A Possible Generalization of Maeda's Conjecture

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Abstract We report on observations we made on computational data that suggest a generalization of Maeda's conjecture regarding the number of Galois orbits of newforms of level $N = 1$, to higher levels. They also suggest a possible formula for this number in many of these cases.

1 Introduction

For $N \geq 1$ and $k \geq 2$ even let $S_k(N)$ denote the space of modular forms of weight k and level $\Gamma_0(N)$ and let $S_k^{\text{new}}(N)$ be its new subspace. In the case of $N = 1$ Maeda made the following conjecture:

Conjecture 1.1. ([HM97]) The Hecke algebra over $\mathbb{Q}$ of $\text{SL}_2(\mathbb{Z})$ is simple (that is, a single number field) whose Galois closure over $\mathbb{Q}$ has Galois group isomorphic to a symmetric group $\mathfrak{S}_n$ (with $n = \dim S_k(\text{SL}_2(\mathbb{Z}))$).

This conjecture implies the following, weaker one:

Conjecture 1.2. $S_k(1)$ consists of a single newform Galois orbit for all k.

We believe we can propose a few generalizations of the latter one, based on computational data we have accumulated and that we will present in the next section:

- Fix N. Then the number of newform Galois orbits in $S_k^{\text{new}}(N)$, as a function of k, is bounded.
- Fix N. The number of non-CM newform Galois orbits in $S_k^{\text{new}}(N)$ as a function of k is eventually constant.

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- Assuming the above guess is true, let **nonCM**(N) be the constant to which the number of non-CM newform orbits in $S_k^{\text{new}}(N)$ eventually stabilizes. Then **nonCM**$(\cdot)$ is a multiplicative function, i.e.

$$\textbf{nonCM}(NM) = \textbf{nonCM}(N)\,\textbf{nonCM}(M)$$

  for all coprime integers N and M.

In Section 2 we introduce notation and formulate our precise conjectures. In Section 3 we have a brief discussion on the non-CM newform Galois orbits, which is the mysterious part, and we continue in Sect. 4 with an overview of the theory of Hecke characters and CM forms, which we use to get a grasp on the number of CM newform Galois orbits. Finally in Sect. 5 we give the table with the patterns we have observed that led us to formulate our conjectures.

2 Setup-Notation

Let N, k be positive integers, k even. We will denote by $S_k(N)$ the space of cusp forms of level $\Gamma_0(N)$ and weight k and by $S_k^{\text{new}}(N)$ its new-subspace. Every element f of $S_k(N)$ has a Fourier expansion

$$\sum_{n \geq 1} c_n(f)\,q^n \ ,$$

where $q = e^{2\pi i z}$, which is called the q-expansion of f. The space $S_k(N)$ admits an action by the $\mathbb{Z}$-algebra of the Hecke operators T_n which preserves the new subspace. We will denote this algebra by $\mathbb{T}_k(N)$ and by $\mathbb{T}_k^{\text{new}}(N)$ its quotient through which it acts on the new subspace. Finally $S_k^{\text{new}}(N)$ admits a basis of simultaneous eigenvectors for all the operators of $\mathbb{T}_k(N)$, which we can assume to be normalized, i.e that $c_1(f) = 1$ for each of these eigenvectors f.

It is a well-known fact (see e.g. [Shi71, Theorem 3.52]) for such a normalized eigenvector f that the extension

$$K_f := \mathbb{Q}\big(\{c_n(f)\}_n\big)/\mathbb{Q}$$

is finite, and that the coefficients $c_n(f)$ are algebraic integers. In view of this we have an action of $G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on these f that commutes with the one of $\mathbb{T}_k(N)$ and therefore it preserves a given basis of normalized eigenforms. It also actually preserves a basis of normalized eigenforms of $S_k^{\text{new}}(N)$. We will call the orbits of this last action the newform Galois orbits of level N and weight k, and denote their number by **ORB**(N, k).

The purpose of this article is to report on computations regarding **ORB**(N, k) for $N > 1$. We need to introduce some other notions and notation first.

Let $\chi\colon (\mathbb{Z}/D\mathbb{Z})^{\times} \to \mathbb{C}^{\times}$ be a Dirichlet character. If f is a modular form of level N and weight k then

$$f \otimes \chi := \sum_{n \geq 1} \chi(n)\, c_n(f)\, q^n$$

is also a modular form of level (dividing) ND^2 and weight k (but not necessarily of trivial Nebentypus, even if f is!). If $f = f \otimes \chi$ for some nontrivial quadratic character χ then f is called a complex multiplication form (CM form for short). Notice that if f is a CM form then f^{σ} is also a CM form for all $\sigma \in G_{\mathbb{Q}}$. Therefore we can talk about CM and non-CM newform Galois orbits. For a fixed level N and weight k we will denote their number by $\mathbf{CM}(N, k)$ and $\mathbf{nonCM}(N, k)$ respectively. Obviously the following holds:

$$\mathbf{ORB}(N, k) = \mathbf{CM}(N, k) + \mathbf{nonCM}(N, k)\,.$$

After systematically computing[1] these values we were led to formulate the following conjectures:

Conjecture 2.1. Fix an integer $N \geq 1$. Then $\mathbf{ORB}(N, k)$, as a function of k, is bounded.

Conjecture 2.2. Fix an integer $N \geq 1$. Then $\mathbf{nonCM}(N, k)$, as a function of k, is eventually constant. If we let $\mathbf{nonCM}(N)$ be this constant then we have that:

$$\mathbf{nonCM}(NM) = \mathbf{nonCM}(N)\,\mathbf{nonCM}(M)$$

for all $(N, M) = 1$, i.e. $\mathbf{nonCM}(\,\cdot\,)$ is a multiplicative function.

Remark 2.3. Notice that Conjecture 2.2 combined with Corollary 4.6 implies Conjecture 2.1.

In Section 5 we provide tables with the stationary values $\mathbf{nonCM}(N)$ that $\mathbf{nonCM}(N, k)$ seem to converge to as functions of k. In particular one sees there that $\mathbf{nonCM}(p) = 2$ which, along with the fact that there are no CM newforms for squarefree levels (see Sect. 4 for details), suggests that $\mathbf{ORB}(p, k) = 2$ for sufficiently large k, for all primes p. Furthermore a positive answer to Conjecture 2.2 would then imply the following:

Conjecture 2.4. Fix an integer $N = p_1 \cdots p_s \geq 1$, where the p_i's are pairwise distinct prime numbers and $s \geq 0$. Then $\mathbf{ORB}(N, k) = 2^s$ for sufficiently large k.

Of course one can get many more conjectural formulas like the one of Conjecture 2.4 for levels containing higher prime powers by combining the suggested multiplicativity of $\mathbf{nonCM}$ along with observations from the tables of Sect. 5. Observing for example that $\mathbf{nonCM}(4) = 1$ and, as before, that $\mathbf{nonCM}(p) = 2$ for every prime, then assuming Conjecture 2.2 is true we get:

$$\mathbf{nonCM}(2^2 p) = 2 \qquad \text{for all primes } p > 2\,.$$

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[1] A complete account of all the computational data we used can be found at
`http://math.uni.lu/~tsaknias/maeda_tables.pdf`

3 Non-CM Forms

In the early versions of this manuscript we had only included Conjecture 2.1 along with the observation that, when stable, the number of newform Galois orbits seemed to behave multiplicatively in many cases (like the squarefree level case for example, which is the subject of Conjecture 2.4). After we emailed these to William Stein he informed us that he had conducted similar computations in the past and suggested that one should look separately the cases of CM and non-CM forms. Moreover he suggested that the decomposition of the non-CM part should be forced by the Atkin-Lehner operators.

In the squarefree level case this agrees with Conjecture 2.4: As we have already mentioned, there are no CM forms for squarefree levels. The number of Atkin-Lehner operators is s, where $N = p_1 \cdots p_s$ and therefore 2^s possible combinations of eigenvalues for all the operators considered together. In relevant computations we performed we verified that each combination does occur for exactly one non-CM Galois orbit (for sufficiently large weight k).

Conceptually, the Atkin-Lehner operators seem to be a good candidate for explaining that **nonCM**(N, k) is (eventually) independent of k: They are the ones that cut out the newspace of $S_k(N)$ and the number of decomposition factors they split $S_k^{\mathrm{new}}(N)$ into seems to be eventually constant as a function of k too.

For non-squarefree levels though things get more complicated. The Atkin-Lehner operators are not enough to distinguish between all non-CM Galois orbits, i.e. the same combination of Atkin-Lehner eigenvalues occurs for more than one non-CM Galois orbit. For example, in the case $N = 16 = 2^4$ the computational data suggest that there are eventually 6 newform Galois orbits for all sufficiently big weights k, none of which is CM. There is only one Atkin-Lehner operator, w_{16}, which cuts the newsubspace into two pieces. These two pieces break further into 3 irreducible ones each, which correspond to the newform Galois orbits. This suggests that a further refinement of the Atkin-Lehner decomposition of $S_k^{\mathrm{new}}(N)$ is needed in order to distinguish all the different non-CM orbits.

On the other hand however, the distinction between CM and non-CM Galois orbits seems essential: The CM Galois orbits seem to be the reason **ORB**(N, k) is eventually periodic (whenever this happens) instead of eventually constant as a function of k and they also seem to be the "error term" that is obstructing the number of Galois orbits to behave multiplicatively.

4 CM Forms

In this section we describe the connection between CM forms and Hecke characters of imaginary quadratic fields (hence the name CM forms). Our goal is to eventually explain why **CM**(N, k) is bounded and periodic. Most of the material in this section is well known facts that we apply in the case of interest to us, i.e. CM forms of

trivial Nebentypus (and therefore even weight). The interested reader should consult [Rib77, §3], [Miy06, §4.8] and [Neu99, Chap. VII, §3].

Let K be an imaginary quadratic field, $\mathcal{O} = \mathcal{O}_K$ its ring of integers and $\mathfrak{m}$ an ideal of K. Let us also denote by $J^{\mathfrak{m}}$ the group of fractional ideals of $\mathcal{O}_K$ that are coprime to $\mathfrak{m}$. A Hecke character ψ of K, of modulus $\mathfrak{m}$, is then a group homomorphism $\psi \colon J^{\mathfrak{m}} \to \mathbb{C}^*$ such that there exists a character $\psi_f \colon (\mathcal{O}_K/\mathfrak{m})^* \to \mathbb{C}^*$ and a group homomorphism $\psi_\infty \colon K_{\mathbb{R}}^* \to \mathbb{C}^*$, where $K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R}$, such that:

$$\psi((\alpha)) = \psi_f(\alpha)\,\psi_\infty(\alpha) \qquad \text{for all } \alpha \in K \ . \tag{1}$$

One calls ψ_f and ψ_∞ the finite and the infinite type of ψ respectively. The conductor of ψ is the conductor of ψ_f and we will call ψ primitive if the conductor is equal to its modulus. The relation (1) implies that $\psi_f(\varepsilon)\,\psi_\infty(\varepsilon) = 1$ for all units $\varepsilon \in \mathcal{O}^*$. The converse is also true: Given a pair (ψ_f, ψ_∞) satisfying this relation for all units $\varepsilon \in \mathcal{O}^*$ then there is a Hecke character they are attached to it ([Neu99, Chap. VII, §3, Exercise 5]). This Hecke character however might not be unique:

Lemma 4.1. *The number of different Hecke characters associated with a given pair* (ψ_f, ψ_∞) *is equal to the class number h_K of K.*

Proof. Let $\psi \colon J^{\mathfrak{m}} \to \mathbb{C}^*$ be a Hecke character associated with the given pair. Let $P^{\mathfrak{m}}$ denote the subgroup of $J^{\mathfrak{m}}$ consisting of principal ideals. Then $J^{\mathfrak{m}}/P^{\mathfrak{m}} \cong J/P$, the class group of K. This is an abelian group so by the structure theorem of abelian groups it is isomorphic to a product $\prod_{i=1}^{s} C_{t_i}$ of cyclic groups with $\prod_i t_i = h_K$. For every i pick an ideal $\mathfrak{a}_i$ generating the group C_{t_i}. Then $\mathfrak{a}_i^{t_i}$ is a principal ideal (π_i). We therefore have:

$$\psi^{t_i}(\mathfrak{a}_i) = \psi\left(\mathfrak{a}_i^{t_i}\right) = \psi((\pi_i)) = \psi_f(\pi_i)\,\psi_\infty(\pi_i) \ .$$

This means that for every generator $\mathfrak{a}_i$ we have t_i choices and therefore we have $\prod_i t_i = h_K$ different Hecke characters associated with the same pair (ψ_f, ψ_∞). □

This reduces the problem of counting Hecke characters to counting compatible pairs (ψ_f, ψ_∞).

From this point on we specialize to the case where K is an imaginary quadratic field. We will denote by D its discriminant and by $(D|.)$ the Kronecker symbol associated to it. The only possible group homomorphisms ψ_∞ are then of the form σ^u, where σ is one of the two conjugate complex embeddings of K and u is a non negative integer. The following theorem associates a (CM) modular form a Hecke character ψ of K ([Miy06, Theorem 4.8.2]):

Theorem 4.2. *Given ψ of infinity type σ^u and finite type ψ_f with modulus $\mathfrak{m}$. Assume $u > 0$ and let $\mathbf{N}\mathfrak{m}$ be the norm of $\mathfrak{m}$. We then have that*

$$f = \sum_n \left(\sum_{\mathbf{N}\mathfrak{a}=n} \psi(\mathfrak{a})\right) q^n$$

is a cuspidal eigenform in

$$S_{u+1}(N, \chi) \ ,$$

where $N = |D|\mathbf{N}\mathfrak{m}$ and $\chi(m) = (D|m)\psi_f(m)$ for all integers m.

The eigenform associated with the Hecke character is new if and only if ψ is primitive (see Remark 3.5 in [Rib77]).

Let us introduce some terminology: We will say that a Hecke character ψ is (N, k)-suitable if it satisfies the following:

- ψ is primitive.
- $|D|\mathbf{N}\mathfrak{m} = N$, where D is the discriminant of K and $\mathfrak{m}$ is the modulus of ψ.
- The infinity type of ψ is σ^{k-1}.
- $\psi_f(m) = (D|m)$ for all integers m.

We will also say that a compatible pair (ψ_f, ψ_∞) is (N, k)-*suitable*, if the Hecke characters attached to it are (N, k)-suitable. An obvious consequence of Theorem 4.2 is that the number of CM newforms in $S_k(\Gamma_0(N))$ is equal to the number of (N, k)-suitable Hecke characters. Since the number of (N, k)-suitable Hecke characters is determined in view of Lemma 4.1 by the number of (N, k)-suitable compatible pairs, we will focus on counting these instead. It is obvious that given a weight k and a level, there is only one infinity type that could give an (N, k)-suitable pair, namely σ^{k-1}. With this in mind we are left with counting the number of finite types ψ_f that are compatible with σ^{k-1} and give an (N, k)-suitable pair.

First of all, notice that only a finite number of imaginary quadratic fields can give an (N, k)-suitable compatible pair: The discriminant D of K must divide N. Let $\mathfrak{m}$ be an ideal of K such that $|D|\mathbf{N}\mathfrak{m} = N$. This allows only a finite number of ideals as the modulus of an (N, k)-suitable pair.

Remark 4.3. Notice that, so far, the bound on the number of possible fields and the bound on the number of possible ideals for every possible field depend only N. This already gives a bound on the number of CM newforms of level N and weight k that is independent of k.

Let now $\psi_f \colon (\mathcal{O}_K/\mathfrak{m})^* \to \mathbb{C}^*$ be a finite type of modulus $\mathfrak{m}$, where $\mathfrak{m}$ is an ideal of K such that $|D|\mathbf{N}\mathfrak{m} = N$. Consider the composition $\mathbb{Z} \to (\mathcal{O}_K/\mathfrak{m}) \to \mathbb{C}$ where the second map is ψ_f. The (N, k)-suitability of (ψ_f, ψ_∞) then implies that this composition is the Kronecker symbol $(D|.)$ which has conductor $|D|$ and we thus get an injection:

$$\mathbb{Z} / |D|\mathbb{Z} \hookrightarrow (\mathcal{O}_K/\mathfrak{m}) \ .$$

This implies in particular that $|D|$ divides $\mathbf{N}\mathfrak{m}$ and therefore that $D^2 \mid N$, restricting further the possible (N, k)-suitable compatible pairs.

The group $(\mathcal{O}/\mathfrak{m})^*$ is an abelian group whose structure can be determined for example by the theorems in [Nak79]. These imply in particular that $(\mathbb{Z}/|D|\mathbb{Z})^*$ can be seen as a subgroup of this group and, since it is abelian, as a quotient of it as well. Since the finite type is completely determined on integers by the Kronecker symbol, to completely describe it one only has to describe its values on the generators of $(\mathcal{O}/\mathfrak{m})^*/(\mathbb{Z}/|D|\mathbb{Z})^*$ which is an abelian group too. It turns out that one

has $\#(\mathcal{O}/\mathfrak{m})^*/\varphi(|D|)$ different finite types whose values on integers match those of the Kronecker symbol. Out of all these choices, one must consider only those for which ψ_f is of conductor $\mathfrak{m}$, in order for the corresponding cuspforms to be new (see Remark 3.5 in [Rib77]).

The last condition one has to check is the compatibility with the infinity type on the integral units of K. Recall that this implies:

$$\psi_f(\varepsilon) = \psi_\infty^{-1}(\varepsilon) = \sigma^{1-k}(\varepsilon) \qquad \text{for all } \varepsilon \in \mathcal{O}_K^*$$

All imaginary quadratic fields contain at least the following integral units: $\{1, -1\}$. In both cases the compatibility condition is automatically satisfied no matter what the weight is. We stress one more time that up to this point, our treatment of possible finite types is independent of the weight k. The only imaginary quadratic fields K that contain more integral units are the following:

- $K = \mathbb{Q}(\sqrt{-1})$. In this case O_K contains 4 integral units:

$$\{1, -1, i, -i\} = \langle i \rangle .$$

  We have that $\psi_f^2(i) = \psi_f(i^2) = (D| - 1) = -1$, so $\psi_f(i) = \pm i$. On the other hand
  $$\psi_f(i) = \sigma(i)^{1-k} = i^{1-k} .$$

  If $\psi_f(i) = i$ we then get that $k \equiv 0 \mod 4$. If $\psi_f(i) = -i$ we get that $k \equiv 2 \mod 4$.
- $K = \mathbb{Q}(\sqrt{-3})$. In this case O_K contains 6 integral units:

$$\{1, -1, w, w^2, -w, -w^2\} = \langle -w \rangle ,$$

  where w is a primitive cubic root of unity.
  We have that $\psi_f^3(-w) = \psi_f((-w)^3) = (D| - 1) = -1$, so $\psi_f(-w) = w^s$, where $s = 0, 1, 2$. On the other hand

$$\psi_f(-w) = \sigma(-w)^{1-k} = (-w)^{1-k} .$$

We thus come to the following conclusion:

Proposition 4.4. *Fix a positive integer N and let k be a varying even integer. Then:*

(1) *The number of CM newforms in $S_k^{\text{new}}(N)$ with complex multiplication field $\mathbb{Q}(\sqrt{-d})$, where $d \neq 1, 3$, is independent of the weight k, i.e. it is constant as a function of k.*

(2) *The number of CM newforms in $S_k^{\text{new}}(N)$ with complex multiplication field $\mathbb{Q}(\sqrt{-1})$, depends only on $k \mod 4$, i.e. has at most two possible values and it is periodic as a function of k.*

(3) *The number of CM newforms in $S_k^{\text{new}}(N)$ with complex multiplication field $\mathbb{Q}(\sqrt{-3})$, depends only on $k \mod 6$, i.e. has at most three possible values and it is periodic as a function of k.*

Two immediate corollaries are the following:

Corollary 4.5. *Fix a positive integer N. The number of CM newforms in $S_k^{\text{new}}(N)$ is a bounded function of k.*

Corollary 4.6. *Fix a positive integer N. Then $\mathbf{CM}(N, k)$ is a bounded function of k.*

Remark 4.7. Hopefully it is clear by how we counted the possible CM newforms for a given level and weight that the way CM newforms group in Galois orbits depends on the degree of the field extension containing the values of ψ_f and on the extension defined by the generating values for ψ mentioned in the proof of Lemma 4.1. The variety of possible combinations however is preventing us from giving a precise recipe for the orbits or just their number.

In the next subsection we study the case $N = p^2$ where p is a prime number. We describe the possible number of CM newforms and we are also able to explain how they group into Galois orbits due to the simplicity of this situation.

4.1 $N = p^2$, p **Prime**

Let ψ be a (p^2, k)-suitable Hecke character of the imaginary quadratic field

$$\mathbb{Q}\left(\sqrt{-d}\,\right),$$

of discriminant D, with modulus $\mathfrak{m}$. As we have seen in the previous section, the (p^2, k)-suitability of ψ implies that $D^2 | N$. Since

$$D = \begin{cases} -d, & \text{if } d \equiv 3 \mod 4 \\ -4d, & \text{otherwise.} \end{cases}$$

we get that $D = -p$ and $p \equiv 3 \mod 4$. So if $p \not\equiv 3 \mod 4$, then $\mathbf{CM}(p^2, k) = 0$ for all even k.

For the rest of the section we therefore assume $p \equiv 3 \mod 4$, $d = p$ and $D = -p$. The (p^2, k)-suitbility of ψ implies that $\mathbf{N}\mathfrak{m} = p^2/|D| = p$. In particular $\mathfrak{m}$ is the prime ideal of $\mathbb{Q}(\sqrt{-p})$ above p. Since $\mathbf{N}\mathfrak{m} = \#(\mathcal{O}/\mathfrak{m})$ and $\mathcal{O}/\mathfrak{m}$ is an extension of $\mathbb{F}_p$ we get that $\mathcal{O}_K/\mathfrak{m} \cong \mathbb{F}_p$ and in particular $(\mathcal{O}_K/\mathfrak{m})^* \cong (\mathbb{Z}/p\mathbb{Z})^*$. This implies that ψ_f is uniquely determined by $(D|.)$. Since the infinity type is uniquely determined by the weight k we have that there is at most one (p^2, k)-suitable compatible pair depending on whether the integral units compatibility is satisfied or not. As we have seen in the previous section, for $p \neq 3$ it is immediately satisfied. In the case $p = 3$ one has that $1 = w^3 \equiv w \mod \mathfrak{m}$ and therefore since

$$w^{k-1} = \psi_\infty(w) = \psi_f^{-1}(w) = 1$$

we get that $k \equiv 1 \mod 3$. Furthermore $\mathbb{Q}(\sqrt{-3})$ has class number 1 so when $k \equiv 1 \mod 3$ the unique $(9, k)$-suitable compatible pair is associated to a unique Hecke character that takes values in $\mathbb{Q}(\sqrt{-3})$. This will in turn give us a unique CM newform with coefficients in $\mathbb{Q}$, and therefore a unique CM newform Galois orbit. To sum it up:

$$\mathbf{CM}(9, k) = \begin{cases} 1, & \text{if } k \equiv 4 \mod 6 \\ 0, & \text{otherwise.} \end{cases}$$

For $p \neq 3$ the pair (ψ_f, ψ_∞) is always compatible independently of the weight k. Let h_K be the class number of $\mathbb{Q}(\sqrt{-p})$. Lemma 4.1 then implies that there exist h_K different Hecke characters and therefore, by Theorem 4.2, we get that there exist h_K many CM newforms. The way these CM newforms group into Galois orbits for every weight k depends on h_K. To see this one has to go into more detail in the proof of Lemma 4.1. For simplicity we assume that the class group of K is cyclic (which is true for all the p in our computational range but certainly not true in general). In this case a Hecke character ψ is completely determined by the value of $\psi(\mathfrak{a})$, which must satisfy the relation

$$\psi^{h_K}(\mathfrak{a}) = \pi^{k-1}.$$

where $\mathfrak{a}$ is an ideal away from $\mathfrak{m}$ generating the class group and $\pi \in K$ is such that $\mathfrak{a}^{h_K} = (\pi)$ and $\psi_f(\pi) = 1$. The possible values for $\psi(\mathfrak{a})$ are then

$$\left(\pi^{k-1}\right)^{1/h_K} \zeta_{h_K}^i,$$

where ζ_{h_K} is a primitive h_K-th root of unity. The field L_ψ where ψ takes its values is then $\mathbb{Q}(\psi(\mathfrak{a}))$ and the field L_F of the corresponding CM newform is a totally real subfield of $\mathbb{Q}(\psi(\mathfrak{a}))$. Their relative degree is $[L_\psi : L_f] = 2$. Lets us give a more detailed description for some primes p:

- $p = 23$. In this case $h_K = 3$, so we get 3 CM newforms. Let ψ be one of the 3 $(23^2, k)$-suitable Hecke characters. For $k \not\equiv 4 \mod 6$ the degree of L_ψ is 6 and therefore the degree of L_f is 3. In this case all 3 CM newforms fall into a single Galois orbit. In the case $k \equiv 4 \mod 6$ one of the Hecke characters has field $L_\psi = K$ and the other two have field $L_\psi = K(\zeta)$ which is of degree 4. In this case we get one CM form with rational coefficients that forms a Galois orbit by itself and two with coefficients in a real quadratic extension, that group together in another Galois orbit. To summarize:

$$\mathbf{CM}(23^2, k) = \begin{cases} 2, & \text{if } k \equiv 4 \mod 6 \\ 1, & \text{otherwise.} \end{cases}$$

Clearly the same holds for any prime $p \equiv 3 \mod 4$ for which K has class number $h_K = 3$.

- $p = 47$. In this case $h_K = 5$, so we get 5 CM newforms and we have:

$$\mathbf{CM}\big(47^2, k\big) = \begin{cases} 2, & \text{if } k \equiv 6 \mod 10 \\ 1, & \text{otherwise.} \end{cases}$$

  Again the same formula holds for any prime $p \equiv 3 \mod 4$ such that $h_K = 5$.
- Let $p \equiv 3 \mod 4$ such that $h_k = \ell$, $\ell > 2$ a rational prime. In this case there are ℓ CM newforms and we have:

$$\mathbf{CM}\big(p^2, k\big) = \begin{cases} 2, & \text{if } k \equiv \ell + 1 \mod 2\ell \\ 1, & \text{otherwise.} \end{cases}$$

5 Observations

Tables 1 and 2 contain the suggested values for **nonCM**(N) for $1 \leq N \leq 200$ and the number of CM newform Galois orbits for the same level. For every level we have computed **nonCM**(N, k) and **CM**(N, k) for all weights up to at least 30. For all N in this range it is clear from the data we have collected that **nonCM**(N, k) eventually stabilizes to some **nonCM**(N). On the other hand **CM**(N, k) has a few cases where it does not stabilize (which we expected after the analysis of Sect. 4) but its behavior is clear even from the first few weights. We also provide the factorization of the level N into prime powers which helps to observe the multiplicativity of **nonCM**(N). To compute **CM**(N, k) and **nonCM**(N, k) for level N and weight k we used standard commands of MAGMA [BCP97], version 2.17 and higher. About some special cases:

- For level 9 we eventually have that **nonCM**$(9) = 4$, **CM**$(9, k) = 0$ for $k \equiv 0, 2$ mod 6 and **CM**$(9, k) = 1$ for $k \equiv 4 \mod 6$.
- For level 27 we eventually have that **nonCM**$(27) = 4$, **CM**$(27, k) = 1$ for $k \equiv 0, 2 \mod 6$ and **CM**$(27, k) = 0$ for $k \equiv 4 \mod 6$.
- For level 36 we eventually have that **nonCM**$(36) = 4$, **CM**$(36, k) = 1$ for $k \equiv 0, 2 \mod 6$ and **CM**$(36, k) = 0$ for $k \equiv 4 \mod 6$.
- For level 108 we eventually have that **nonCM**$(108) = 4$, **CM**$(108, k) = 1$ for $k \equiv 0, 2 \mod 6$ and **CM**$(108, k) = 2$ for $k \equiv 4 \mod 6$.

Table 1 Number of newform Galois orbits for level $1 \le N \le 100$

N	factors	nonCM	CM
1	1	1	0
2	2^1	2	0
3	3^1	2	0
4	2^2	1	0
5	5^1	2	0
6	$2^1 3^1$	4	0
7	7^1	2	0
8	2^3	2	0
9	3^2	4	0,1,0
10	$2^1 5^1$	4	0
11	11^1	2	0
12	$2^2 3^1$	2	0
13	13^1	2	0
14	$2^1 7^1$	4	0
15	$3^1 5^1$	4	0
16	2^4	6	0
17	17^1	2	0
18	$2^1 3^2$	8	1,0,1
19	19^1	2	0
20	$2^2 5^1$	2	0
21	$3^1 7^1$	4	0
22	$2^1 11^1$	4	0
23	23^1	2	0
24	$2^3 3^1$	4	0
25	5^2	6	0
26	$2^1 13^1$	4	0
27	3^3	4	1,0,1
28	$2^2 7^1$	2	0
29	29^1	2	0
30	$2^1 3^1 5^1$	8	0
31	31^1	2	0
32	2^5	4	1
33	$3^1 11^1$	4	0
34	$2^1 17^1$	4	0
35	$5^1 7^1$	4	1,0,1
36	$2^2 3^2$	4	0
37	37^1	4	0
38	$2^1 19^1$	2	0
39	$3^1 13^1$	8	0
40	$2^3 5^1$	2	0
41	41^1	2	0
42	$2^1 3^1 7^1$	8	0
43	43^1	4	0
44	$2^2 11^1$	2	0
45	$3^2 5^1$	8	0
46	$2^1 23^1$	4	0
47	47^1	2	0
48	$2^4 3^1$	12	0
49	7^2	7	1
50	$2^1 5^2$	12	0
51	$3^1 17^1$	4	0
52	$2^2 13^1$	2	0
53	53^1	2	0
54	$2^1 3^3$	8	0
55	$5^1 11^1$	4	0
56	$2^3 7^1$	4	0
57	$3^1 19^1$	4	0
58	$2^1 29^1$	4	0
59	59^1	2	0
60	$2^2 3^1 5^1$	4	0
61	61^1	2	0
62	$2^1 31^1$	4	0
63	$3^2 7^1$	8	0
64	2^6	16	1
65	$5^1 13^1$	4	0
66	$2^1 3^1 11^1$	8	0
67	67^1	2	0
68	$2^2 17^1$	2	0
69	$3^1 23^1$	4	0
70	$2^1 5^1 7^1$	8	0
71	71^1	2	0
72	$2^3 3^2$	8	0
73	73^1	2	0
74	$2^1 37^1$	4	0
75	$3^1 5^2$	12	0
76	$2^2 19^1$	2	0
77	$7^1 11^1$	4	0
78	$2^1 3^1 13^1$	8	0
79	79^1	2	0
80	$2^4 5^1$	12	0
81	3^4	5	0
82	$2^1 41^1$	4	0
83	83^1	2	0
84	$2^2 3^1 7^1$	4	0
85	$5^1 17^1$	4	0
86	$2^1 43^1$	4	0
87	$3^1 29^1$	4	0
88	$2^3 11^1$	4	0
89	89^1	2	0
90	$2^1 3^2 5^1$	16	0
91	$7^1 13^1$	4	0
92	$2^2 23^1$	2	0
93	$3^1 31^1$	4	0
94	$2^1 47^1$	4	0
95	$5^1 19^1$	4	0
96	$2^5 3^1$	8	0
97	97^1	2	0
98	$2^1 7^2$	14	0
99	$3^2 11^1$	8	0
100	$2^2 5^2$	6	0

Table 2 Number of newform Galois orbits for level $101 \leq N \leq 200$

N	factors	nonCM	CM
101	101^1	2	0
102	$2^1 3^1 17^1$	8	0
103	103^1	2	0
104	$2^3 13^1$	4	0
105	$3^1 5^1 7^1$	8	0
106	$2^1 53^1$	4	0
107	107^1	2	0
108	$2^2 3^3$	4	1,2,1
109	109^1	2	0
110	$2^1 5^1 11^1$	8	0
111	$3^1 37^1$	4	0
112	$2^4 7^1$	12	0
113	113^1	2	0
114	$2^1 3^1 19^1$	8	0
115	$5^1 23^1$	4	0
116	$2^2 29^1$	2	0
117	$3^2 13^1$	8	0
118	$2^1 59^1$	4	0
119	$7^1 17^1$	4	0
120	$2^3 3^1 5^1$	8	0
121	11^2	9	1
122	$2^1 61^1$	4	0
123	$3^1 41^1$	4	0
124	$2^2 31^1$	2	0
125	5^3	4	0
126	$2^1 3^2 7^1$	16	0
127	127^1	2	0
128	2^7	8	0
129	$3^1 43^1$	4	0
130	$2^1 5^1 13^1$	8	0
131	131^1	2	0
132	$2^2 3^1 11^1$	4	0
133	$7^1 19^1$	4	0
134	$2^1 67^1$	4	0
135	$3^3 5^1$	8	0
136	$2^3 17^1$	4	0
137	137^1	2	0
138	$2^1 3^1 23^1$	8	0
139	139^1	2	0
140	$2^2 5^1 7^1$	4	0
141	$3^1 47^1$	4	0
142	$2^1 71^1$	4	0
143	$11^1 13^1$	4	0
144	$2^4 3^2$	24	1
145	$5^1 29^1$	4	0
146	$2^1 73^1$	4	0
147	$3^1 7^2$	14	0
148	$2^2 37^1$	2	0
149	149^1	2	0
150	$2^1 3^1 5^2$	24	0
151	151^1	2	0
152	$2^3 19^1$	4	0
153	$3^2 17^1$	8	0
154	$2^1 7^1 11^1$	8	0
155	$5^1 31^1$	4	0
156	$2^2 3^1 13^1$	4	0
157	157^1	2	0
158	$2^1 79^1$	4	0
159	$3^1 53^1$	4	0
160	$2^5 5^1$	8	0
161	$7^1 23^1$	4	0
162	$2^1 3^4$	10	0
163	163^1	2	0
164	$2^2 41^1$	2	0
165	$3^1 5^1 11^1$	8	0
166	$2^1 83^1$	4	0
167	167^1	2	0
168	$2^3 3^1 7^1$	8	0
169	13^2	9	0
170	$2^1 5^1 17^1$	8	0
171	$3^2 19^1$	8	0
172	$2^2 43^1$	2	0
173	173^1	2	0
174	$2^1 3^1 29^1$	8	0
175	$5^2 7^1$	12	0
176	$2^4 11^1$	12	0
177	$3^1 59^1$	4	0
178	$2^1 89^1$	4	0
179	179^1	2	0
180	$2^2 3^2 5^1$	8	0
181	181^1	2	0
182	$2^1 7^1 13^1$	8	0
183	$3^1 61^1$	4	0
184	$2^3 23^1$	4	0
185	$5^1 37^1$	4	0
186	$2^1 3^1 31^1$	8	0
187	$11^1 17^1$	4	0
188	$2^2 47^1$	2	0
189	$3^3 7^1$	8	0
190	$2^1 5^1 19^1$	8	0
191	191^1	2	0
192	$2^6 3^1$	32	0
193	193^1	2	0
194	$2^1 97^1$	4	0
195	$3^1 5^1 13^1$	8	0
196	$2^2 7^2$	7	0
197	197^1	2	0
198	$2^1 3^2 11^1$	16	0
199	199^1	2	0
200	$2^3 5^2$	12	0

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