# GALOIS ACTION ON $\bar{\mathbb{Q}}$-ISOGENY CLASSES OF ABELIAN $L$-SURFACES WITH QUATERNIONIC MULTIPLICATION

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Abstract. We construct a projective Galois representations attached to an abelian $L$-surface with quaternionic multiplication, describing the Galois action on its Tate module. We prove that such representation characterizes the Galois action on the isogeny class of the abelian $L$-surface, seen as a set of points of certain Shimura curves.

## 1. Introduction

Let $L$ be a number field. An abelian variety $A/\bar{L}$ is called a *abelian $L$-variety* if for each $\sigma \in \mathrm{Gal}(\bar{L}/L)$ there exists an isogeny $\mu_\sigma : A^\sigma \to A$ such that $\psi \circ \mu_\sigma = \mu_\sigma \circ \psi^\sigma$ for all $\psi \in \mathrm{End}(A)$. The current interest on abelian $L$-varieties began, with $L = \mathbb{Q}$, when K. Ribet observed that absolutely simple factors of the modular Jacobians $J_1(N)$ are in fact abelian $\mathbb{Q}$-varieties [2]. Actually, after the proof of Serre's conjecture on representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ [3, 3.2.4?], every so-called *building blocks* (namely those $\mathbb{Q}$-varieties whose endomorphism algebra is a central division algebra over a totally real number field $F$ with Schur index $t = 1$ or $t = 2$ and $t[F : \mathbb{Q}] = \dim A$), is an absolutely simple factors up to isogeny of a modular Jacobian $J_1(N)$.

In the dimension one case, we can think elliptic $L$-curves as generalizations of elliptic curves defined over $L$. Indeed, given an elliptic $L$-curve $E$ (without Complex Multiplication) we can define a projective representation

$$\rho_E^\ell : \mathrm{Gal}(\bar{L}/L) \longrightarrow \mathrm{GL}_2(\mathbb{Q}_\ell)/\mathbb{Q}_\ell^\times, \tag{1.1}$$

generalizing the classical representation on the Tate module of an elliptic curve over $L$.

In the dimension two case, we have a similar situation with the so-called *fake elliptic curves* or *abelian surfaces with quaternionic multiplication (QM)*, namely, pairs $(A, \imath)$ where $A$ is an abelian surface and $\imath$ is an embedding of a quaternion order $\mathcal{O}$ into its endomorphism ring. If we set $A = E \times E$ with $\mathcal{O} = \mathrm{M}_2(\mathbb{Z})$ and the obvious embedding into $\mathrm{End}(E \times E)$, we recover the classical dimension one setting. In this paper we shall construct a map $\rho_{(A,\imath)}$, attached to an abelian $L$-surface with QM $(A, \imath)$, of the form

$$\rho_{(A,\imath)} : \mathrm{Gal}(\bar{L}/L) \longrightarrow (\mathcal{O} \otimes \mathbb{A}_f)^\times / (\mathrm{End}(A, \imath) \otimes \mathbb{Q})^\times$$

where $\mathbb{A}_f$ is the ring of finite adeles and $\mathrm{End}(A, \imath)$ is the set of endomorphisms commuting every element of the image of $\imath$. The $\ell$-adic component of $\rho_{(A,\imath)}$ will coincide with the projective representation of (1.1) in the classical scenario. As well as in this classical setting, the map $\rho_{(A,\imath)}$ will depend on the choice of a basis of all Tate modules $T_\ell A$. We will see that this choice is equivalent to the choice of an isomorphism of $\mathcal{O}$-modules $\varphi : A_{tor} \to B/\mathcal{O}$ (Lemma 2.2), where $A_{tor}$ is the set of torsion points of $A$ and $B = \mathcal{O} \otimes \mathbb{Q}$ is the corresponding quaternion algebra over $\mathbb{Q}$. Hence, once the choice of $\varphi$ is provided, we will denote the corresponding map as $\rho_{(A,\imath,\varphi)}$.

It is easy to deduce that, given an abelian $L$-surface with QM $(A,\imath)$, any pair $(A',\imath')$ $\bar{L}$-isogenious to $(A,\imath)$ is also an abelian $L$-surface with QM. This implies that the $\bar{L}$-isogeny class of $(A,\imath)$ has a well defined action of $\mathrm{Gal}(\bar{L}/L)$. The main goal of this paper is to describe such Galois action by means of the map $\rho_{(A,\imath)}$. In order to do that, we will consider the $\bar{L}$-isogeny class of $(A,\imath)$ as a set of $\bar{L}$-rational points of the certain Shimura curve $X_\Gamma$. By means of the moduli interpretation of $X_\Gamma$, the Galois action on the isogeny class will be translated into a Galois action on this set of infinitely many points. We will characterize this set of points as a double coset space (Proposition 3.3) and finally, in Theorem 4.2, we give a description of the Galois action by means of the map $\rho_{(A,\imath)}$.

The article is structured as follows: In §2 we introduce abelian $L$-surfaces with QM and the map $\rho_{(A,\imath,\varphi)}$, we give some of its properties distinguishing between the Complex Multiplication (CM) case and the non-CM case. In §3 we introduce Shimura curves $X_\Gamma$ and their moduli interpretation, moreover, we characterize the isogeny class of an abelian surface with QM as a double coset space with a concrete embedding into $X_\Gamma$. In §4 we present our main result describing the Galois action on the isogeny class of an abelian $L$-surface with QM $(A,\imath)$ by means of the map $\rho_{(A,\imath,\varphi)}$. In §5 we discuss about distinct moduli problems that the Shimura curve $X_\Gamma$ solves and the reinterpretation of our main result in these new terms. Finally, in §6 we give a complete description of the CM case characterizing the map $\rho_{(A,\imath,\varphi)}$ via Class Field Theory.

1.1. **Notation.** Let $\hat{\mathbb{Z}}$ denote the completion of $\mathbb{Z}$, hence $\hat{\mathbb{Z}}=\varprojlim(\mathbb{Z}/N\mathbb{Z})$. Let $\mathbb{A}_f$ denote the ring of finite adeles, namely $\mathbb{A}_f=\hat{\mathbb{Z}}\otimes\mathbb{Q}$. Note that $\mathbb{Q}/\mathbb{Z}=\varinjlim\mathbb{Z}/N\mathbb{Z}$, therefore

$$\begin{aligned}\mathrm{End}(\mathbb{Q}/\mathbb{Z}) &= \mathrm{Hom}(\varinjlim(\mathbb{Z}/N\mathbb{Z}),\mathbb{Q}/\mathbb{Z})=\varprojlim\mathrm{Hom}(\mathbb{Z}/N\mathbb{Z},\mathbb{Q}/\mathbb{Z})\\ &= \varprojlim\mathrm{Hom}(\mathbb{Z}/N\mathbb{Z},\mathbb{Z}/N\mathbb{Z})=\varprojlim(\mathbb{Z}/N\mathbb{Z})=\hat{\mathbb{Z}}.\end{aligned}$$

Let $B$ be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $D$, and let $\mathcal{O}$ be an Eichler order in $B$. Let $G$ be the group scheme over $\mathbb{Z}$ such that $G(R)=(\mathcal{O}^{opp}\otimes R)^\times$ for all rings $R$, where $\mathcal{O}^{opp}$ is the opposite algebra to $\mathcal{O}$. Note that the group $G(\mathbb{A}_f)$ does not depend on the Eichler order $\mathcal{O}$ chosen since it is maximal locally for all but finitely many places.

Write $\hat{\mathcal{O}}=\mathcal{O}\otimes\hat{\mathbb{Z}}$, then me have the isomorphism $\hat{\mathcal{O}}=\varprojlim(\mathcal{O}/N\mathcal{O})$. Moreover $\varinjlim(\mathcal{O}/N\mathcal{O})=B/\mathcal{O}$ as $\mathcal{O}$-modules. By the same argument as above, this implies that $\mathrm{End}_{\mathcal{O}}(B/\mathcal{O})=\hat{\mathcal{O}}^{opp}$. Hence we can identify $G(\mathbb{A}_f)=(\mathrm{End}_{\mathcal{O}}(B/\mathcal{O})\otimes\mathbb{Q})^\times$.

Throughout this paper, we will denote $\mathrm{End}^0:=\mathrm{End}\otimes\mathbb{Q}$.

## 2. Abelian $L$-surfaces with Quaternionic Multiplication

An *abelian surface with QM by* $\mathcal{O}$ is a pair $(A,\imath)$ where $A$ is an abelian surface and $\imath$ is an embedding $\mathcal{O}\hookrightarrow\mathrm{End}(A)$, optimal in sense that $\imath(B)\cap\mathrm{End}(A)=\imath(\mathcal{O})$. If the order $\mathcal{O}$ is clear by the context, we will call them just *QM-abelian surfaces.* Let us consider the subring

$$\mathrm{End}(A,\imath)=\{\lambda\in\mathrm{End}(A):\ \lambda\circ\imath(o)=\imath(o)\circ\lambda,\ \text{for all } o\in\mathcal{O}\}.$$

If $A$ is defined over $\mathbb{C}$, $\mathrm{End}(A,\imath)^0$ can be either $\mathbb{Q}$ or an imaginary quadratic field $K$, in this last situation we say that $(A,\imath)$ has *complex multiplication (CM).*

**Definition 2.1.** Let $M/L/\mathbb{Q}$ be number fields. A *abelian $L$-surface with QM by* $\mathcal{O}$ is an abelian surface with QM by $\mathcal{O}$ $(A,\imath)$ over $M$ such that, for all $\sigma\in\mathrm{Gal}(M/L)$, there exist an isogeny $\mu_\sigma:A^\sigma\to A$, such that $\mu_\sigma\circ\imath(o)^\sigma=\imath(o)\circ\mu_\sigma$ for all $o\in\mathcal{O}$.

Given an abelian $L$-surface with QM we shall construct a map

$$\rho_{(A,\imath,\varphi)}:\mathrm{Gal}(\bar{L}/L)\longrightarrow G(\mathbb{A}_f)/\mathrm{End}^0(A,\imath)^\times$$

that describes the Galois action on the Tate module. In order to do this, we will fix an isomorphism $\varphi : A_{tor} \to B/\mathcal{O}$. The following result shows that to choose such an isomorphism $\varphi$ is equivalent to choose a basis $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ of the Tate module $\hat{T}(A) = \mathrm{Hom}(A_{tor}, \mathbb{Q}/\mathbb{Z})$.

**Lemma 2.2.** *Given a basis $\{\varphi_i\}_{i=1\cdots 4}$ of $\hat{T}(A)$, there exists a $\mathbb{Z}$-basis $\{e_i\}_{i=1\cdots 4}$ of $\mathcal{O}$ such that*

$$\varphi : A_{tor} \longrightarrow B/\mathcal{O}; \qquad P \longmapsto \sum_{i=1\cdots 4} \varphi_i(P)e_i$$

*is a $\mathcal{O}$-module isomorphism. Analogously, given a $\mathcal{O}$-module isomorphism $\varphi : A_{tor} \to B/\mathcal{O}$, any $\mathbb{Z}$-basis $\{e_i\}_{i=1\cdots 4}$ of $\mathcal{O}$ provides a basis $\{\varphi_i\}_{i=1\cdots 4}$ of $\hat{T}(A)$ satisfying $\varphi(P) = \sum_{i=1}^4 \varphi_i(P)e_i$.*

*Proof.* Since $\{\varphi_i\}_{i=1\cdots 4}$ is a basis, the map $A_{tor} \to (\mathbb{Q}/\mathbb{Z})^4$ is an isomorphism. Then there exists a unique sequence $\{P_n\}_{n\in\mathbb{N}} \subset A_{tor}$ such that $\varphi_i(P_n) = \frac{1}{n}\delta_1^i$. Since $A$ has QM by $\mathcal{O}$, we know that $A[n] \simeq \mathcal{O}/n\mathcal{O}$. Thus there exists $e_j \in \mathcal{O}$ such that $\varphi_i(\imath(e_j)P_n) = \frac{1}{n}\delta_j^i$ (in particular $e_1 = 1$). We claim that $\{e_i\}_{i=1\cdots 4}$ form a $\mathbb{Z}$-basis for $\mathcal{O}$. Indeed, for any $\alpha \in \mathcal{O}$, there exists $n_i \in \mathbb{Z}$ such that $\varphi_i(\imath(\alpha)P_n) = \frac{n_i}{n}$, thus

$$\varphi_i(\imath(\alpha)P_n) - \varphi_i\left(\imath\left(\sum_{j=1\cdots 4} n_j e_j\right)P_n\right) = \frac{n_i}{n} - \sum_{j=1\cdots 4} \frac{n_j}{n}\delta_i^j = 0.$$

Since $\{\varphi_i\}_{i=1\cdots 4}$ form a basis and $P_n$ generates $A[n]$ as $\mathcal{O}$-module, we conclude $\alpha = \sum_{j=1\cdots 4} n_j e_j$ and $\{e_i\}_{i-1\cdots 4}$ form a $\mathbb{Z}$-basis for $\mathcal{O}$.

Finally, we consider the well defined morphism $\varphi$ and let $\alpha = \sum_{j=1\cdots 4} n_j e_j \in \mathcal{O}$, then

$$\varphi(\imath(\alpha)P_n) = \sum_{i,j=1\cdots 4} n_j \varphi_i\left(\imath(e_j)P_n\right)e_i = \sum_{i,j=1\cdots 4} n_j \frac{1}{n}\delta_i^j e_i = \frac{1}{n}\sum_{i=1\cdots 4} n_i e_i = \alpha\varphi(P_n).$$

Since $P_n$ generates $A[n]$ as a $\mathcal{O}$-module, we conclude that $\varphi$ is a $\mathcal{O}$-module isomorphism.

Analogously, given a $\mathcal{O}$-module isomorphism $\varphi : A_{tor} \to B/\mathcal{O}$ and given a $\mathbb{Z}$-basis $\{e_i\}_{i=1\cdots 4}$ of $\mathcal{O}$, we define the morphism $\varphi_i(P) = x_i$, where $\varphi(P) = \sum_{i=1\cdots 4} x_i e_i$. It is clear that $\{\varphi_i\}_{i=1\cdots 4}$ provides a $\mathbb{Z}$-basis of $\mathrm{Hom}(A_{tor}, \mathbb{Q}/\mathbb{Z})$. □

Since $A/M$ is an abelian $L$-surface with QM, we can fix isogenies $\mu_\sigma : A^\sigma \to A$, for all $\sigma \in \mathrm{Gal}(\bar{L}/L)$. We denote $\bar{\rho}_{(A,\imath,\varphi)}(\sigma)$ the element in $\mathrm{End}(B/\mathcal{O})$ satisfying

$$\varphi(\mu_\sigma(P^\sigma)) = \varphi(P)\bar{\rho}_{(A,\imath,\varphi)}(\sigma), \text{ for all } P \in A_{tor}.$$

We compute that, for all $\alpha \in \mathcal{O}$,

$$\begin{aligned}(\alpha\varphi(P))\,\bar{\rho}_{(A,\imath,\varphi)}(\sigma) &= \varphi(\mu_\sigma((\imath(\alpha)P)^\sigma)) = \varphi(\mu_\sigma(\imath(\alpha)^\sigma(P^\sigma))) = \varphi(\imath(\alpha)\mu_\sigma(P^\sigma)) \\ &= \alpha\varphi(\mu_\sigma(P^\sigma)) = \alpha\left(\varphi(P)\bar{\rho}_{(A,\imath,\varphi)}(\sigma)\right).\end{aligned}$$

Thus $\bar{\rho}_{(A,\imath,\varphi)}(\sigma) \in \mathrm{End}_{\mathcal{O}}(B/\mathcal{O})$. Since $\mu_\sigma$ has finite kernel, we deduce that $\bar{\rho}_{(A,\imath,\varphi)}(\sigma) \in \mathrm{End}^0_{\mathcal{O}}(B/\mathcal{O})^\times$. Once we identify $\mathrm{End}^0_{\mathcal{O}}(B/\mathcal{O})^\times$ with $G(\mathbb{A}_f)$, we have a map

$$\bar{\rho}_{(A,\imath,\varphi)} : \mathrm{Gal}(\bar{L}/L) \longrightarrow G(\mathbb{A}_f).$$

This map may depend on the choice of the isogenies $\mu_\sigma$, nevertheless we can consider the quotient $G(\mathbb{A}_f)/\mathrm{End}^0(A,\imath)^\times$, where $\mathrm{End}^0(A,\imath)^\times$ is embedded in $G(\mathbb{A}_f)$ by means of the natural embedding

$$\varphi^* : \mathrm{End}^0(A,\imath)^\times \hookrightarrow G(\mathbb{A}_f) = \mathrm{End}^0_{\mathcal{O}}(B/\mathcal{O})^\times; \qquad \varphi^*(\lambda) = \lambda^* = \varphi \circ \lambda \circ \varphi^{-1}.$$

Hence the composition with the quotient map, gives rise to a map of the form:

$$\rho_{(A,\imath,\varphi)} : \mathrm{Gal}(\bar{L}/L) \longrightarrow G(\mathbb{A}_f)/\mathrm{End}^0(A,\imath)^\times.$$

**Lemma 2.3.** *The map $\rho_{(A,\imath,\varphi)}$ is independent on the choice of the isogenies $\mu_\sigma$.*

*Proof.* Let $\mu'_\sigma : A^\sigma \to A$ be another isogeny. Then $\lambda_\sigma := \frac{1}{\deg(\mu_\sigma)}\mu'_\sigma \circ \mu_\sigma^\vee \in \mathrm{End}^0(A,\imath)^\times$. We denote by $\rho'_{(A,\imath,\varphi)}$ the element in $G(\mathbb{A}_f)$ obtained using $\mu'_\sigma$ instead of $\mu_\sigma$. Thus we have:

$$\begin{aligned}\varphi(P)\bar\rho'_{(A,\imath,\varphi)}(\sigma) &= \varphi(\mu'_\sigma(P^\sigma)) = \varphi(\lambda_\sigma(\mu_\sigma(P^\sigma))) = \varphi(\mu_\sigma(P^\sigma))\lambda_\sigma^* \\ &= \varphi(P)\bar\rho_{(A,\imath,\varphi)}(\sigma)\lambda_\sigma^*.\end{aligned}$$

Thus $\bar\rho'_{(A,\imath,\varphi)}(\sigma) = \bar\rho_{(A,\imath,\varphi)}(\sigma)\lambda_\sigma^*$ and

$$\bar\rho'_{(A,\imath,\varphi)}(\sigma)\mathrm{End}^0(A,\imath)^\times = \bar\rho_{(A,\imath,\varphi)}(\sigma)\mathrm{End}^0(A,\imath)^\times,$$

which proves our assertion. □

Note that $G(\mathbb{A}_f)/\mathrm{End}^0(A,\imath)^\times$ is a group in the non-CM case. Nevertheless, in the CM case, $\mathrm{End}(A,\imath)^0 = K$ an imaginary quadratic field, hence $K^\times$ it is not normal in $B(\mathbb{A}_f)^\times$.

The embedding $\mathrm{End}(A,\imath) \hookrightarrow \mathcal{O}^{opp}$ gives rise to an embedding of groups, $\mathrm{End}(A,\imath) \hookrightarrow G(R)$, for all rings $R$. Let us denote by $N_A/\mathbb{Z}$ be the normalizer group scheme of $\mathrm{End}(A,\imath)$ in $G$, namely, the group scheme over $\mathbb{Z}$ such that $N_A(R)$ is the normalizer of $\mathrm{End}(A,\imath)$ in $G(R)$. Note that $N_A = G$, in the non-CM case. Moreover, if $(A,\imath)$ has CM by the imaginary quadratic field $K$, then $N_A(\mathbb{Q}) = K^\times \cup jK^\times$, with $j^2 \in \mathbb{Q}^\times$ and $jk = \bar{k}j$ for all $k \in K^\times$. In any case $N_A(\mathbb{A}_f)/\mathrm{End}^0(A,\imath)^\times$ is now a group.

**Lemma 2.4.** *The map $\rho_{(A,\imath,\varphi)}$ factors through*

$$\rho_{(A,\imath,\varphi)} : \mathrm{Gal}(\bar{L}/L) \stackrel{\rho^N_{(A,\imath,\varphi)}}{\longrightarrow} N_A(\mathbb{A}_f)/\mathrm{End}^0(A,\imath)^\times \hookrightarrow G(\mathbb{A}_f)/\mathrm{End}^0(A,\imath)^\times$$

*Moreover, the map $\rho^N_{(A,\imath,\varphi)}$ is a group homomorphism.*

*Proof.* On the one side, for all $\sigma \in \mathrm{Gal}(\bar{L}/L)$ and $\lambda \in \mathrm{End}(A,\imath)$ we have

$$\bar\rho_{(A,\imath,\varphi)}(\sigma)\lambda^*\bar\rho_{(A,\imath,\varphi)}(\sigma)^{-1} \in \mathrm{End}(A,\imath)^0.$$

Indeed,

$$\begin{aligned}(\deg\mu_\sigma)\varphi(P)\bar\rho_{(A,\imath,\varphi)}(\sigma)\lambda^*\bar\rho_{(A,\imath,\varphi)}(\sigma)^{-1} &= (\deg\mu_\sigma)\varphi(\lambda(\mu_\sigma(P^\sigma)))\bar\rho_{(A,\imath,\varphi)}(\sigma)^{-1} \\ &= \varphi(\mu_\sigma^\vee(\lambda(\mu_\sigma(P^\sigma)))^{\sigma^{-1}}) \\ &= \varphi((\mu_\sigma^\vee\lambda\mu_\sigma)^{\sigma^{-1}}(P)) \\ &= \varphi(P)\left((\mu_\sigma^\vee\lambda\mu_\sigma)^{\sigma^{-1}}\right)^*,\end{aligned}$$

where clearly $(\mu_\sigma^\vee\lambda\mu_\sigma)^{\sigma^{-1}} \in \mathrm{End}^0(A,\imath)$. Therefore $\bar\rho_{(A,\imath,\varphi)}(\sigma) \in N_A(\mathbb{A}_f)$.

On the other side, one checks that

$$\bar\rho_{(A,\imath,\varphi)}(\sigma\tau)^{-1}\bar\rho_{(A,\imath,\varphi)}(\sigma)\bar\rho_{(A,\imath,\varphi)}(\tau)$$

acts on $\hat{T}A \otimes \mathbb{Q} := (\prod'_p T_pA) \otimes \mathbb{Q}$ in the same way as does

$$c_{(A,\imath)}(\sigma,\tau) = (1/\deg(\mu_{\sigma\tau}))\mu_\sigma\mu_\tau^\sigma\mu_{\sigma\tau}^\vee \in (\mathrm{End}(A,\imath)\otimes_{\mathbb{Z}}\mathbb{Q})^\times = \mathrm{End}^0(A,\imath)^\times.$$

In particular, the quotient $\rho_{(A,\imath,\varphi)}(\sigma)$ is a group homomorphism. □

**Remark 2.5.** Assume that the discriminant $D = 1$, thus the quaternion algebra $B = \mathrm{M}_2(\mathbb{Q})$. An abelian surface with QM by $\mathcal{O} = \mathrm{M}_2(\mathbb{Z})$ is the product $A = E \times E$, where $E$ is an elliptic curve. In the particular case that $E$ is defined over $L$ (thus clearly $A = E \times E$ is an abelian $L$-surface with QM), the representation $\rho_{(A,\imath,\varphi)}$ is just the quotient modulo $\mathrm{End}^0(A,\imath)^\times = \mathrm{End}^0(E)^\times$ of the classical action on the Tate module

$$\rho_E : \mathrm{Gal}(\bar{L}/L) \longrightarrow \mathrm{GL}_2(\hat{\mathbb{Z}}) = \prod_\ell \mathrm{GL}_2(\mathbb{Z}_\ell) \hookrightarrow \mathrm{GL}_2(\mathbb{A}_f).$$

## 3. Shimura curves and isogeny classes

Assume that $\mathcal{O}_0$ is a maximal order in $B$, let $\Gamma$ be an open subgroup of $\hat{\mathcal{O}}_0^\times = G(\hat{\mathbb{Z}})$. We say that two $\mathcal{O}_0$-module isomorphisms $\varphi, \varphi' : A_{tor} \stackrel{\simeq}{\to} B/\mathcal{O}_0$ are $\Gamma$-*equivalent* if there exists a $\gamma \in \Gamma$ such that $\varphi' = \varphi\gamma$. The *Shimura curve* $X_\Gamma$, is the compactification of the coarse moduli space of triples $(A, \imath, \bar{\varphi})$, where $(A, \imath)$ are abelian surfaces with QM by $\mathcal{O}_0$ and $\bar{\varphi}$ is a $\Gamma$-equivalence class of $\mathcal{O}_0$-module isomorphisms $\varphi : A_{tor} \stackrel{\simeq}{\to} B/\mathcal{O}_0$. The curve $X_\Gamma$ is defined over some number field $L_\Gamma$. If $k$ is a field of characteristic zero, given $P \in X_\Gamma(\bar{k})$ corresponding to the isomorphism class of a triple $(A, \imath, \bar{\varphi})/\bar{k}$, its Galois conjugate $P^\sigma \in X_\Gamma(\bar{k})$, for $\sigma \in \mathrm{Gal}(\bar{k}/k)$, corresponds to the isomorphism class of $(A^\sigma, \imath^\sigma, \bar{\varphi}^\sigma)$, where

$$\varphi^\sigma : A^\sigma_{tor} \stackrel{\simeq}{\longrightarrow} B/\mathcal{O}_0; \qquad \varphi^\sigma(Q^\sigma) = \varphi(Q).$$

Thus, a $k$-rational point $P$ on $X_\Gamma$ corresponds to the isomorphism class of a triple $(A, \imath, \bar{\varphi})/\bar{k}$ which is isomorphic to all its $\mathrm{Gal}(\bar{k}/k)$-conjugates.

The complex points of the Shimura curve are in correspondence with the double coset space

$$X_\Gamma(\mathbb{C}) = (\Gamma_\infty\Gamma\backslash G(\mathbb{A}))/G(\mathbb{Q}) \cup \{\text{cusps}\}, \quad \Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \right\}.$$

The triple $(A_g, \imath_g, \bar{\varphi}_g)$ over $\mathbb{C}$ corresponding to $g = (g_\infty, g_f) \in G(\mathbb{A})$ is $A_g := (B \otimes \mathbb{R})_{g_\infty}/I_{g_f}$, where $I_{g_f} = \hat{\mathcal{O}}_0 g_f \cap B$ and $(B \otimes \mathbb{R})_{g_\infty} = \mathrm{M}_2(\mathbb{R})$ with complex structure $h_{g_\infty}$

$$h_{g_\infty} : \mathbb{C} \to \mathrm{M}_2(\mathbb{R});\ i \mapsto g_\infty^{-1} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} g_\infty;$$

the embedding $\imath_g : \mathcal{O}_0 \to \mathrm{End}(A_g)$, is given by $\imath_g(\alpha)(b \otimes z) = \alpha b \otimes z$; and $\bar{\varphi}_g$ is the $\Gamma$-equivalence class of $\varphi_g : (A_g)_{tor} = B/I_{g_f} \to B/\mathcal{O}_0$, $\varphi_g(b) = bg_f^{-1}$. We compute that

$$\begin{aligned}
\mathrm{End}^0(A_g, \imath_g)^\times &= \{\gamma \in \mathrm{Aut}_B(B \otimes \mathbb{R}) :\ \gamma I_{g_f} \otimes \mathbb{Q} = I_{g_f} \otimes \mathbb{Q} \text{ and } \gamma h_{g_\infty} = h_{g_\infty}\gamma\} \\
&= \{\gamma \in G(\mathbb{R}) :\ \gamma B = B \text{ and } \gamma h_{g_\infty}\gamma^{-1} = h_{g_\infty}\} \\
&= \{\gamma \in G(\mathbb{Q}) :\ \gamma h_{g_\infty}\gamma^{-1} = h_{g_\infty}\} \\
&= \{\gamma \in G(\mathbb{Q}) :\ g_\infty\gamma g_\infty^{-1} \in \Gamma_\infty\}.
\end{aligned}$$

**Remark 3.1.** In most of the literature, objects classified by the Shimura curve $X_\Gamma$ are triples $(A, \imath, \bar{\psi})$, where $(A, \imath)$ is an abelian surface with QM by $\mathcal{O}_0$ as above and $\bar{\psi}$ is a $\Gamma$-equivalence class of $\mathcal{O}_0$-module isomorphisms $\psi : \hat{T}(A) = \mathrm{Hom}(A_{tor}, \mathbb{Q}/\mathbb{Z}) \stackrel{\simeq}{\to} \hat{\mathcal{O}}_0$. It is clear that this interpretation is equivalent to ours, since for any $\varphi : A_{tor} \stackrel{\simeq}{\to} B/\mathcal{O}_0$ we have the corresponding isomorphism

$$\psi : \hat{T}(A) = \mathrm{Hom}(A_{tor}, \mathbb{Q}/\mathbb{Z}) \stackrel{\simeq}{\longrightarrow} \mathrm{Hom}(B/\mathcal{O}_0, \mathbb{Q}/\mathbb{Z}) \simeq \hat{\mathcal{O}}_0.$$

**Remark 3.2.** In the particular case that $\Gamma = \Gamma_N = \ker(G(\hat{\mathbb{Z}}) \to G(\mathbb{Z}/N\mathbb{Z}))$, to give a $\Gamma$-equivalence class of isomorphisms $\varphi : A_{tor} \to B/\mathcal{O}_0$ is equivalent to give an isomorphism $\varphi_N : A[N] \to \mathcal{O}_0/N\mathcal{O}_0$, namely, a level-$N$-structure. This is the classical Shimura curve situation.

We say that two triples $(A, \imath, \bar{\varphi})$ and $(A', \imath', \bar{\varphi}')$ are isogenious if there exist an isogeny $\phi : A' \to A$, satisfying $\phi \circ \imath'(\alpha) = \imath(\alpha) \circ \phi$, for all $\alpha \in \mathcal{O}$. We denote by $\phi : (A', \imath') \to (A, \imath)$ the isogeny with the corresponding compatibility with respect to $\imath$ and $\imath'$.

Let $P \in X_\Gamma(\mathbb{C})$ be a point corresponding to $(A, \imath, \bar{\varphi})$. Let us denote by $[P]$ the $\mathbb{C}$-isogeny class of $(A, \imath, \bar{\varphi})/\mathbb{C}$ in $X_\Gamma$, namely, the set of points $Q \in X_\Gamma(\mathbb{C})$ parametrizing triples $(A', \imath', \bar{\varphi}')/\mathbb{C}$ where $(A', \imath')$ is isogenious to $(A, \imath)$.

**Proposition 3.3.** *Let* $P = [g] = [g_\infty, 1] \in (\Gamma_\infty\Gamma\backslash G(\mathbb{A}))/G(\mathbb{Q}) \subseteq X_\Gamma(\mathbb{C})$. *Then we have the following bijection*

$$\psi_{g_\infty} : \Gamma\backslash G(\mathbb{A}_f)/\mathrm{End}^0(A_g, \imath_g)^\times \xrightarrow{\simeq} [P]; \qquad g_f \longmapsto [g_\infty, g_f].$$

*Proof.* The non-CM case is described in [1, Lemma 1], we give here a proof that works in any case. Recall that $(A_{g_\infty}, \imath_{g_\infty}, \bar\varphi_{g_\infty})$ is the triple corresponding to $P$. For any $g_f \in G(\mathbb{A}_f)$, there exists $n \in \mathbb{Z}$ such that $I_{g_f} n \subseteq \mathcal{O}_0$, therefore we have the isogeny

$$A_{g_\infty g_f} = (B \otimes \mathbb{R})_{g_\infty}/I_{g_f} \longrightarrow (B \otimes \mathbb{R})_{g_\infty}/\mathcal{O}_0 = A_{g_\infty}, \quad b \longmapsto nb,$$

which is clearly compatible with the embeddings $\imath_{g_\infty}$ and $\imath_{g_\infty g_f}$ since the inclusion $I_{g_f} n \subseteq \mathcal{O}_0$ is a monomorphism of $\mathcal{O}_0$-modules. This implies $[g_\infty, g_f] \in [P]$, for all $g_f \in G(\mathbb{A}_f)$.

Conversely, any isogeny $(A_{g'_\infty g_f}, \imath_{g'_\infty g_f}) \to (A_{g_\infty}, \imath_{g_\infty})$ induces an equality of complex structures $(B \otimes \mathbb{R})_{g'_\infty} = (B \otimes \mathbb{R})_{g_\infty}$, that implies that $g'_\infty = \Gamma_\infty g_\infty$. Therefore the corresponding point $[g'_\infty, g_f]$ has a representant of the form $[g_\infty, g'_f]$ in the double coset space $(\Gamma_\infty\Gamma\backslash G(\mathbb{A}))/G(\mathbb{Q})$.

We conclude that we have a surjective map

$$\Gamma\backslash G(\mathbb{A}_f) \longrightarrow [P]; \qquad g_f \longmapsto [g_\infty, g_f],$$

and the result follows from the fact that $[g_\infty, g_f] = [g_\infty, g'_f]$ in $(\Gamma_\infty\Gamma\backslash G(\mathbb{A}))/G(\mathbb{Q})$ if and only if there exists $\beta \in G(\mathbb{Q})$ such that $g_f = g'_f\beta$ and $g_\infty\beta \in \Gamma_\infty g_\infty$, hence $\beta \in \mathrm{End}^0(A_{g_\infty}, \imath_{g_\infty})^\times$. □

**Remark 3.4.** Let us consider the natural map

$$X_\Gamma \supseteq (\Gamma_\infty\Gamma\backslash G(\mathbb{A}))/G(\mathbb{Q}) \xrightarrow{\psi} \Gamma_\infty\backslash G(\mathbb{R})/G(\mathbb{Q}), \qquad [g_\infty, g_f] \longmapsto [g_\infty].$$

Then it is clear that, if $P = [g_\infty, g_f]$, the isogeny class $[P]$ is the fiber of $\psi$ over $[g_\infty]$.

## 4. Galois action on isogeny classes

Assume now that $(A, \imath)$ is an abelian $L$-surface with QM by $\mathcal{O}_0$, let $(A, \imath, \bar\varphi)$ be a triple corresponding to the point $P \in X_\Gamma(\bar{\mathbb{Q}})$. First we show that any $(A', \imath')$ isogenous to $(A, \imath)$ is an abelian $L$-surface with QM.

**Lemma 4.1.** *Let* $(A, \imath)/\bar L$ *be an abelian* $L$*-surface with QM and assume that* $(A', \imath')/\bar L$ *is isogenous to* $(A, \imath)$. *Then* $(A', \imath')/\bar L$ *is an abelian* $L$*-surface with QM.*

*Proof.* Let $\sigma \in \mathrm{Gal}(\bar L/L)$. Since $(A, \imath)$ is a $L$ abelian surface with QM, there exists an isogeny $(A^\sigma, \imath^\sigma) \xrightarrow{\mu_\sigma} (A, \imath)$. Fix an isogeny $(A', \imath') \xrightarrow{\phi} (A, \imath)$. Thus by conjugating $\phi$ by $\sigma$ and composing with $\phi^\vee \circ \mu_\sigma$, one obtains

$$((A')^\sigma, (\imath')^\sigma) \xrightarrow{\phi^\sigma} (A^\sigma, \imath^\sigma) \xrightarrow{\mu_\sigma} (A, \imath) \xrightarrow{\phi^\vee} (A', \imath').$$

Hence $(A', \imath')/\bar L$ is an abelian $L$-surface with QM. □

Note that, since $P$ and so $(A, \imath)$ are defined over $\bar{\mathbb{Q}}$, the $\mathbb{C}$-isogeny class coincide with the $\bar{\mathbb{Q}}$-isogeny class $[P]$. Moreover, the above lemma implies that $\mathrm{Gal}(\bar L/L)$ acts on $[P]$. Indeed, if $Q \in [P]$ corresponds to $(A', \imath', \bar\varphi')$ and $\sigma \in \mathrm{Gal}(\bar L/L)$, then $Q^\sigma$ parametrizes $((A')^\sigma, (\imath')^\sigma, (\bar\varphi')^\sigma)$. Since $(A', \imath')$ is an L-abelian surface with QM by the lemma, there exists $\mu'_\sigma : ((A')^\sigma, (\imath')^\sigma) \to (A', \imath')$. This implies $((A')^\sigma, (\imath')^\sigma)$ is isogenous to $(A, \imath)$, hence $Q^\sigma \in [P]$. The main theorem of this section relates this action with the map $\rho_{(A,\imath,\varphi)}$ introduced in §2 by means of the characterization of $[P]$ given in Proposition 3.3.

**Theorem 4.2.** *Assume that $P = [g_\infty, 1] \in X_\Gamma$ corresponds to a triple $(A, \imath, \bar{\varphi})$, where $(A, \imath)/\bar{\mathbb{Q}}$ is an abelian $L$-surface with QM and $\bar{\varphi}$ is the $\Gamma$-equivalent class of the natural isomorphism*

$$\varphi : A_{tor} = ((B \otimes \mathbb{R})_{g_\infty}/\mathcal{O}_0)_{tor} \longrightarrow B/\mathcal{O}_0,$$

*Then the map $\rho_{(A,\imath,\varphi)} : \mathrm{Gal}(\bar{L}/L) \longrightarrow G(\mathbb{A}_f)/\mathrm{End}^0(A, \imath)^\times$ constructed by means of $\varphi$ satisfies*

$$\psi_{g_\infty}(g_f)^\sigma = \psi_{g_\infty}(g_f \rho_{(A,\imath,\varphi)}(\sigma)) \in [P],$$

*for all $g_g \in G(\mathbb{A}_f)$ and $\sigma \in \mathrm{Gal}(\bar{L}/L)$.*

**Remark 4.3.** Note that, by Lemma 2.4, the image $\rho_{(A,\imath,\varphi)}(\sigma)$ lies in the commutator of $\mathrm{End}^0(A, \imath)$ in $G(\mathbb{A}_f)$, thus the product $g_f \rho_{(A,\imath,\varphi)}(\sigma)$ is well defined in $\Gamma\backslash G(\mathbb{A}_f)/\mathrm{End}^0(A, \imath)$.

*Proof.* Recall that the abelian surface corresponding to $\psi_{g_\infty}(g_f)$ is given by the complex torus $A_{g_f} = (B \otimes \mathbb{R})_{g_\infty}/I_{g_f}$, where $I_{g_f} = B \cap \hat{\mathcal{O}}_0 g_f$. Moreover, considering a representative of $\Gamma g_f \mathrm{End}^0(A, \imath)^\times$ such that $g_f^{-1} \in \hat{\mathcal{O}}_0$, the $\mathcal{O}_0$-stable isogeny between $A$ and $A_{g_f}$ is given by

$$\phi_{g_f} : A = (B \otimes \mathbb{R})_{g_\infty}/\mathcal{O}_0 \longrightarrow (B \otimes \mathbb{R})_{g_\infty}/I_{g_f} = A_{g_f}, \quad b \longmapsto b.$$

Also recall that a representative of $\bar{\varphi}_{g_f}$ is given by

$$\varphi_{g_f} : (A_{g_f})_{tor} = ((B \otimes \mathbb{R})_{g_\infty}/I_{g_f})_{tor} = B/I_{g_f} \longrightarrow B/\mathcal{O}_0, \quad b \longmapsto b g_f^{-1}.$$

Thus one checks that $\varphi_{g_f} \circ \phi_{g_f} = \varphi g_f^{-1} : A_{tor} \to B/\mathcal{O}_0$.

For any $\sigma \in \mathrm{Gal}(\bar{L}/L)$, the point $\psi_{g_\infty}(g_f)^\sigma$ corresponds to the triple $(A_{g_f}^\sigma, \imath_{g_f}^\sigma, \bar{\varphi}_{g_f}^\sigma)$. We have the following isogenies

$$(A_{g_f}^\sigma, \imath_{g_f}^\sigma) \xleftarrow{\phi_{g_f}^\sigma} (A^\sigma, \imath^\sigma) \xrightarrow{\mu_\sigma} (A, \imath) \xrightarrow{\phi_{g_f}} (A_{g_f}, \imath_{g_f}),$$

thus $(A_{g_f}^\sigma, \imath_{g_f}^\sigma)$ and $(A, \imath)$ are linked by the isogeny $\phi_{g_f}^\sigma \circ \mu_\sigma^\vee$. This implies that, for all $P \in A_{tor}$, we have $\varphi(P)\psi_{g_\infty}^{-1}(\psi_{g_\infty}(g_f)^\sigma)^{-1} = \varphi_{g_f}^\sigma(\phi_{g_f}^\sigma(\mu_\sigma^\vee(P)))$, hence

$$\begin{aligned} \varphi(\mu_\sigma(P^\sigma))\psi_{g_\infty}^{-1}(\psi_{g_\infty}(g_f)^\sigma)^{-1} &= \deg(\mu_\sigma)\varphi_{g_f}^\sigma(\phi_{g_f}^\sigma(P^\sigma)) \\ \varphi(P)\rho_{(A,\imath,\varphi)}(\sigma)\psi_{g_\infty}^{-1}(\psi_{g_\infty}(g_f)^\sigma)^{-1} &= \deg(\mu_\sigma)\varphi_{g_f}^\sigma(\phi_{g_f}(P)^\sigma) \\ &= \deg(\mu_\sigma)\varphi_{g_f}(\phi_{g_f}(P)) \\ &= \deg(\mu_\sigma)\varphi(P)g_f^{-1} \end{aligned}$$

We conclude $g_f \rho_{(A,\imath,\varphi)}(\sigma) = \psi_{g_\infty}^{-1}(\psi_{g_\infty}(g_f)^\sigma)$, thus $\psi_{g_\infty}(g_f \rho_{(A,\imath,\varphi)}(\sigma)) = \psi_{g_\infty}(g_f)^\sigma$. □

## 5. Change of moduli interpretation

In section §2, we defined an abelian $L$-surface $(A, \imath)$ with QM by any Eichler order $\mathcal{O}$ and defined the corresponding representation $\rho_{(A,\imath,\varphi)}$ attached to a fixed $\mathcal{O}$-module isomorphism $\varphi : A_{tor} \to B/\mathcal{O}$. Nevertheless, we used a maximal order $\mathcal{O}_0$ to define the Shimura curve $X_\Gamma$ and to describe its moduli interpretation as the space classifying triples $(A_0, \imath_0, \bar{\varphi}_0)$, where $(A_0, \imath_0)$ has QM by $\mathcal{O}_0$. In this section we shall change this moduli interpretation for some of this Shimura curves $X_\Gamma$ in order to classify abelian surfaces with QM by $\mathcal{O}$.

Thus from now on $\mathcal{O}$ will be an Eichler order in $B$ of level $N$ and $\mathcal{O}_0$ a maximal order such that $\mathcal{O} \subseteq \mathcal{O}_0$. Fix the embedding $\lambda : \mathcal{O} \hookrightarrow \mathcal{O}_0$. Let $\Gamma$ be now an open subgroup of $\hat{\mathcal{O}}^\times = (\mathcal{O} \otimes \hat{\mathbb{Z}})^\times$. Since $\hat{\mathcal{O}}^\times$ is an open subset of $G(\hat{\mathbb{Z}}) = \hat{\mathcal{O}}_0^\times$ by means of $\lambda$, the subgroup $\Gamma$ is also an open subgroup of $G(\hat{\mathbb{Z}})$. Thus we can consider the Shimura curve $X_\Gamma$.

**Proposition 5.1.** *We have an equivalence of moduli interpretations for the Shimura curve $X_\Gamma$. It either classifies:*

(i) *Triples $(A_0, \imath_0, \bar\varphi_0)$, where $(A_0, \imath_0)$ is an abelian surface with QM by $\mathcal{O}_0$ and $\bar\varphi_0$ is a $\Gamma$-equivalence class of $\mathcal{O}_0$-module isomorphisms $\varphi_0 : (A_0)_{tor} \to B/\mathcal{O}_0$.*

(ii) *Triples $(A, \imath, \bar\varphi)$, where $(A, \imath)$ is an abelian surface with QM by $\mathcal{O}$ and $\bar\varphi$ is a $\Gamma$-equivalence class of $\mathcal{O}$-module isomorphisms $\varphi : A_{tor} \to B/\mathcal{O}$.*

In order to prove this proposition we will need the following lemma. Note that the embedding $\lambda : \mathcal{O} \hookrightarrow \mathcal{O}_0$ gives rise to a morphism $\lambda : B/\mathcal{O} \to B/\mathcal{O}_0$.

**Lemma 5.2.** *There exists a one-to-one correspondence between triples $(A, \imath, \varphi)$, where $(A, \imath)$ is an abelian surface with QM by $\mathcal{O}$ and $\varphi$ is a $\mathcal{O}$-module isomorphism $\varphi : A_{tor} \to B/\mathcal{O}$, and triples $(A_0, \imath_0, \varphi_0)$, where $(A_0, \imath_0)$ is an abelian surface with QM by $\mathcal{O}_0$ and $\varphi_0$ is a $\mathcal{O}_0$-module isomorphism $\varphi_0 : (A_0)_{tor} \to B/\mathcal{O}_0$. A triple $(A, \imath, \varphi)$ corresponds to $(A_0, \imath_0, \varphi_0)$ if there exists an isogeny $\phi : A \to A_0$, such that $\varphi_0 \circ \phi = \lambda \circ \varphi$ and $\phi \circ \imath(\alpha) = \imath_0(\lambda(\alpha)) \circ \phi$, for all $\alpha \in \mathcal{O}$.*

*Proof.* Given $(A, \imath, \varphi)$, let us consider the subgroup $C := \varphi^{-1}(\ker(B/\mathcal{O} \overset{\lambda}{\to} B/\mathcal{O}_0)) \subset A_{tor}$. Therefore, we can construct the abelian surface $A_0 = A/C$ and the corresponding isogeny $\phi : A \to A_0$. Since $\mathcal{O} \subseteq \mathcal{O}_0$, for all $\alpha \in \mathcal{O}$, we have $\alpha(\ker\lambda) \subseteq \ker\lambda$, hence, $\imath(\alpha)C \subseteq C$ and the embedding $\imath$ gives rise to an embedding $\imath_0 : \mathcal{O} \hookrightarrow \mathrm{End}(A_0)$. Moreover, the $\mathcal{O}$-module isomorphim $\varphi$ gives rise to a $\mathcal{O}$-module isomorphism $\varphi_0$ that fits into the following commutative diagram:

$$\begin{array}{ccc} A_{tor} & \xrightarrow{\varphi} & B/\mathcal{O} \\ \phi\downarrow & & \downarrow\lambda \\ (A_0)_{tor} = A_{tor}/C & \xrightarrow{\varphi_0} & B/\mathcal{O}_0 \end{array}$$

Hence $\varphi_0 \circ \phi = \lambda \circ \varphi$. Moreover, the fact that $(A_0)_{tor} \simeq B/\mathcal{O}_0$ as $\mathcal{O}$-modules implies that $\imath_0$ can be extended to an embedding $\imath_0 : \mathcal{O}_0 \hookrightarrow \mathrm{End}(A_0)$. Thus $(A_0, \imath_0)$ has QM by $\mathcal{O}_0$. We have constructed the triple $(A_0, \imath_0, \varphi_0)$ corresponding to $(A, \imath, \varphi)$.

Finally, given $(A_0, \imath_0, \varphi_0)$, let us consider $C^\vee := \varphi_0^{-1}(\ker(B/\mathcal{O}_0 \overset{\lambda^\vee}{\to} B/\mathcal{O}))$, where $\lambda^\vee : B/\mathcal{O}_0 \to B/\mathcal{O}$ is given by $\lambda^\vee(b) = [\mathcal{O}_0 : \mathcal{O}]\lambda^{-1}(b)$. We define $A := A_0/C^\vee$, thus the isogeny with kernel $C^\vee$ is the dual isogeny of some $\phi : A \to A_0$ that fits into the above commutative diagram for some $\mathcal{O}$-module isomorphism $\varphi$. We construct the triple $(A, \imath, \varphi)$ corresponding to $(A_0, \imath_0, \varphi_0)$ as in the previous situation.

□

Due to this previous lemma we can easily prove the above proposition:

*Proof of Proposition 5.1.* We know that the Shimura curve $X_\Gamma$ classify triples $(A_0, \imath_0, \bar\varphi_0)$ as in $(i)$. By the above Lemma, given a representative $\varphi_0$ of the $\Gamma$-equivalence class $\bar\varphi_0$, there exists a triple $(A, \imath, \varphi)$, where $\varphi$ is a $\mathcal{O}$-module isomorphism. It is clear that the $\Gamma$-equivalence class $\bar\varphi_0$ corresponds to a $\Gamma$-equivalence class $\bar\varphi$. □

**Definition 5.3.** A *triple with QM by $\mathcal{O}$* is a triple $(A, \imath, \varphi)$, where $(A, \imath)$ is an abelian surface with QM by $\mathcal{O}$ and $\varphi$ is a $\mathcal{O}$-module isomorphism $\varphi : A_{tor} \to B/\mathcal{O}$. A *$L$-triple with QM by $\mathcal{O}$* is a triple $(A, \imath, \varphi)$ with QM by $\mathcal{O}$ such that $(A, \imath)$ is an abelian $L$-surface with QM.

We denote the one-to-one correspondence of Lemma 5.2 by

$$\Lambda_{\mathcal{O}}^{\mathcal{O}_0} : \{\text{Triples with QM by } \mathcal{O}\} \longrightarrow \{\text{Triples with QM by } \mathcal{O}_0\}$$

Note that, given a triple $(A, \imath, \varphi)$ with QM by $\mathcal{O}$, one can construct the projective representation

$$\rho_{(A,\imath,\varphi)} : \mathrm{Gal}(\bar L/L) \longrightarrow G(\mathbb{A}_f)/\mathrm{End}^0(A, \imath)^\times.$$

The following result relates the representations attached to triples associated by the correspondence $\Lambda_{\mathcal{O}}^{\mathcal{O}_0}$.

**Lemma 5.4.** *Let $(A,\imath,\varphi)$ be a $L$-triple with QM by $\mathcal{O}$. Assume that $\Lambda_{\mathcal{O}}^{\mathcal{O}_0}(A,\imath,\varphi)=(A_0,\imath_0,\varphi_0)$, then we have that $(A_0,\imath_0,\varphi_0)$ is a $L$-triple with QM by $\mathcal{O}_0$ and*

$$\rho_{(A,\imath,\varphi)}=\rho_{(A_0,\imath_0,\varphi_0)},$$

*when we identify $G(\mathbb{A}_f)=\mathrm{End}^0_{\mathcal{O}}(B/\mathcal{O})^\times=\mathrm{End}^0_{\mathcal{O}_0}(B/\mathcal{O}_0)^\times$ by means of $\lambda$.*

*Proof.* We know that there exists an isogeny $\phi:A\to A_0$, such that $\varphi_0\circ\phi=\lambda\circ\varphi$ and $\phi\circ\imath(\alpha)=\imath_0(\lambda(\alpha))\circ\phi$, for all $\alpha\in\mathcal{O}$. Since $(A,\imath)$ is an abelian $L$-surface with QM, there exists isogenies $\mu_\sigma:A^\sigma\to A$, for all $\sigma\in\mathrm{Gal}(\bar{L}/L)$, such that $\mu_\sigma\circ\imath(o)^\sigma=\imath(o)\circ\mu_\sigma$ for all $o\in\mathcal{O}$. The composition

$$\mu_\sigma^0:A_0^\sigma\stackrel{(\phi^\sigma)^\vee}{\longrightarrow}A^\sigma\stackrel{\mu_\sigma}{\longrightarrow}A\stackrel{\phi}{\longrightarrow}A_0,$$

satisfies

$$\begin{aligned}\mu_\sigma^0\circ\imath_0(\lambda(o))^\sigma &= \phi\circ\mu_\sigma\circ(\phi^\sigma)^\vee\circ\imath_0(\lambda(o))^\sigma=\phi\circ\mu_\sigma\circ\imath(o)^\sigma\circ(\phi^\sigma)^\vee\\ &= \phi\circ\imath(o)\circ\mu_\sigma\circ(\phi^\sigma)^\vee=\imath_0(\lambda(o))\circ\mu_\sigma^0,\quad\text{for all } o\in\mathcal{O},\end{aligned}$$

thus $\mu_\sigma^0\circ\imath_0(\alpha)^\sigma=\imath_0(\alpha)\circ\mu_\sigma^0$ for all $\alpha\in\mathcal{O}_0$. This implies that $(A_0,\imath_0)$ is an abelian $L$-surface with QM.

Moreover, for all $\sigma\in\mathrm{Gal}(\bar{L}/L)$ and $P\in(A_0)_{tor}$, we have

$$\begin{aligned}\varphi_0(\mu_\sigma^0(P^\sigma)) &= \varphi_0(\phi\circ\mu_\sigma\circ(\phi^\sigma)^\vee(P^\sigma))=\lambda(\varphi(\mu_\sigma(\phi^\vee(P)^\sigma)))\\ &= \lambda(\varphi(\phi^\vee(P)))\rho_{(A,\imath,\varphi)}(\sigma)=\deg(\phi)\varphi_0(P)\rho_{(A,\imath,\varphi)}(\sigma).\end{aligned}$$

This implies $\rho_{(A,\imath,\varphi)}(\sigma)=\rho_{(A_0,\imath_0,\varphi_0)}(\sigma)$. □

**Remark 5.5.** As a consequence of this lemma, we obtain that Theorem 4.2 also applies if we change the maximal order $\mathcal{O}_0$ by a not necessarily maximal Eichler order $\mathcal{O}$, considering the moduli interpretation $(ii)$ of Proposition 5.1.

## 6. Complex Multiplication abelian $K$-surfaces with QM

In this section we shall deal with the Complex Multiplication (CM) case. Hence, only for this section, we assume that the abelian surface $(A,\imath)$ with QM by $\mathcal{O}$ has also CM by $K$, so is to say $\mathrm{End}^0(A,\imath)=K$ an imaginary quadratic field. The following result describes the projective representation $\rho^N_{(A,\imath,\varphi)}$ (and therefore $\rho_{(A,\imath,\varphi)}$) in the CM case:

**Proposition 6.1.** *Assume that $\mathrm{End}(A,\imath)=\mathcal{O}_K$, an order in $K$. We have the following results*

- (i) *Any abelian surface $(A',\imath')$ with QM by $\mathcal{O}$ and CM by $K$ is isogenious to $(A,\imath)$.*
- (ii) *We can chose a representative of the isomorphism class of $(A,\imath)$ defined over $\bar{\mathbb{Q}}$. Moreover, it is an abelian $\mathbb{Q}$-surface with QM.*
- (iii) *Given any isomorphism $\varphi:A_{tor}\to B/\mathcal{O}$, the restriction to $\mathrm{Gal}(\bar{K}/K)$ of the corresponding projective representation $\rho^N_{(A,\imath,\varphi)}$ factors through the inverse of the Artin map $\mathrm{Art}:\mathbb{A}_{K,f}^\times/K^\times\longrightarrow\mathrm{Gal}(K^{ab}/K)$, namely, we have the following commutative diagram*

$$\begin{array}{ccccc}\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) & & \xrightarrow{\quad\rho^N_{(A,\imath,\varphi)}\quad} & & N_A(\mathbb{A}_f)/K^\times\\ \uparrow & & & & \uparrow\\ \mathrm{Gal}(\bar{K}/K) & \twoheadrightarrow & \mathrm{Gal}(K^{ab}/K) & \xrightarrow{\mathrm{Art}^{-1}} & \mathbb{A}_{K,f}^\times/K^\times.\end{array}$$

(iv) *The representation* $\rho^N_{(A,\imath,\varphi)}$ *factors through* $\mathrm{Gal}(K^{ab}/K) \rtimes \mathrm{Gal}(K/\mathbb{Q})$ *sending the generator of* $\mathrm{Gal}(K/\mathbb{Q})$ *to the class* $jK^\times$*, where* $j \in N_A(\mathbb{Q})$ *is any element satisfying* $j^2 \in \mathbb{Q}^\times$ *and* $jk = \bar{k}j$*, for all* $k \in K^\times$*.*

*Proof.* We have just seen that abelian surfaces with QM by $\mathcal{O}$ over $\mathbb{C}$ are classified by the non-cuspidal points of $X_{\hat{\mathcal{O}}^\times}$. Assume that $(A,\imath)$ with CM by $K$ corresponds to $[g_\infty, g_f] \in (\Gamma_\infty \hat{\mathcal{O}}^\times \backslash G(\mathbb{A}))/G(\mathbb{Q})$. Thus the complex structure on its tangent space is given by an embedding $h_{g_\infty} : \mathbb{C} \hookrightarrow B^{opp} \otimes \mathbb{R}$. On the other side, we have the natural embedding $\psi_{(A,\imath)} : \mathrm{End}(A,\imath) \hookrightarrow \mathrm{End}_{\mathcal{O}}(H_1(A,\mathbb{Z})) = \mathrm{End}_{\mathcal{O}}(I_{g_f}) = \mathcal{O}^{opp}$, giving rise to $\psi_{(A,\imath)} : \mathcal{O}_K \hookrightarrow \mathcal{O}^{opp}$, satisfying

$$\psi_{(A,\imath)}(K) \cap \mathcal{O}^{opp} = \mathrm{End}^0(A,\imath) \cap \mathrm{End}_{\mathcal{O}}(t_A) = \mathrm{End}(A,\imath) = \psi_{(A,\imath)}(\mathcal{O}_K).$$

Since any pair $\psi_{(A,\imath)}(k)$, $k \in K$, and $h_{g_\infty}(z)$, $z \in \mathbb{C}$ commute in $\mathrm{End}_{\mathcal{O}}(H_1(A,\mathbb{Z}) \otimes \mathbb{R}) = B^{opp} \otimes \mathbb{R}$, we have that $h_{g_\infty}$ is the extension of scalars of $\psi_{(A,\imath)}$.

Given another pair $(A',\imath')$ with CM by K corresponding to $[g'_\infty, g'_f]$, by Skolem-Noether, there exists $g \in G(\mathbb{Q})$, such that $\psi_{(A',\imath')} = g^{-1}\psi_{(A,\imath)}g$. This implies that $g_{g'_\infty} = g^{-1}h_{g_\infty}g$ and $g'_\infty = g_\infty g$, hence $[g'_\infty, g'_f] = [g_\infty, g'_f g^{-1}]$. We conclude that $(A',\imath')$ is isogenious to $(A,\imath)$ by Proposition 3.3. This proves $(i)$.

Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ and assume that $(A,\imath)$ corresponds to the double coset $[g_\infty, 1]$. The pair $(A^\sigma, \imath^\sigma)$ satisfies $\mathrm{End}(A,\imath) = \mathrm{End}(A^\sigma,\imath^\sigma)$, thus it correspond to a point $[g_\infty, g_f] \in X_{\hat{\mathcal{O}}^\times}$ by $(i)$. Fix a $\mathcal{O}$-module isomorphism $\varphi : A_{tor} \to B/\mathcal{O}$. The choice of a representative $g_f$ such that $I_{g_f} \subseteq \mathcal{O}$ fixes a $\mathcal{O}$-module isomorphism $\varphi' : A^\sigma_{tor} \to B/\mathcal{O}$ and an isogeny $\mu_\sigma : (A^\sigma, \imath^\sigma) \to (A,\imath)$ such that $\varphi(\mu_\sigma(Q)) = \varphi'(Q)g_f$. Since $A^\sigma_{tor}$ is isomorphic to $A_{tor}$ by means of the $\mathcal{O}$-module isomorphism $P \mapsto P^\sigma$, we have that $\varphi(\mu_\sigma(P^\sigma)) \in \varphi(P)\hat{\mathcal{O}}^\times g_f$. We compute that, for every $\lambda \in \mathrm{End}(A,\imath)$,

$$\mu_\sigma^\vee(\lambda(\mu_\sigma(P^\sigma)))^{\sigma^{-1}} = (\mu_\sigma^\vee \circ \lambda \circ \mu_\sigma)^{\sigma^{-1}}(P), \qquad (\mu_\sigma^\vee \circ \lambda \circ \mu_\sigma)^{\sigma^{-1}} \in \mathrm{End}(A,\imath),$$

Therefore the map $\varphi(P) \mapsto \varphi(\mu_\sigma(P^\sigma))$ is in the commutator of $\mathcal{O}_K$ in $\hat{\mathcal{O}}$. This implies that we can choose a representative of $[g_f] \in \hat{\mathcal{O}}^\times \backslash G(\mathbb{A}_f)$ in the commutator $N_A(\mathbb{A}_f)$ of $\mathrm{End}(A,\imath)$ in $G(\mathbb{A}_f)$. Hence we can suppose that $g_f \in \hat{\mathcal{O}}^\times \cap N_A(\mathbb{A}_f) \backslash N_A(\mathbb{A}_f)/K^\times$. This double coset space is the semi-direct product of $\mathbb{Z}/2\mathbb{Z}$ and $\hat{\mathcal{O}}_K^\times \backslash \mathbb{A}^\times_{K,f}/K^\times = \mathrm{Cl}(\mathcal{O}_K)$, which is of couse finite. Thus the set of isomorphism classes $\{(A^\sigma, \imath^\sigma) : \sigma \in \mathrm{Aut}(\mathbb{C})\}$ is finite. This implies that $(A,\imath)$ admits a model over $\bar{\mathbb{Q}}$. By $(ii)$ we deduce that $(A,\imath)$ is an abelian $\mathbb{Q}$-surface with QM, this proves $(ii)$.

Part $(iii)$ follows directly from Theorem 4.2 and Shimura's Reciprocity Law [4, Main Theorem I].

Finally, note first that the class $jK^\times$ is an element in $N_A(\mathbb{A}_f)/K^\times$ of order 2 and $N_A(\mathbb{A}_f)/K^\times = A^\times_{K,f}/K^\times \rtimes \langle jK^\times \rangle$. Let $\sigma_c \in \mathrm{Aut}(\mathbb{C})$ denote complex multiplication automorphism. Since $(A,\imath)$ is defined over $\bar{\mathbb{Q}}$, the automorphism $\sigma_c$ acts as an element in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\bar{\mathbb{Q}}$-rational points of $(A,\imath)$. Recall that $A = (B \otimes \mathbb{R})_{g_\infty}/\mathcal{O}$, where the complex structure on $(B \otimes \mathbb{R})_{g_\infty}$ is given by $h_{g_\infty}$, the scalar extension of $\psi_{(A,\imath)} : \mathcal{O}_K \hookrightarrow \mathcal{O}^{opp}$. Complex conjugation must be the unique automorphism $\gamma$ on $(B \otimes \mathbb{R})$ such that $\gamma \circ h_{g_\infty}(z) = h_{g_\infty}(z^{\sigma_c}) \circ \gamma$, for all $z \in \mathbb{C}^\times$. Therefore $\gamma$ corresponds to conjugate by $j$ since $\psi_{(A,\imath)}(k)j^{-1}bj = j^{-1}\psi_{(A,\imath)}(k^{\sigma_c})bj$, for all $b \in B$, $k \in K^\times$. This implies that $A^{\sigma_c} = (B \otimes \mathbb{R})_{g_\infty j}/\mathcal{O} = (B \otimes \mathbb{R})_{g_\infty}/\mathcal{O}j$ and we have the following diagram:

$$\begin{array}{ccccc}
A_{tor} & \xrightarrow{P \mapsto P^{\sigma_c}} & A^{\sigma_c}_{tor} & \xrightarrow{\mu_{\sigma_c}} & A_{tor} \\
\downarrow{\scriptstyle\varphi} & & \downarrow & & \downarrow{\scriptstyle\varphi} \\
B/\mathcal{O} & \xrightarrow[b \mapsto bj]{} & B/\mathcal{O}j & \xrightarrow[b \mapsto b]{} & B/\mathcal{O}
\end{array}$$

Thus $\varphi(\mu_{\sigma_c}(P^{\sigma_c})) = \varphi(P)j$, which implies $\rho_{(A,\imath)}(\sigma_c) = jK^\times$. Since $\rho^N_{(A,\imath,\varphi)}$ maps exhaustively $\mathrm{Gal}(\bar{K}/K)$ to $A^\times_{K,f}/K^\times$ by $(iii)$, and $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})/\mathrm{Gal}(\bar{K}/K) \simeq \mathrm{Gal}(K/\mathbb{Q})$, which is generated by the image of $\sigma_c$, part $(iv)$ follows. □

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