Group of automorphisms of a Shimura curve

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1 Introduction

Let B be the indefinite quaternion algebra over $\mathbb{Q}$ of discriminant D. Let $\mathcal{O}$ be a maximal order in B. Let X_D be the Shimura curve over $\mathbb{Q}$ attached to $\mathcal{O}$, whose set of complex points is given by

$$X_D(\mathbb{C}) = (\hat{\mathcal{O}}^\times \backslash \hat{B}^\times \times \mathbb{P})/B^\times,$$

where $\mathbb{P} = \mathbb{C} \backslash \mathbb{R}$. As it is well known, such X_D is equipped with a natural group of involutions called the Atkin-Lehner group $W(D)$, where each involution ω_n is indexed by the divisors $n \mid D$. Thus the Atkin-Lehner group $W(D)$ as a subgroup of the group of automorphisms $\mathrm{Aut}(X_0(D,N))$ is a 2-elementary abelian group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$, where $r = \#\{p \mid D\}$. The following proposition characterizes $\mathrm{Aut}(X_D)$ also as a 2-elementary abelian group, in case that the genus is at least 2.

Proposition 1.1. [3, Proposition 1.5] *Let $U \subseteq W(D)$ be a subgroup and let X_D/U denote the quotient curve. If the genus of X_D/U is at least 2, then all automorphisms of X_D/U are defined over $\mathbb{Q}$ and*

$$\mathrm{Aut}(X_D/U) = (\mathbb{Z}/2\mathbb{Z})^s$$

for some $s \geq r - \mathrm{rank}_{\mathbb{F}_2}(U)$.

Remark 1.2. This proposition is also true if we consider Shimura curves with non trivial level.

The following conjecture predicts the structure of $\mathrm{Aut}(X_D/U)$:

Conjecture 1.3. *Let $U \subseteq W(D)$ and X_D/U be as above. Then,*

$$\mathrm{Aut}(X_D/U) = (\mathbb{Z}/2\mathbb{Z})^{r-\mathrm{rank}_{\mathbb{F}_2}(U)}.$$

The following result, due to Kontogeorgis and one of the autors, proves Conjecture 1.3 for almost all $D \leq 1500$.

Proposition 1.4. [3, Proposition 3.5] *For $D \leq 1500$, the only automorphisms of X_D are the Atkin-Lehner involutions, provided $g(X_D) \geq 2$ and $D \neq 493, 583, 667, 697, 943$.*

The aim of this paper is to prove Conjecture 1.3 in the specific case $D = 667$ where the techniques of [3] turn out to be insufficient. The new technique we use is the theory of specialization at singular fibers of certain *special* points in X_D called CM or *Heegner points*.

2 Specialization of Heegner points

Let K be an imaginary quadratic field and fix an embedding $\varphi : K \hookrightarrow B$. We denote by $\mathrm{CM}_D(K)$ the set of Heegner points with CM by K. Its description as a set of points in $X_D(\mathbb{C})$ follows by choosing $\tau \in \mathbb{P}$, one of the two fixed points by $K^\times \subset B^\times$, then

$$\mathrm{CM}_D(K) = (\hat{\mathcal{O}}^\times \backslash \hat{B}^\times \times B^\times \tau)/B^\times = (\hat{\mathcal{O}}^\times \backslash \hat{B}^\times / K^\times) \times \tau \subset (\hat{\mathcal{O}}^\times \backslash \hat{B}^\times \times \mathbb{P})/B^\times = X_D(\mathbb{C}),$$

here the second equality has been obtained since $\{b \in B^\times,\ b\tau = \tau\} = K^\times$.

In order to describe the specialization of such points, we must choose a concrete model of X_D over Spec($\mathbb{Z}$). There exists a proper integral model $\mathcal{X}_D/\mathrm{Spec}(\mathbb{Z})$ that suitably extend the moduli interpretation of X_D to arbitrary schemes. Such model has smooth reduction at $q \nmid D$ and, by the theory of Cerednik-Drinfeld [1] [2], the set of singular points of the singular fiber $\mathcal{X}_D$ at $p \mid D$ is in bijective correspondence with the double coset $\mathrm{Pic}(\mathcal{O}^p) := (\hat{\mathcal{O}}^p)^\times \backslash (\hat{B}^p)^\times / (B^p)^\times$, where B^p is a definite quaternion algebra over $\mathbb{Q}$ of discriminant D/p and $\mathcal{O}^p$ in any Eichler order in B^p of level p. Denote such correspondence by

$$\lambda_p : (\mathcal{X}_D)_p^{\mathrm{sing}} \xrightarrow{\simeq} \mathrm{Pic}(\mathcal{O}^p).$$

The set $\mathrm{Pic}(\mathcal{O}^p)$ classifies oriented Eichler orders in B^p of level p. Given a singular point $P \in \mathcal{X}_D$, the thickness of P is given by $\#(\lambda_p(P))^\times/2$.

We proceed to describe the results of one of the authors in [4] on the specialization of Heegner points with singular specialization on the special fibers of $\mathcal{X}_D$ at $p \mid D$. Assume from now on that p ramifies in K. Then, by [4, Theorem 1.1], there exist a natural embedding $K \hookrightarrow B^p$ induced by φ and an Eichler order $\mathcal{O}^p$ of level p in B^p such that $K \cap \mathcal{O} \simeq K \cap \mathcal{O}^p := R$.

Let us consider the double coset $\mathrm{CM}^p(K) = (\hat{\mathcal{O}}^p)^\times \backslash (\hat{B}^p)^\times / K^\times$ and the natural projection map:

$$\pi^p : \mathrm{CM}^p(K) = (\hat{\mathcal{O}}^p)^\times \backslash (\hat{B}^p)^\times / K^\times \longrightarrow (\hat{\mathcal{O}}^p)^\times \backslash (\hat{B}^p)^\times / (B^p)^\times = \mathrm{Pic}(\mathcal{O}^p)$$

Let us also consider the double coset $\mathrm{CM}^p(K) = \hat{\mathcal{O}}^\times \backslash \hat{B}^\times / K^\times$ and, if I_D and $I_{D/p}$ are the set of ideals in $\mathbb{Z}$ coprime to D and D/p respectively, the natural surjective maps

$$\rho_K : \mathrm{CM}(K) \longrightarrow I_D, \quad \rho_K^p : \mathrm{CM}^p(K) \longrightarrow I_{D/p},$$

where $\rho_K([g])$ and $\rho_K^p([g^p])$ are the conductors on K of $K \cap g\hat{\mathcal{O}}g^{-1}$ and $K \cap g^p \hat{\mathcal{O}}^p (g^p)^{-1}$, respectively. From now on we are going to identify $\rho_K^{-1}(c) \subset \mathrm{CM}(K)$ as a subset of $\mathrm{CM}_D(K) \subset X_D(\mathbb{C})$ by means of the isomorphism $\mathrm{CM}(K) \simeq \mathrm{CM}_D(K)$.

The following theorem is due to Shimura and characterizes the field of definition of any $P \in \mathrm{CM}_D(K)$.

Theorem 2.1. *[7] Let R_c be the unique order of K of conductor c. If $\mathbb{Q}(P)$ is the field of definition of any $P \in \rho_K^{-1}(c)$, then $K \cdot \mathbb{Q}(P) = H_c^K$ the ring class field of R_c.*

Finally, the following result characterizes the specialization of any Heegner point in $\mathrm{CM}_D(K)$.

Theorem 2.2. *[4, Theorem 1.1] If p ramifies in K, any $P \in \mathrm{CM}_D(K)$ has singular specialization in $(\mathcal{X}_D)_p$. Moreover, for a given $c \in I_D$, there exist a bijection $\rho_K^{-1}(c) \xrightarrow{\simeq} (\rho_K^p)^{-1}(c)$, giving rise to a 1 to 1 correspondence*

$$\theta_p : \mathrm{CM}_D(K) \simeq \mathrm{CM}(K) \xrightarrow{1:1} \mathrm{CM}^p(K),$$

such that $\pi^p(\theta_p(P)) = \lambda_p(\tilde{P})$, where $\tilde{P} \in (\mathcal{X}_D)_p^{\mathrm{sing}}$ is the specialization of $P \in \mathrm{CM}_D(K)$.

Remark 2.3. It is easy to check that the set $(\rho_K^p)^{-1}(c)$ is in correspondence with the set of optimal embeddings of R_c into any order in $\mathrm{Pic}(\mathcal{O}^p)$. Moreover, the map π^p maps an optimal embedding to the isomorphism class of its target in $\mathrm{Pic}(\mathcal{O}^p)$.

3 Group of automorphisms of a Shimura curve

Finally, we proceed to proof the following proposition, which is a concrete example of Conjecture 1.3.

Proposition 3.1. *If $D = 667$, $\mathrm{Aut}(X_D) = W(D)$.*

Before offering the proof, let us invoke first two results due to A. Ogg.

Lemma 3.2. [5] *Let L be a field and $\mu(L)$ its group of roots of unity. Let $\rho = \max\{1, \mathrm{char}(L)\}$ be the characteristic exponent of L. Let C be an irreducible curve defined over L and $P \in C(L)$ a regular point on it. Let G be a finite group of L-automorphisms acting on C and fixing the point P. Then there is a homomorphism $f : G \to \mu(L)$ whose kernel is a ρ-group.*

Proposition 3.3. *[6, §1] Let $m \mid D$, $m > 0$. Then, if $K_m = \mathbb{Q}(\sqrt{-m})$, set $\mathfrak{F}_{\omega_m}$ of fixed points of the Atkin-Lehner involution ω_m acting on X_D is*

$$\mathfrak{F}_{\omega_m} = \begin{cases} \rho_{K_1}^{-1}(1) \sqcup \rho_{K_2}^{-1}(1) & \text{if m} = 2 \\ \rho_{K_m}^{-1}(1) \sqcup \rho_{K_m}^{-1}(2) & \text{if m} \equiv 3 \mod 4 \\ \rho_{K_m}^{-1}(1) & \text{otherwise.} \end{cases}$$

Proof of Proposition 3.1. In our case $D = 667$, thus by Theorem 3.3 the set of fixed points of ω_D is $\mathfrak{F}_{\omega_D} = \rho_{K_D}^{-1}(1) \sqcup \rho_{K_D}^{-1}(2)$. We computed that the ring class fields $H_1^{K_D}$ and $H_2^{K_D}$ have degree 4 and 12, respectively. Hence by Theorem 2.1, points in $\rho_{K_D}^{-1}(1)$ and points in $\rho_{K_D}^{-1}(2)$ are defined over different fields. Therefore, $\phi(P) \notin \rho_{K_D}^{-1}(2)$, for all $P \in \rho_{K_D}^{-1}(1)$ and all $\phi \in \mathrm{Aut}(X_D)$. Moreover, since $\mathrm{Aut}(X_D)$ is abelian,

$$\omega_D(\phi(P)) = \phi(\omega_D(P)) = \phi(P),$$

for all $\phi \in \mathrm{Aut}(X_0^D)$ and $P \in \mathfrak{F}_{\omega_D}$. Hence $\phi(\mathfrak{F}_{\omega_D}) = \mathfrak{F}_{\omega_D}$ and, thus, $\phi(\rho_{K_D}^{-1}(1)) = \rho_{K_D}^{-1}(1)$.

Let $p = 29$ and write $(\mathcal{X}_D)_p$ for the special fiber of $\mathcal{X}_D$ at p. Since p ramifies in K_D, any $P \in \mathrm{CM}_D(K_D)$ has singular specialization. By Theorem 2.2, the set $\rho_{K_D}^{-1}(1) \subset \mathrm{CM}_D(K_D)$ is in bijective correspondence with $(\rho_{K_D}^p)^{-1}(1)$, which in turn classifies optimal embeddings of the maximal order of K_D into any Eichler order in $\mathrm{Pic}(\mathcal{O}^p)$. We computed the optimal embeddings $(\rho_{K_D}^p)^{-1}(1)$ and their images through the map π^p, and we obtained that $(\rho_{K_D}^p)^{-1}(1) = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$, $\pi^p(\varphi_1) = \mathcal{O}_1$, $\pi^p(\varphi_2) = \pi^p(\varphi_4) = \mathcal{O}_2$, $\pi^p(\varphi_3) = \mathcal{O}_3$, $\mathcal{O}_1 \neq \mathcal{O}_3$ and $\#\mathcal{O}_1/2 = \#\mathcal{O}_3/2 = 2$, $\#\mathcal{O}_2/2 = 1$. By Theorem 2.2, this implies that $\rho_{K_D}^{-1}(1) = \{P_1, P_2, P_3, P_4\}$ specialize to, $\tilde{P}_2 = \tilde{P}_4$, $\tilde{P}_1 \neq \tilde{P}_3 \neq \tilde{P}_2$ with thicknesses $e_{\tilde{P}_1} = e_{\tilde{P}_3} = 2$ and $e_{\tilde{P}_2} = 1$.

Applying Lemma 3.2 to $C = X_D$, $L = \bar{\mathbb{Q}}$, where $\rho = 1$, f is injective and $G \simeq (\mathbb{Z}/2\mathbb{Z})^t$, we deduce that $G = 1$ or $\mathbb{Z}/2\mathbb{Z}$, since $\mu(\bar{\mathbb{Q}})$ must be cyclic. This implies that there can only exists at most one $\phi \in \mathrm{Aut}(X_D)$ such that $\phi(P) = P$ for a given $P \in X_D(\overline{\mathbb{Q}})$.

Notice that $\mathrm{Aut}(X_D)$ fixes the subset of points in $\rho_{K_D}^{-1}(1)$ that specialize to singular points with the same thicknesses. Thus $\mathrm{Aut}(X_D)$ fixes $\{P_1, P_3\}$ (also $\{P_2, P_4\}$). Since the unique automorphism that fixes P_1 or P_3 is ω_D by Ogg's Lemma, we deduce that $\mathrm{Aut}(X_D) = (\mathbb{Z}/2\mathbb{Z})^2 = W(D)$. □

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