Weighted lattice polynomials

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Abstract

We define the concept of weighted lattice polynomial functions as lattice polynomial functions constructed from both variables and parameters. We provide equivalent forms of these functions in an arbitrary bounded distributive lattice. We also show that these functions include the class of discrete Sugeno integrals and that they are characterized by a median based decomposition formula.

Key words: weighted lattice polynomial, lattice polynomial, bounded distributive lattice, discrete Sugeno integral.

1 Introduction

In lattice theory, *lattice polynomials* have been defined as well-formed expressions involving variables linked by the lattice operations $\wedge$ and $\vee$ in an arbitrary combination of parentheses; see for instance Birkhoff [2, §II.5] and Grätzer [4, §I.4]. In turn, such expressions naturally define *lattice polynomial functions*. For example,

$$p(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee x_3$$

is a 3-ary (ternary) lattice polynomial function.

The concept of lattice polynomial function can be straightforwardly generalized by fixing some variables as "parameters", like in the 2-ary (binary) polynomial function

$$p(x_1, x_2) = (c \vee x_1) \wedge x_2,$$

where c is a constant element of the underlying lattice.

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In this paper we investigate those "parameterized" polynomial functions, which we shall call *weighted lattice polynomial* (w.l.p.) functions. More precisely, we show that, in any bounded distributive lattice, w.l.p. functions can be expressed in disjunctive and conjunctive normal forms and we further investigate these forms in the special case when the lattice is totally ordered. We also show that w.l.p. functions include the discrete Sugeno integral [9], which has been extensively studied and used in the setting of nonlinear aggregation and integration. Finally, we prove that w.l.p. functions can be characterized by means of a median based system of functional equations.

Throughout, we let L denote an arbitrary bounded distributive lattice with lattice operations $\wedge$ and $\vee$. We denote respectively by 0 and 1 the bottom and top elements of L. For any integer $n \geqslant 1$, we set $[n] := \{1, \ldots, n\}$ and, for any $S \subseteq [n]$, we denote by $\mathbf{e}_S$ the characteristic vector of S in $\{0,1\}^n$, that is, the n-dimensional vector whose ith component is 1, if $i \in S$, and 0, otherwise. Finally, since L is bounded,

$$\bigvee_{x \in \varnothing} x = 0 \quad \text{and} \quad \bigwedge_{x \in \varnothing} x = 1.$$

2 Weighted lattice polynomial functions

Before introducing the concept of w.l.p. function, let us recall the definition of lattice polynomial functions; see for instance Grätzer [4, §I.4].

Definition 1 *The class of lattice polynomial functions from L^n to L is defined as follows:*

(1) For any $k \in [n]$, the projection $(x_1, \ldots, x_n) \mapsto x_k$ is a lattice polynomial function from L^n to L.
(2) If p and q are lattice polynomial functions from L^n to L, then $p \wedge q$ and $p \vee q$ are lattice polynomial functions from L^n to L.
(3) Every lattice polynomial function from L^n to L is constructed by finitely many applications of the rules (1) and (2).

We now recall that, in a distributive lattice, any lattice polynomial function can be written in disjunctive and conjunctive normal forms, that is, as a join of meets and dually; see for instance Birkhoff [2, §II.5].

Proposition 2 *Let $p : L^n \to L$ be any lattice polynomial function. Then there are integers $k, l \geqslant 1$ and families $\{A_j\}_{j=1}^k$ and $\{B_j\}_{j=1}^l$ of nonempty subsets of $[n]$ such that*

$$p(\mathbf{x}) = \bigvee_{j=1}^{k} \bigwedge_{i \in A_j} x_i = \bigwedge_{j=1}^{l} \bigvee_{i \in B_j} x_i.$$

Equivalently, there are nonconstant set functions $\alpha : 2^{[n]} \to \{0,1\}$ *and* $\beta : 2^{[n]} \to \{0,1\}$, *with* $\alpha(\varnothing) = 0$ *and* $\beta(\varnothing) = 1$, *such that*

$$p(\mathbf{x}) = \bigvee_{\substack{S\subseteq[n]\\ \alpha(S)=1}} \bigwedge_{i\in S} x_i = \bigwedge_{\substack{S\subseteq[n]\\ \beta(S)=0}} \bigvee_{i\in S} x_i.$$

As mentioned in the introduction, the concept of lattice polynomial function can be generalized by fixing some variables as parameters. Based on this observation, we naturally introduce the class of w.l.p. functions as follows.

Definition 3 *The class of w.l.p. functions from* L^n *to* L *is defined as follows:*

(1) For any $k \in [n]$ *and any* $c \in L$, *the projection* $(x_1, \ldots, x_n) \mapsto x_k$ *and the constant function* $(x_1, \ldots, x_n) \mapsto c$ *are w.l.p. functions from* L^n *to* L.
(2) If p *and* q *are w.l.p. functions from* L^n *to* L, *then* $p \wedge q$ *and* $p \vee q$ *are w.l.p. functions from* L^n *to* L.
(3) Every w.l.p. function from L^n *to* L *is constructed by finitely many applications of the rules (1) and (2).*

Remark 4 *Thus defined, w.l.p. functions are simply, in the universal algebra terminology, those functions which are definable by polynomial expressions; see for instance Kaarli and Pixley [5] and Lausch and Nöbauer [6]. Furthermore, these functions are clearly nondecreasing in each variable.*

Using Proposition 2, we can easily see that any w.l.p. function can be written in disjunctive and conjunctive normal forms (see also Lausch and Nöbauer [6] and Ovchinnikov [8]).

Proposition 5 *Let* $p : L^n \to L$ *be any w.l.p. function. Then there are integers* $k, l \geqslant 1$, *parameters* $a_1, \ldots, a_k, b_1, \ldots, b_l \in L$, *and families* $\{A_j\}_{j=1}^k$ *and* $\{B_j\}_{j=1}^l$ *of subsets of* $[n]$ *such that*

$$p(\mathbf{x}) = \bigvee_{j=1}^{k} \Big(a_j \wedge \bigwedge_{i\in A_j} x_i\Big) = \bigwedge_{j=1}^{l} \Big(b_j \vee \bigvee_{i\in B_j} x_i\Big).$$

Equivalently, there exist set functions $\alpha : 2^{[n]} \to L$ *and* $\beta : 2^{[n]} \to L$ *such that*

$$p(\mathbf{x}) = \bigvee_{S\subseteq[n]} \Big(\alpha(S) \wedge \bigwedge_{i\in S} x_i\Big) = \bigwedge_{S\subseteq[n]} \Big(\beta(S) \vee \bigvee_{i\in S} x_i\Big).$$

It follows from Proposition 5 that any n-ary w.l.p. function is entirely determined by 2^n parameters.

Remark 6 *Proposition 5 naturally includes the lattice polynomial functions.*

To see this, it suffices to consider nonconstant set functions $\alpha : 2^{[n]} \to \{0,1\}$ *and* $\beta : 2^{[n]} \to \{0,1\}$*, with* $\alpha(\varnothing) = 0$ *and* $\beta(\varnothing) = 1$*.*

3 Disjunctive and conjunctive normal forms

We now investigate the link between a given w.l.p. function and the parameters that define it.

Let us denote by $p^{\vee}_{\alpha}$ (resp. $p^{\wedge}_{\beta}$) the w.l.p. function disjunctively (resp. conjunctively) defined by the set function $\alpha : 2^{[n]} \to L$ (resp. $\beta : 2^{[n]} \to L$), that is,

$$p^{\vee}_{\alpha}(\mathbf{x}) := \bigvee_{S\subseteq[n]} \Big(\alpha(S) \wedge \bigwedge_{i\in S} x_i\Big),$$
$$p^{\wedge}_{\beta}(\mathbf{x}) := \bigwedge_{S\subseteq[n]} \Big(\beta(S) \vee \bigvee_{i\in S} x_i\Big).$$

Of course, the set functions α and β are not uniquely determined. For instance, both expressions $x_1 \vee (x_1 \wedge x_2)$ and x_1 represent the same lattice polynomial function.

For any w.l.p. function $p : L^n \to L$, define the set functions $\alpha_p : 2^{[n]} \to L$ and $\beta_p : 2^{[n]} \to L$ as $\alpha_p(S) := p(\mathbf{e}_S)$ and $\beta_p(S) := p(\mathbf{e}_{[n]\setminus S})$ for all $S \in [n]$. Since p is nondecreasing, α_p is isotone and β_p is antitone.

Lemma 7 *For any w.l.p. function* $p : L^n \to L$ *we have* $p = p^{\vee}_{\alpha_p} = p^{\wedge}_{\beta_p}$*.*

Proof. Let us establish the first equality. The other one can be proved similarly.

By Proposition 5, there exists a set function $\alpha : 2^{[n]} \to L$ such that $p = p^{\vee}_{\alpha}$. It follows that

$$\alpha_p(T) = \bigvee_{S\subseteq T} \alpha(S) \qquad (T \subseteq [n]).$$

Therefore, we have

$$
\begin{aligned}
p^{\vee}_{\alpha_p}(\mathbf{x}) &= \bigvee_{T\subseteq[n]} \Big(\alpha_p(T) \wedge \bigwedge_{i\in T} x_i\Big) \;=\; \bigvee_{T\subseteq[n]} \Big(\bigvee_{S\subseteq T} \alpha(S) \wedge \bigwedge_{i\in T} x_i\Big)\\
&= \bigvee_{T\subseteq[n]} \bigvee_{S\subseteq T} \Big(\alpha(S) \wedge \bigwedge_{i\in T} x_i\Big) \;=\; \bigvee_{S\subseteq[n]} \bigvee_{T\supseteq S} \Big(\alpha(S) \wedge \bigwedge_{i\in T} x_i\Big)\\
&= \bigvee_{S\subseteq[n]} \Big(\alpha(S) \wedge \bigvee_{T\supseteq S} \bigwedge_{i\in T} x_i\Big) \;=\; \bigvee_{S\subseteq[n]} \Big(\alpha(S) \wedge \bigwedge_{i\in S} x_i\Big)\\
&= p(\mathbf{x}). \qquad \square
\end{aligned}
$$

It follows from Lemma 7 that any n-ary w.l.p. function is entirely determined by its restriction to $\{0,1\}^n$.

Assuming that L is a chain (that is, L is totally ordered), we now describe the class of all set functions that disjunctively (or conjunctively) define a given w.l.p. function.

Proposition 8 *Assume that L is a chain. Let $p : L^n \to L$ be any w.l.p. function and consider two set functions $\alpha : 2^{[n]} \to L$ and $\beta : 2^{[n]} \to L$.*

(1) We have $p^{\vee}_{\alpha} = p$ if and only if $\alpha^*_p \leqslant \alpha \leqslant \alpha_p$, where the set function $\alpha^*_p : 2^{[n]} \to L$ is defined as

$$
\alpha^*_p(S) = \begin{cases} \alpha_p(S), & \text{if } \alpha_p(S) > \alpha_p(S \setminus \{i\}) \text{ for all } i \in S,\\ 0, & \text{otherwise.} \end{cases}
$$

(2) We have $p^{\wedge}_{\beta} = p$ if and only if $\beta_p \leqslant \beta \leqslant \beta^*_p$, where the set function $\beta^*_p : 2^{[n]} \to L$ is defined as

$$
\beta^*_p(S) = \begin{cases} \beta_p(S), & \text{if } \beta_p(S) < \beta_p(S \setminus \{i\}) \text{ for all } i \in S,\\ 1, & \text{otherwise.} \end{cases}
$$

Proof. Let us prove the first assertion. The other one can be proved similarly.

($\Rightarrow$) Assume $p^{\vee}_{\alpha} = p$ and fix $S \subseteq [n]$. On the one hand, we have

$$
0 \leqslant \alpha(S) \leqslant \bigvee_{K\subseteq S} \alpha(K) = \alpha_p(S).
$$

On the other hand, if $\alpha_p(S) > \alpha_p(S \setminus \{i\})$ for all $i \in S$, then $\alpha(S) = \alpha_p(S)$. Indeed, otherwise, since L is a chain, there would exist $K^* \subsetneq S$ such that

$$
\alpha_p(S) = \bigvee_{K\subseteq S} \alpha(K) = \alpha(K^*) \leqslant \alpha_p(K^*) < \alpha_p(S),
$$

which is a contradiction.

($\Leftarrow$) By Lemma 7, we have $p = p^{\vee}_{\alpha_p}$. Fix $S \subseteq [n]$ and assume there is $i \in S$ such that $\alpha_p(S) = \alpha_p(S \setminus \{i\})$. Then

$$\Big(\alpha_p(S \setminus \{i\}) \wedge \bigwedge_{j \in S\setminus\{i\}} x_j\Big) \vee \Big(\alpha_p(S) \wedge \bigwedge_{j \in S} x_j\Big) = \Big(\alpha_p(S \setminus \{i\}) \wedge \bigwedge_{j \in S\setminus\{i\}} x_j\Big)$$

and hence $\alpha_p(S)$ can be replaced with any lower value without altering $p^{\vee}_{\alpha_p}$. Hence $p^{\vee}_{\alpha_p} = p^{\vee}_{\alpha}$. $\square$

Example 9 *Assuming that L is a chain, the possible disjunctive expressions of $x_1 \vee (x_1 \wedge x_2)$ as a 2-ary w.l.p. function are given by*

$$x_1 \vee (c \wedge x_1 \wedge x_2) \qquad (c \in L).$$

For $c = 0$, we retrieve x_1 and, for $c = 1$, we retrieve $x_1 \vee (x_1 \wedge x_2)$.

We note that, from among all the set functions that disjunctively (or conjunctively) define a given w.l.p. function p, only α_p (resp. β_p) is isotone (resp. antitone). Indeed, suppose for instance that α is isotone. Then, for any $S \subseteq [n]$, we have

$$\alpha(S) = \bigvee_{K \subseteq S} \alpha(K) = \alpha_p(S),$$

that is, $\alpha = \alpha_p$.

4 The discrete Sugeno integral

Certain w.l.p. functions have been considered in the area of nonlinear aggregation and integration. The best known instances are given by the discrete *Sugeno integral*, which is a particular discrete integration with respect to a *fuzzy measure* (see Sugeno [9,10]). For a recent survey on the discrete Sugeno integral, see Dubois et al. [3].

In this section we show the relationship between the discrete Sugeno integral and the w.l.p. functions. To this end, we introduce the Sugeno integral as a function from L^n to L. Originally defined when L is the real interval $[0, 1]$, the Sugeno integral has different equivalent representations (see Section 5). Here we consider its disjunctive normal representation [9], which enables us to extend the original definition of the Sugeno integral to the more general case where L is any bounded distributive lattice.

Definition 10 *An L-valued* fuzzy measure *on $[n]$ is an isotone set function $\mu : 2^{[n]} \to L$ such that $\mu(\varnothing) = 0$ and $\mu([n]) = 1$.*

Definition 11 *Let μ be an L-valued fuzzy measure on $[n]$. The* Sugeno integral *of a function $\mathbf{x} : [n] \to L$ with respect to μ is defined by*

$$\mathcal{S}_\mu(\mathbf{x}) := \bigvee_{S\subseteq[n]} \Big(\mu(S) \wedge \bigwedge_{i\in S} x_i\Big).$$

Surprisingly, it appears immediately that any function $f : L^n \to L$ is an n-ary Sugeno integral if and only if it is a w.l.p. function fulfilling $f(\mathbf{e}_\varnothing) = 0$ and $f(\mathbf{e}_{[n]}) = 1$. Moreover, as the following proposition shows, any w.l.p. function can be easily expressed in terms of a Sugeno integral.

Recall that, when n is odd, $n = 2k - 1$, the n-ary *median* function is defined in any distributive lattice as the following lattice polynomial function (see for instance Barbut and Monjardet [1, Chap. IV])

$$\operatorname{median}(\mathbf{x}) = \bigvee_{\substack{S\subseteq[2k-1]\\|S|=k}} \bigwedge_{i\in S} x_i = \bigwedge_{\substack{S\subseteq[2k-1]\\|S|=k}} \bigvee_{i\in S} x_i.$$

Proposition 12 *For any w.l.p. function $p : L^n \to L$, there exists a fuzzy measure $\mu : 2^{[n]} \to L$ such that*

$$p(\mathbf{x}) = \operatorname{median}\Big(p(\mathbf{e}_\varnothing), \mathcal{S}_\mu(\mathbf{x}), p(\mathbf{e}_{[n]})\Big).$$

Proof. Let $\mu : 2^{[n]} \to L$ be the fuzzy measure which coincides with α_p on $2^{[n]}$ except at $\varnothing$ and $[n]$. Then, we have

$$\begin{aligned}
&\operatorname{median}\Big(p(\mathbf{e}_\varnothing), \mathcal{S}_\mu(\mathbf{x}), p(\mathbf{e}_{[n]})\Big)\\
&= \Bigg(\alpha_p(\varnothing) \vee \bigvee_{\substack{S\subseteq[n]\\S\neq\varnothing, S\neq[n]}} \Big(\mu(S) \wedge \bigwedge_{i\in S} x_i\Big) \vee \Big(\bigwedge_{i\in[n]} x_i\Big)\Bigg) \wedge \alpha_p([n])\\
&= \bigvee_{S\subseteq[n]} \Big(\alpha_p(S) \wedge \bigwedge_{i\in S} x_i\Big)\\
&= p(\mathbf{x}). \qquad \square
\end{aligned}$$

Corollary 13 *Consider a function $f : L^n \to L$. The following assertions are equivalent:*

(1) f is a Sugeno integral.
(2) f is an idempotent w.l.p. function, i.e., such that $f(x, \ldots, x) = x$ for all $x \in L$.
(3) f is a w.l.p. function fulfilling $f(\mathbf{e}_\varnothing) = 0$ and $f(\mathbf{e}_{[n]}) = 1$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) Immediate consequence of Proposition 12. $\square$

Remark 14 *As the definition of the w.l.p. functions almost coincide with that of the Sugeno integral, certain properties of the Sugeno integral can be applied as-is or in a slightly extended form to the w.l.p. functions. For instance, Proposition 8 was already known for the Sugeno integral (see Marichal [7]).*

5 A representation theorem

Combining Proposition 12 with the well-known representations of the Sugeno integral, we easily deduce equivalent representations for the w.l.p. functions.

When L is a chain, for any permutation σ on $[n]$, we define the subset

$$\mathcal{O}_\sigma := \{\mathbf{x} \in L^n \mid x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\}.$$

Theorem 15 *Let $p : L^n \to L$ be any w.l.p. function. For any $\mathbf{x} \in L^n$, we have*

$$p(\mathbf{x}) = \bigvee_{S\subseteq[n]} \Big(\alpha_p(S) \wedge \bigwedge_{i\in S} x_i\Big) = \bigwedge_{S\subseteq[n]} \Big(\alpha_p(N \setminus S) \vee \bigvee_{i\in S} x_i\Big).$$

Moreover, assuming that L is a chain, for any permutation σ on $[n]$ and any $\mathbf{x} \in \mathcal{O}_\sigma$, setting $S_\sigma(i) := \{\sigma(i), \ldots, \sigma(n)\}$ for all $i \in [n]$, we have

$$\begin{aligned} p(\mathbf{x}) &= \bigvee_{i=1}^{n+1} \Big(\alpha_p(S_\sigma(i)) \wedge x_{\sigma(i)}\Big) = \bigwedge_{i=0}^{n} \Big(\alpha_p(S_\sigma(i+1)) \vee x_{\sigma(i)}\Big) \\ &= \mathrm{median}\Big(x_1, \ldots, x_n, \alpha_p(S_\sigma(1)), \alpha_p(S_\sigma(2)), \ldots, \alpha_p(S_\sigma(n+1))\Big), \end{aligned}$$

with the convention that $x_{\sigma(0)} = 0$, $x_{\sigma(n+1)} = 1$, and $S_\sigma(n+1) = \varnothing$.

Proof. The first part has been established in Lemma 7. The second part follows from Proposition 12 and the following representations of the Sugeno integral. For any L-valued fuzzy measure μ on $[n]$, we have (see for instance [7])

$$\begin{aligned} \mathcal{S}_\mu(\mathbf{x}) &= \bigvee_{i=1}^{n} \Big(\mu(S_\sigma(i)) \wedge x_{\sigma(i)}\Big) = \bigwedge_{i=1}^{n} \Big(\mu(S_\sigma(i+1)) \vee x_{\sigma(i)}\Big) \\ &= \mathrm{median}\Big(x_1, \ldots, x_n, \mu(S_\sigma(2)), \mu(S_\sigma(3)), \ldots, \mu(S_\sigma(n))\Big). \end{aligned}$$

$\square$

Remark 16 *It follows from Theorem 15 that, when the order of the coordinates of* $\mathbf{x}$ *is known, then* $p(\mathbf{x})$ *is entirely determined by* $(n+1)$ *parameters (instead of* 2^n*).*

6 The median based decomposition formula

Given a function $f : L^n \to L$ and an index $k \in [n]$, we define the functions $f_k^0 : L^n \to L$ and $f_k^1 : L^n \to L$ as

$$
\begin{aligned}
f_k^0(\mathbf{x}) &= f(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n),\\
f_k^1(\mathbf{x}) &= f(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_n).
\end{aligned}
$$

Clearly, if f is a w.l.p. function, so are f_k^0 and f_k^1.

Now consider the following system of n functional equations, which we will refer to as the *median based decomposition formula*:

$$f(\mathbf{x}) = \mathrm{median}\Big(f_k^0(\mathbf{x}), x_k, f_k^1(\mathbf{x})\Big) \qquad (k \in [n]) \tag{1}$$

This functional system expresses that, for any index k, the variable x_k can be totally isolated in $f(\mathbf{x})$ by means of a median calculated over the variable x_k and the two functions f_k^0 and f_k^1, which are independent of x_k.

In this final section we establish that this system characterizes the n-ary w.l.p. functions.

Theorem 17 *The solutions of the median based decomposition formula (1) are exactly the* n*-ary w.l.p. functions.*

Proof. Recall that the ith variable ($i \in [n]$) of a function $f : L^n \to L$ is said to be *effective* if there are two n-vectors in L^n, differing only in the ith component, on which f takes on different values.

The proof that every function $f : L^n \to L$ satisfying system (1) is a w.l.p. function is done by induction on the number of effective variables of f. If f has a single effective variable x_k then, using the kth equation of (1), we immediately see that f is a w.l.p. function. The inductive step in then based on the straightforward fact that if f satisfies (1) then, for any $i \in [n]$, the functions f_i^0 and f_i^1 also satisfy (1).

Let us now show that any w.l.p. function $p : L^n \to L$ fulfills system (1). Let $\mathcal{P}_n$ be the set of nondecreasing functions $f : L^n \to L$ fulfilling (1). Clearly, $\mathcal{P}_n$

contains all the projection and constant functions from L^n to L. Moreover, we can readily see that if $f, g \in \mathcal{P}_n$ then $f \wedge g \in \mathcal{P}_n$ and $f \vee g \in \mathcal{P}_n$. It follows that $\mathcal{P}_n$ contains all the w.l.p. functions from L^n to L. $\square$

Corollary 18 *For any w.l.p. function $p : L^n \to L$ and any $k \in [n]$, we have*

$$p(x_1, \ldots, x_{k-1}, p(\mathbf{x}), x_{k+1}, \ldots, x_n) = p(\mathbf{x}).$$

Proof. Using Theorem 17 and the fact that p is nondecreasing, we immediately obtain

$$p(x_1, \ldots, x_{k-1}, p(\mathbf{x}), x_{k+1}, \ldots, x_n) = \mathrm{median}\Big(p_k^0(\mathbf{x}), p(\mathbf{x}), p_k^1(\mathbf{x})\Big) = p(\mathbf{x}). \qquad \square$$

7 Conclusion

We have introduced the concept of weighted lattice polynomial functions, which generalize the lattice polynomial functions by allowing some variables to be fixed as parameters. We have observed that these functions include the class of discrete Sugeno integrals, which have been extensively used not only in aggregation function theory but also in fuzzy set theory. Finally, we have provided a median based system of functional equations that completely characterizes the weighted lattice polynomial functions.

Just as special Sugeno integrals (such as the weighted minima, the weighted maxima, and their ordered versions) have already been investigated and axiomatized (see Dubois et al. [3]), certain subclasses of weighted lattice polynomial functions deserve to be identified and investigated in detail. This is a topic for future research.

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