A LEVEL RAISING RESULT FOR MODULAR GALOIS REPRESENTATIONS MODULO PRIME POWERS.

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ABSTRACT. In this work we provide a level raising theorem for mod λ^n modular Galois representations. It allows one to see such a Galois representation that is modular of level N, weight 2 and trivial Nebentypus as one that is modular of level Np, for a prime p coprime to N, when a certain local condition at p is satisfied. It is a generalization of a result of Ribet concerning mod ℓ Galois representations.

1. INTRODUCTION

Let N and k be positive integers, $S_k(\Gamma_0(N))$ be the space of modular forms of level N and weight k, and $\mathbb{T}_k(N)$ be the $\mathbb{Z}$-algebra of Hecke operators acting faithfully on this space. Let also R be a complete Noetherian local ring with maximal ideal $\mathfrak{m}_R$ and residue field of characteristic $\ell > 0$. A (weak) eigenform of level N and k with with coefficients in R is then defined to be a ring homomorphism $\theta : \mathbb{T}_k(N) \longrightarrow R$ (One can find a discussion on the various notions of modularity modulo prime powers as well as a comparison between them in [CKW11]). We will denote by $\bar{\theta}$ its composition with $R \longrightarrow R/\mathfrak{m}_R$, i.e. the residual reduction of θ. Then one has the following theorem of Carayol (Theorem 3 in [Car94]):

Theorem 1.1 (Carayol)**.** *Let $k \geq 2$ and $N > 4$ or assume that 6 is invertible in R (i.e. that $\ell \geq 5$). If the representation attached to $\bar{\theta}$ is absolutely irreducible, then one can attach a Galois representation $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(R)$ to θ in the following sense: For every prime $q \nmid N\ell$, ρ is unramified at q and*

$$\mathrm{tr}(\rho(\mathrm{Frob}_q)) = \theta(T_q).$$

A representation that arises in the way described by the previous theorem is called *modular*. If one wants to explicitly mention a specific eigenform θ due to which the representation ρ is modular one can say that ρ is *attached to* or *associated with* θ.

One can then ask if the converse is true: Given a Galois representation $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(R)$, when is it modular? Furthermore can one have a hold on what the level and weight of this eigenform will be?

Let p be a rational prime. Then one has a natural inclusion map

$$S_k(\Gamma_0(N)) \oplus S_k(\Gamma_0(N)) \longrightarrow S_k(\Gamma_0(Np))$$

whose image is called the p-old subspace. This subspace is stable under the action of $\mathbb{T}_k(Np)$ and so is its orthogonal complement through the so-called Peterson product. We call this complementary subspace the p-new subspace and we denote by $\mathbb{T}_k^{p-\mathrm{new}}(Np)$ the quotient of $\mathbb{T}_k(Np)$ that acts faithfully on it. We will call this quotient the p-new quotient of $\mathbb{T}_k(Np)$. There is also the p-old

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quotient that is defined in the obvious way. Fix another rational prime ℓ. Assume $\mathcal{O}$ is the ring of integers of a number field and λ a prime above ℓ. Here we prove the following level raising result:

Theorem 1.2. *Let $n \geq 2$ be an integer and $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathcal{O}/\lambda^n)$ be a continuous Galois representation that is modular, associated with a Hecke map $\theta : \mathbb{T}_2(N) \longrightarrow \mathcal{O}/\lambda^n$, and residually absolutely irreducible. Let also p be a prime such that $(\ell N, p) = 1$ and assume that $\mathrm{tr}(\rho(\mathrm{Frob}_p)) \equiv \pm(p+1) \bmod \lambda^n$. Then ρ is also associated with a Hecke map $\theta' : \mathbb{T}_2(Np) \longrightarrow \mathcal{O}/\lambda^n$ which is new at p, i.e. θ' factors through the $\mathbb{T}_2^{p-new}(Np)$.*

Remark: For $n = 1$ this is Theorem 1 of [Rib90b].

Remark: The theorem does not exclude the case $\ell | N$.

Remark: As with the case $n = 1$, one can also prove the theorem in the case $p = \ell$ by assuming the condition $\theta(T_p) \equiv \pm(p+1) \bmod \lambda^n$ instead of the one involving the trace of the representation.

Remark: Notice that even if the Hecke map that makes ρ modular in the first place lifts to characteristic 0, i.e. comes from a classical eigenform, there is no guarantee that the Hecke map of level new at p that one obtains in the end lifts too.

Corollary 1.3. *Let ρ be as in Theorem 1.2. Then there exist infinitely many primes p (coprime to N) such that ρ is modular of level Np, new at p.*

Proof. Immediate consequence of Lemma 7.1 in [Rib90a]. □

In what follows we set $\mathbb{T}_N := \mathbb{T}_2(N)$ and $\mathbb{T}_{Np} := \mathbb{T}_2(Np)$. We will also denote the p-th Hecke operator in $\mathbb{T}_{Np}$ by U_p in order to emphasize the different way of acting compared to the one in $\mathbb{T}_N$.

2. Jacobians of modular curves

In this section we gather the necessary results from [Rib90b] that we will need in the proof of the main result.

Let N be a positive integer. Let $X_0(N)_{\mathbb{C}}$ be the modular curve of level N and $J_0(N) := \mathrm{Pic}^0(X_0(N))$ its Jacobian. There is a well defined action of the Hecke operators T_n on $X_0(N)$ and hence, by functoriality, on $J_0(N)$ too. The dual of $J_0(N)$ carries an action of the Hecke algebra as well and can be identified with $S_2(\Gamma_0(N))$. This implies that one has a faithful action of $\mathbb{T}_N$ on $J_0(N)$.

Let now p be a prime not dividing N. In the same way one has an action of Hecke operators on $X_0(Np)$ and its Jacobian $J_0(Np)$ and the latter admits a faithful action of $\mathbb{T}_{Np}$. The interpretation of $X_0(N)$ and $X_0(Np)$ allows us to define the two natural degeneracy maps $\delta_1, \delta_p : X_0(Np) \longrightarrow X_0(N)$ and their pullbacks $\delta_1^*, \delta_p^* : J_0(N) \longrightarrow J_0(Np)$.

There is a map

$$\alpha : J_0(N) \times J_0(N) \longrightarrow J_0(Np), \qquad (x, y) \mapsto \delta_1^*(x) + \delta_p^*(y). \tag{1}$$

whose image is by definition the p-old subvariety of $J_0(Np)$. We will denote this by A. This map α is *almost* Hecke-equivariant:

$$\alpha \circ T_q = T_q \circ \alpha \text{ for every prime } q \neq p, \tag{2}$$

$$\alpha \circ \begin{pmatrix} T_p & p \\ -1 & 0 \end{pmatrix} = U_p \circ \alpha \tag{3}$$

Of course, the first one makes sense only if one interprets the operator T_q as acting diagonally on $J_0(N) \times J_0(N)$. Consider also the kernel **Sh** of the map $J_0(N) \longrightarrow J_1(N)$ induced by $X_1(N) \longrightarrow$

$X_0(N)$. If we inject it into $J_0(N) \times J_0(N)$ via $x \mapsto (x, -x)$ then its image, which we will denote by Σ, is the kernel of the previous map α (see Proposition 1 in [Rib90b]). Furthermore **Sh**, and therefore Σ too, are annihilated by the operators $\eta_r = T_r - (r+1) \in \mathbb{T}_N$ for all primes $r \nmid Np$. (see Proposition 2 in [Rib90b]).

We make a small parenthesis here to introduce a useful notion.

Definition 2.1. *A maximal ideal $\mathfrak{m}$ of the Hecke algebra $\mathbb{T}_N$ is called Eisenstein if it contains the operator $T_r - (r+1)$ for almost all primes r.*

We need a few more definitions and facts (see Corollary in [Rib90b] and the discussion after that):

Let Δ be the kernel of $\begin{pmatrix} 1+p & T_p \\ T_p & 1+p \end{pmatrix} \in M^{2\times 2}(\mathbb{T}_N)$ acting on $J_0(N) \times J_0(N)$. Δ is finite and comes equipped with a perfect $\mathbb{G}_m$-valued skew-symmetric pairing ω. Furthermore Σ is a subgroup of Δ, self orthogonal, and $\Sigma \subset \Sigma^{\perp} \subset \Delta$. One can also see Δ/Σ and therefore its subroup $\Sigma^{\perp}/\Sigma$, as a subgroup of A.

Let B be the p-new subvariety of $J_0(Np)$. It is a complement of A, i.e. $A + B = J_0(Np)$ and $A \cap B$ is finite. The Hecke algebra acts on it faithfully through its p-new quotient and it turns out (see Theorem 2 in [Rib90b]) that

$$A \cap B \cong \Sigma^{\perp}/\Sigma. \tag{4}$$

as groups, with the isomorphism given by the map α.

3. Proof of Theorem 1.2

Let $\theta : \mathbb{T}_N \longrightarrow \mathcal{O}/\lambda^n$ be the eigenform associated with ρ, $\bar{\theta} : \mathbb{T}_N \longrightarrow \mathcal{O}/\lambda$ its reduction mod λ (which is associated with $\overline{\rho}$, the mod λ reduction of ρ) and let I and $\mathfrak{m}$ be the kernels of θ and $\bar{\theta}$ respectively. It will be enough to find a weak modular form $\theta' : \mathbb{T}_2(Np) \longrightarrow \mathcal{O}/\lambda^n$ that agrees with θ on T_q for all primes $q \neq p$ (i.e. they define the same Galois representation) and factors through $\mathbb{T}_2^{p-\mathrm{new}}(Np)$(i.e. new at p). In what follows we will be writing $\mathrm{Ann}(M)$ instead of $\mathrm{Ann}_{\mathbb{T}_N}(M)$ to denote the annihilator of a $\mathbb{T}_N$-module M.

Let us begin with the following auxiliary result:

Lemma 3.1. *$\mathfrak{m}$ is the only maximal ideal of $\mathbb{T}_N$ containing I.*

Proof. We will equivalently show that $\mathbb{T}_N/I$ is local. The proof actually works for any Artinian ring injecting into a local ring.

By the definition of I, $\mathbb{T}_N/I$ injects in $\mathcal{O}/\lambda^n$. Since $\mathbb{T}_N/I$ is Artinian it decomposes into the product of its localizations at its prime (actually maximal) ideals, which are finitely many, say $s \geq 1$. The set containing the identity e_i of each component then forms a complete set (i.e. $\sum_{i=1}^{s} e_i = 1$) of pairwise orthogonal (i.e. $e_i e_j = 0$ for $1 \leq i \neq j \leq s$) non-trivial (i.e. $e_i \neq 0, 1$) idempotents for $\mathbb{T}_N/I$. The set $\{\bar{e}_1, \ldots, \bar{e}_s\}$ of their image through the injection of $\mathbb{T}_N/I$ into $\mathcal{O}/\lambda^n$ is clearly a complete set of pairwise orthogonal non-trivial idempotents too. This implies that $\mathcal{O}/\lambda^n$ is isomorphic to $\prod_{i=1}^{s} \bar{e}_i(\mathcal{O}/\lambda^n)$. But this cannot happen unless $s = 1$ since $\mathcal{O}/\lambda^n$ is local. Since $s = 1$ we get that $\mathbb{T}_N/I$ is local. $\square$

We define:

$$V_I = J_0(N)[I],$$
$$V_{\mathfrak{m}} = J_0(N)[\mathfrak{m}]$$

We have that $\mathfrak{m} \subseteq \mathrm{Ann}(V_{\mathfrak{m}})$ by the definition of $V_{\mathfrak{m}}$. But $\mathfrak{m}$ is maximal so $\mathfrak{m} = \mathrm{Ann}(V_{\mathfrak{m}})$. We also have that $\mathrm{Ann}(V_I) \subseteq \mathrm{Ann}(V_{\mathfrak{m}}) = \mathfrak{m}$, so $\mathfrak{m}$ is in the support of $\mathrm{Ann}(V_I)$. Since the representation $\overline{\rho}$, which is the reduction of ρ and it is associated to $\bar{\theta}$, is irreducible we get that $\mathfrak{m}$ is not Eisenstein (See for example Theorem 5.2c in [Rib90a]). Since $I \subseteq \mathrm{Ann}(V_I)$, Lemma 3.1 implies that $\mathrm{Supp}(V_I)$ is the singleton $\{\mathfrak{m}\}$.

As in [Rib90b] we will consider the case where $\mathrm{tr}(\rho(\mathrm{Frob}_p)) \equiv -(p+1) \bmod \lambda^n$. The other case where $\mathrm{tr}(\rho(\mathrm{Frob}_p)) \equiv p+1 \bmod \lambda^n$ is treated in exactly the same case, with some minor alterations which we explicitly mention. Since ρ is modular, associated with θ, this translates to

$$\theta(T_p) \equiv -(p+1) \bmod \lambda^n. \tag{5}$$

Now consider the composite map

$$J_0(N) \to J_0(N) \times J_0(N) \xrightarrow{\alpha} A \subseteq J_0(Np),$$

where the first map is the diagonal embedding (in the case of $\mathrm{tr}(\rho(\mathrm{Frob}_p)) \equiv p+1 \bmod \lambda^n$ we pick the anti-diagonal map) and the second is the map α defined in the previous section. By abuse of notation, we will also denote by V_I the image of V_I in $J_0(N) \times J_0(N)$ via the diagonal embedding. We then claim that its intersection with Σ is zero: Assume that it is not, and denote it by V_I'. It is easy to see that V_I' is preserved by the action of $\mathbb{T}_N$ so it can be seen as a $\mathbb{T}_N$-module: For an $(x,x) \in V_I'$ we have (using realtion (2))

$$\alpha(T_q(x,x)) = T_q(\alpha(x,x)) = T_q(0) = 0 \qquad \text{for primes } q \neq p \tag{6}$$

and (using relation (5))

$$\alpha(T_p(x,x)) = \alpha(T_p(x), T_p(x)) = \alpha(-(p+1)x, -(p+1)x) = -(p+1)\alpha(x,x) = 0.$$

In the case where $\theta(T_p) \equiv p+1 \bmod \lambda^n$, the elements of V_I' are of the form $(x,-x)$ but the reasoning is the same. Since Σ is annihilated by almost all operators $T_r - (r+1)$, V_I' is annihilated by almost all of them too. This implies that every maximal ideal containing $\mathrm{Ann}(V_I')$ is Eisenstein. But $\mathrm{Ann}(V_I) \subseteq \mathrm{Ann}(V_I')$ so V_I has an Eisenstein ideal in its support. On the other hand the only maximal ideal in the support of V_I is $\mathfrak{m}$ which is non-Eisenstein, so we get a contradiction. One can therefore see V_I as a subgroup of A and we will abuse notation to denote its image through the above map by V_I too. We have the following Lemma:

Lemma 3.2. *V_I is stable under the action of $\mathbb{T}_{Np}$ and the action is given by a ring homomorphism $\theta' : \mathbb{T}_{Np} \longrightarrow \mathcal{O}/\lambda^n$.*

Proof. This is nothing but a straightforward calculation:

First note that the action of $\mathbb{T}_N$ on V_I factors through T_N/I so we obtain a map $\theta(\mathbb{T}_N/I) \longrightarrow \mathrm{End}(V_I)$. Let y be a non-trivial element of the image of V_I in A. Then there exists $x \in V_I$ such that $\alpha(x,x) = y$. Let now q be a prime other than p. In view of relation (2) and we have that:

$$T_q(y) = T_q(\alpha(x,x)) = \alpha(T_q(x), T_q(x)) = \alpha(\theta(T_q)x, \theta(T_q)x) = \theta(T_q)\alpha(x,x) = \theta(T_q)y.$$

For $q = p$ we have (using relation (3) and (5)):

$$U_p(y) = U_p(\alpha(x,x)) = \alpha(\begin{pmatrix} T_p & p \\ -1 & 0 \end{pmatrix}(x,x)^{\mathrm{T}}) = \alpha(T_p(x) + px, -x) =$$
$$= \alpha(\theta(T_p)x + px, -x) = \alpha(-x,-x) = -\alpha(x,x) = -y$$

It turns out that y is an eigenvector and that the action of $\mathbb{T}_{Np}$ on it defines a ring homomorphism $\theta' : \mathbb{T}_{Np} \longrightarrow \mathcal{O}/\lambda^n$ via:

$$\theta'(T_q) = \theta(T_q) \qquad \text{for all primes } q \neq p \text{ and}$$
$$\theta'(U_p) = -1$$

To treat the other case one has to keep in mind for the formulas above that $y = \alpha(x, -x)$ and proceed in the same way to get the same result except that $U_p(y) = y$ this time and therefore $\theta'(U_p) = 1$. □

Remark: Since the θ and θ' actually agree on almost all primes, it is clear that they are associated with the same Galois representation, so θ' is the candidate map we were looking for.

To finish of the proof of the main result it remains to show that the map factors through the p-new quotient of the Hecke algebra. To this end, it is enough to show that V_I, when viewed as a subgroup of $J_0(Np)$ is a subgroup of $(A \cap B)$. We again proceed according to Ribet. It is easy to see that V_I, when considered as a subgroup of $J_0(N) \times J_0(N)$, is a subgroup of Δ. Let $\bar{V}_I$ be the image of V_I in $\Delta/\Sigma^{\perp}$. Then, in view of (4), we just need to show that $\bar{V}_I$ is trivial.

First notice that $\bar{V}_I$ is preserved by the action of $\mathbb{T}_N$. For this it is enough to check that if $z \in V_I \cap \Sigma^{\perp}$ then $T_q(z) \in \Sigma^{\perp}$ and $T_p(z) \in \Sigma^{\perp}$ (clearly they will also be in V_I). Let $x \in \Sigma$. We then have the following: $\omega(x, T_q(z)) = \omega(T_q^{\vee}(x), z)$. Now the subalgebras of generated by T_q and $T_q^{\vee}$ are isomorphic (see p444 in [Rib90a]). Since the subalgebra generated by T_q preserves Σ as shown in (6), we get that $T_q^{\vee}(x) \in \Sigma$ and therefore that $\omega(T_q^{\vee}(x), z) = 0$. Finally, using (5) again, $\omega(x, T_p(z)) = \omega(x, -(p+1)z) = -(p+1)\omega(x, z) = 0$.

Now according to Ribet in the proof of Lemma 2 in [Rib90b], $\Delta/\Sigma^{\perp}$ is dual to Σ which is annihilated by almost all operators $T_r - (r+1)$, so $\Delta/\Sigma^{\perp}$, and therefore $\bar{V}_I$, is annihilated by them too. This implies that any maximal ideal containing $\mathrm{Ann}(\bar{V}_I)$ is Eisenstein. Recall that V_I is not Eisenstein. Now assume for contradiction that $\bar{V}_I$ is non-zero. Since $\mathrm{Ann}(\bar{V}_I)$ contains $\mathrm{Ann}(V_I)$, we get that the support of V_I also contains Eisenstein ideals. This is the desired contradiction that completes the proof of Theorem 1.2.

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References

[Car94] Henri Carayol. Formes modulaires et représentations galoisiennes à valeurs dans un anneau local complet. In *p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991)*, volume 165 of *Contemp. Math.*, pages 213–237. Amer. Math. Soc., Providence, RI, 1994.

[CKW11] I. Chen, I. Kiming, and Gabor Wiese. On modular galois representations modulo prime powers. *http://arxiv.org/abs/1105.1918v1*, 2011.

[DS74] Pierre Deligne and Jean-Pierre Serre. Formes modulaires de poids 1. *Ann. Sci. École Norm. Sup. (4)*, 7:507–530 (1975), 1974.

[Rib90a] Kenneth. A. Ribet. On modular representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ arising from modular forms. *Invent. Math.*, 100(2):431–476, 1990.

[Rib90b] Kenneth A. Ribet. Raising the levels of modular representations. In *Séminaire de Théorie des Nombres, Paris 1987–88*, volume 81 of *Progr. Math.*, pages 259–271. Birkhäuser Boston, Boston, MA, 1990.

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