\|^3}$$

where in the gravitational case k is always positive, in the electrostatic case its sign depends on the signs of the charges of the involved particles. But also oscillatory systems are given by a central force. E.g. the ideal pendulum is described by the central force $F(q) = -k \cdot q$ (even defined on all $\mathbb{R}^3$).

We denote by $q(t)$ a trajectory of a particle in the central force field. The variable t should be interpreted as time. We adopt the physicists' convention to denote time derivatives by $\dot{q} = \frac{dq}{dt}$. In this way we obtain the speed or velocity $v = \dot{q}$ as first derivative and the acceleration $a = \ddot{q}$ as second derivative.

Recall Newton's law of motion

$$F = m\ddot{q} = m\dot{v} = ma. \tag{6}$$

The task is to find the trajectory $q(t)$ starting from an initial position at time $t = 0$, i.e. $q(0) = q_0 \in \mathbb{R}^3$.

The important observation is that central forces are rotational symmetric, meaning

$$\forall A \in \mathrm{SO}(3),\ \forall q \in \mathbb{R}^3 : \quad F(A \cdot q) = A \cdot F(q). \tag{7}$$

This says that it does not matter whether we rotate before evaluation of the force or after evaluation [1]. Hence, if $q(t)$ is a solution of the equation of motion (6) then $Aq(t)$ is also a solution. We obtain an action of SO(3) on the set of solutions. If one takes the initial condition into account the solution $Aq(t)$ is a solution with initial value Aq_0. Hence, the subgroup of rotations leaving the initial vector q_0 invariant will act on the space of solution of the initial value problem. Knowing that a set of solutions admits a group action of a certain group already gives quite a lot of informations on the set, without knowing any of the solutions individually. This we will explain in more detail in the following.

The *angular momentum* is defined as

$$I := q \wedge mv. \tag{8}$$

Here $\wedge$ stands for the vector product in $\mathbb{R}^3$ (Sometimes also written as $\times$). Recall that the vector product of two parallel vectors will vanish. If we evaluate the angular momentum with respect to a given solution $q : [0,\infty[\to \mathbb{R}^3$ it will depend (a priori) on the time t,

$$I(q(t)) = q(t) \wedge mv(t) \tag{9}$$

(note $v(t) = \dot{q}(t)$). We will show that $I(q(t))$ will be constant with respect to t. The proof will use the fact that the F is a central force. For this goal we consider the (total) derivative of (9) with respect to t

$$\begin{aligned} \frac{d}{dt} I(q(t)) &= \dot{q}(t) \wedge mv(t) + q(t) \wedge m\dot{v}(t) \\ &= v(t) \wedge mv(t) + q(t) \wedge F(q(t)) = 0. \end{aligned}$$

Here we used Newton's Law, and the fact that F is central, hence $F(q)$ is parallel to q, and all vector product expressions will vanish.

This means that for a fixed solution $q(t)$ with initial value $q(0) = q_0$ the value of $I(q(t))$ will remain constant "along" the solution, i.e. $I(q(t)) = I(q(0)) = I(q_0)$. The angular momentum I is called an *integral of motion*. That it is a constant was forced by the existence of the rotational symmetry. Let me stress the fact, that its value will depend on the solution chosen.

For our system we have another integral of motion. Again it has to do with a symmetry. It is the invariance under time translation. If we release, let us say a particle in the field, at time t_0 or at $t_0 + c$ will not change the outcome (considered in the shifted time scale). It is the *energy*

$$E(q,v) := \frac{1}{2} mv^2 + U(q), \qquad F(q) = -\operatorname{grad} U(q), \tag{10}$$

which will be conserved. Here $U(q)$ is any function $\mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}$, such that $F(q) = -\operatorname{grad} U(q)$. It is called a potential of F. Not every force F admits a potential, but central forces do. U will be unique up to adding a constant. Important is that our force does not depend on the time t, this corresponds to the fact that the system is time translation invariant. We calculate

$$\frac{d}{dt} E = m \cdot v \cdot \dot{v} + (\operatorname{grad} U) \cdot \dot{q} = (m\dot{v} - F) \cdot v = 0.$$

1. In more fancy notation one might call F equivariant instead of invariant.

Here we used again Newton's law $F = m\dot{v}$. Consequently, the energy will stay constant along the particle trajectory.

In physics there is another conservation law the *conservation of momentum* $p = mv$. It corresponds to translation invariance in space. Our system is not translation invariant as the center is a singled out point. And obviously (in the gravitational field) a particle prepared with zero velocity at a point q_0 will start with zero momentum, but it will fall with higher and higher speed into the center. Hence, the momentum mv will increase.

Note that the movement is governed by a second order equation (Newton's law), which implies that both q_0 and v_0 need to be given to fix a solution. This corresponds to 6 parameters. From the fact that we have four independent integrals of motions (three components of I and in addition E) this reduces to two degrees of freedom.

There is a general treatment of continuous symmetries and related integrals of motions. This is the famous Noether Theorem [16].

4 Lie algebras

For the moment let $\mathbb{K}$ be a field with characteristics different from 2. Below we will only consider $\mathbb{R}$ or $\mathbb{C}$.

Definition 4. A *Lie algebra* over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space L endowed with a product (denoted by brackets)

$$[\ ,\ ] : L \times L \to L$$

which has the following properties (for all $x, y, z \in L$, $\lambda \in \mathbb{K}$)

1. bilinearity : $[x + \lambda z, y] = [x, y] + \lambda[z, y]$,
2. anti-symmetry : $[x, y] = -[y, x]$,
3. Jacobi identity : $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

Examples

1. Any vector space L can be considered as a Lie algebra by defining $[x, y] := 0$ for all $x, y \in L$. Such a Lie algebra is called abelian Lie algebra.
2. If $L = \mathbb{R}^3$, then taking $[x, y] := x \wedge y$, with the vector product $\wedge$ yields a Lie algebra.
3. If L is any associative algebra (where the product is denoted by $\cdot$), then the commutator $[x, y] := x \cdot y - y \cdot x$ defines a Lie algebra structure on the vector space L.
4. The set of smooth vector fields over a smooth manifold is a Lie algebra by taking the Lie bracket as Lie product.

4.1 Lie groups – Lie algebras

Lie groups are (at least in general) nonlinear objects. Whereas Lie algebras are linear objects. Linear objects are in general easier to deal with. In the finite-dimensional case of a Lie group there is a way to assign to every Lie group a corresponding Lie algebra. Here I will not introduce the general construction, but concentrate on some examples where the Lie groups are matrix groups. Let

$\mathrm{GL}(n,\mathbb{C})$ be the general linear group of invertible $n \times n$ complex-valued matrices and G any closed (continuous) subgroup, and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

$$\mathfrak{g} := \mathrm{Lie}\ G := \{X \in \mathrm{Mat}(n \times n, \mathbb{K}) \mid \forall t \in \mathbb{R} : \mathrm{e}^{tX} \in G\}. \tag{11}$$

Recall that the series

$$\mathrm{e}^{tX} = \sum_{k=0}^{\infty} \frac{t^k}{k!} X^k \tag{12}$$

converges for every $X \in \mathrm{Mat}(n \times n, \mathbb{K})$ and every $t \in \mathbb{R}$. The set Lie G is a vector subspace of all $n \times n$ matrices. The matrix commutator

$$[X, Y] := X \cdot Y - Y \cdot X \tag{13}$$

endows the space with a Lie algebra structure.

Example 5. In the case of the full general linear group, its Lie algebra $\mathfrak{gl}(n,\mathbb{K})$ is the set of all $n \times n$ matrices (non-invertible ones included).

Example 6. In the special case of $\mathrm{SO}(n)$, or $\mathrm{O}(n)$ we obtain as Lie algebra

$$\mathfrak{so}(n) = \mathfrak{o}(n) = \{X \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid {}^{\mathrm{t}}X + X = 0\}, \tag{14}$$

the skew-symmetric matrices. Indeed $A \in \mathrm{O}(n)$ if an only if is ${}^{\mathrm{t}}A \cdot A = I_n$. Using (11) we get as criterion for $X \in \mathfrak{o}(n)$ that

$$I_n = {}^{\mathrm{t}}(\mathrm{e}^{tX}) \cdot \mathrm{e}^{tX} = \mathrm{e}^{t({}^{\mathrm{t}}X + X)} \quad \forall t \in \mathbb{R}.$$

This is true if and only if ${}^{\mathrm{t}}X + X = 0$. Hence the left hand side of (14) is indeed the Lie algebra of $\mathrm{O}(n)$. For $\mathrm{SO}(n)$ we have to impose the additional condition that $\det(A) = 1$. But

$$\det(\mathrm{e}^{tX}) = \mathrm{e}^{t \cdot \mathrm{tr}(X)} = 1$$

is automatic as all skew-symmetric matrices have trace zero. In particular, we see by this example that the same Lie algebra might correspond to different Lie groups. In this case it comes from the fact that the elements of $\mathrm{O}(n)$ with determinant -1 cannot be "reached" via e^{tA}.

In the general functorial correspondence, the Lie algebra to a (finite-dimensional) Lie group is constructed as the set of left-invariant vector fields with Lie bracket given by the bracket of vector field. Equivalently, the Lie algebra can be realized on the tangent space of the Lie group (considered as differentiable manifold) at the unit element. In fact one can define a general exponential in a suitable manner. In the case of the above matrix groups the construction yield the same structure.

From the functorial point one can conclude that if the Lie group G has a representation on a vector space H, i.e. a group homomorphism $\varphi : G \to \mathrm{GL}(H)$, one can construct a Lie homomorphism $\mathrm{Lie}(\varphi) : \mathfrak{g} \to gl(H)$. If H is finite-dimensional and in the matrix case then

$$\mathrm{Lie}(\varphi)(X) = \frac{d}{dt} \varphi(\mathrm{e}^{tX})_{|t=0}. \tag{15}$$

The obtained representation Lie(φ) is the *infinitesimal* version of the representation.

An interesting point is, given a representation of the Lie algebra $\mathfrak{g}$ whether we can find a representation of the Lie group G yielding the given one as infinitesimal representation. This means can we *integrate* the representation. In general these are hard problems. In physics quite often one is satisfied with the infinitesimal version. In fact, quite often the fundamental physical model is only described locally on the infinitesimal level.

Another problem is that in infinite dimensions there is no such close relationship between the group and the algebra. There is no exponential map. In fact it is not even clear what manifold structure we should take on G. We will now turn to the the infinite dimensional situation and will only discuss Lie algebras.

5 From finite to infinite dimensional systems

In the examples we considered so far the systems were finite-dimensional. This means that we have a finite-dimensional system of parameters describing the system. The system of independent symmetries is also only "finite-dimensional". This means on the level of the Lie group it is a finite-dimensional Lie group and on the infinitesimal level it is a finite-dimensional Lie algebra. But there are other systems, for example systems related to fields, partial differential equations, etc., where we have infinitely many independent degrees of freedom. In this section I only want to indicate some important examples where in the infinite dimensional case symmetries also give important clues about the system.

5.1 KdV (Korteweg – de Vries) equation

The KdV equation describes wave propagation on the surface of shallow water like it appears in a canal. Let x be the essential spatial coordinate along the canal and t the time. The amplitude $u(x,t)$ fulfills the nonlinear partial differential equation

$$u_t - 6uu_x + u_{xxx} = 0. \tag{16}$$

Here as usual u_i denotes the partial derivative of u with respect to the direction i (either t or x). John Scott Russell, a Scottish naval engineer, observed in 1834 on a horse ride along a canal solitary traveling waves. They were traveling with different speed, overtaking each other, and keeping after the collision their form. This was a nonlinear phenomena. Russell himself was trying to analyze these solitary waves. But he could not convince all his contemporaries. In 1895 Korteweg and de Vries studied the phenomena mathematically in a satisfactory way and showed that Russell was right.

The described waves are nowadays called *solitons*, a name which comes from *solitary waves*. The deeper reason for this behaviour is that the KdV equation has infinitely many integrals of motions. The fact that they have to be conserved forces the soliton behaviour. Again these integrals of motions are related to symmetries. They are related to the Virasoro algebra which will be introduced in the next section [1].

5.2 Conformal Field Theory

Conformal Field Theories (CFTs) are field theories which are invariant under *conformal transformations*. A prominent example where two-dimensional CFTs play an important role is in string theory. There is huge amount of literature on the subject, both from the physics and from the mathematics side. We will not need the field theoretical aspects in the following. Hence I will not say much here. I refer to [17] for a mathematical treatment.

Here I mention only the following. Let $\mathbb{R}^n$ be the n-dimensional space equipped with the metric (g_{ab}). The *conformal coordinate transformations* are those coordinate transformations which leave the metric invariant up to a multiplicative (maybe varying) positive factor ρ. To be more precise, let $\xi = (\xi^a)$ be a system of coordinates and $\nu(\xi) = (\nu^b)$ new coordinates, then $\nu(\xi)$ is a conformal transformation if there exists $\rho : \mathbb{R}^n \to \mathbb{R}^{>0}$ such that

$$\sum_{a',b'} \frac{\partial \xi^{a'}}{\partial \nu_a} \frac{\partial \xi^{b'}}{\partial \nu_b} g_{a'b'} = \rho(\xi) g_{ab}. \tag{17}$$

There is a fundamental difference between space dimension $n = 2$ and $n > 2$. If $n > 2$ the "conformal group" is finite-dimensional, consisting essentially of translations, rotations, dilatations, and inversions. In particular, the Lie algebra corresponding to infinitesimal transformations, will be finite-dimensional. In dimension two if we describe $\mathbb{R}^2$ as complex plane $\mathbb{C}$ all analytic or anti-analytic maps are conformal maps. Hence, we have an infinite space of infinitesimal conformal coordinate transformations. It is the Witt algebra (Virasoro algebra without central term) introduced in the next section which is related to the infinitesimal transformations. For a discussion of the local-global problem in this context see [17].

6 Examples of infinite dimensional Lie algebras

In the following I will introduce the basic examples of infinite dimensional Lie algebras. In particular they appear in string theory and conformal field theory. But not only there. Typically they show up if we have systems of infinitely many degrees of freedoms. We already had a short look on the KdV partial differential equation.

6.1 Witt algebra

We start from the circle S^1. Consider $\mathrm{Diff}_+(S^1)$ the orientation preserving diffeomorphisms of the circle. It is an infinite dimensional group. To make it to some kind of a "Lie group" we have to consider topological questions. As we are not interested in the group but only in the algebra we will not go into details here. Instead we refer to the book [2] for a careful treatment.

Morally the Lie algebra of $\mathrm{Diff}_+(S^1)$ would be the set $\mathrm{Vect}(S^1)$ of vector fields on S^1. That is a complicated object, so we consider only the so called polynomial vector fields $\mathrm{Vect}_{poly}(S^1)$ of S^1. Strictly speaking, one should use the naming "vector fields with only finitely many Fourier modes". We take as basis

$$e_n := -\mathrm{i}\, \mathrm{e}^{(\mathrm{i} n\varphi)} \frac{d}{d\varphi}, \qquad n \in \mathbb{Z}. \tag{18}$$

The Lie bracket is the Lie bracket of vector fields. A direct calculation yields

$$[e_n, e_m] = (m - n) e_{n+m}. \tag{19}$$

This algebra is the Witt algebra over $\mathbb{R}$. In conformal field theory we have to consider its complexification

$$\mathcal{W} := \mathrm{Vect}_{\mathrm{poly}}(S^1) \otimes \mathbb{C}. \tag{20}$$

We can extend the vector fields on S^1 to the punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. In terms of the complex variable $z = r\mathrm{e}^{\mathrm{i}\varphi}$ the complexified basis elements are

$$e_n = z^{n+1}\frac{d}{dz}, \qquad n \in \mathbb{Z}. \tag{21}$$

The Lie product (19) remains unchanged. This (complex) algebra is the Witt algebra $\mathcal{W}$. Sometimes physicists call it also the Virasoro algebra without central extension [2], (now over $\mathbb{C}$).

After the complexification we can give a different (but related) interpretation of the algebra. We consider the Riemann sphere S^2 as Riemann surface with its complex structure. This means that $S^2 = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the compactified complex plane $\mathbb{C} = \mathbb{R}^2$. Equivalently, it is the complex projective line $\mathbb{P}^1(\mathbb{C})$.

Let z be the (quasi) global coordinate z (quasi, because it is not defined at ∞). Let $w = 1/z$ be the local coordinate at ∞. A global meromorphic vector field v on S^2 will be given on the corresponding subsets where z resp. w are defined as

$$v = \left(v_1(z)\frac{d}{dz}\,,\; v_2(w)\frac{d}{dw}\right), \qquad v_2(w) = -v_1(z(w))w^2. \tag{22}$$

It is clear that from the knowledge of v_1 the whole vector field v will follow. Hence, we will usually just write down v_1 and in fact identify the vector field v with its local representing function v_1, which we will denote by the same letter. For the bracket we calculate

$$[v,u] = \left(v\frac{d}{dz}u - u\frac{d}{dz}v\right)\frac{d}{dz}. \tag{23}$$

All meromorphic vector fields constitute a Lie algebra. The subspace of those meromorphic vector fields which are holomorphic outside of $\{0,\infty\}$ is a Lie subalgebra. Its elements can be given as

$$v(z) = f(z)\frac{d}{dz} \tag{24}$$

where f is a meromorphic function on S^2, which is holomorphic outside $\{0,\infty\}$. Those are exactly the Laurent polynomials $\mathbb{C}[z,z^{-1}]$. Consequently, this subalgebra has the set $\{e_n, n \in \mathbb{Z}\}$ as basis elements. The Lie product is the same and it can be identified with the Witt algebra $\mathcal{W}$.

The subalgebra of global holomorphic vector fields is $\langle e_{-1}, e_0, e_1\rangle_{\mathbb{C}}$. It is isomorphic to the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.

2. In fact the story with the naming is rather confusing. It gives another example of the principle “use a person for the name which has no or only minor relation to the object”. Witt considered the corresponding algebra in the context of characteristics p, and there the algebra looks different. See the book [2] for certain remarks on the issue.

The algebra $\mathcal{W}$ is more than just a Lie algebra. It is a graded Lie algebra. If we set for the degree $\deg(e_n) := n$ then

$$\mathcal{W} = \bigoplus_{n\in\mathbb{Z}} \mathcal{W}_n, \qquad \mathcal{W}_n = \langle e_n \rangle_{\mathbb{C}}. \tag{25}$$

Obviously, $\deg([e_n, e_m]) = \deg(e_n) + \deg(e_m)$. Note that $[e_0, e_n] = n\, e_n$, which says that the degree decomposition is the eigen-space decomposition with respect to the adjoint action of e_0 on $\mathcal{W}$. I close this section by pointing out that algebraically $\mathcal{W}$ can also be given as Lie algebra of derivations of the algebra of Laurent polynomials $\mathbb{C}[z, z^{-1}]$.

6.2 Current algebras

Now I will introduce another important class of symmetry algebras. They appear as so-called *gauge algebras*. The starting point is a finite-dimensional complex Lie algebra $\mathfrak{g}$. One considers $\mathfrak{g}$-valued meromorphic maps on S^2 which are holomorphic outside of $\{0, \infty\}$. They can be given algebraically as elements of the set

$$\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]. \tag{26}$$

With the Lie product

$$[x \otimes f, y \otimes g] := [x, y] \otimes (f \cdot g), \tag{27}$$

the space $\overline{\mathfrak{g}}$ becomes an infinite dimensional Lie algebra. Again we have a graded structure by setting $\deg(x \otimes z^n) := n$. These Lie algebras are called *current algebras* or sometimes *loop algebras*.

7 Representations

We return to our Witt algebra $\mathcal{W}$. Representations (also called $\mathcal{W}$-modules) are important. As explained, if we have a symmetry algebra, let us say of a PDE, then the solution space carries a representation of the symmetry algebra. Hence, knowing the representations of $\mathcal{W}$ will help to understand the space of solutions of a system which has $\mathcal{W}$ as symmetry algebra.

Some naturally defined representations are given by the forms $\mathcal{F}^\lambda$ of weight λ. Here λ is either an integer or a half-integer. Again we consider meromorphic forms which are holomorphic outside of $\{0, \infty\}$. Such elements can be described by $f(z)(dz)^\lambda$. The elements $e \in \mathcal{W}$ operate on $\mathcal{F}^\lambda$ by taking the Lie derivative

$$\left(e(z)\frac{d}{dz}\right).\,(f(z)(dz)^\lambda) := \left(e(z)\frac{df}{dz}(z) + \lambda \cdot f(z)\frac{de}{dz}(z)\right)(dz)^\lambda. \tag{28}$$

The space $\mathcal{F}^\lambda$ becomes a Lie module over $\mathcal{W}$

$$[e, h].f = e.(h.f) - h.(e.f). \tag{29}$$

In $\mathcal{F}^\lambda$ we introduce the basis

$$f_n^\lambda(z) = z^{n-\lambda}(dz)^\lambda, \quad n \in \mathbb{Z}. \tag{30}$$

In case that λ is a half-integer the index n will run over the half-integers. In terms of the basis the module structure calculates to

$$e_n \,.\, f_m^\lambda = (m + \lambda n) f_{n+m}^\lambda. \tag{31}$$

Note that for $\lambda = -1$ we obtain back $\mathcal{W}$.

In this way we get nice, naturally defined, infinite-dimensional representations. The problem is that in quantum physics we need representations which are generated from ground states, the vacuum states. We need creation and annihilation operators. The above introduced representation spaces are not of this type, there is no ground state.

7.1 Semi-infinite wedge representations

To obtain representations which behave in the required manner we consider something similar to exterior forms in the finite dimensional case. We start with $\mathcal{F}^\lambda$ for a fixed weight λ and its system of basis elements f_n^λ. To avoid cumbersome notation we drop the λ. Next we consider formal expressions of the type

$$\Phi = f_{i_1} \wedge f_{i_2} \wedge \cdots \wedge f_r \wedge f_{r+1} \wedge \cdots \tag{32}$$

with strictly increasing indices

$$i_1 < i_2 < i_3 < \cdots < r < r+1 < \cdots .$$

Starting from a certain position (which depends on the individual Φ) it is strictly increasing by 1. The *semi-infinite wedge space* $\mathcal{H}^\lambda$ is the vector space freely generated by all such formal expressions. Also the term *fermionic Fock space* is used. Special elements are given by

$$\phi_T = f_T \wedge f_{T+1} \wedge f_{T+2} \wedge \cdots . \tag{33}$$

These elements are called *vacuum of level* T.

We extend the action of $e \in \mathcal{W}$ from $\mathcal{F}^\lambda$ to $\mathcal{H}^\lambda$ by Leibniz rule

$$\begin{aligned} e\,.\,\Phi = (e\,.\,f_{i_1}) \wedge \quad & f_{i_2} \wedge \cdots \wedge \quad f_r \wedge f_{r+1} \wedge \cdots \\ + \quad f_{i_1} \wedge & (e\,.\,f_{i_2}) \wedge \cdots \wedge \quad f_r \wedge f_{r+1} \wedge \cdots \\ & \vdots \\ + \quad f_{i_1} \wedge & f_{i_2} \wedge \cdots \wedge (e\,.\,f_r) \wedge f_{r+1} \wedge \cdots \\ & \vdots \end{aligned} \tag{34}$$

Now the symbol $\wedge$ comes into play. If in the result the increasing order is violated we interchange the entries accordingly and take up sign changes. If two entries are the same the result will be zero.

Next we study the effect of the action of the basis elements e_k, $k \in \mathbb{Z}$ of $\mathcal{W}$ in more detail. As long as $k \neq 0$ there will be only finitely many summands in (34) as for a fixed k the regular tail (indices increasing by 1) will take care that finally all the appearing summands will be zero. Just to give examples

$$e_1\,.\,\Phi_0 = (e_1\,.\,f_0) \wedge f_1 \cdots + \cdots = (\lambda f_1) \wedge f_1 + \cdots = 0$$

$$e_{-1}\,.\,\Phi_0 = (e_{-1}\,.\,f_0) \wedge f_1 \cdots + f_0 \wedge (e_{-1}\,.\,f_1) \wedge \cdots = -\lambda f_{-1} \wedge f_1 \wedge \cdots$$

In fact we obtain that

$$e_n \Phi_T = 0, \qquad \forall n \geq 1. \tag{35}$$

This means that the subalgebra $\mathcal{W}_+ = \langle e_k \mid k \geq 1 \rangle$ annihilates the vacuum Φ_T.

The only problem is the action of e_0. Recall that $e_0 \,.\, f_n = n f_n$. This means the entry f_n reproduces itself with a certain scalar. Hence we obtain

$$e_o \,.\, \Phi_0 = (\sum_{k=0}^{\infty} k)\, \Phi_0, \tag{36}$$

which is clearly not defined. As physicists say, one has to regularize the action, or one has to subtract $\infty \cdot \Phi_0$. But this has to be done in a coherent manner. One way is to put

$$\infty := \sum_{k=0}^{\infty} k \quad \text{and then} \qquad \sum_{m}^{\infty} k - \infty = -\sum_{k=0}^{m-1} k. \tag{37}$$

This heuristic formula can be made mathematically perfectly rigorous (see [4]). If we denote the modified action with $\tilde{e}_0$, we get for the convention (37)

$$\tilde{e}_0 \Phi_0 = 0 \cdot \Phi_0, \quad \tilde{e}_0 \Phi_T = \left(-\sum_{k=0}^{m-1} k \right) \Phi_T. \tag{38}$$

The actions of the other e_k will not change.

Now the action is well-defined but we have the problem that it is not a representation of the Witt algebra anymore. This we will show at an example In $\mathcal{W}$ we have the relation $[e_{-2}, e_2] = 4e_0$. Let us check whether for $\lambda = 0$ (as would be necessary) we have

$$e_{-2} \,.\, (e_2 \,.\, \Phi_0) - e_2 \,.\, (e_{-2} \,.\, \Phi_0) = 4(\tilde{e}_0 \,.\, \Phi_0).$$

As Φ_0 is annihilated by e_2 the first summand on the left hand side will vanish. Also by the definition of the action of $\tilde{e}_0$ the right hand side will vanish. The second summand on the left hand side gives

$$-e_2 \,.\, (e_{-2} \,.\, \Phi_0) = \Phi_0.$$

Hence, obviously both side do not coincide. But in any case the difference of both sides will be only a scalar multiple of the identity. This means it is a projective action. It might be the hope that if we add an element to the algebra which commutes with all other elements and define that the additional element acts by a fixed scalar times the identity that we obtain back a honest action of the extended algebra. Indeed in the described case this work. The extended algebra will be the centrally extended Witt algebra which is the Virasoro algebra to be discussed in the next section. I like to stress the fact, that we were forced to pass to a centrally extended Witt algebra due to the fact that we need to regularize a naturally given action.

Also I have to point out that the expression for "∞" in (37) was to a certain extend arbitrary. We could have started with e.g. $k = 1$ in the infinite sum. Then also the modification of the action will be different. This ambiguity will appear in the next section again.

8 The Virasoro algebra

The *Virasoro algebra* $\mathcal{V}$ is a one-dimensional non-trivial central extension of the Witt algebra $\mathcal{W}$. As vector space it is the direct sum $\mathcal{V} = \mathbb{C} \oplus \mathcal{W}$. If we set for $x \in \mathcal{W}$, $\hat{x} := (0, x)$, and $t := (1, 0)$ then its basis elements are $\hat{e}_n$, $n \in \mathbb{Z}$ and t. The Lie product is given by

$$[\hat{e}_n, \hat{e}_m] = (m-n)\hat{e}_{n+m} - \frac{1}{12}(n^3 - n)\delta_n^{-m}\, t, \qquad [\hat{e}_n, t] = [t, t] = 0, \tag{39}$$

for all $n, m \in \mathbb{Z}$ [3]. If we set $\deg(\hat{e}_n) := \deg(e_n) = n$ and $\deg(t) := 0$ then $\mathcal{V}$ becomes a graded algebra. The algebra $\mathcal{W}$ will only be a subspace, not a subalgebra of $\mathcal{V}$. But it will be in a natural way a quotient.

There is a general theory of central extensions valid for every Lie algebra. The crucial point is the expression coming with the central element t. Let us denote this by c. The c will depend bilinearly on e_n and e_m. In the above case it will be

$$c(e_n, e_m) = -\frac{1}{12}(n^3 - n)\delta_n^{-m}. \tag{40}$$

Using the condition that $\mathcal{V}$ should be a Lie algebra, c has to be an anti-symmetric bilinear form on $\mathcal{W}$, fulfilling the Lie algebra cohomology two-cocycle condition

$$c([x, y], z) + c([y, z], x) + c([z, x], y) = 0. \tag{41}$$

A cocycle c is called a coboundary if there exists a linear map $\varphi : \mathcal{W} \to \mathbb{C}$ such that

$$c(x, y) = \varphi([x, y]). \tag{42}$$

Two cocycles are cohomologous if their difference is a coboundary. The quotient space cocycles modulo coboundaries is called cohomology space and denoted by $\mathrm{H}^2(\mathfrak{g}, \mathbb{C})$.

There is a notion of equivalence of central extensions of a given Lie algebra $\mathfrak{g}$ (in our example $\mathfrak{g} = \mathcal{W}$). Instead of taking $\hat{x}$ to be $(0, x)$ as element corresponding to $x \in \mathfrak{g}$ we could also take $\hat{x} = (\varphi(x), x)$ with $\varphi : \mathfrak{g} \to \mathbb{C}$ a linear map. In this way we obtain what is called an equivalent central extension. Equivalent central extensions are always isomorphic. How we introduced them they correspond just to a change of basis of certain type. On the level of the cohomology this correspond to the fact that the cocycles differ by a coboundary. The cohomology space $\mathrm{H}^2(\mathfrak{g}, \mathbb{C})$ classifies central extensions up to equivalence.

In fact, one can show that for the Witt algebra $\mathrm{H}^2(\mathcal{W}, \mathbb{C})$ is one-dimensional. This means that up to equivalence and rescaling the central element there is only one non-trivial central extension of the Witt algebra $\mathcal{W}$ and this is the Virasoro algebra $\mathcal{V}$. One can even show that $\mathcal{V}$ is *the universal central extension* meaning that every one-dimensional central extension is a surjective image of $\mathcal{V}$. Also note that one always has the trivial central extension which is just the direct sum of the Lie algebra with the one-dimensional abelian Lie algebra.

Now we can resolve the problem with the ambiguity of the regularisation. If we choose a different "∞" then we will obtain a cocycle which will be different, e.g. from the one given in (39). But it will define an equivalent central extension.

3. Here δ_k^l is the Kronecker delta which is equal to 1 if $k = l$, otherwise zero.

In any case the semi-infinite wedge representation will define a cocycle after regularization. Hence up to equivalence (meaning a change of basis) the given projective representation of $\mathcal{W}$ will give a linear representation of the Virasoro algebra.

9 Affine Lie algebras

Also for the current algebra $\overline{\mathfrak{g}}$ associated to a finite-dimensional Lie algebra $\mathfrak{g}$ there exists central extensions. As discussed in the previous section central extensions are given by Lie algebra two-cocycles ψ. Let β be an invariant, symmetric bilinear form for $\mathfrak{g}$, then a two-cocycle for $\overline{\mathfrak{g}}$ is given by

$$\psi(x \otimes z^n, y \otimes z^m) = \beta(x,y) \cdot m \cdot \delta_n^{-m}. \tag{43}$$

If $\mathfrak{g}$ is a simple Lie algebra, then up to equivalence and rescaling there is only one non-trivial central extension. Note also that in this case there is only one (up to rescaling) β, which is given by the Cartan-Killing form.

It has to be remarked that the central extension in the simple case will become trivial if restricted to the finite-dimensional Lie algebra $\mathfrak{g}$, as the corresponding cohomology space $\mathrm{H}^2(\mathfrak{g},\mathbb{C}) = \{0\}$, due to the second Whitehead lemma in Lie algebra cohomology.

9.1 Sugawara construction

As already mentioned in this write-up the symmetry given by $\overline{\mathfrak{g}}$ (or the centrally extended algebra $\widehat{\mathfrak{g}}$) corresponds to gauge symmetry of the conformal field theory. Let $\mathfrak{g}$ be either simple or abelian. In this case there is a relation between the gauge symmetry algebra and the centrally extended conformal symmetry algebra, the Virasoro algebra. The relation is via the Sugawara construction. Let V be an admissible representation of $\widehat{\mathfrak{g}}$. This means

1. the central element operates as $c \cdot id$, (c is called the central charge of the representation)
2. every element $v \in V$ will be annihilated by the elements of $\widehat{\mathfrak{g}}$ of sufficiently high degree (the degree depends on v).

Let u_i, $i = 1,\dots,\dim\mathfrak{g}$ be a basis of $\mathfrak{g}$ and u^i the dual basis with respect to β. The k-th *Sugawara operator* is defined as

$$T_k := \frac{1}{2}\sum_{j\in\mathbb{Z}}\sum_{i=1}^{\dim\mathfrak{g}} :(u_i \otimes z^{-j})(u^i \otimes z^{j+k}): \ . \tag{44}$$

This looks like a formal sum, but here the elements $(u_i\otimes z^{-j})$ have to be considered as operators on V. Furthermore $:\ a\cdot b\ :$ denotes normal ordering of the operators a and b. This means that if the degree of the element a is greater than the degree of b (here in our case if $-j > j+k$) then the product has to be switched to $b \cdot a$. In this way the annihilation operators will be brought to the right to act first. By the normal ordering $T_k v$ will be well-defined for all v and all k.

Let κ be the dual Coxeter number. If you do not know its definition, do not mind. It will be a non-negative number. If you are interested in getting the definition, see [3]. Recall that c was the central charge. Then one shows that

$$[T_n, T_k] = (c+\kappa)(k-n)T_{n+k} + \delta_n^{-k}\frac{n^3-n}{12}\dim\mathfrak{g}\cdot(c+k)\cdot id. \qquad (45)$$

For $c+\kappa \neq 0$ we can rescale the operators and obtain a representation of the Virasoro algebra with central charge

$$-\frac{c\cdot\dim\mathfrak{g}}{c+\kappa}. \qquad (46)$$

Here as in the case of the semi-infinite wedge representations a regularization procedure was necessary to make the representation work. Without the normal ordering prescription the T_0 would not be a well-defined operator. Hence, we are forced again to consider the Virasoro algebra and not only the Witt algebra. Again there is the ambiguity due to the precise form of the normal ordering which can be changed. This ambiguity corresponds to a cohomologous change of the cocycle.

I like to close this section by mentioning that the affine Lie algebras are in the classification of Kac and Moody, the untwisted affine Kac-Moody algebras. If you want to learn more about them you should study the book of Kac [3].

10 Krichever-Novikov type algebras

The Witt, Virasoro, and affine Lie algebra case is the genus zero situation, i.e. where we consider the Riemann sphere (which is the unique Riemann surface of genus zero), and our objects are allowed to have poles at the two points $\{0,\infty\}$. From the application in CFT we have to look for generalizations to (compact) Riemann surfaces of arbitrary genus. Also one needs to allow poles at a finite, but arbitrary number of marked points. In the interpretation of CFT they correspond to incoming fields and outgoing fields. Hence, the set of marked points will have a natural decomposition into two disjoint subsets.

More precisely, let Σ be a compact Riemann surface of genus g, and $N,K\in\mathbb{N}$ with $N\geq 2$ and $1\leq K<N$. Fix

$$I = (P_1,\ldots,P_K), \quad \text{and} \quad O = (Q_1,\ldots,Q_{N-K}) \qquad (47)$$

disjoint ordered tuples of distinct points ("marked points", "punctures") on the Riemann surface. In particular, we assume $P_i \neq Q_j$ for every pair (i,j). The points in I are called the *in-points*, the points in O the *out-points*.

The Figures 4, 5, 6 show different geometric situations.

The global objects, algebras, structures, ... will be meromorphic objects on the Riemann surface Σ which are holomorphic outside of the points in A.

The grading of the Witt and Virasoro algebra is very important for the construction of certain type of representations (e.g. the vacuum representations). For our generalized situation, we do not have an honest grading, but only an almost-grading (for the definition I refer to the quoted work below). The almost-grading will be induced by the splitting of A into I and O. Krichever and Novikov introduced it in the case where A has only two points. In this case the splitting is uniquely given. I gave the generalization to arbitrary (finite) sets A and splittings in Refs. [10], [11], [9], [12], [13]. In the case of two points my description gives

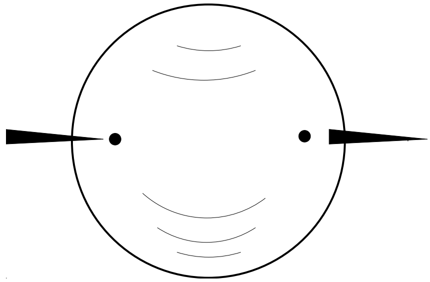

Figure 4: Riemann surface of genus zero with one incoming and one outgoing point.

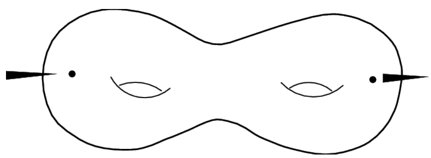

Figure 5: Riemann surface of genus two with one incoming and one outgoing point.

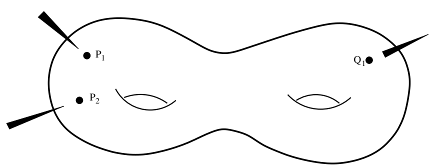

Figure 6: Riemann surface of genus two with two incoming and one outgoing point.

back the Krichever-Novikov description. If one specializes even more to $g = 0$ and A consisting of two points, which we might assume to be $\{0\}$ and $\{\infty\}$, i.e.

$$\Sigma = \mathbb{P}^1(\mathbb{C}) = S^2, \quad I = \{z = 0\}, \quad O = \{z = \infty\} \tag{48}$$

then we obtain back the well-known algebras of Conformal Field Theory discussed in the previous sections, like the Witt, Virasoro, and affine algebra. Here a lot of things could be added, but for this I have to refer to the forthcoming book [8]. For the application of Krichever-Novikov type algebras to CFT see also [15], [14] and [20].

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