A BICATEGORY OF REDUCED ORBIFOLDS FROM THE POINT OF VIEW OF DIFFERENTIAL GEOMETRY

MATTEO TOMMASINI

Abstract. We describe a bicategory (**Red Orb**) of reduced orbifolds in the framework of differential geometry (i.e. without any explicit reference to notions of Lie groupoids or differentiable stacks, but only using orbifold atlases, local lifts and changes of charts). In order to construct such a bicategory we first define a 2-category (**Red Atl**) whose objects are reduced orbifold atlases (on paracompact, second countable, Hausdorff topological spaces). The definition of morphisms is obtained as a slight modification of a definition by A. Pohl, while the definition of 2-morphisms and compositions of them is new in this setup. By using the bicalculus of fractions described by D. Pronk, we are able to construct from such a 2-category the bicategory (**Red Orb**). We prove that it is weakly equivalent to the bicategory of reduced orbifolds described in terms of proper, effective, étale Lie groupoids by D. Pronk and I. Moerdijk and to the 2-category of reduced orbifolds described by several authors in the past in terms of a suitable class of differentiable Deligne-Mumford stacks . Finally, we give a simple description of the homotopy category of (**Red Orb**).

Contents

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Introduction

A well known issue in mathematics is that of modelling geometric objects where points have non-trivial groups of automorphisms. In topology and differential geometry the standard approach to these objects (when the groups associated to every point are finite) is through orbifolds. This concept was formalized for the first time by Ikiro Satake in 1956 in [Sa] with some different hypothesis than the current ones, although the informal idea dates back at least to Henri Poincaré (for example, see [Poi]). Currently there are at least 3 main approaches to orbifolds:

(a) via orbifold atlases (see for example [CR] and [Po] for the reduced case, but take into account the fact that [CR] and [Po] give no compatible descriptions for what concerns morphisms of orbifolds),
(b) via a class of Lie groupoids (see for example [Pr], [M] and [MM]),
(c) via a class of C^∞-Deligne-Mumford stacks (see for example [J1] and [J2]).

On the one hand, the approaches in (a) give rise to 1-categories; on the other hand, the approach in (b) gives rise to a bicategory (i.e. almost a 2-category, except that we don't require that compositions of 1-morphisms is associative, but only that it is associative up to canonical 2-morphisms) and the approach in (c) gives rise to a 2-category. It was proved in [Pr] that (b) and (c) are weakly equivalent as bicategories. Since (b) and (c) are compatible approaches, then one shall argue that there should also exist a non-trivial structure of 2-category or bicategory having as objects orbifold atlases or equivalence classes of them (i.e. orbifold structures). Moreover, one would like to describe such a structure so that it is compatible with (b) and (c).

In the present paper we manage to get such results for the class of all reduced orbifolds, i.e. orbifolds that are locally modelled on open connected sets of some $\mathbb{R}^n$, modulo finite groups acting smoothly and effectively on them. In order to do that, we proceed as follows.

(1) We describe a 2-category (**Red Atl**) whose objects are reduced orbifold atlases on any paracompact, second countable, Hausdorff topological space. The definition of morphisms is obtained as a slight modification of an analogous definition given by Anke Pohl in [Po], while the notion of 2-morphisms (and compositions of them) is new in this setup (see definitions 1.12 and 1.19). Such definitions are useful for differential geometers mainly because they don't require any previous knowledge of Lie groupoids and/or differentiable stacks. (**Red Atl**) is a 2-category (see proposition 3.4), but it is still not the structure that we are trying to get since in this 2-category different orbifold atlases that represent the same orbifold structure are not related by an isomorphism neither by an equivalence.
(2) We recall briefly the definition of the 2-category (**PEÉ Gpd**) whose objects are proper, effective, étale Lie groupoids and we describe a 2-functor F from (**Red Atl**) to (**PEÉ Gpd**) (see theorem 4.15).
(3) In [Pr] Dorette Pronk proved that the set $\mathbf{W}_{\mathbf{PEÉ\,Gpd}}$ of all weak equivalences (also known as Morita equivalences or essential equivalences) in (**PEÉ Gpd**) admits a right calculus of fractions. Roughly speaking, this amounts to saying that it is possible to construct a bicategory $(\mathbf{PEÉ\,gpd})\left[\mathbf{W}^{-1}_{\mathbf{PEÉ\,Gpd}}\right]$ and a pseudofunctor

$$G : (\mathbf{PEÉ\,Gpd}) \longrightarrow (\mathbf{PEÉ\,gpd})\left[\mathbf{W}^{-1}_{\mathbf{PEÉ\,Gpd}}\right]$$

that sends each weak equivalence to an equivalence and that is universal with respect to this property (see proposition 5.3). The bicategory obtained in this way is the bicategory we mentioned in (b) above if we restrict to the case of reduced orbifolds. Using the same technique, we are able to identify in $(\mathbf{Red\,Atl})$ a set $\mathbf{W}_{\mathbf{Red\,Atl}}$ of 1-morphisms (that we call again "weak equivalences", see definition 6.1) and to prove that it admits a right calculus of fractions. Roughly speaking, such weak equivalences are in bijection with weak equivalences of Lie groupoids via the 2-functor F (we refer to proposition 6.5 for the precise statement). Since we have a right calculus of fractions, we are able to construct a bicategory $(\mathbf{Red\,Atl})$ and a pseudofunctor

$$H : (\mathbf{Red\,Atl}) \longrightarrow (\mathbf{Red\,Orb}) := (\mathbf{Red\,Atl})\left[\mathbf{W}^{-1}_{\mathbf{Red\,atl}}\right]$$

that sends each weak equivalence to an equivalence and that is universal with respect to this property (see proposition 6.8). Objects in this new bicategory are again reduced orbifold atlases; a morphism from an atlas $\mathcal{U}$ to an atlas $\mathcal{V}$ is a triple consisting of an atlas $\mathcal{U}'$, a weak equivalence $\mathcal{U}' \to \mathcal{U}$ and a morphism $\mathcal{U}' \to \mathcal{V}$ (in other terms, a morphism is given first by replacing the source with a new atlas that is weakly equivalent to the original one, then by considering a morphism from the new atlas to the fixed target). We refer to (6.7) for the description of 2-morphisms in this bicategory.

(4) We are able to prove that the pseudofunctor $G \circ F$ sends each weak equivalence of orbifold atlases to an equivalence. Therefore, by the universal property of H, there exists a unique pseudofunctor L making the following diagram commute

$$\begin{array}{ccc}
(\mathbf{Red\,Atl}) & \xrightarrow{\quad F \quad} & (\mathbf{PE\acute{E}\,Gpd}) \\
\Big\downarrow H & \curvearrowright & \Big\downarrow G \\
(\mathbf{Red\,Orb}) & \xrightarrow{\quad L \quad} & (\mathbf{PE\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{PE\acute{E}\,Gpd}}\right].
\end{array}$$

(5) Then we are able to prove (see theorem 6.9) that L is a weak equivalence of bicategories (and an equivalence of bicategories if one uses the axiom of choice). This proves that the approach described in $(\mathbf{Red\,Orb})$ is compatible with the approach (b) to reduced orbifolds in terms of proper, effective, étale Lie groupoids. Since (b) and (c) are equivalent approaches by [Pr], this allows us to prove that $(\mathbf{Red\,Orb})$ is weakly equivalent to the 2-category of effective orbifolds described as a full 2-subcategory of the category of C^{∞}-Deligne-Mumford stacks (see theorem 7.1).

In the last part of this paper we will also provide a simple description of a 1-category $(\mathbf{Ho})$ whose objects are reduced orbifold structures (i.e. classes of equivalent reduced orbifold atlases on a topological space), whose morphisms are suitable classes of morphisms (see definition 8.4) and that contains the category of C^{∞}-manifolds as a full subcategory. We prove (see theorem 8.8) that $(\mathbf{Ho})$ is equivalent to the homotopy category of $(\mathbf{Red\,Orb})$ (i.e. the 1-category obtained from $(\mathbf{Red\,Orb})$ by identifying any pair of morphisms connected by an invertible 2-morphism).

Some problems remain open:

(i) we have described a bicategory structure by restricting to the case of reduced orbifolds. Is it possible to give an analogous description also in the more general case of non-reduced orbifolds? Since (b) and (c) are defined also for non-reduced orbifolds, in principle this should be possible, but it seems that it will require more work (for example, the notion of morphisms and 2-morphisms will be much more complicated in that setup).

(ii) Our approach is compatible with the one in terms of Lie groupoids and in terms of C^∞-Deligne-Mumford stacks, but it seems that the 1-category (**Ho**) that we obtain is not equivalent to the 1-category described in [CR]. To be more precise, the description given in [CR] holds also for non-reduced orbifolds; if we restrict to the full subcategory of reduced orbifolds, then the objects are exactly the same of (**Ho**), but the morphisms (described there as "good maps", i.e. equivalence classes of "compatible systems") appear to be a strict subset of the morphisms in (**Ho**). The most important obstruction for having a complete agreement is the request in [CR, §4.4] that a compatible system should give a bijection between a collection of open *connected* sets that cover the topological space in the source and a collection of open connected sets covering (part of) the topological space in the target. This implies that the trivial map from a manifold consisting of 2 disjoint copies of a unit ball in some $\mathbb{R}^n$ to a manifold consisting of the same unit ball (both considered as trivial orbifolds) can't be a morphism according to [CR]. On the other hand, any such morphism is obviously a morphism of manifolds, hence it is also a morphism of orbifolds in (**Ho**). Therefore, in order to have a complete compatibility between the 1-category in [CR] and (**Red Orb**) (or, equivalently, Lie groupoids or C^∞-Deligne-Mumford stacks) one should relax a bit the axioms used in [CR].

1. Reduced orbifold atlases

Let us review some basic definitions about orbifolds.

Definition 1.1. [MP, §1] Let X be a paracompact, second countable, Hausdorff topological space and let $U \subseteq X$ be open and non-empty. Then a *reduced orbifold chart* (also known as *reduced uniformizing system*) *of dimension* n for U is the datum of:

- a *connected* open subset $\widetilde{U}$ of $\mathbb{R}^n$;
- a *finite* group G of diffeomorphisms of $\widetilde{U}$;
- a continuous, surjective and G-invariant map $\pi : \widetilde{U} \to U$, which induces an homeomorphism between $\widetilde{U}/G$ and U, where we give to $\widetilde{U}/G$ the quotient topology.

Remark 1.2. We will always assume that G *acts effectively*; the orbifolds that have this property are usually called *reduced* or *effective*. Since we will only consider reduced orbifolds, we will sometimes omit the name "reduced". Some of the current literature on orbifolds assumes that $\widetilde{U}$ is only a connected smooth manifold of dimension n instead of an open connected subset of $\mathbb{R}^n$. This makes a difference for the definition of charts, but the arising notion of orbifold is not affected by that. To be more precise, to any orbifold atlas (see below) where the $\widetilde{U}$'s are connected smooth manifolds of dimension n, one can associate easily another orbifold atlas where the $\widetilde{U}$'s are open connected subsets of $\mathbb{R}^n$ and the 2 orbifold atlases give rise to the same orbifold structure (see below).

Definition 1.3. [Po, §2.1] Let us fix any 2 charts $(\widetilde{U}, G, \pi)$ and $(\widetilde{U}', G', \pi')$ for non-empty open subsets U, U' of X. Then a *change of charts* with source $(\widetilde{U}, G, \pi)$ and target $(\widetilde{U}', G', \pi')$ is any diffeomorphism $\lambda : \hat{U} \to \hat{U}'$ defined between *connected* open sets $\hat{U} \subseteq \widetilde{U}$ and $\hat{U}' \subseteq \widetilde{U}'$, such that $\pi' \circ \lambda = \pi$. In particular, for consistency of notations, we will say that also the empty map $\varnothing \to \varnothing$ is a change of charts. If λ is any change of charts, we denote by $\operatorname{dom} f$ its domain and by $\operatorname{cod} f$ its codomain. If

$x \in \operatorname{dom} f$, then $\operatorname{germ}_x f$ denotes the germ of f at x. An *embedding* is any change of charts λ as before, such that $\operatorname{dom}\lambda = \widetilde{U}$. Two charts as before are called *compatible* if for each pair $\widetilde{x} \in \widetilde{U}$, $\widetilde{x}' \in \widetilde{U}'$ with $\pi(\widetilde{x}) = \pi'(\widetilde{x}')$ there exists a change of charts λ with $\widetilde{x} \in \operatorname{dom}\lambda$ and $\widetilde{x}' \in \operatorname{cod}\lambda$.

Remark 1.4. Let us suppose that we have any change of charts λ from $(\widetilde{U}, G, \pi)$ to $(\widetilde{U}', G', \pi')$; let us also fix any pair of points $\widetilde{x} \in \operatorname{dom}\lambda$ and $\widetilde{x}' \in \operatorname{cod}\lambda$ with $\pi(\widetilde{x}) = \pi'(\widetilde{x}')$ and any open neighbourhood $\overline{A}$ of $\widetilde{x}$ in $\operatorname{dom}\lambda$. Then there exists an open neighbourhood $\widetilde{A} \subseteq \overline{A}$ of $\widetilde{x}$ such that $(\widetilde{A}, G_{\widetilde{x}}, \pi|_{\widetilde{A}})$ is a chart (where $G_{\widetilde{x}}$ is the stabilizer of $\widetilde{x}$ in G); in particular, $\lambda|_{\widetilde{A}}$ is an embedding of charts. Up to composing λ with an element of G' we can also assume that $\lambda(\widetilde{x}) = \widetilde{x}'$.

Definition 1.5. [MP, §1] Let X be a paracompact, second countable, Hausdorff topological space; a *reduced orbifold atlas of dimension* n on X is any family $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i\in I}$ of reduced orbifolds charts of dimension n, such that:

(i) the family $\{\pi_i(\widetilde{U}_i)\}_{i\in I}$ is an open cover of X;

(ii) for every pair $(i, i') \in I \times I$ and for every $x \in \pi_i(\widetilde{U}_i) \cap \pi_{i'}(\widetilde{U}_{i'})$ there exists a chart $(\widetilde{U}, G, \pi)$ such that $x \in \pi(\widetilde{U}) \subseteq \pi_i(\widetilde{U}_i) \cap \pi_{i'}(\widetilde{U}_{i'})$ and there exists a pair of embeddings of $(\widetilde{U}, G, \pi)$ in $(\widetilde{U}_i, G_i, \pi_i)$ and $(\widetilde{U}_{i'}, G_{i'}, \pi_{i'})$ respectively.

Given any orbifold atlas $\mathcal{U}$ as before and any pair $(i, i') \in I \times I$, we denote by $\mathcal{C}h(\mathcal{U}, i, i')$ the set of all changes of charts λ from $(\widetilde{U}_i, G_i, \pi_i)$ to $(\widetilde{U}_{i'}, G_{i'}, \pi_{i'})$ and we set $\mathcal{C}h(\mathcal{U}) := \coprod_{(i,i')\in I\times I} \mathcal{C}h(\mathcal{U}, i, i')$.

Remark 1.6. By remark 1.4, (ii) is equivalent to imposing that any 2 charts of $\mathcal{U}$ are compatible. In the literature there is also a more restrictive notion of atlas where (ii) is replaced by

(ii)$'$ same condition as (ii) but we require that the chart $(\widetilde{U}, G, \pi)$ belongs to the atlas $\mathcal{U}$.

The atlases that are determined by (ii)$'$ are a strict subset of those determined by (ii); nonetheless the orbifold structures (see below) that one gets are the same. In particular, the chart $(\widetilde{U}, G, \pi)$ will always belong to the maximal atlas $\mathcal{U}^{\max}$ associated to $\mathcal{U}$ (see below).

Definition 1.7. [MP, §1] Let $\mathcal{U}$ and $\mathcal{U}'$ be reduced orbifold atlases for the same topological space X. We say that they are *equivalent* if their union is again an orbifold atlas for X, i.e. if and only if any chart of $\mathcal{U}$ is compatible with any chart of $\mathcal{U}'$. This gives a partial order on the set of atlases for X; a *reduced orbifold structure* of dimension n on X is any maximal orbifold atlas in that set, or equivalently, any equivalence class with respect to compatibility of atlases. A *reduced orbifold* of dimension n is any pair $(X, [\mathcal{U}])$ consisting of a paracompact, second countable, Hausdorff topological space X and a reduced orbifold structure $[\mathcal{U}]$ on X. Any atlas in $[\mathcal{U}]$ is called an orbifold atlas for $(X, [\mathcal{U}])$.

Definition 1.8. [CR, §4.1] Let $f : X \to Y$ be any continuous map between topological spaces and let $U \subseteq X$ and $V \subseteq Y$ be open subsets such that $f(U) \subseteq V$. Let us suppose that there exists charts $(\widetilde{U}, G, \pi)$ for U and $(\widetilde{V}, H, \phi)$ for V. Then a *local lift* of f with respect to these 2 charts is any smooth map $\widetilde{f} : \widetilde{U} \to \widetilde{V}$ such that $\phi \circ \widetilde{f} = f \circ \pi$.

Definition 1.9. Let us fix any orbifold atlas $\mathcal{U}$ as before and let P be any subset of $\mathcal{C}h(\mathcal{U})$. We say that P is *a good subset of* $\mathcal{C}h(\mathcal{U})$ if the following property hold:

(P1) for each $\lambda \in \mathcal{C}h(\mathcal{U})$ and for each $\widetilde{x} \in \operatorname{dom}\lambda$ there exists $\hat{\lambda} \in P$ such that $\widetilde{x} \in \operatorname{dom}\hat{\lambda}$ and $\operatorname{germ}_{\widetilde{x}} \lambda = \operatorname{germ}_{\widetilde{x}} \hat{\lambda}$.

Since P is a subset of $\mathcal{C}h(\mathcal{U})$, for each $(i,i')\in I\times I$ we write for simplicity $P(i,i'):=P\cap\mathcal{C}h(\mathcal{U},i,i')$ and $P(i,-):=\coprod_{i'\in I}P(i,i')$.

Remark 1.10. In the notations of [Po], (P1) is the condition that P *generates* the pseudogroup $\mathcal{C}h(\mathcal{U})$ (inside the larger pseudogroup $\Psi(\mathcal{U})$ defined in [Po]). In [Po] there are other two technical conditions (axioms of "quasi-pseudogroup"), but they are implied by (P1) in our case, so we omit them. Under this remark, our definition of morphism (see below) will be almost equivalent to that stated in [Po, definition 4.10]; the only difference will be that the morphisms of the form $\nu(\lambda)$ in [Po] are required to be embeddings between orbifold charts (small enough), while we only require that they are changes of charts. Anyway, it is easy to see that the two definitions give rise to the same notion of morphism between orbifold atlases.

Definition 1.11. Let $\mathcal{U}=\{(\widetilde{U}_i,G_i,\pi_i)\}_{i\in I}$ and $\mathcal{V}=\{(\widetilde{V}_j,H_j,\phi_j)\}_{j\in J}$ be orbifold atlases for X and Y respectively. Then a *representative of a morphism* from $\mathcal{U}$ to $\mathcal{V}$ is any tuple $\hat{f}:=(f,\overline{f},\{\widetilde{f}_i\}_{i\in I},P_f,\nu_f)$ that satisfies the following conditions:

(Q1) $f:X\to Y$ is any continuous map;
(Q2) $\overline{f}:I\to J$ is any set map;
(Q3) for each $i\in I$, the map $\widetilde{f}_i$ is a local lift of f with respect to the orbifold charts$(\widetilde{U}_i,G_i,\pi_i)\in\mathcal{U}$ and $(\widetilde{V}_{\overline{f}(i)},H_{\overline{f}(i)},\phi_{\overline{f}(i)})\in\mathcal{V}$;
(Q4) P is any good subset of $\mathcal{C}h(\mathcal{U})$;
(Q5) $\nu_f:P\to\mathcal{C}h(\mathcal{V})$ is any set map that assigns to each $\lambda\in P(i,i')$ a change of charts $\nu_f(\lambda)\in\mathcal{C}h(\mathcal{V},\overline{f}(i),\overline{f}(i'))$, such that:
   (a) $\operatorname{dom}\nu_f(\lambda)$ is an open set containing $\widetilde{f}_i(\operatorname{dom}\lambda)$,
   (b) $\operatorname{cod}\nu_f(\lambda)$ is an open set containing $\widetilde{f}_{i'}(\operatorname{cod}\lambda)$,
   (c) $\widetilde{f}_{i'}\circ\lambda=\nu_f(\lambda)\circ\widetilde{f}_i|_{\operatorname{dom}\lambda}$,
   (d) for all $i\in I$, for all $\lambda,\lambda'\in P_f(i,-)$ and for all $\widetilde{x}_i\in\operatorname{dom}\lambda\cap\operatorname{dom}\lambda'$ with $\operatorname{germ}_{\widetilde{x}_i}\lambda=\operatorname{germ}_{\widetilde{x}_i}\lambda'$, we have

$$\operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\nu_f(\lambda)=\operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\nu_f(\lambda'),$$

   (e) for all $(i,i',i'')\in I^3$, for all $\lambda\in P_f(i,i')$, for all $\mu\in P_f(i',i'')$ and for all $\widetilde{x}_i\in\lambda^{-1}(\operatorname{cod}\lambda\cap\operatorname{dom}\mu)$ we have

$$\operatorname{germ}_{\widetilde{f}_{i'}(\lambda(\widetilde{x}_i))}\nu_f(\mu)\cdot\operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\nu_f(\lambda)=\operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\nu_f(\eta)$$

   where η is any element of $P(i,i'')$ such that $\operatorname{germ}_{\widetilde{x}_i}\eta=\operatorname{germ}_{\widetilde{x}_i}\mu\circ\lambda$ (it exists by (P1)),
   (f) for all $i\in I$, for all $\lambda\in P(i,i)$ and for all $\widetilde{x}_i\in\operatorname{dom}\lambda$ such that $\operatorname{germ}_{\widetilde{x}_i}\lambda=\operatorname{germ}_{\widetilde{x}_i}\operatorname{id}_{\widetilde{U}_i}$, we have

$$\operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\nu_f(\lambda)=\operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\operatorname{id}_{\widetilde{V}_{\overline{f}(i)}}.$$

Definition 1.12. Let us fix two orbifold atlases $\mathcal{U}=\{(\widetilde{U}_i,G_i,\pi_i)\}_{i\in I}$ and $\mathcal{V}=\{(\widetilde{V}_j,H_j,\phi_j)\}_{j\in J}$ for X and Y respectively and let

$$\hat{f}=\left(f,\overline{f},\left\{\widetilde{f}_i\right\}_{i\in I},P_f,\nu_f\right)\quad\text{and}\quad\hat{f}':=\left(f',\overline{f}',\left\{\widetilde{f}'_i\right\}_{i\in I},P'_f,\nu'_f\right)$$

be two representatives of orbifold maps from $\mathcal{U}$ to $\mathcal{V}$. We say that $\hat{f}$ is *equivalent* to $\hat{f}'$ if and only if $f=f'$, $\overline{f}=\overline{f}'$, $\widetilde{f}_i=\widetilde{f}'_i$ for all $i\in I$, and

$$\operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\nu_f(\lambda)=\operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\nu'_f(\lambda')$$

for all $i \in I$, for all $\lambda \in P_f(i,-)$, $\lambda' \in P'_f(i,-)$ and for all $\widetilde{x}_i \in \operatorname{dom}\lambda \cap \operatorname{dom}\lambda'$ with $\operatorname{germ}_{\widetilde{x}_i}\lambda = \operatorname{germ}_{\widetilde{x}_i}\lambda'$. This defines an equivalence relation (it is reflexive by (Q5d)). The equivalence class of $\hat{f}$ will be denoted by

$$[\hat{f}] = \left(f, \overline{f}, \left\{\widetilde{f}_i\right\}_{i\in I}, [P_f, \nu_f]\right) : \mathcal{U} \longrightarrow \mathcal{V}$$

and it is called an *orbifold map from $\mathcal{U}$ to $\mathcal{V}$ over the continuous map $f : X \to Y$.*

Definition 1.13. Let us fix any orbifold atlas $\mathcal{U} := \{(\widetilde{U}_i, G_i, \pi_i)\}_{i\in I}$ on a topological space X and let $\mathcal{U}^{\max}$ be the maximal atlas associated to $\mathcal{U}$. Then there is an obvious morphism $\iota_{\mathcal{U}}$ over id_X from $\mathcal{U}$ to $\mathcal{U}^{\max}$ given by considering every chart in $\mathcal{U}$ as a chart in $\mathcal{U}^{\max}$ and by considering every change of charts in $\mathcal{U}$ as a change of charts in $\mathcal{U}^{\max}$ (in particular, we can choose as good subset P the set $\mathcal{C}h(\mathcal{U}) \subseteq \mathcal{C}h(\mathcal{U}^{\max})$).

We need to define compositions of morphisms of orbifold atlases. In order to do that we follow [Po, construction 5.9], with the only difference due to remark 1.10.

Construction 1.14. Let us fix any triple of orbifold atlases

$$\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i\in I}, \quad \mathcal{V} := \{(\widetilde{V}_j, H_j, \phi_j)\}_{j\in J}, \quad \mathcal{W} = \{(\widetilde{W}_l, K_l, \xi_l)\}_{l\in L}$$

for 3 topological spaces X, Y and Z respectively. Let us also fix morphisms

$$[\hat{f}] = \left(f, \overline{f}, \left\{\widetilde{f}_i\right\}_{i\in I}, [P_f, \nu_f]\right) : \mathcal{U} \longrightarrow \mathcal{V},$$
$$[\hat{g}] = \left(g, \overline{g}, \left\{\widetilde{g}_j\right\}_{j\in J}, [P_g, \nu_g]\right) : \mathcal{V} \longrightarrow \mathcal{W}.$$

Then we define a composition

$$[\hat{g}] \circ [\hat{f}] := [\hat{h}] = \left(g\circ f, \overline{g}\circ\overline{f}, \left\{\widetilde{g}_{\overline{f}(i)} \circ \widetilde{f}_i\right\}_{i\in I}, [P_h, \nu_h]\right) : \mathcal{U} \longrightarrow \mathcal{W}.$$

Here we construct the class $[P_h, \nu_h]$ as follows: first of all we fix representatives (P_f, ν_f) for $[P_f, \nu_f]$ and (P_g, ν_g) for $[P_g, \nu_g]$. Then let us fix any $i \in I$, any $\lambda \in P_f(i,-)$ and any point $\widetilde{x}_i \in \operatorname{dom}\lambda$. Since P_g is a good subset of $\mathcal{C}h(\mathcal{V})$, then by (P1) there exists $\gamma_{\widetilde{f}_i(\widetilde{x}_i),\nu_f(\lambda)} \in P_g$ and an open set

$$\widetilde{V}_{\widetilde{f}_i(\widetilde{x}_i),\nu_f(\lambda)} \subseteq \operatorname{dom}\nu_f(\lambda) \cap \operatorname{dom}\gamma_{\widetilde{f}_i(\widetilde{x}_i),\nu_f(\lambda)} \subseteq \widetilde{V}_{\overline{f}(i)}$$

such that $\widetilde{f}_i(\widetilde{x}_i) \in \widetilde{V}_{\widetilde{f}_i(\widetilde{x}_i),\nu_f(\lambda)}$ and

$$\left(\nu_f(\lambda)\right)\Big|_{\widetilde{V}_{\widetilde{f}_i(\widetilde{x}_i),\nu_f(\lambda)}} = \left(\gamma_{\widetilde{f}_i(\widetilde{x}_i),\nu_f(\lambda)}\right)\Big|_{\widetilde{V}_{\widetilde{f}_i(\widetilde{x}_i),\nu(\lambda)}}.$$

For each pair $(\lambda, \widetilde{x}_i)$ as before, let us consider the open set

$$\widetilde{U}_{\widetilde{x}_i,\lambda} := \widetilde{f}_i^{-1}\left(\widetilde{V}_{\widetilde{f}_i(\widetilde{x}_i),\nu_f(\lambda)}\right) \cap \operatorname{dom}\lambda \subseteq \widetilde{U}_i.$$

Then for each $i \in I$ and for each $\lambda \in P_f(i,-)$ we choose any set of points

$$\{\widetilde{x}_i^e\}_{e\in E(\lambda)} \subseteq \widetilde{U}_i$$

such that:

(i) $\widetilde{V}_{\widetilde{f}_i(\widetilde{x}_i^e),\nu_f(\lambda)} \neq \widetilde{V}_{\widetilde{f}_i(\widetilde{x}_i^{e'}),\nu_f(\lambda)}$ for all $e \neq e'$ in $E(\lambda)$;

(ii) the set $\{\widetilde{U}_{\widetilde{x}_i^e,\lambda}\}_{e\in E(\lambda)}$ is a covering for $\operatorname{dom}\lambda$.

Then we set:

$$P_h := \Big\{ \lambda|_{\widetilde{U}_{\widetilde{x}_i^e,\lambda}}, \quad \forall\, i \in I,\, \forall\, \lambda \in P_f(i,-),\, \forall\, e \in E(\lambda) \Big\}.$$

Since P_f is a good subset of $\mathcal{C}h(\mathcal{U})$, then by (ii) we get that also P_h is a good subset of $\mathcal{C}h(\mathcal{U})$. Then for each $\lambda_{\widetilde{U}_{\widetilde{x}_i^e,\lambda}} \in P_h$ we set

$$\nu_f^{\mathrm{ind}} \left(\lambda|_{\widetilde{U}_{\widetilde{x}_i^e,\lambda}} \right) := \gamma_{\widetilde{f}_i(\widetilde{x}_i^e),\nu_f(\lambda)}.$$

Property (i) implies that ν_f^{ind} is a well-defined map from P_h to $\mathcal{C}h(\mathcal{V})$ and a direct computation proves that $(P_h, \nu_f^{\mathrm{ind}}) \in [P_f, \nu_f]$. Then we simply define

$$\nu_h \Big(\lambda|_{\widetilde{U}_{\widetilde{x}_i^e,\lambda}} \Big) := \nu_g \Big(\gamma_{\widetilde{f}_i(\widetilde{x}_i^e),\nu_f(\lambda)} \Big) = \nu_g \circ \nu_f^{\mathrm{ind}} \Big(\lambda|_{\widetilde{U}_{\widetilde{x}_i^e,\lambda}} \Big)$$

for every $\lambda|_{\widetilde{U}_{\widetilde{x}_i^e,\lambda}} \in P_h$. It is easy to verify that ν_h satisfies properties (Q5a)-(Q5d). The construction of P_h and ν_h depends on some choices, but it can be proved that the equivalence class $[P_h, \nu_h]$ does not depend on such choices. In this way we have defined a notion of composition of morphisms of orbifold atlases.

Lemma 1.15. *The composition of morphisms of orbifold atlases is associative.*

The proof is obvious for what concerns the composition of maps of the form $f, \overline{f}$ and $\widetilde{f}_i$; the proof of the associativity on the pairs of the form $[P_f, \nu_f]$ is straightforward, so we omit it.

Definition 1.16. Let us fix any continuous function $f : X \to Y$ and let $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$ and $\mathcal{V} := \{(\widetilde{V}_j, H_j, \phi_j)\}_{j \in J}$ be orbifold atlases for X and Y respectively. Moreover, let us fix 2 morphisms from $\mathcal{U}$ to $\mathcal{V}$ over f:

$$[\hat{f}^m] := \left(f, \overline{f}^m, \left\{ \widetilde{f}_i^m \right\}_{i \in I}, \left[P_f^m, \nu_f^m \right] \right), \quad \text{for } m = 1, 2.$$

Then a *representative of a* 2*-morphism* from $[\hat{f}^1]$ to $[\hat{f}^2]$ is any set of data:

$$\delta := \left\{ \left(\widetilde{U}_i^a, \delta_i^a \right) \right\}_{i \in I, a \in A(i)}$$

such that:

(R1) for all $i \in I$ the set $\{\widetilde{U}_i^a\}_{a \in A(i)}$ is an open covering of $\widetilde{U}_i$ (we allow the possibility that some of the $\widetilde{U}_i^a$'s are empty sets);

(R2) for all $i \in I$ and for all $a \in A(i)$, δ_i^a is a change of charts in $\mathcal{V}$ with

$$\widetilde{f}_i^1 \left(\widetilde{U}_i^a \right) \subseteq \operatorname{dom} \delta_i^a \subseteq \widetilde{V}_{\overline{f}^1(i)}, \quad \widetilde{f}_i^2 \left(\widetilde{U}_i^a \right) \subseteq \operatorname{cod} \delta_i^a \subseteq \widetilde{V}_{\overline{f}^2(i)};$$

(R3) for all $i \in I$, for all $a \in A(i)$ and for all $\widetilde{x}_i \in \widetilde{U}_i^a$ we have

$$\widetilde{f}_i^2(\widetilde{x}_i) = \delta_i^a \circ \widetilde{f}_i^1(\widetilde{x}_i); \tag{1.1}$$

(R4) for all $i \in I$, for all $a, a' \in A(i)$ and for all $\widetilde{x}_i \in \widetilde{U}_i^a \cap \widetilde{U}_i^{a'}$ (if non empty) we have

$$\operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta_i^a = \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta_i^{a'}$$

(note that (R3) only proves that δ_i^a coincides with $\delta_i^{a'}$ on the set $\widetilde{f}_i^1(\widetilde{U}_i^a \cap \widetilde{U}_i^{a'})$, that in general is not an open set since $\widetilde{f}_i^1$ can be non-open);

(R5) for all $(i,i')\in I\times I$, for all $(a,a')\in A(i)\times A(i')$, for all $\lambda\in\mathcal{C}h(\mathcal{U},i,i')$, and for all $\widetilde{x}_i\in\operatorname{dom}\lambda\cap\widetilde{U}_i^a$ such that $\lambda(\widetilde{x}_i)\in\widetilde{U}_{i'}^{a'}$ there exist

$$\left(P_f^m,\nu_f^m\right)\in\left[P_f^m,\nu_f^m\right],\quad \lambda^m\in P_f^m(i,i')\quad\text{for } m=1,2 \tag{1.2}$$

such that

$$\widetilde{x}_i\in\operatorname{dom}\lambda^m,\quad \operatorname{germ}_{\widetilde{x}_i}\lambda^m=\operatorname{germ}_{\widetilde{x}_i}\lambda\quad\text{for } m=1,2, \tag{1.3}$$

$$\operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)}\nu_f^2(\lambda^2)\cdot\operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)}\delta_i^a=\operatorname{germ}_{\widetilde{f}_{i'}^1(\lambda(\widetilde{x}_i))}\delta_{i'}^{a'}\cdot\operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)}\nu_f^1(\lambda^1). \tag{1.4}$$

Remark 1.17. Both the left hand side and the right hand side of (1.4) are well-defined. Indeed,

$$\widetilde{f}_i^2(\widetilde{x}_i)=\delta_i^a\circ\widetilde{f}_i^1(\widetilde{x}_i)$$

by (1.1) and

$$\nu_f^1(\lambda^1)\Big(\widetilde{f}_i^1(\widetilde{x}_i)\Big)=\widetilde{f}_{i'}^1\Big(\lambda^1(\widetilde{x}_i)\Big)=\widetilde{f}_{i'}^1\Big(\lambda(\widetilde{x}_i)\Big)$$

by definition λ^1 and by (Q5c).

Remark 1.18. Let us suppose that there exist data as in (1.2) that satisfy conditions (1.3) and (1.4). Let us suppose that $(P_f'^m,\nu_f'^m,\lambda'^m)$ for $m=1,2$ is another set of data as (1.2) that satisfies condition (1.3). Then by definition 1.12 we conclude that

$$\operatorname{germ}_{\widetilde{f}_i^m(\widetilde{x}_i)}\nu_f^m(\lambda^m)=\operatorname{germ}_{\widetilde{f}_i^m(\widetilde{x}_i)}\nu_f'^m(\lambda'^m)\quad\text{for } m=1,2,$$

so (1.4) is verified also by the new set of data. Therefore, (R5) is equivalent to:

(R5)′ for all $(i,i')\in I\times I$, for all $(a,a')\in A(i)\times A(i')$, for all $\lambda\in\mathcal{C}h(\mathcal{U},i,i')$, for all $\widetilde{x}_i\in\operatorname{dom}\lambda\cap\widetilde{U}_i^a$ such that $\lambda(\widetilde{x}_i)\in\widetilde{U}_{i'}^{a'}$ and for all data (1.2) that satisfy (1.3), we have that (1.4) holds.

Definition 1.19. Let us fix any continuous function $f:X\to Y$ and let $\mathcal{U}$ and $\mathcal{V}$ be orbifold atlases for X and Y respectively. Moreover, let us fix 2 morphisms $[\hat{f}^1]$, $[\hat{f}^2]$ from $\mathcal{U}$ to $\mathcal{V}$ over f and let us fix 2 representatives of 2-morphisms from $[\hat{f}^1]$ to $[\hat{f}^2]$:

$$\delta:=\left\{\left(\widetilde{U}_i^a,\delta_i^a\right)\right\}_{i\in I,a\in A(i)},\quad \overline{\delta}:=\left\{\left(\widetilde{U}_i^{\overline{a}},\overline{\delta}_i^{\overline{a}}\right)\right\}_{i\in I,\overline{a}\in\overline{A}(i)}.$$

Then we say that δ is *equivalent* to $\overline{\delta}$ if and only if for all $i\in I$, for all pairs $(a,\overline{a})\in A(i)\times\overline{A}(i)$ and for all $\widetilde{x}_i\in\widetilde{U}_i^a\cap\widetilde{U}_i^{\overline{a}}$ (if non-empty) we have

$$\operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)}\delta_i^a=\operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)}\overline{\delta}_i^{\overline{a}}.$$

This definition gives rise to an equivalence relation (it is reflexive by (R4)). We denote by $[\delta]:[\hat{f}^1]\Rightarrow[\hat{f}^2]$ the class of any δ as before and we say that $[\delta]$ is a 2-*morphism* from $[\hat{f}^1]$ to $[\hat{f}^2]$.

Remark 1.20. In principle the definition of 2-morphism can be given even if $[\hat{f}^1]$ and $[\hat{f}^2]$ are defined from the same source but with different targets $\mathcal{V}^1,\mathcal{V}^2$, provided that $\mathcal{V}^1$ and $\mathcal{V}^2$ are equivalent orbifold atlases. Actually, in this case we can consider the compositions of $[\hat{f}^1]$ and of $[\hat{f}^1]$ with the "inclusions" $\iota_{\mathcal{V}^1}$ and $\iota_{\mathcal{V}^2}$ of $\mathcal{V}_1$ and $\mathcal{V}_2$ in their common maximal atlas $\mathcal{V}^{\max}$ (see definition 1.13). We prefer to give anyway the definition of 2-morphism only in the case when the target of $[\hat{f}^1]$ and of $[\hat{f}^2]$ are the same because we will need to match the axioms of a 2-category.

2. Vertical and horizontal compositions between 2-morphisms

Construction 2.1. Let us fix any pair of orbifold atlases $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$, $\mathcal{V} = \{(\widetilde{V}_j, H_j, \phi_j)\}_{j \in J}$ for X and Y respectively, any continuous map $f : X \to Y$ and any triple of morphisms from $\mathcal{U}$ to $\mathcal{V}$ over f:

$$[\hat{f}^m] := \left(f, \overline{f}^m, \left\{\widetilde{f}_i^m\right\}_{i \in I}, \left[P_f^m, \nu_f^m\right]\right) \quad \text{for} \quad m = 1, 2, 3.$$

In addition, let us fix any 2-morphism $[\delta] : [\hat{f}^1] \Rightarrow [\hat{f}^2]$ and any 2-morphism $[\sigma] : [\hat{f}^2] \Rightarrow [\hat{f}^3]$. We want to define a *vertical composition* $[\sigma] \odot [\delta] : [\hat{f}^1] \Rightarrow [\hat{f}^3]$; in order to do that, let us fix representatives

$$\delta = \left\{\left(\widetilde{U}_i^a, \delta_i^a\right)\right\}_{i \in I, a \in A(i)}, \quad \sigma = \left\{\left(\widetilde{U}_i^b, \sigma_i^b\right)\right\}_{i \in I, b \in B(i)}$$

for $[\delta]$ and $[\sigma]$ respectively. For each $i \in I$ and for each $(a, b) \in A(i) \times B(i)$ we set

$$\widetilde{U}_i^{a,b} := \widetilde{U}_i^a \cap \widetilde{U}_i^b, \quad \widetilde{V}_i^{a,b} := (\delta_i^a)^{-1}\left(\operatorname{cod} \delta_i^a \cap \operatorname{dom} \sigma_i^b\right), \quad \theta_i^{a,b} := \sigma_i^b \circ \delta_i^a|_{\widetilde{V}_i^{a,b}}$$

and we define:

$$\theta := \left\{\left(\widetilde{U}_i^{a,b}, \theta_i^{a,b}\right)\right\}_{i \in I, (a,b) \in A(i) \times B(i)}.$$

Remark 2.2. Note that some of the $U_i^{a,b}$'s can be empty, but this creates no problems for (R1); moreover, some of the $\theta_i^{a,b}$'s can be the changes of charts of the form $\varnothing \to \varnothing$, but again this gives no problems for definition 1.3. If one prefers not to deal with empty changes of charts, then for each $i \in I$ we can simply restrict to the set of those (a, b)'s in $A(i) \times B(i)$ such that $\widetilde{U}_i^{a,b} \neq \varnothing$; for any such (a, b) we have automatically that also $\widetilde{V}_i^{a,b}$ is non-empty. An analogous remark applies also for construction 2.5 below.

Lemma 2.3. *The collection θ so defined is a representative of 2-morphism from $[\hat{f}^1]$ to $[\hat{f}^3]$.*

See the appendix for the proof. A straightforward proof shows that the class of θ does not depend on the choices of representatives δ for $[\delta]$ and σ for $[\sigma]$. Therefore, it makes sense to give the following definition.

Definition 2.4. Given any pair $[\delta]$, $[\sigma]$ as before, we define their *horizontal composition* as:

$$[\sigma] \odot [\delta] := [\theta] : [\hat{f}^1] \Longrightarrow [\hat{f}^3].$$

Construction 2.5. Let us fix any triple of orbifold atlases

$$\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}, \quad \mathcal{V} := \{(\widetilde{V}_j, H_j, \phi_j)\}_{j \in J}, \quad \mathcal{W} = \{(\widetilde{W}_l, K_l, \xi_l)\}_{l \in L}$$

for X, Y and Z respectively. Let us also fix any pair of morphisms

$$[\hat{f}^m] := \left(f, \overline{f}^m, \left\{\widetilde{f}_i^m\right\}_{i \in I}, \left[P_f^m, \nu_f^m\right]\right) : \mathcal{U} \longrightarrow \mathcal{V} \quad \text{for} \quad m = 1, 2,$$

$$[\hat{g}^m] := \left(g, \overline{g}^m, \left\{\widetilde{g}_j^m\right\}_{j \in J}, \left[P_g^m, \nu_g^m\right]\right) : \mathcal{V} \longrightarrow \mathcal{W} \quad \text{for} \quad m = 1, 2.$$

Moreover, let us suppose that we have fixed any 2-morphism $[\delta] : [\hat{f}^1] \Rightarrow [\hat{f}^2]$ and any 2-morphism $[\eta] : [\hat{g}^1] \Rightarrow [\hat{g}^2]$. Our aim is to define an horizontal composition

$[\eta] * [\delta] : [\hat{g}^1] \circ [\hat{f}^1] \Rightarrow [\hat{g}^2] \circ [\hat{f}^2]$. In order to do that, we fix any representative (P^1_g, ν^1_g) for $[P^1_g, \nu^1_g]$ and any representative

$$\overline{\delta} := \left\{\left(\widetilde{U}^{\overline{a}}_i, \delta^{\overline{a}}_i\right)\right\}_{i \in I, \overline{a} \in \overline{A}(i)}$$

for $[\delta]$. For any $i \in I$ and any $\overline{a} \in \overline{A}(i)$ we have that $\overline{\delta}^{\overline{a}}_i \in \mathcal{C}h(\mathcal{V})$. Since P^1_g is a good subset of $\mathcal{C}h(\mathcal{U})$, then there exists a set

$$\{\delta^a_i\}_{a \in A(i,\overline{a})} \subseteq P^1_g$$

such that

$$\operatorname{dom} \overline{\delta}^{\overline{a}}_i \subseteq \bigcup_{a \in A(i,\overline{a})} \operatorname{dom} \delta^a_i$$

and such that for each $\widetilde{x}_i \in \operatorname{dom} \overline{\delta}^{\overline{a}}_i$ there exists $a \in A(i,\overline{a})$ such that $\widetilde{f}^1_i(\widetilde{x}_i) \in \operatorname{dom} \delta^a_i$ and $\operatorname{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)} \delta^{\overline{a}}_i = \operatorname{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)} \delta^a_i$. For each $a \in A(i,\overline{a})$ we set $\widetilde{U}^a_i := \widetilde{U}^{\overline{a}}_i \cap (\widetilde{f}^1_i)^{-1}(\operatorname{dom} \delta^a_i)$; then for each $i \in I$ we set $A(i) := \coprod_{\overline{a} \in \overline{A}(i)} A(i,\overline{a})$ and we consider the representative

$$\delta := \left\{\left(\widetilde{U}^a_i, \delta^a_i\right)\right\}_{i \in I, a \in A(i)} \in [\delta].$$

We choose also any representative $\eta := \{(\widetilde{V}^c_j, \eta^c_j)\}_{j \in J, c \in C(j)}$ for $[\eta]$. Then for each $i \in I, a \in A(i)$ and $c \in C(\overline{f}^2(i))$ we set

$$\widetilde{U}^{a,c}_i := \widetilde{U}^a_i \cap \left(\widetilde{f}^2_i\right)^{-1} \left(\operatorname{cod} \delta^a_i \cap \widetilde{V}^c_{\overline{f}^2(i)}\right),$$

$$\widetilde{W}^{a,c}_i := \left(\nu^1_g(\delta^a_i)\right)^{-1} \left(\operatorname{cod} \nu^1_g(\delta^a_i) \cap \operatorname{dom} \eta^c_{\overline{f}^2(i)}\right),$$

$$\gamma^{a,c}_i := \eta^c_{\overline{f}^2(i)} \circ \nu^1_g(\delta^a_i)\Big|_{\widetilde{W}^{a,c}_i}$$

and we define

$$\gamma := \left\{\left(\widetilde{U}^{a,c}_i, \gamma^{a,c}_i\right)\right\}_{i \in I, (a,c) \in A(i) \times C(\overline{f}^2(i))}.$$

Lemma 2.6. *The collection γ so defined is a representative of a 2-morphism from $[\hat{g}^1] \circ [\hat{f}^1]$ to $[\hat{g}^2] \circ [\hat{f}^2]$.*

See the appendix for the proof. A direct check proves that the class of γ does not depend on the representatives (P^1_g, ν^1_g), δ and η chosen for $[P^1_g, \nu^1_g]$, $[\delta]$ and $[\eta]$ respectively (with the condition that for each i and a the change of charts δ^a_i belongs to P^1_g as before). So it makes sense to give the following definition.

Definition 2.7. Given any pair $[\delta]$, $[\eta]$ as before, we define their *vertical composition* as:

$$[\eta] * [\delta] := [\gamma] : [\hat{g}^1] \circ [\hat{f}^1] \Longrightarrow [\hat{g}^2] \circ [\hat{f}^2].$$

3. The 2-category (**Red Atl**)

Definition 3.1. Given any reduced orbifold atlas $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$ on a space X, we define the *identity* of $\mathcal{U}$ as the morphism

$$\mathrm{id}_{\mathcal{U}} := \left(\mathrm{id}_X, \mathrm{id}_I, \left\{\mathrm{id}_{\widetilde{U}_i}\right\}_{i \in I}, [\mathcal{C}h(\mathcal{U}), \nu_{\mathrm{id}}]\right) : \mathcal{U} \longrightarrow \mathcal{U}$$

where ν_{id} is the identity on $\mathcal{C}h(\mathcal{U})$. Given any pair of reduced orbifold atlases $\mathcal{U}$ and $\mathcal{V}$ and any morphism $[\hat{f}] = (f, \overline{f}, \{\widetilde{f}_i\}_{i \in I}, [P_f, \nu_f])$ from $\mathcal{U}$ to $\mathcal{V}$, we define the 2-*identity* $i_{[\hat{f}]}$ as the class of

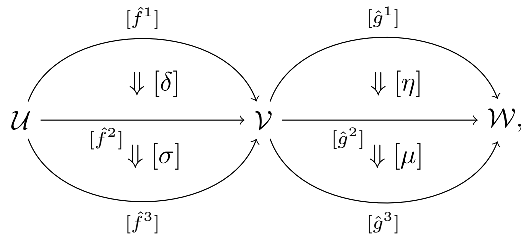
$$\left\{\left(\widetilde{U}_i, \mathrm{id}_{\widetilde{V}_{\overline{f}(i)}}\right)\right\}_{i \in I}.$$

Moreover, for each orbifold atlas $\mathcal{U}$ on a topological space X, we set $i_{\mathcal{U}} := i_{\mathrm{id}_{\mathcal{U}}}$.

A direct check proves that:

Lemma 3.2. *The morphisms and* 2*-morphisms of the form id and i are the identities with respect to* $\circ$ *and* $\odot$ *respectively. Moreover, any* 2*-morphism is invertible with respect to* $\odot$.

Lemma 3.3. *Given any diagram as follows*



we have

$$\Big([\mu] \odot [\eta]\Big) * \Big([\sigma] \odot [\delta]\Big) = \Big([\mu] * [\sigma]\Big) \odot \Big([\eta] * [\delta]\Big).$$

See the appendix for the proof.

Proposition 3.4. *The definitions of reduced orbifold atlases, morphisms and* 2*-morphisms, compositions* $\circ, \odot, *$ *and identities give rise to a* 2*-category, that we will denote by* (**Red Atl**).

Proof. In order to construct a 2-category, we define some data as follows.

(1) The class of objects is just the set of all reduced orbifold atlases $\mathcal{U}$ for every paracompact, second countable, Hausdorff topological space X (if any).
(2) If $\mathcal{U}$ and $\mathcal{V}$ are atlases for X and Y respectively, we define a small category (**Red Atl**)$(\mathcal{U}, \mathcal{V})$ as follows: the space of objects is the set of all morphisms $[\hat{f}] : \mathcal{U} \to \mathcal{V}$ (if any) over all continuous maps $f : X \to Y$; for any pair of compatible systems $[\hat{f}]$ and $[\hat{g}]$ for f and g respectively, we define:

$$((\mathbf{Red\,Atl})(\mathcal{U}, \mathcal{V}))\,([\hat{f}], [\hat{g}]) := \begin{cases} \text{all 2-morphisms } [\hat{f}] \Rightarrow [\hat{g}] & \text{if } f = g, \\ \varnothing & \text{otherwise.} \end{cases}$$

The composition in any such category is the vertical composition $\odot$, that is clearly associative; the identity over any object $[\hat{f}]$ is just $i_{[\hat{f}]}$. By lemma 3.2 we get that actually any such category is a groupoid, i.e. a category where all morphisms are invertible.

(3) For every triple $\mathcal{U}, \mathcal{V}, \mathcal{W}$ of objects, we define the functor "composition"

$$(\textbf{Red Atl})(\mathcal{U}, \mathcal{V}) \times (\textbf{Red Atl})(\mathcal{V}, \mathcal{W}) \longrightarrow (\textbf{Red Atl})(\mathcal{U}, \mathcal{W})$$

as $\circ$ on any pair of morphisms and as $*$ on any pair of horizontally composable 2-morphisms. We want to prove that this gives rise to a functor. It is easy to see that identities are preserved, so one needs only to prove that compositions are preserved, i.e. that the *interchange law* (see [B, proposition 1.3.5]) is satisfied. This is exactly the statement of lemma 3.3.

All the other necessary proofs that (**Red Atl**) is a 2-category are trivial, so we omit them. □

4. From reduced orbifold atlases to proper, effective, étale Lie groupoids

We recall briefly some notions of Lie groupoids.

Definition 4.1. [L, definition 2.11] A *Lie groupoid* is the datum of two smooth manifolds R, U and five smooth maps:

- $s, t : R \rightrightarrows U$ such that both s and t are submersions (so that the fiber products of the form $R\,{}_t\!\times_s \cdots\ {}_t\!\times_s R$ (for finitely many terms) *exist*); these two maps are usually called *source* and *target* of the Lie groupoid;
- $m : R\,{}_t\!\times_s R \to R$, called *multiplication*;
- $i : R \to R$, known as *inverse* of the Lie groupoid;
- $e : U \to R$, called *identity*;

which satisfy the following axioms:

(S1) $s \circ e = 1_U = t \circ e$;
(S2) if we denote by pr_1 and pr_2 the two projections from the fibred product $R\,{}_t\!\times_s R$ to R, then we have $s \circ m = s \circ pr_1$ and $t \circ m = t \circ pr_2$;
(S3) the two morphisms $m \circ (1_R \times m)$ and $m \circ (m \times 1_R)$ from $R\,{}_t\!\times_s R\,{}_t\!\times_s R$ to R are equal;
(S4) the two morphisms $m \circ (e \circ s, 1_R)$ and $m \circ (1_R, e \circ t)$ from R to R are both equal to the identity of R;
(S5) $i \circ i = 1_R$, $s \circ i = t$ (and therefore $t \circ i = s$); moreover, we require that $m \circ (1_R, i) = e \circ s$ and $m \circ (i, 1_R) = e \circ t$:

In other terms, a Lie groupoid is an internal groupoid (see [BCE+, §3.1]) in the category of smooth manifolds, such that s and t are submersions. For simplicity, we will denote any Lie groupoid as before by $R \overset{s}{\underset{t}{\rightrightarrows}} U$. In the literature one can also find the notations (U, R, s, t, m, e, i), $R\,{}_s\!\times_t R \overset{m}{\to} R \overset{i}{\to} R \overset{s}{\underset{t}{\rightrightarrows}} U \overset{e}{\to} R$ and $G = (G_0, G_1)$ (where G_0 is the set U and G_1 is the set R in our notations).

Definition 4.2. [M, §2.1] Given two Lie groupoids $R \overset{s}{\underset{t}{\rightrightarrows}} U$ and $R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'$, a *morphism* between them is any pair (ψ, Ψ), where $\psi : U \to U'$ and $\Psi : R \to R'$ are smooth morphisms, which together commute with all structure morphisms of the two Lie groupoid. In other words, we ask that $s' \circ \Psi = \psi \circ s$, $t' \circ \Psi = \psi \circ t$, $\Psi \circ e = e' \circ \psi$, $\Psi \circ m = m' \circ (\Psi \times \Psi)$ and $\Psi \circ i = i' \circ \Psi$.

Definition 4.3. [PS, definition 2.3] Let us suppose we have fixed 2 morphisms of Lie groupoid $(\psi^m, \Psi^m) :\ (R \overset{s}{\underset{t}{\rightrightarrows}} U) \to (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')$ for $m = 1, 2$. Then a 2-*morphism* (also known as *natural transformation*) $\alpha : (\psi^1, \Psi^1) \Rightarrow (\psi^2, \Psi^2)$ is the datum of any smooth map $\alpha : U \to R'$ such that the following conditions hold:

(T1) $s' \circ \alpha = \psi^1$ and $t' \circ \alpha = \psi^2$;
(T2) $m' \circ (\alpha \circ s, \Psi^2) = m' \circ (\Psi^1, \alpha \circ t)$.

There are well known notions of compositions of morphisms, vertical and horizontal compositions of 2-morphisms and identities; we refer to [T, §2.2] for more details. In particular, we have:

Proposition 4.4. [PS, §2.1] *The data of Lie groupoids, morphisms, and natural transformations between them (together with compositions and identities) form a 2-category, known as* (**Lie Gpd**).

Definition 4.5. [M, §1.2 and 1.5] A Lie groupoid $R \overset{s}{\underset{t}{\rightrightarrows}} U$ is *proper* if the map $(s,t) : R \to U \times U$ is proper; it is étale if the maps s and t are both étale (i.e. local diffeomorphisms).

Definition 4.6. [M, example 1.5] Let $R \overset{s}{\underset{t}{\rightrightarrows}} U$ be an étale Lie groupoid, let us fix any point $u \in U$ and let us define $R_u := (s,t)^{-1}\{(u,u)\}$, called the *isotropy subgroup of* u. Since both s and t are étale, for every point r in R_u we can find a sufficiently small open neighbourhood W_r of r where both s and t are invertible. Then to every point r in R_u we can associate the set map:

$$\tilde{r} := t \circ (s|_{W_r})^{-1} : s(W_r) \to t(W_r);$$

this is a diffeomorphism between two open neighbourhoods of u and it fixes u. Then we can define a set map (actually, a group homomorphism) χ_u from R_u to the group of germs of smooth maps defined around u and which fix u. Then we say that $R \overset{s}{\underset{t}{\rightrightarrows}} U$ is *effective* (or *reduced*) if χ_u is injective for every u in U.

Definition 4.7. We define the 2-categories (**É Gpd**), (**PÉ Gpd**) and (**PEÉ Gpd**) as full 2-subcategories of (**Lie Gpd**) by restricting to étale Lie groupoids, respectively proper étale Lie groupoids, respectively proper, effective, étale Lie groupoids. Morphisms and 2-morphisms are simply restricted according to that. The aim of this section is to describe a 2-functor F from (**Red Atl**) to (**PEÉ Gpd**).

Construction 4.8. (adapted from [Po, construction 2.4] and from [Pr2, §4.4]) Let us fix any reduced orbifold atlas $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$ of dimension n. Then we define $F_0(\mathcal{U}) := (R \overset{s}{\underset{t}{\rightrightarrows}} U)$ as the following Lie groupoid.

- The manifold U is defined as $\coprod_{i \in I} \widetilde{U}_i$, with the natural smooth structure given by the fact that each $\widetilde{U}_i$ is an open subset of $\mathbb{R}^n$.
- As a set, we define

$$R := \left\{ \operatorname{germ}_{\widetilde{x}_i} \lambda, \quad \forall i \in I, \forall \lambda \in \mathcal{C}h(\mathcal{U}, i, -), \forall \widetilde{x}_i \in \operatorname{dom} \lambda \right\}.$$

For each $i \in I$ and for each $\lambda \in \mathcal{C}h(\mathcal{U}, i, -)$ we set

$$R(\lambda) := \left\{ \operatorname{germ}_{\widetilde{x}_i} \lambda, \quad \forall \, \widetilde{x}_i \in \operatorname{dom} \lambda \right\} \subseteq R.$$

Then the topological and differentiable structure on R are given by the germ topology and by the germ differentiable structure, i.e. we require that for each $i \in I$ and each $\lambda \in \mathcal{C}h(\mathcal{U}, i, -)$ the bijection

$$\gamma(\lambda) : \begin{cases} R(\lambda) & \to \quad \operatorname{dom} \lambda \subseteq \widetilde{U}_i \subseteq \mathbb{R}^n \\ \operatorname{germ}_{\widetilde{x}_i} \lambda & \mapsto \quad \widetilde{x}_i \end{cases} \tag{4.1}$$

is a diffeomorphism.
- The structure maps are defined as follows:

$$s(\operatorname{germ}_{\widetilde{x}_i} \lambda) := \widetilde{x}_i, \quad t(\operatorname{germ}_{\widetilde{x}_i} \lambda) := \lambda(\widetilde{x}_i),$$
$$m(\operatorname{germ}_{\widetilde{x}_i} \lambda, \operatorname{germ}_{\lambda(\widetilde{x}_i)} \mu) := \operatorname{germ}_{\lambda(\widetilde{x}_i)} \mu \cdot \operatorname{germ}_{\widetilde{x}_i} \lambda = \operatorname{germ}_{\widetilde{x}_i} \mu \circ \lambda,$$
$$i(\operatorname{germ}_{\widetilde{x}_i} \lambda) := \operatorname{germ}_{\lambda(\widetilde{x}_i)} \lambda^{-1}, \quad e(\widetilde{x}_i) := \operatorname{germ}_{\widetilde{x}_i} \operatorname{id}_{\widetilde{U}_i}.$$

A direct check proves that s and t are both étale, that m, e, i are smooth and that axioms (S1)-(S5) are satisfied, so $F_0(\mathcal{U})$ is an étale Lie groupoid.

Lemma 4.9. *For every reduced orbifold atlas $\mathcal{U}$, the Lie groupoid $F_0(\mathcal{U})$ belongs to* (**PEÉ Gpd**).

Proof. For each $u \in U$ the map χ_u defined before is obviously injective, hence $F_0(\mathcal{U})$ is effective; so we need only to prove that $(s,t): R \to U \times U$ is proper. Let us fix any compact set $K \subseteq U \times U$ and let $\{q^m\}_{m\in\mathbb{N}}$ be any sequence in $(s,t)^{-1}(K) \subseteq R$. Up to passing to a subsequence we can always assume that the sequence $(s,t)(q^m) \in K$ converges to some point $(\widetilde{x}_i, \overline{x}_{i'}) \in \widetilde{U}_i \times \widetilde{U}_{i'}$, so for $m > m_1$ we can assume that $(s,t)(q^m) \in \widetilde{U}_i \times \widetilde{U}_{i'}$ and we can write $(s,t)(q^m) =: (\widetilde{x}_i^m, \overline{x}_i^m)$. By definition of R, we have that for $m > m_1$ there exists a (non-unique) change of charts λ^m from $(\widetilde{U}_i, G_i, \pi_i)$ to $(\widetilde{U}_{i'}, G_{i'}, \pi_{i'})$ such that

$$\widetilde{x}_i^m \in \operatorname{dom}\lambda^m, \quad q^m = \operatorname{germ}_{\widetilde{x}_i}\lambda^m, \quad \lambda^m(\widetilde{x}_i^m) = \overline{x}_i^m.$$

Since $\mathcal{U}$ is an orbifold atlas, then there exists a change of charts λ from $(\widetilde{U}_i, G_i, \pi_i)$ to $(\widetilde{U}_{i'}, G_{i'}, \pi_{i'})$ such that $\widetilde{x}_i \in \operatorname{dom}\lambda$. For $m > m_2 \geq m_1$ we have that $\widetilde{x}_i^m \in \operatorname{dom}\lambda$. Then for each such m there exists a chart $(\widetilde{U}^m, G^m, \pi_i|_{\widetilde{U}^m})$ such that $\widetilde{x}_i^m \in \widetilde{U}^m \subseteq \operatorname{dom}\lambda \cap \operatorname{dom}\lambda^m \subseteq \widetilde{U}_i$. For $m > m_2$ we have that both λ and λ^m (suitably restricted) can be considered as embeddings from $(\widetilde{U}^m, G^m, \pi_i|_{\widetilde{U}^m})$ to $(\widetilde{U}_{i'}, G_{i'}, \pi_{i'})$. So by [MP, lemma A.1] there exists a unique $g^m \in G_{i'}$ such that $\lambda^m|_{\widetilde{U}^m} = g^m \circ \lambda|_{\widetilde{U}^m}$. Since $G_{i'}$ is a finite set, then after passing to a subsequence we can assume that g^m is the same for all $m > m_2$; we denote such a map by g. Then by definition of differentiable structure on R we have

$$\lim_{m\to\infty} q^m = \lim_{m\to\infty} \operatorname{germ}_{\widetilde{x}_i^m}\lambda^m = \lim_{m\to\infty} \operatorname{germ}_{\widetilde{x}_i^m} g \circ \lambda = \operatorname{germ}_{\widetilde{x}_i} g \circ \lambda.$$

So this proves that $(s,t)^{-1}(K)$ is compact, so (s,t) is proper. □

Construction 4.10. (adapted from [Po, proposition 4.7]) Let us fix any pair of reduced orbifold atlases $\mathcal{U} := \{(\widetilde{U}_i, G_i, \pi_i)\}_{i\in I}$ and $\mathcal{V} := \{(\widetilde{V}_j, H_i, \phi_j)\}_{j\in J}$ for X and Y respectively and any morphism $[\hat{f}] : \mathcal{U} \to \mathcal{V}$ with representative given by

$$\hat{f} := \left(f, \overline{f}, \left\{\widetilde{f}_i\right\}_{i\in I}, P_f, \nu_f\right).$$

We set

$$F_0(\mathcal{U}) =: \left(R \overset{s}{\underset{t}{\rightrightarrows}} U\right), \quad F_0(\mathcal{V}) =: \left(R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'\right),$$

where

$$U := \textstyle\coprod_{i\in I} \widetilde{U}_i, \quad R := \left\{\operatorname{germ}_{\widetilde{x}_i}\lambda, \quad \forall\, i \in I, \forall\, \lambda \in \mathcal{C}h(\mathcal{U}, i, -), \forall\, \widetilde{x}_i \in \operatorname{dom}\lambda\right\},$$

$$U' := \textstyle\coprod_{j\in J} \widetilde{V}_j, \quad R' := \left\{\operatorname{germ}_{\widetilde{y}_j}\mu, \quad \forall\, j \in J, \forall\, \mu \in \mathcal{C}h(\mathcal{V}, j, -), \forall\, \widetilde{y}_j \in \operatorname{dom}\mu\right\}.$$

Then we define a set map $\psi : U \to U'$ as

$$\psi|_{\widetilde{U}_i} := \widetilde{f}_i : \widetilde{U}_i \longrightarrow \widetilde{V}_{\overline{f}(i)} \subseteq U'$$

for all $i \in I$. Now let r be any point in R and let $\widetilde{x}_i := s(r) \in \widetilde{U}_i$ for some $i \in I$. Since P_f is a good subset of $\mathcal{C}h(\mathcal{U})$, then there is a (non-unique) $\lambda \in P_f(i, -)$ such that $r = \operatorname{germ}_{\widetilde{x}_i}\lambda$. We set

$$\Psi(r) := \operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)} \nu_f(\lambda) \in R'.$$

If λ' is another element of $P_f(i,-)$ such that $r = \mathrm{germ}_{\widetilde{x}_i} \lambda'$, then property (Q5d) for $\hat{f}$ proves that $\Psi(r)$ has the same expression, so Ψ is a well-defined set map from R to R'. A direct check proves that both ψ and Ψ are smooth and that the pair (ψ, Ψ) satisfies definition 4.2, so it is a morphism from $F_0(\mathcal{U})$ to $F_0(\mathcal{V})$. Now let us suppose that

$$\hat{f}' := \left(f, \overline{f}, \left\{\widetilde{f}_i\right\}_{i\in I}, P'_f, \nu'_f\right)$$

is another representative of $[\hat{f}]$. Then by definition 1.12 we get that the morphism from $F_0(\mathcal{U})$ to $F_0(\mathcal{V})$ associated to $\hat{f}'$ coincides with (ψ, Ψ). Therefore it makes sense to set

$$F_1([\hat{f}]) := (\psi, \Psi) : F_0(\mathcal{U}) \longrightarrow F_0(\mathcal{V}).$$

Construction 4.11. Now let us fix any pair of atlases $\mathcal{U}$ and $\mathcal{V}$ for X and Y respectively and any pair of morphisms $[\hat{f}^1], [\hat{f}^2] : \mathcal{U} \to \mathcal{V}$ over a continuous function $f : X \to Y$, with representatives

$$\hat{f}^m := \left(f, \overline{f}^m, \left\{\widetilde{f}^m_i\right\}_{i\in I}, P^m_f, \nu^m_f\right) \quad \text{for } m = 1, 2.$$

Let us also fix any 2-morphism $[\delta] : [\hat{f}^1] \Rightarrow [\hat{f}^2]$ and any representative

$$\delta := \left\{\left(\widetilde{U}^a_i, \delta^a_i\right)\right\}_{i\in I, a\in A(i)}$$

for it. Let us set

$$F_0(\mathcal{U}) =: \left(R \overset{s}{\underset{t}{\rightrightarrows}} U\right), \; F_0(\mathcal{V}) =: \left(R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'\right), \; F_1([\hat{f}^m]) =: (\psi^m, \Psi^m) \text{ for } m = 1, 2.$$

Then let us define a set map $\underline{\delta} : U = \coprod_{i\in I} \widetilde{U}_i \to R'$ as

$$\underline{\delta}(\widetilde{x}_i) := \mathrm{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)} \delta^a_i$$

for every $i \in I$, for every $a \in A$ and for every $\widetilde{x}_i \in \widetilde{U}^a_i$; this is well-defined by property (R4) for δ. We claim that $\underline{\delta}$ is a 2-morphism from (ψ^1, Ψ^1) to (ψ^2, Ψ^2). Clearly $\underline{\delta}$ is smooth, indeed on each open subset of U of the form $\widetilde{U}^a_i$ we have that $\underline{\delta}$ coincides with the composition of $\widetilde{f}_i$ (that is smooth by definition of local lift) and of the inverse of the chart $\gamma(\delta^a_i)$ for R' (see (4.1)). Moreover, let us fix any $i \in I$, any $a \in A(i)$ and any $\widetilde{x}_i \in \widetilde{U}^a_i$. Then

$$s' \circ \underline{\delta}(\widetilde{x}_i) = s'\left(\mathrm{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)} \delta^a_i\right) = \widetilde{f}^1_i(\widetilde{x}_i) = \psi^1(\widetilde{x}_i),$$

$$t' \circ \underline{\delta}(\widetilde{x}_i) = t'\left(\mathrm{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)} \delta^a_i\right) = \delta^a_i \circ \widetilde{f}^1_i(\widetilde{x}_i) \overset{(R3)}{=} \widetilde{f}^2_i(\widetilde{x}_i) = \psi^2(\widetilde{x}_i),$$

so $\underline{\delta}$ satisfies axiom (T1). Now let us fix any point $r \in R$ and let us set $\widetilde{x}_i := s(r)$, $\overline{x}_{i'} := t(r)$ for a unique pair $(i, i') \in I \times I$. Since both P^1_f and P^2_f are good subsets of $\mathcal{C}h(\mathcal{U})$, then for $m = 1, 2$ there exist $\lambda^m \in P^m_f(i, i')$ such that $\mathrm{germ}_{\widetilde{x}_i} \lambda^m = r$. By property (R1) there exist $a \in A(i)$ and $a' \in A(i')$ such that $\widetilde{x}_i \in \widetilde{U}^a_i$ and $\overline{x}_{i'} \in \widetilde{U}^{a'}_{i'}$. Then:

$$\big(m' \circ (\underline{\delta} \circ s, \Psi^2)\big)(r) = m' \left(\underline{\delta}(\widetilde{x}_i), \Psi^2(\mathrm{germ}_{\widetilde{x}_i} \lambda^2)\right) =$$
$$= m' \left(\mathrm{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)} \delta^a_i, \mathrm{germ}_{\widetilde{f}^2_i(\widetilde{x}_i)} \nu^2_f(\lambda^2)\right) = \mathrm{germ}_{\widetilde{f}^2_i(\widetilde{x}_i)} \nu^2_f(\lambda^2) \cdot \mathrm{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)} \delta^a_i \stackrel{(R5)'}{=}$$
$$\stackrel{(R5)'}{=} \mathrm{germ}_{\widetilde{f}^1_{i'}(\lambda^1(\widetilde{x}_i))} \delta^{a'}_{i'} \cdot \mathrm{germ}_{\widetilde{f}^1_i} \nu^1_f(\lambda^1) = m' \left(\Psi^1(\mathrm{germ}_{\widetilde{x}_i} \lambda^1), \underline{\delta}(\lambda^1(\widetilde{x}_i))\right) =$$
$$= \big(m' \circ (\Psi^1, \underline{\delta} \circ t)\big)(r).$$

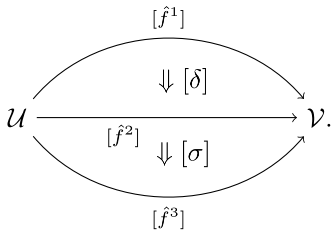
So $\underline{\delta}$ satisfies also axiom (T2), therefore $\underline{\delta}$ is a 2-morphism from $F_1([\hat{f}^1])$ to $F_1([\hat{f}^2])$. By definition 1.19 we get that $\underline{\delta}$ depends only on $[\delta]$ and not on the representative δ chosen for that class. So it makes sense to set:

$$F_2([\delta]) := \underline{\delta} : F_1([\hat{f}^1]) \Longrightarrow F_1([\hat{f}^2]).$$

A direct check proves that:

Lemma 4.12. *For every pair of composable morphisms $[\hat{f}] : \mathcal{U} \to \mathcal{V}$, $[\hat{g}] : \mathcal{V} \to \mathcal{W}$ we have $F_1([\hat{g}] \circ [\hat{f}]) = F_1([\hat{g}]) \circ F_1([\hat{f}])$. For every morphism $[\hat{f}]$ between orbifold atlases we have $F_2(i_{[\hat{f}]}) = i_{F_1([\hat{f}])}$; for every orbifold atlas $\mathcal{U}$ we have $F_1(\mathrm{id}_{\mathcal{U}}) = \mathrm{id}_{F_0(\mathcal{U})}$.*

Lemma 4.13. *Let us fix any diagram as follows:*



Then $F_2([\sigma] \odot [\delta]) = F_2([\sigma]) \odot F_2([\delta])$.

Proof. Let us set representatives

$$\hat{f}^m := \left(f, \overline{f}^m, \left\{\widetilde{f}^m_i\right\}_{i \in I}, P^m_f, \nu^m_f\right) \quad \text{for} \quad m = 1, 2, 3$$

for $[\hat{f}^m]$ for $m = 1, 2, 3$ and representatives

$$\delta = \left\{\left(\widetilde{U}^a_i, \delta^a_i\right)\right\}_{i \in I, a \in A(i)}, \quad \sigma = \left\{\left(\widetilde{U}^b_i, \sigma^b_i\right)\right\}_{i \in I, b \in B(i)}$$

for $[\delta]$ and $[\sigma]$ respectively. Then let us set $F_2([\delta]) =: \underline{\delta}$ and $F_2([\sigma]) =: \underline{\sigma}$; let us fix any $i \in I$, any $(a, b) \in A(i) \times B(i)$ and any $\widetilde{x}_i \in \widetilde{U}^{a,b}_i = \widetilde{U}^a_i \cap \widetilde{U}^b_i$. Then

$$\Big(F_2([\sigma]) \odot F_2([\delta])\Big)(\widetilde{x}_i) = (\underline{\sigma} \odot \underline{\delta})(\widetilde{x}_i) = m' \circ (\underline{\delta}, \underline{\sigma})(\widetilde{x}_i) =$$
$$= m' \left(\mathrm{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)} \delta^a_i, \mathrm{germ}_{\widetilde{f}^2_i(\widetilde{x}_i)} \sigma^b_i\right) = \mathrm{germ}_{\widetilde{f}^2_i(\widetilde{x}_i)} \sigma^b_i \cdot \mathrm{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)} \delta^a_i =$$
$$= \mathrm{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)} \sigma^b_i \circ \delta^a_i = F_2\Big([\sigma] \odot [\delta]\Big)(\widetilde{x}_i).$$

□

Lemma 4.14. *Let us fix any diagram as follows:*

$$\mathcal{U} \underset{[\hat{f}^2]}{\overset{[\hat{f}^1]}{\Downarrow [\delta]}} \mathcal{V} \underset{[\hat{g}^2]}{\overset{[\hat{g}^1]}{\Downarrow [\eta]}} \mathcal{W}.$$

Then $F_2([\eta] * [\delta]) = F_2([\eta]) * F_2([\delta])$.

Proof. Let us set representatives

$$\hat{f}^m := \left(f, \overline{f}^m, \left\{\widetilde{f}_i^m\right\}_{i \in I}, P_f^m, \nu_f^m\right) \quad \text{for} \quad m = 1, 2,$$
$$\hat{g}^m := \left(g, \overline{g}^m, \left\{\widetilde{g}_j^m\right\}_{j \in J}, P_g^m, \nu_g^m\right) \quad \text{for} \quad m = 1, 2$$

for $[\hat{f}^m]$ and $[\hat{g}^m]$ respectively and representatives

$$\delta = \left\{\left(\widetilde{U}_i^a, \delta_i^a\right)\right\}_{i \in I, a \in A(i)}, \quad \eta = \left\{\left(\widetilde{V}_j^c, \eta_j^c\right)\right\}_{j \in J, c \in C(j)}$$

for $[\delta]$ and $[\eta]$ respectively, with δ as in construction 2.5. Then let us set:

$$F_0(\mathcal{U}) =: \left(R \overset{s}{\underset{t}{\rightrightarrows}} U\right), \quad F_0(\mathcal{V}) =: \left(R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'\right), \quad F_0(\mathcal{W}) =: \left(R'' \overset{s''}{\underset{t''}{\rightrightarrows}} U''\right),$$
$$F_1([\hat{f}^m]) =: (\psi^m, \Psi^m), \quad F_1([\hat{g}^m]) =: (\phi^m, \Phi^m) \quad \text{for } m = 1, 2,$$
$$F_2([\delta]) =: \underline{\delta}, \quad F_2([\eta]) =: \underline{\eta}.$$

Let us fix any $i \in I$, any $(a, c) \in A(i) \times C(\overline{f}^2(i))$ and any point

$$\widetilde{x}_i \in \widetilde{U}_i^{a,c} = \widetilde{U}_i^a \cap \left(\widetilde{f}_i^2\right)^{-1} \left(\operatorname{cod} \delta_i^a \cap \widetilde{V}^c_{\overline{f}^2(i)}\right).$$

Then we have

$$\Big(F_2([\eta]) * F_2([\delta])\Big)(\widetilde{x}_i) = \left(m'' \circ \left(\Phi_1 \circ \underline{\delta}, \underline{\eta} \circ \psi_2\right)\right)(\widetilde{x}_i) =$$
$$= m'' \left(\Phi_1(\operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta_i^a), \underline{\eta}(\widetilde{f}_i^2(\widetilde{x}_i))\right) =$$
$$= m'' \left(\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i)} \nu_g^1(\delta_i^a), \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)} \widetilde{f}_i^2(\widetilde{x}_i)} \eta^c_{\overline{f}^2(i)}\right) =$$
$$= \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)} \widetilde{f}_i^2(\widetilde{x}_i)} \eta^c_{\overline{f}^2(i)} \cdot \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i)} \nu_g^1(\delta_i^a) =$$
$$= \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i)} \eta^c_{\overline{f}^2(i)} \circ \nu_g^1(\delta_i^a) = F_2\Big([\eta] * [\delta]\Big)(\widetilde{x}_i).$$

□

Lemmas 4.9, 4.12, 4.13 and 4.14 prove that:

Theorem 4.15. *The data* $F := (F_0, F_1, F_2)$ *define a 2-functor from* (**Red Atl**) *to* (**PEÉ Gpd**).

We state 2 properties of F that we are going to use soon.

Lemma 4.16. *(adapted from* [Po, proposition 4.9]*) Let us fix any pair of orbifold atlases* $\mathcal{U}, \mathcal{V}$ *for* X *and* Y *respectively. Let us set* $F_0(\mathcal{U}) =: (R \overset{s}{\underset{t}{\rightrightarrows}} U)$ *and* $F_0(\mathcal{V}) =: (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')$. *Let us also fix any morphism* $(\psi, \Psi) : (R \overset{s}{\underset{t}{\rightrightarrows}} U) \to (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')$. *Then there exists a unique morphism* $[\hat{f}] : \mathcal{U} \to \mathcal{V}$ *such that* $F_1([\hat{f}]) = (\psi, \Psi)$.

Proof. Let us suppose that $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i\in I}$ and $\mathcal{V} = \{(\widetilde{V}_j, H_j, \phi_j)\}_{j\in J}$. Since each $\widetilde{U}_i$ is connected by definition of orbifold atlas, then the morphism $\psi : U \to U'$ provides a set map $\overline{f} : I \to J$ such that $\psi(\widetilde{U}_i) \subseteq \widetilde{V}_{\overline{f}(i)}$. For each $i \in I$ we set $\widetilde{f}_i := \psi|_{\widetilde{U}_i} : \widetilde{U}_i \to \widetilde{V}_{\overline{f}(i)}$. Then we define continuous maps

$$\pi : U = \coprod_{i\in I} \widetilde{U}_i \longrightarrow X, \quad \pi' : U' = \coprod_{j\in j} \widetilde{V}_j \longrightarrow Y$$

as $\pi|_{\widetilde{U}_i} := \pi_i$ and $\pi'|_{\widetilde{V}_j} := \phi_j$. It is easy to see that both π and π' are open and surjective; moreover, for each $r \in R$ we have $\pi \circ s(r) = \pi \circ t(r)$; analogously for each $r' \in R'$ we have $\pi' \circ s'(r') = \pi' \circ t'(r')$. Let us also define a set map $f : X \to Y$ as

$$f(\pi_i(\widetilde{x}_i)) := \pi' \circ \psi(\widetilde{x}_i) = \phi_{\overline{f}(i)} \circ \widetilde{f}_i(\widetilde{x}_i) \quad \forall\, i \in I, \forall\, \widetilde{x}_i \in \widetilde{U}_i.$$

This is well-defined; indeed let us suppose that $\pi_i(\widetilde{x}_i) = \pi_{i'}(\overline{x}_{i'})$ for some $i' \in I$ and $\overline{x}_{i'} \in \widetilde{U}_{i'}$. Then by definition of orbifold atlas there exists a change of charts λ from $(\widetilde{U}_i, G_i, \pi_i)$ to $(\widetilde{U}_{i'}, G_{i'}, \pi_{i'})$ with $\widetilde{x}_i \in \operatorname{dom}\lambda$ and $\lambda(\widetilde{x}_i) = \overline{x}_{i'}$. Therefore we have

$$\begin{aligned} f(\pi_i(\widetilde{x}_i)) &= \pi' \circ \psi(\widetilde{x}_i) = \pi' \circ \psi \circ s(\operatorname{germ}_{\widetilde{x}_i} \lambda) = \pi' \circ s' \circ \Psi(\operatorname{germ}_{\widetilde{x}_i} \lambda) = \\ &= \pi' \circ t' \circ \Psi(\operatorname{germ}_{\widetilde{x}_i} \lambda) = \pi' \circ \psi \circ t(\operatorname{germ}_{\widetilde{x}_i} \lambda) = \pi' \circ \psi \circ \lambda(\widetilde{x}_i) = \\ &= \pi' \circ \psi(\overline{x}_{i'}) = f(\pi_{i'}(\overline{x}_{i'})), \end{aligned}$$

so f is well-defined. Since π is open and surjective and since $f \circ \pi = \pi' \circ \psi$, then we conclude that f is continuous. Now let us fix any $i \in I$, any $\widetilde{x}_i \in \widetilde{U}_i$ and any $\lambda \in \mathcal{C}h(\mathcal{U}, i, -)$ with $\widetilde{x}_i \in \operatorname{dom}\lambda$. By construction of R' and since (ψ, Ψ) commutes with s and t, there exists $\mu \in \mathcal{C}h(\mathcal{V}, \overline{f}(i), -)$ such that

$$\Psi(\operatorname{germ}_{\widetilde{x}_i} \lambda) = \operatorname{germ}_{\psi(\widetilde{x}_i)} \mu = \operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)} \mu.$$

Now let us consider the open set $\widetilde{U}(\lambda, \widetilde{x}_i, \mu) := \operatorname{dom}\lambda \cap \widetilde{f}_i^{-1}(\operatorname{dom}\mu)$. Then for each $\widetilde{x}'_i \in \widetilde{U}(\lambda, \widetilde{x}_i, \mu)$ we have

$$\Psi(\operatorname{germ}_{\widetilde{x}'_i} \lambda) = \operatorname{germ}_{\widetilde{f}_i(\widetilde{x}'_i)} \mu.$$

Then let us set

$$P_f := \left\{ \lambda|_{\widetilde{U}(\lambda,\widetilde{x}_i,\mu)}, \quad \forall\, i \in I, \forall\, \lambda \in \mathcal{C}h(\mathcal{U}, i, -), \forall\, \widetilde{x}_i \in \operatorname{dom}\lambda \right\};$$

if we have 2 (or more) collections $(\lambda, \widetilde{x}_i, \mu)$ and $(\lambda', \widetilde{x}'_i, \mu')$ such that

$$\lambda|_{\widetilde{U}(\lambda,\widetilde{x}_i,\mu)} = \lambda'|_{\widetilde{U}(\lambda',\widetilde{x}'_i,\mu')}, \tag{4.2}$$

then we simply make arbitrarily a choice of a collection $(\lambda, \widetilde{x}_i, \mu)$ associated to the morphism (4.2) in P_f. In particular, for each $\lambda|_{\widetilde{U}(\lambda,\widetilde{x}_i,\mu)} \in P_f$ we can set $\nu_f(\lambda|_{\widetilde{U}(\lambda,\widetilde{x}_i,\mu)}) := \mu$. Then it is easy to see that

$$\hat{f} := \left(f, \overline{f}, \left\{ \widetilde{f}_i \right\}_{i\in I}, P_f, \nu_f \right)$$

is a representative of a morphism from $\mathcal{U}$ to $\mathcal{V}$. Since the changes of charts of the form μ are not uniquely determined, then $\hat{f}$ is not unique, but the class $[\hat{f}]$ is so. A direct check proves that $F_1([\hat{f}]) = (\psi, \Psi)$; moreover it is easy to see that if $[\hat{f}^1], [\hat{f}^2] : \mathcal{U} \to \mathcal{V}$ are such that $F_1([\hat{f}^1]) = F_1([\hat{f}^2])$, then $[\hat{f}^1] = [\hat{f}^2]$. This suffices to complete the proof. $\square$

Lemma 4.17. *Let us fix any pair of orbifold atlases $\mathcal{U},\mathcal{V}$ for 2 topological spaces X and Y respectively and any pair of morphisms $[\hat{f}^m]:\mathcal{U}\rightarrow\mathcal{V}$ for $m=1,2$ with representatives*

$$\hat{f}^m := \left(f,\overline{f},\left\{\widetilde{f}_i^m\right\}_{i\in I},P_f^m,\nu_f^m\right) \quad \textit{for } m=1,2.$$

Let us set

$$F_0(\mathcal{U}) =: \left(R \overset{s}{\underset{t}{\rightrightarrows}} U\right),\; F_0(\mathcal{V}) := \left(R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'\right),\; F_1([\hat{f}^m]) =: (\psi^m,\Psi^m) \; \textit{for } m=1,2.$$

Let us also fix any 2-morphism $\alpha:(\psi^1,\Psi^1)\Rightarrow(\psi^2,\Psi^2)$. Then there exists a unique 2-morphism $[\delta]:[\hat{f}^1]\Rightarrow[\hat{f}^2]$ such that $F_2([\delta])=\alpha$.

Proof. By definition 4.3 α is a smooth map from U to R' such that $s'\circ\alpha=\psi^1$; so for each $i\in I$ and for each $\overline{x}_i\in\widetilde{U}_i\subseteq U$ we can choose a change of charts $\delta_i^{\overline{x}_i}$ such that

$$\alpha(\overline{x}_i) = \mathrm{germ}_{\psi^1(\overline{x}_i)}\,\delta_i^{\overline{x}_i} = \mathrm{germ}_{\widetilde{f}_i^1(\overline{x}_i)}\,\delta_i^{\overline{x}_i}. \tag{4.3}$$

For each $\overline{x}_i\in\widetilde{U}_i$ we consider the set

$$R'(\delta_i^{\overline{x}_i}) := \left\{\mathrm{germ}_{\widetilde{y}_{\overline{f}^1(i)}}\,\delta_i^{\overline{x}_i}, \quad \forall\,\widetilde{y}_{\overline{f}^1(i)}\in\mathrm{dom}\,\delta_i^{\overline{x}_i}\right\}\subseteq R'.$$

By construction 4.8 for R', $R'(\delta_i^{\overline{x}_i})$ is open in R'; moreover the map γ defined on that set by

$$\gamma\left(\mathrm{germ}_{\widetilde{y}_{\overline{f}^1(i)}}\,\delta_i^{\overline{x}_i}\right) := \widetilde{y}_{\overline{f}^1(i)}$$

is a diffeomorphism to an open set of some $\mathbb{R}^n$. We set $\widetilde{U}_i^{\overline{x}_i}:=\alpha^{-1}(R'(\delta_i^{\overline{x}_i}))\cap\widetilde{U}_i$. For each $i\in I$ we choose any collection $\{\overline{x}_i^a\}_{a\in A(i)}$ such that the set $\{\widetilde{U}_i^{\overline{x}_i^a}\}_{a\in A(i)}$ is an open covering of $\widetilde{U}_i$. For simplicity of notations, we set $\delta_i^a:=\delta_i^{\overline{x}_i^a}$ and $\widetilde{U}_i^a:=\widetilde{U}_i^{\overline{x}_i^a}$. We claim that the collection $\delta:=\{(\widetilde{U}_i^a,\delta_i^a)\}_{i\in I,a\in A(i)}$ is a representative of a 2-morphism from $[\hat{f}^1]$ to $[\hat{f}^2]$.

Let us fix any $i\in I$, any $a\in A(i)$ and any $\widetilde{x}_i\in\widetilde{U}_i^a$; by definition of this set we have

$$\alpha(\widetilde{x}_i) = \mathrm{germ}_{\psi^1(\widetilde{x}_i)}\,\delta_i^a = \mathrm{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)}\,\delta_i^a \tag{4.4}$$

(in other terms, (4.3) holds not only for the point $\overline{x}_i^a$, but also in an open neighbourhood of that point). By definition 4.3 we have $t'\circ\alpha=\psi^2$; so for each $\widetilde{x}_i\in\widetilde{U}_i^a$ we have

$$\widetilde{f}_i^2(\widetilde{x}_i) = \psi^2(\widetilde{x}_i) = t'\circ\alpha(\widetilde{x}_i) = t'\left(\mathrm{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)}\,\delta_i^a\right) = \delta_i^a\circ\widetilde{f}_i^1(\widetilde{x}_i),$$

so in particular

$$\widetilde{f}_i^1(\widetilde{U}_i^a)\subseteq\mathrm{dom}\,\delta_i^a, \quad \widetilde{f}_i^2(\widetilde{U}_i^a)\subseteq\mathrm{cod}\,\delta_i^a;$$

therefore properties (R1), (R2) and (R3) are verified for δ. If a,a' are indices in $A(i)$ and $\widetilde{x}_i\in\widetilde{U}_i^a\cap\widetilde{U}_i^{a'}$, then by (4.4) we have

$$\mathrm{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)}\,\delta_i^a = \alpha(\widetilde{x}_i) = \mathrm{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)}\,\delta_i^{a'},$$

so (R4) holds. Now let us fix any $(i,i')\in I\times I$, any $(a,a')\in A(i)\times A(i')$, any $\lambda\in\mathcal{C}h(\mathcal{U},i,i')$ and any $\widetilde{x}_i\in\mathrm{dom}\,\lambda\cap\widetilde{U}_i^a$ such that $\lambda(\widetilde{x}_i)\in\widetilde{U}_{i'}^{a'}$. Since P_f^1 and P_f^2

are both good subsets of $\mathcal{C}h(\mathcal{U})$, then for $m = 1, 2$ there exists $\lambda^m \in P^m_f(i, i')$ such that $\widetilde{x}_i \in \operatorname{dom}\lambda^m$ and $\operatorname{germ}_{\widetilde{x}_i}\lambda^m = \operatorname{germ}_{\widetilde{x}_i}\lambda$. We recall (see construction 4.10) that

$$\Psi^m(\operatorname{germ}_{\widetilde{x}_i}\lambda) = \operatorname{germ}_{\widetilde{f}^m_i(\widetilde{x}_i)}\nu^m_f(\lambda^m) \quad \text{for } m = 1, 2.$$

Therefore:

$$\operatorname{germ}_{\widetilde{f}^2_i(\widetilde{x}_i)}\nu^2_f(\lambda^2)\cdot\operatorname{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)}\delta^a_i = m'\left(\operatorname{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)}\delta^a_i, \operatorname{germ}_{\widetilde{f}^2_i(\widetilde{x}_i)}\nu^2_f(\lambda^2)\right) \stackrel{(4.4)}{=}$$
$$\stackrel{(4.4)}{=} \left(m'\circ(\alpha\circ s, \Psi^2)\right)(\operatorname{germ}_{\widetilde{x}_i}\lambda^2) \stackrel{(T2)}{=} \left(m'\circ(\Psi^1, \alpha\circ t)\right)(\operatorname{germ}_{\widetilde{x}_i}\lambda^1) =$$
$$= m'\left(\operatorname{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)}\nu^1_f(\lambda), \operatorname{germ}_{\widetilde{f}^1_{i'}(\lambda^1(\widetilde{x}_i))}\delta^{a'}_{i'}\right) = \operatorname{germ}_{\widetilde{f}^1_{i'}(\lambda(\widetilde{x}_i))}\delta^{a'}_{i'}\cdot\operatorname{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)}\nu^1_f(\lambda).$$

So also property (R5) holds. Therefore δ is a representative of a 2-morphism from $[\hat{f}^1]$ to $[\hat{f}^2]$. Different choices of the sets $\{\overline{x}^a_i\}_a$ and $\{\delta^a_i\}_a$ give rise to different δ's, but their equivalence class $[\delta]$ is the same. Now it is easy to see that $F_2([\delta]) = \alpha$; moreover, a direct computation proves that if $[\delta^1]$ and $[\delta^2]$ are such that $F_2([\delta^1]) = F_2([\delta^2])$, then $[\delta^1] = [\delta^2]$. This suffices to conclude. □

So we have proved that for every pair of reduced orbifold atlases $\mathcal{U}, \mathcal{V}$ the functor

$$(F_1, F_2)(\mathcal{U}, \mathcal{V}) : (\mathbf{Red\,Alt})(\mathcal{U}, \mathcal{V}) \longrightarrow (\mathbf{PE\acute{E}\,Gpd})(F_0(\mathcal{U}), F_0(\mathcal{V}))$$

is a bijection on objects and morphisms (i.e. on 1-morphisms and 2-morphisms of (**Red Atl**) and of (**PEÉ Gpd**)). F_0 is not injective (see remark 8.3 below); it is surjective only up to "weak equivalences" (see lemma 6.6 below).

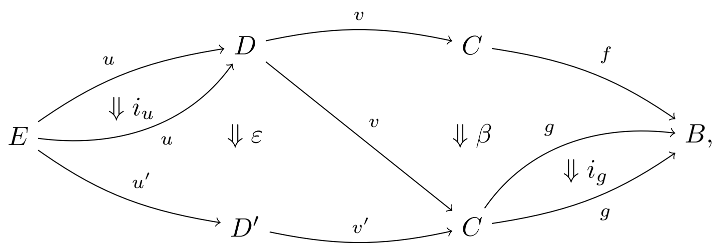
5. Weak equivalences in (**É Gpd**)

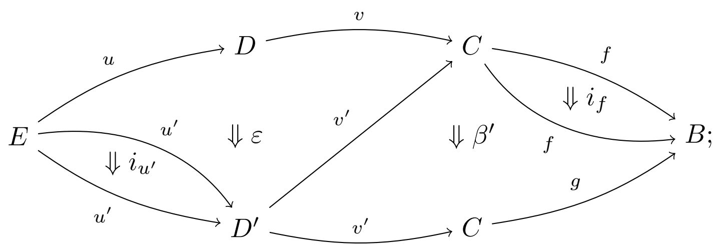
Definition 5.1. [Pr, conditions 2.1] Given any 2-category or bicategory $\mathscr{C}$, a subset $\mathbf{W}$ of 1-morphisms of $\mathscr{C}$ is said *to admit a right calculus of fractions* if it satisfies the following conditions

(U1) all equivalences are in $\mathbf{W}$;
(U2) $\mathbf{W}$ is closed under compositions;
(U3) for every morphism $w : A \to B$ in $\mathbf{W}$ and every morphism $f : C \to B$, there exist a pair of morphisms $v : D \to C$ in $\mathbf{W}$ and $g : D \to A$ and an invertible 2-morphism $\alpha : w \circ g \Rightarrow f \circ v$;
(U4) (i) given any morphism $w : B \to A$ in $\mathbf{W}$, any pair of morphisms $f, g : C \to B$ and any $\alpha : w \circ f \Rightarrow w \circ g$, there exists a morphism $v : D \to C$ in $\mathbf{W}$ and a 2-morphism $\beta : f \circ v \Rightarrow g \circ v$ such that $\alpha * i_v = i_w * \beta$;
(ii) if α as in (i) is invertible, then so is β;
(iii) if $(v' : D' \to C, \beta' : f \circ v' \Rightarrow g \circ v')$ is another pair with the same properties of (v, β) in (i), then there exist 2 morphisms $u : E \to D$, $u' : E \to D'$ and an invertible 2-morphism $\varepsilon : v \circ u \Rightarrow v' \circ u'$ such that $v \circ u$ and $v' \circ u'$ are both in $\mathbf{W}$ and

$$\left(i_g * \varepsilon\right) \odot \left(\beta * i_u\right) = \left(\beta' * i_{u'}\right) \odot \left(i_f * \varepsilon\right).$$

The relevant diagrams for the previous identity are:





(U5) if $w : A \to B$ is a morphism in $\mathbf{W}$, $v : A \to B$ is any morphism and there exists an invertible 2-morphism $\alpha : v \Rightarrow w$, then v belongs to $\mathbf{W}$.

We recall (see [Pr]) that given any 2-category or bicategory $\mathscr{C}$ and any set $\mathbf{W}$ as before, there exists a bicategory $\mathscr{C}[\mathbf{W}^{-1}]$ (called *bicategory of fractions*) together with a pseudofunctor $G : \mathscr{C} \to \mathscr{C}[\mathbf{W}^{-1}]$ that sends each element of $\mathbf{W}$ to an equivalence and that is universal with respect to such property (in the notations of [Pr] G is called *bifunctor*, but this notation is no more in use). We refer to [Pr, §2.2, 2.3, 2.4] for more details on the construction of bicategories of fractions and to [PW, §1] for a general overview on bicategories and pseudofunctors.

We recall also (see [M, §2.4]) that a morphism $(\psi, \Psi) : (R \overset{s}{\underset{t}{\rightrightarrows}} U) \to (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')$ between Lie groupoids is a *weak equivalence* (also known as *Morita equivalence* or *essential equivalence*) iff the following 2 conditions hold:

(V1) the smooth map $t' \circ \pi^1 : R' {}_{s'} \times_{\psi} U \to U'$ is a surjective submersion (here π^1 is the projection $R' {}_{s'} \times_{\psi} U \to R'$ and the fiber product exists since s is a submersion);

(V2) the following square is cartesian (it is commutative by definition 4.2)

$$\begin{array}{ccc} R & \xrightarrow{\Psi} & R' \\ {\scriptstyle (s,t)}\big\downarrow & & \big\downarrow{\scriptstyle (s',t')} \\ U \times U & \xrightarrow{(\psi \times \psi)} & U' \times U'. \end{array} \tag{5.1}$$

Any two Lie groupoids $R \overset{s}{\underset{t}{\rightrightarrows}} U$ and $R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'$ are said to be *weakly equivalent* (or *Morita equivalent* or *essentially equivalent*) iff there exists a third Lie groupoid $R'' \overset{s''}{\underset{t''}{\rightrightarrows}} U''$ and two weak equivalences as follows

$$\Big(R \overset{s}{\underset{t}{\rightrightarrows}} U\Big) \xleftarrow{(\psi,\Psi)} \Big(R'' \overset{s''}{\underset{t''}{\rightrightarrows}} U''\Big) \xrightarrow{(\phi,\Phi)} \Big(R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'\Big).$$

This is actually an equivalence relation, see for example [MM, chapter 5]. We denote by $\mathbf{W}_{\mathbf{\acute{E}\,Gpd}}$ the set of all weak equivalences in $(\mathbf{\acute{E}\,Gpd})$, i.e. the set of all weak equivalences between étale Lie groupoids. Then we have:

Proposition 5.2. [Pr, §4.1] *The set $\mathbf{W}_{\mathbf{\acute{E}\,Gpd}}$ admits a right calculus of fractions, so there exists a bicategory $(\mathbf{\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{\acute{E}\,Gpd}}\right]$ and a pseudofunctor*

$$\overline{G}:(\mathbf{\acute{E}\,Gpd})\longrightarrow(\mathbf{\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{\acute{E}\,Gpd}}\right]$$

that sends each weak equivalence between étale Lie groupoids to an equivalence and that is universal with respect to such property.

We denote by $\mathbf{W}_{\mathbf{P\acute{E}\,Gpd}}$ and $\mathbf{W}_{\mathbf{PE\acute{E}\,Gpd}}$ the set of all weak equivalences between proper, étale Lie groupoids, respectively between proper, effective, étale Lie groupoids. Then we have:

Proposition 5.3. *The sets $\mathbf{W}_{\mathbf{P\acute{E}\,Gpd}}$ and $\mathbf{W}_{\mathbf{PE\acute{E}\,Gpd}}$ admit a right calculus of fractions, so there exist pseudofunctors*

$$\widetilde{G}:(\mathbf{P\acute{E}\,Gpd})\longrightarrow(\mathbf{P\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{P\acute{E}\,Gpd}}\right],$$

$$G:(\mathbf{PE\acute{E}\,Gpd})\longrightarrow(\mathbf{PE\acute{E}\,gpd})\left[\mathbf{W}^{-1}_{\mathbf{PE\acute{E}\,Gpd}}\right]$$

that send each weak equivalence between proper, (effective) étale Lie groupoids to an equivalence and that are universal with respect to such property. Moreover, both $(\mathbf{P\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{P\acute{E}\,Gpd}}\right]$ and $(\mathbf{PE\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{PE\acute{E}\,Gpd}}\right]$ are a full 2-subcategories of $(\mathbf{\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{\acute{E}\,Gpd}}\right]$ and we have a commutative diagram as follows:

$$\begin{array}{ccccc}
(\mathbf{PE\acute{E}\,Gpd}) & \hookrightarrow & (\mathbf{P\acute{E}\,Gpd}) & \hookrightarrow & (\mathbf{\acute{E}\,Gpd}) \\
\Big\downarrow G & \curvearrowright & \Big\downarrow \widetilde{G} & \curvearrowright & \Big\downarrow \overline{G} \\
(\mathbf{PE\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{PE\acute{E}\,Gpd}}\right] & \hookrightarrow & (\mathbf{P\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{P\acute{E}\,Gpd}}\right] & \hookrightarrow & (\mathbf{\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{\acute{E}\,Gpd}}\right],
\end{array}$$

where each map without a name is simply an embedding as full 2-subcategories or full bi-subcategories.

Proof. By [MM, proposition 5.26], if $R\overset{s}{\underset{t}{\rightrightarrows}}U$ and $R'\overset{s'}{\underset{t'}{\rightrightarrows}}U'$ are weakly equivalent Lie groupoids, then the first Lie groupoid is proper if and only if the second one is so. Moreover, by [MM, example 5.21(2)] if $R\overset{s}{\underset{t}{\rightrightarrows}}U$ and $R'\overset{s'}{\underset{t'}{\rightrightarrows}}U'$ are both étale and they are weakly equivalent, then the first Lie groupoid is effective if and only if the second one is so (note that being étale is not preserved by weak equivalences).

Now axioms (U1), (U2) and (U5) are easily verified for the set $\mathbf{W}_{\mathbf{PE\acute{E}\,Gpd}}$. Let us consider (U3), so let us fix any weak equivalence $w:B\rightarrow A$ and any morphism $f:C\rightarrow B$ with A,B,C all proper, effective, étale Lie groupoids. By proposition 5.2 we get that (U3) holds in $(\mathbf{\acute{E}\,Gpd})$ (we simply ignore the fact that A,B and C are proper and effective). Therefore there exists an étale Lie groupoid D, a weak equivalence $v:D\rightarrow C$, a morphism of Lie groupoids $f:D\rightarrow A$ and an invertible 2-morphism $\alpha:w\circ g\Rightarrow f\circ v$. Now D and C are étale Lie groupoids that are weakly equivalent and C is proper and effective; so also D is proper and effective. Therefore axiom (U3) holds for the set $\mathbf{W}_{\mathbf{PE\acute{E}\,Gpd}}$. An analogous proof shows that also (U4) holds for $\mathbf{W}_{\mathbf{PE\acute{E}\,Gpd}}$. The proof for the set $\mathbf{W}_{\mathbf{P\acute{E}\,Gpd}}$ is the same. Therefore we have a right calculus of fractions for those 2 sets and there exist pseudofunctors $\widetilde{G}, G$ as in the claim. The last part of the claim is straightforward by looking at the explicit construction of the bicategories of fractions in [Pr]. □

6. Weak equivalences in (**Red Atl**)

We introduce also a notion of weak equivalence in (**Red Atl**) as follows. Using the 2-functor F, such definition will match with the definition of weak equivalence of Lie groupoids (see proposition 6.5 below).

Definition 6.1. Let us fix any pair of reduced orbifold atlases $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i\in I}$, and $\mathcal{V}$ on X and Y respectively and any morphism

$$[\hat{w}] = \big(w, \overline{w}, \{\widetilde{w}_i\}_{i\in I}, [P_w, \nu_w]\big) : \mathcal{U} \longrightarrow \mathcal{V}.$$

Then we say that $[\hat{w}]$ is a *weak equivalence of orbifold atlases* iff the following 2 conditions hold:

(W1) the continuous map $w : X \to Y$ is an homeomorphism;
(W2) for each $i \in I$ the chart $(\widetilde{U}_i, G_i, w\circ\pi_i)$ on Y is compatible with the atlas $\mathcal{V}$ (equivalently, it belongs to the maximal atlas $\mathcal{V}^{\max}$ associated to $\mathcal{V}$).

Given any two atlases $\mathcal{U}^1, \mathcal{U}^2$, we say that they are *weakly equivalent* if and only if there is an atlas $\mathcal{U}'$ and a pair of weak equivalences of orbifold atlases as follows:

$$\mathcal{U}^1 \xleftarrow{\;[\hat{w}^1]\;} \mathcal{U}' \xrightarrow{\;[\hat{w}^2]\;} \mathcal{U}^2.$$

In that case we write $\mathcal{U}^1 \sim \mathcal{U}^2$ (we will prove in proposition 6.7 below that actually this is an equivalence relation).

Remark 6.2. We recall that in any bicategory it makes sense to define equivalences. In the 2-category (**Red Atl**) a morphism $[\hat{w}] : \mathcal{U} \to \mathcal{V}$ is an *equivalence* iff there exists a morphism $[\hat{v}] : \mathcal{V} \to \mathcal{U}$ and invertible 2-morphisms $[\varepsilon] : \mathrm{id}_{\mathcal{U}} \Rightarrow [\hat{v}]\circ[\hat{w}]$ and $[\eta] : [\hat{w}]\circ[\hat{v}] \Rightarrow \mathrm{id}_{\mathcal{V}}$ that satisfy the triangle identities (see [Mac, page. 42]). Then it is easy to see that every equivalence of orbifold atlases is a weak equivalence.

The following 2 lemmas are on the same line of [Po, propositions 5.3 and 6.2]; the significant differences are given by a slightly different notion of morphism between orbifold atlases (see remark 1.10) and by the fact that we allow more general weak equivalences than the "unit weak equivalences" in [Po].

Lemma 6.3. *If $[\hat{w}] : \mathcal{U} \to \mathcal{V}$ is a weak equivalence of orbifold atlases, then $F_1([\hat{w}])$ is a weak equivalence of groupoids.*

Proof. Let us suppose that

$$\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i\in I}, \quad \mathcal{V} = \{(\widetilde{V}_j, H_j, \phi_j)\}_{j\in J};$$

let us fix any representative

$$\hat{w} := \big(w, \overline{w}, \{\widetilde{w}_i\}_{i\in I}, P_w, \nu_w\big)$$

for $[\hat{w}]$ and let us set:

$$F_0(\mathcal{U}) =: \Big(R \overset{s}{\underset{t}{\rightrightarrows}} U\Big), \quad F_0(\mathcal{V}) =: \Big(R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'\Big), \quad F_1([\hat{w}]) =: (\psi, \Psi).$$

Then let us fix any $i \in I$ and any $\widetilde{x}_i \in \widetilde{U}_i$. By definition of morphism, we have

$$w \circ \pi_i = \phi_{\overline{w}(i)} \circ \widetilde{w}_i; \tag{6.1}$$

by definition of weak equivalence, the chart $(\widetilde{U}_i, G_i, w \circ \pi_i)$ is compatible with the atlas $\mathcal{V}$, so in particular it is compatible with $(\widetilde{V}_{\overline{w}(i)}, H_{\overline{w}(i)}, \phi_{\overline{w}(i)})$. Therefore by (6.1) applied to $\widetilde{x}_i$, there exists a change of charts λ from $(\widetilde{U}_i, G_i, w \circ \pi_i)$ to $(\widetilde{V}_{\overline{w}(i)}, H_{\overline{w}(i)}, \phi_{\overline{w}(i)})$, such that $\widetilde{x}_i \in \operatorname{dom} \lambda$. In particular,

$$\phi_{\overline{w}(i)} \circ \lambda = w \circ \pi_i|_{\operatorname{dom} \lambda}. \tag{6.2}$$

Then let us consider the map

$$\gamma := \widetilde{w}_i \circ \lambda^{-1} : \operatorname{cod} \lambda \longrightarrow \widetilde{V}_{\overline{w}(i)}.$$

For each $\widetilde{y} \in \operatorname{cod} \lambda$ we have

$$\phi_{\overline{w}(i)} \circ \gamma(\widetilde{y}) = \phi_{\overline{w}(i)} \circ \widetilde{w}_i \circ \lambda^{-1}(\widetilde{y}_{\overline{w}(i)}) \overset{(6.1)}{=} w \circ \pi_i \circ \lambda^{-1}(\widetilde{y}) \overset{(6.2)}{=} \phi_{\overline{w}(i)}(\widetilde{y}).$$

So for each such $\widetilde{y}$ as before, there exists a (in general non-unique) $h \in H_{\overline{w}(i)}$ such that $\gamma(\widetilde{y}) = h(\widetilde{y})$. So by [MM, lemma 2.11] we have that there is a unique $\overline{h} \in H_{\overline{w}(i)}$ such that $\gamma = \overline{h}|_{\operatorname{cod} \lambda}$. Therefore,

$$\widetilde{w}_i|_{\operatorname{dom} \lambda} = \gamma \circ \lambda = \overline{h} \circ \lambda.$$

So we have proved that for each $i \in I$ the map $\widetilde{w}_i$ coincides locally with a diffeomorphism, hence it is étale; therefore $\psi = \coprod_{i \in I} \widetilde{w}_i$ is étale. Hence, also the induced morphism (see property (V1))

$$\pi^1 : R' \,{}_{s'}\times_{\psi} U \longrightarrow R'$$

is étale. Since t' is also étale, we conclude that $t' \circ \pi^1$ is étale, so in particular it is a submersion. Let us prove also that it is surjective. Let us fix any point $\widetilde{y}_j \in \widetilde{V}_j \subseteq U'$; since w is an homeomorphism, then it makes sense to define $x := w^{-1}(\phi_j(\widetilde{y}_j))$. Let $(\widetilde{U}_i, G_i, \pi_i)$ be any chart in $\mathcal{U}$ and $\widetilde{x}_i \in \widetilde{U}_i$ such that $\pi_i(\widetilde{x}_i) = x$, so

$$w \circ \pi_i(\widetilde{x}_i) = w(x) = \phi_j(\widetilde{y}_j).$$

By definition of morphism between orbifold atlases, this implies

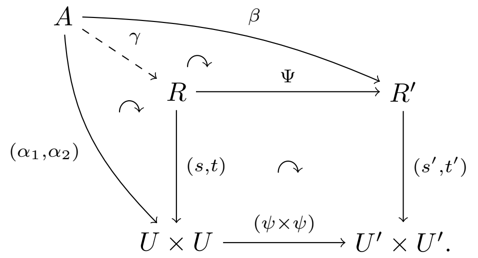
$$\phi_{\overline{w}(i)} \circ \widetilde{w}_i(\widetilde{x}_i) = \phi_j(\widetilde{y}_j).$$

Since $\mathcal{V}$ is an orbifold atlas, there exists a change of charts μ from $(\widetilde{V}_{\overline{w}(i)}, H_{\overline{w}(i)}, \phi_{\overline{w}(i)})$ to $(\widetilde{V}_j, H_j, \phi_j)$ such that $\widetilde{w}_i(\widetilde{x}_i) \in \operatorname{dom} \mu$ and $\mu(\widetilde{w}_i(\widetilde{x}_i)) = \widetilde{y}_j$. Then we have

$$p := \left(\operatorname{germ}_{\widetilde{w}_i(\widetilde{x}_i)} \mu, \widetilde{x}_i\right) \in R' \,{}_{s'}\times_{\psi} U$$

and $t' \circ \pi^1(p) = \widetilde{y}_j$. So we have proved that $t' \circ \pi^1$ is surjective, so (V1) holds.

In order to prove that (ψ, Ψ) is a weak equivalence, we have also to prove that the square (5.1) has the universal property of fiber products. In order to do that, let us fix any smooth manifold A together with any pair of smooth maps $(\alpha_1, \alpha_2) : A \to U \times U$ and $\beta : A \to R'$, such that $(s', t') \circ \beta = (\psi \times \psi) \circ (\alpha_1, \alpha_2)$. We need to prove that there is a unique smooth map $\gamma : A \to R$ making the following diagram commute.



(6.3)

Let us fix any point $a \in A$ and let $(\widetilde{x}_i, \overline{x}_{i'}) := (\alpha_1, \alpha_2)(a)$. Since $(s', t') \circ \beta = (\psi \times \psi) \circ (\alpha_1, \alpha_2)$, then we have that $\beta(a) = \mathrm{germ}_{\widetilde{w}_i(\widetilde{x}_i)} \mu$ for some $\mu \in \mathcal{C}h(\mathcal{V})$ such that $\widetilde{w}_i(\widetilde{x}_i) \in \mathrm{dom}\, \mu$ and $\mu(\widetilde{w}_i(\widetilde{x}_i)) = \widetilde{w}_{i'}(\overline{x}_{i'})$. Since $[\hat{w}]$ is a morphism over w, then this implies:

$$w \circ \pi_i(\widetilde{x}_i) = \phi_{\overline{w}(i)} \circ \widetilde{w}_i(\widetilde{x}_i) = \phi_{\overline{w}(i')} \circ \mu \circ \widetilde{w}_i(\widetilde{x}_i) = \phi_{\overline{w}(i')} \circ \widetilde{w}_{i'}(\overline{x}_{i'}) = w \circ \pi_{i'}(\overline{x}_{i'}).$$

Since w is an homeomorphism, then $\pi_i(\widetilde{x}_i) = \pi_{i'}(\overline{x}_{i'})$, so there exists a change of charts λ from $(\widetilde{U}_i, G_i, \pi_i)$ to $(\widetilde{U}_{i'}, G_{i'}, \pi_{i'})$ such that $\widetilde{x}_i \in \mathrm{dom}\, \lambda$ and $\lambda(\widetilde{x}_i) = \overline{x}_{i'}$. Since P_w is a good subset of $\mathcal{C}h(\mathcal{U})$, then there exists $\hat{\lambda} \in P_w$ such that $\widetilde{x}_i \in \mathrm{dom}\, \hat{\lambda}$ and $\mathrm{germ}_{\widetilde{x}_i} \lambda = \mathrm{germ}_{\widetilde{x}_i} \hat{\lambda}$. Now let us consider any chart $(\widetilde{U}, G, \pi)$ in $\mathcal{U}^{\max}$ around $\widetilde{x}_i$, such that

$$\widetilde{U} \subseteq \mathrm{dom}\, \hat{\lambda} \cap (\widetilde{w}_i)^{-1} (\mathrm{dom}\, \mu) \cap \widetilde{U}' \subseteq \widetilde{U}_i,$$

where $\widetilde{U}'$ is any open neighbourhood of $\widetilde{x}_i$ such that $\widetilde{w}_i|_{\widetilde{U}'}$ is a diffeomorphism (see before). Then both $\nu_w(\hat{\lambda}) \circ \widetilde{w}_i|_{\widetilde{U}}$ and $\mu \circ \widetilde{w}_i|_{\widetilde{U}}$ are embeddings from $(\widetilde{U}, G, w \circ \pi)$ to $(\widetilde{V}_{\overline{w}(i')}, H_{\overline{w}(i')}, \phi_{\overline{w}(i')})$. Therefore, by [MP, proposition A.1] there exists a unique $h \in H_{\overline{w}(i')}$ such that

$$h \circ \nu_w(\hat{\lambda}) \circ \widetilde{w}_i|_{\widetilde{U}} = \mu \circ \widetilde{w}_i|_{\widetilde{U}}.$$

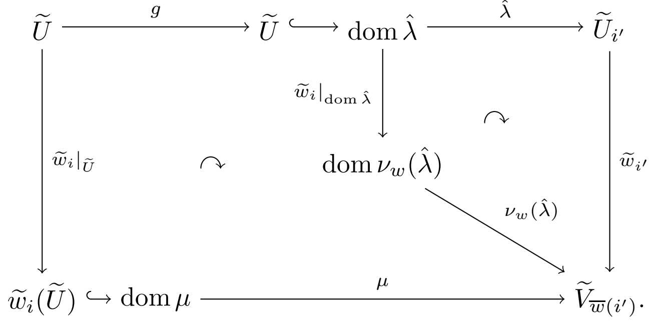
Now

$$\nu_w(\hat{\lambda})(\widetilde{w}_i(\widetilde{x}_i)) \stackrel{(Q5c)}{=} \widetilde{w}_{i'} \circ \hat{\lambda}(\widetilde{x}_i) = \widetilde{w}_{i'}(\overline{x}_{i'}) = \mu(\widetilde{w}_i(\widetilde{x}_i)),$$

so h belongs to the stabilizer of $\widetilde{w}_{i'}(\overline{x}_i)$ in $H_{\overline{w}(i')}$. So by [MP, lemma A.2] there exists a unique g in the stabilizer of $\widetilde{x}_i$ in G, such that

$$\nu_w(\hat{\lambda}) \circ \widetilde{w}_i \circ g = \mu \circ \widetilde{w}_i|_{\widetilde{U}}.$$

By combining this with (Q5a) and (Q5c) we have a commutative diagram as follows



So we have

$$\widetilde{w}_{i'} \circ (\hat{\lambda} \circ g) = \mu \circ \widetilde{w}_i|_{\widetilde{U}}. \tag{6.4}$$

Now $\hat{\lambda} \circ g$ is a change of charts between $(\widetilde{U}_i, G_i, \pi_i)$ and $(\widetilde{U}_{i'}, G_{i'}, \pi_{i'})$; since P_w is a good subset of $\mathcal{C}h(\mathcal{U})$, then there exists $\overline{\lambda} \in P_w$ such that $\widetilde{x}_i \in \operatorname{dom} \overline{\lambda}$ and $\operatorname{germ}_{\widetilde{x}_i} \hat{\lambda} \circ g = \operatorname{germ}_{\widetilde{x}_i} \overline{\lambda}$. By (Q5c) we have

$$\operatorname{germ}_{\widetilde{x}_i} \widetilde{w}_{i'} \circ \overline{\lambda} = \operatorname{germ}_{\widetilde{x}_i} \nu_w(\overline{\lambda}) \circ \widetilde{w}_i. \tag{6.5}$$

By construction we have that $\widetilde{w}_i$ is a diffeomorphism around $\widetilde{x}_i$. Then:

$$\beta(a) = \operatorname{germ}_{\widetilde{w}_i(\widetilde{x}_i)} \mu \overset{(6.4)}{=} \left(\operatorname{germ}_{\widetilde{x}_i} \widetilde{w}_{i'} \circ \hat{\lambda} \circ g\right) \cdot \left(\operatorname{germ}_{\widetilde{x}_i} \widetilde{w}_i\right)^{-1} =$$

$$= \left(\operatorname{germ}_{\widetilde{x}_i} \widetilde{w}_{i'} \circ \overline{\lambda}\right) \cdot \left(\operatorname{germ}_{\widetilde{x}_i} \widetilde{w}_i\right)^{-1} \overset{(6.5)}{=} \operatorname{germ}_{\widetilde{w}_i(\widetilde{x}_i)} \nu_w(\overline{\lambda}) = \Psi\left(\operatorname{germ}_{\widetilde{x}_i} \overline{\lambda}\right).$$

Then we set

$$\gamma(a) := \operatorname{germ}_{\widetilde{x}_i} \overline{\lambda} \in R.$$

This defines a set map $\gamma : A \to R$; the previous construction proves that γ makes (6.3) commute. Moreover, using again the fact that $\widetilde{w}_i$ is locally a diffeomorphism it is easy to prove that for each $a \in A$ the point $\gamma(a)$ defined before is the unique point such that $\Psi \circ \gamma(a) = \beta(a)$ and $(s,t) \circ \gamma(a) = (\alpha_1, \alpha_2)(a)$. Then in order to conclude we have only to prove that γ is locally smooth. Since s is étale, given any point $a \in A$ there exists an open neighbourhood $R(a)$ of $\gamma(a)$ such that $s|_{R(a)}$ is invertible. So the set $A(a) := \alpha_1^{-1}(s(R(a)))$ is an open neighbourhood of a in A and $\gamma|_{A(a)}$ coincides with the smooth map $(s|_{R(a)})^{-1} \circ \alpha_1$. □

Lemma 6.4. *Let $\mathcal{U}$ and $\mathcal{V}$ be reduced orbifold atlases (for X and Y respectively). Let $(\psi, \Psi) : F_0(\mathcal{U}) \to F_0(\mathcal{V})$ be a weak equivalence of Lie groupoids and let $[\hat{w}] : \mathcal{U} \to \mathcal{V}$ be the unique morphism such that $F_1([\hat{w}]) = (\psi, \Psi)$ (see lemma 4.16). Then $[\hat{w}]$ is a weak equivalence of orbifold atlases.*

Proof. First of all, let us denote by

$$[\hat{w}] = \left(w, \overline{w}, \{\widetilde{w}_i\}_{i \in I}, [P_w, \nu_w]\right)$$

the unique morphism obtained in lemma 4.16 from (ψ, Ψ). Then we claim that $w : X \to Y$ is an homeomorphism. In order to prove that, we recall that in the proof of lemma 4.16 we defined

$$w(\pi_i(\widetilde{x}_i)) := \phi_{\overline{w}(i)} \circ \widetilde{w}_i(\widetilde{x}_i) \quad \forall\, i \in I, \forall \widetilde{x}_i \in \widetilde{U}_i$$

and this was well-defined on X; in order to prove that it is an homeomorphism, we have to prove that it is injective, surjective and open. First of all, let us suppose that $\widetilde{x}_i \in \widetilde{U}_i, \overline{x}_{i'} \in \widetilde{U}_{i'}$ are such that $w(\pi_i(\widetilde{x}_i)) = w(\pi_{i'}(\overline{x}_{i'}))$. This implies that

$$\phi_{\overline{w}(i)} \circ \widetilde{w}_i(\widetilde{x}_i) = \phi_{\overline{w}(i')} \circ \widetilde{w}_{i'}(\overline{x}_{i'}).$$

Since $\mathcal{V}$ is an orbifold atlas, then there exist $\mu \in \mathcal{C}h(\mathcal{V})$ with $\widetilde{w}_i(\widetilde{x}_i) \in \operatorname{dom} \mu$ and $\mu(\widetilde{w}_i(\widetilde{x}_i)) = \widetilde{w}_{i'}(\overline{x}_{i'})$. In particular, we have

$$(s', t')\left(\operatorname{germ}_{\widetilde{w}_i(\widetilde{x}_i)} \mu\right) = (\widetilde{w}_i(\widetilde{x}_i), \widetilde{w}_{i'}(\overline{x}_{i'})) = (\psi \times \psi)(\widetilde{x}_i, \overline{x}_{i'}),$$

so

$$\left(\operatorname{germ}_{\widetilde{w}_i(\widetilde{x}_i)} \mu, \widetilde{x}_i, \overline{x}_{i'}\right) \in R' {}_{(s',t')}\times_{(\psi \times \psi)} (U \times U).$$

Since (5.1) is cartesian, then there exists a unique $r \in R$ such that $(s,t)(r) = (\widetilde{x}_i, \overline{x}_{i'})$ and $\Psi(r) = \text{germ}_{\widetilde{w}_i(\widetilde{x}_i)}\,\mu$. This implies that $\pi_i(\widetilde{x}_i) = \pi_{i'}(\overline{x}_{i'})$; so w is injective.

Let us now fix any point $\phi_j(\widetilde{y}_j) \in Y$; since (ψ, Ψ) is a weak equivalence, then $t' \circ \pi^1 : R' \;{}_{s'}\times_{\psi}\; U \to U'$ is surjective. Therefore there exists $(r', \widetilde{x}_i)$ such that $s'(r') = \psi(\widetilde{x}_i) = \widetilde{w}_i(\widetilde{x}_i)$ and such that $t'(r') = \widetilde{y}_j$. Then necessarily $r' = \text{germ}_{\widetilde{w}_i(\widetilde{x}_i)}\,\mu$ for some $\mu \in \mathcal{C}h(\mathcal{V})$ with $\widetilde{w}_i(\widetilde{x}_i) \in \operatorname{dom}\mu$ and $\mu(\widetilde{w}_i(\widetilde{x}_i)) = \widetilde{y}_j$. Therefore

$$\phi_j(\widetilde{y}_j) = \phi_j \circ \mu \circ \widetilde{w}_i(\widetilde{x}_i) = \phi_{\overline{w}(i)} \circ \widetilde{w}_i(\widetilde{x}_i) = w(\pi_i(\widetilde{x}_i)),$$

so this proves that w is surjective. Now $\{\pi_i(\widetilde{U}_i)\}_{i \in I}$ is an open covering of X; so in order to prove that w is open it is sufficient to prove that for each $i \in I$ and for each $U \subseteq \pi_i(\widetilde{U}_i)$ open, we have that $w(U)$ is open. By definition of chart, we have that any such U is equal to $\pi_i(\widetilde{U})$ for some open (invariant) set $\widetilde{U} \subseteq \widetilde{U}_i$, so

$$w(U) = w \circ \pi_i(\widetilde{U}) = \phi_{\overline{w}(i)} \circ \widetilde{w}_i(\widetilde{U}) = \phi_{\overline{w}(i)} \circ \psi(\widetilde{U}).$$

Since (ψ, Ψ) is a weak equivalence between étale Lie groupoids, then by [MM, exercise 5.16(4)] we have that ψ is étale, so in particular it is open. In addition, $\phi_{\overline{w}(i)}$ is open by definition of orbifold chart; therefore $w(U)$ is open; so w is an homeomorphism.

In order to prove that $[\hat{w}]$ is a weak equivalence, we need also to prove that for each $i \in I$ the chart $(\widetilde{U}_i, G_i, w \circ \pi_i)$ on Y is compatible with $\mathcal{V}$. So let us fix any pair of points $(\widetilde{x}_i, \widetilde{y}_j) \in \widetilde{U}_i \times \widetilde{V}_j$ such that $w \circ \pi_i(\widetilde{x}_i) = \phi_j(\widetilde{y}_j)$. Then we have

$$\phi_{\overline{w}(i)} \circ \widetilde{w}_i(\widetilde{x}_i) = w \circ \pi_i(\widetilde{x}_i) = \phi_j(\widetilde{y}_j),$$

so there exist a change of charts μ from $(\widetilde{V}_{\overline{w}(i)}, H_{\overline{w}(i)}, \phi_{\overline{w}(i)})$ to $(\widetilde{V}_j, H_j, \phi_j)$ such that $\widetilde{w}_i(\widetilde{x}_i) \in \operatorname{dom}\mu$. Since μ is a change of charts, then $\phi_j \circ \mu = \phi_{\overline{w}(i)}$. Moreover, since ψ is étale, then $\widetilde{w}_i$ is locally a diffeomorphism. Therefore there exists an open neighbourhood $\widetilde{U}$ of $\widetilde{x}_i$, contained in $\widetilde{w}_i^{-1}(\operatorname{dom}\mu)$, such that $\widetilde{w}_i$ is an embedding if restricted to $\widetilde{U}$. Then $\lambda := \mu \circ \widetilde{w}_i|_{\widetilde{U}}$ is an embedding from $\widetilde{U} \subseteq \widetilde{U}_i$ to $\widetilde{V}_j$. Moreover, we have

$$\phi_j \circ \lambda = \phi_j \circ \mu \circ \widetilde{w}_i|_{\widetilde{U}} = \phi_{\overline{w}(i)} \circ \widetilde{w}_i|_{\widetilde{U}} = w \circ \pi_i|_{\widetilde{U}},$$

therefore λ is a change of charts from $(\widetilde{U}_i, G_i, w \circ \pi_i)$ to $(\widetilde{V}_j, H_j, \phi_j)$ with $\widetilde{x}_i \in \operatorname{dom}\lambda$. So we have proved that the chart $(\widetilde{U}_i, G_i, w \circ \pi_i)$ is compatible with $\mathcal{V}$ for every $i \in I$. This suffices to conclude. □

By combining lemmas 6.3 and 6.4 we get:

Proposition 6.5. *Given any 2 reduced orbifold atlases $\mathcal{U}, \mathcal{V}$, the bijection*

$$F_1(\mathcal{U}, \mathcal{V}) : (\textbf{Red Atl})(\mathcal{U}, \mathcal{V}) \longrightarrow (\textbf{PEÉ Gpd})(F_0(\mathcal{U}), F_0(\mathcal{V}))$$

of lemma 4.16 induces a bijection between weak equivalences of reduced orbifold atlases and weak equivalences of proper, effective, étale Lie groupoids.

Lemma 6.6. *Let us fix any proper, effective, étale Lie groupoid $R \overset{s}{\underset{t}{\rightrightarrows}} U$. Then there exists a reduced orbifold atlas $\mathcal{U}$ and a weak equivalence $(\psi, \Psi) : F_0(\mathcal{U}) \to (R \overset{s}{\underset{t}{\rightrightarrows}} U)$.*

Proof. Given $R \overset{s}{\underset{t}{\rightrightarrows}} U$, the orbifold atlas $\mathcal{U}$ is obtained as in the last part of the proof of theorem 4.1 in [MP]. In [T, lemmas 4.7, 4.8 and 4.9] we proved that we can define a weak equivalence as required. The proofs in [T] were done in the category of complex manifolds, but they can be easily adapted to the case of smooth manifolds, so we omit the details. □

Proposition 6.7. *The relation $\sim$ of definition 6.1 is actually an equivalence relation.*

Proof. $\sim$ is clearly symmetric and reflexive, hence we have only to prove transitivity. So let us suppose that we have $\mathcal{U}^1 \sim \mathcal{U}^2 \sim \mathcal{U}^3$. By lemma 6.3, this implies that $F_0(\mathcal{U}^1)$ is weakly equivalent to $F_0(\mathcal{U}^2)$, and the latter is weakly equivalent to $F_0(\mathcal{U}^3)$. We already mentioned the fact that weak equivalence of Lie groupoids is an equivalence relation, so there exists a Lie groupoid $R \overset{s}{\underset{t}{\rightrightarrows}} U$ and two weak equivalences

$$F_0(\mathcal{U}^1) \xleftarrow{(\phi^1,\Phi^1)} \big(R \overset{s}{\underset{t}{\rightrightarrows}} U\big) \xrightarrow{(\phi^3,\Phi^3)} F_0(\mathcal{U}^3).$$

By [MM, exercise 5.22] we can always choose $R \overset{s}{\underset{t}{\rightrightarrows}} U$ as an étale Lie groupoid. Moreover, since $F_0(\mathcal{U}^1)$ is proper and effective, then by [MM, proposition 5.26 and example 5.21(2)] we get that $R \overset{s}{\underset{t}{\rightrightarrows}} U$ is proper and effective. Then by lemma 6.6 there exists a reduced orbifold atlas $\mathcal{U}$ and a weak equivalence $(\psi,\Psi): F_0(\mathcal{U}) \to (R \overset{s}{\underset{t}{\rightrightarrows}} U)$. Since the set $\mathbf{W}_{\mathbf{PE\acute{E}\,Gpd}}$ admits a right calculus of fractions, then compositions of weak equivalences are weak equivalences (see property (U2)). Therefore by composing we get weak equivalences:

$$F_0(\mathcal{U}^1) \xleftarrow{(\phi^1\circ\psi,\Phi^1\circ\Psi)} F_0(\mathcal{U}) \xrightarrow{(\phi^3\circ\psi,\Phi^3\circ\Psi)} F_0(\mathcal{U}^3).$$

Then by lemma 6.4 we get weak equivalences of reduced orbifold atlases

$$\mathcal{U}^1 \xleftarrow{F_1^{-1}(\phi^1\circ\psi,\Phi^1\circ\Psi)} \mathcal{U} \xrightarrow{F_1^{-1}(\phi^3\circ\psi,\Phi^3\circ\Psi)} \mathcal{U}^3.$$

This proves that $\sim$ is transitive. □

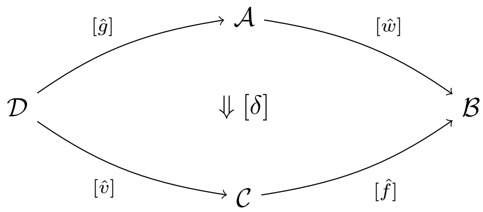
Now let us prove the following fundamental result.

Proposition 6.8. *The set $\mathbf{W}_{\mathbf{Red\,Atl}}$ consisting of all weak equivalences of reduced orbifold atlases admits a right calculus of fractions, so there exists a bicategory* $(\mathbf{Red\,Orb}) := (\mathbf{Red\,Atl})\left[\mathbf{W}^{-1}_{\mathbf{Red\,Alt}}\right]$ *and a pseudofunctor*

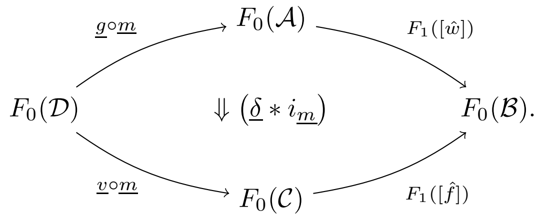
$$H : (\mathbf{Red\,Atl}) \longrightarrow (\mathbf{Red\,Orb})$$

that sends every weak equivalence of reduced orbifold atlases to an equivalence and that is universal with respect to this property.

Proof. Both (U1) and (U2) are easy to verify; let us verify (U3), so let us fix any weak equivalence $[\hat{w}] : \mathcal{A} \to \mathcal{B}$ and any morphism $[\hat{f}] : \mathcal{C} \to \mathcal{B}$ of reduced orbifold atlases. By lemma 6.3 we have that $F_1([\hat{w}])$ is a weak equivalence of groupoids; moreover we have proved in proposition 5.3 that the set $\mathbf{W}_{\mathbf{PE\acute{E}\,Gpd}}$ satisfies (U3). Therefore there exist a proper, effective, étale Lie groupoid $R \overset{s}{\underset{t}{\rightrightarrows}} U$, a weak equivalence $\underline{v} : (R \overset{s}{\underset{t}{\rightrightarrows}} U) \to F_0(\mathcal{C})$, a morphism $\underline{g} : (R \overset{s}{\underset{t}{\rightrightarrows}} U) \to F_0(\mathcal{A})$ and a 2-morphism $\underline{\delta} : F_1([\hat{w}]) \circ \underline{g} \Rightarrow F_1([\hat{f}]) \circ \underline{v}$. By lemma 6.6 there exist a reduced orbifold atlas $\mathcal{D}$ and a weak equivalence of Lie groupoids $\underline{m} : F_0(\mathcal{D}) \to (R \overset{s}{\underset{t}{\rightrightarrows}} U)$. Then $\overline{v} \circ \overline{m}$ is a weak equivalence of Lie groupoids. Lemmas 4.16 and 4.17 prove that there exists unique morphisms $[\hat{v}]$, $[\hat{g}]$ and a unique 2-morphism $[\delta]$ as follows



such that the 2-functor F maps such a diagram to



By lemma 6.4 we have that $[\hat{v}]$ is a weak equivalence of orbifold atlases; by lemma 3.2 we get that $[\delta]$ is an invertible 2-morphism, so (U3) holds.

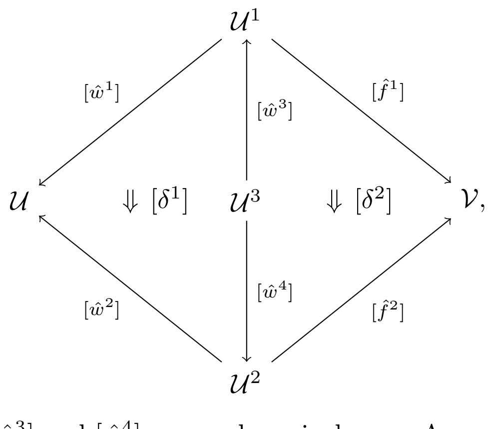
The proof that (U4) holds follows the same ideas described for (U3), so we omit it. Lastly, if $[\delta] : [\hat{v}] \Rightarrow [\hat{w}]$ is a 2-morphism of orbifold atlases, then the underlying continuous maps of $[\hat{v}]$ and $[\hat{w}]$ are the same by definition of 2-morphism. Since being a weak equivalence of reduced orbifold atlases depends only on the underlying continuous map and on the atlases in source and target, then $[\hat{w}]$ is a weak equivalence if and only if $[\hat{v}]$ is so, so (U5) holds for $\mathbf{W}_{\mathbf{Red\,Atl}}$. This suffices to conclude. □

For the explicit description of the bicategories of fractions and of the pseudofunctor H we refer mainly to [Pr]. If we use lemmas 6.5 and 3.2 together with [PS, lemma 8.1], we get the following description of (**Red Orb**), that we are going to use soon.

- The objects of (**Red Orb**) are exactly the objects of (**Red Atl**), i.e. reduced orbifold atlases.
- Given any two atlases $\mathcal{U}, \mathcal{V}$ the 1-morphisms in (**Red Orb**) from the first atlas to the second one consist of all the triples $(\mathcal{U}', [\hat{w}], [\hat{f}])$ where $\mathcal{U}'$ is any reduced orbifold atlas, $[\hat{w}]$ is any weak equivalence and $[\hat{f}]$ is any morphism of orbifold atlases, as follows

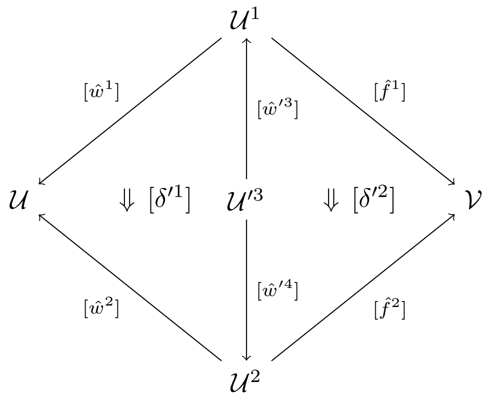
$$\mathcal{U} \xleftarrow{[\hat{w}]} \mathcal{U}' \xrightarrow{[\hat{f}]} \mathcal{V}. \tag{6.6}$$

- Given any pair of objects $\mathcal{U}, \mathcal{V}$ and any pair of morphisms $(\mathcal{U}^i, [\hat{w}^i], [\hat{f}^i]) : \mathcal{U} \to \mathcal{V}$ for $i = 1, 2$, a 2-morphism from $(\mathcal{U}^1, [\hat{w}^1], [\hat{f}^1])$ to $(\mathcal{U}^2, [\hat{w}^2], [\hat{f}^2])$ is an equivalence class of data $(\mathcal{U}^3, [\hat{w}^3], [\hat{w}^4], [\delta^1], [\delta^2])$ in (**Red Atl**) as follows

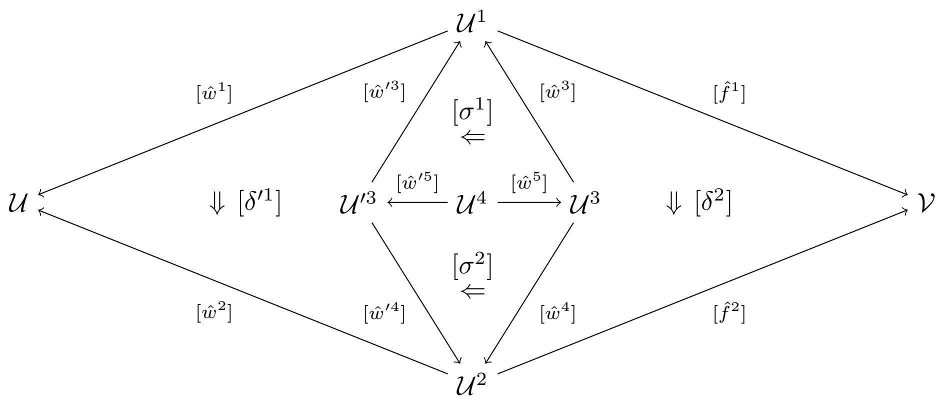


(6.7)

such that both $[\hat{w}^3]$ and $[\hat{w}^4]$ are weak equivalences. Any other set of data



represents the same 2-morphism if and only if there exist a reduced orbifold atlas $\mathcal{U}^4$, weak equivalences $[\hat{w}^5], [\hat{w}'^5]$ and 2-morphisms $[\sigma^1], [\sigma^2]$ as follows



such that

$$\Big(i_{[\hat{w}^2]} * [\sigma^2]\Big) \odot \Big([\delta^1] * i_{[\hat{w}^5]}\Big) \odot \Big(i_{[\hat{w}^1]} * [\sigma^1]\Big) = [\delta'^1] * i_{[\hat{w}'^5]}$$

and

$$\Big(i_{[\hat{f}^2]} * [\sigma^2]\Big) \odot \Big([\delta^2] * i_{[\hat{w}^5]}\Big) \odot \Big(i_{[\hat{f}^1]} * [\sigma^1]\Big) = [\delta'^2] * i_{[\hat{w}'^5]}.$$

We denote by

$$\Big[\mathcal{U}^3,[\hat{w}^3],[\hat{w}^4],[\delta^1],[\delta^2]\Big]:\Big(\mathcal{U}^1,[\hat{w}^1],[\hat{f}^1]\Big)\Longrightarrow\Big(\mathcal{U}^2,[\hat{w}^2],[\hat{f}^2]\Big)$$

the class of any such data.

- For the construction of compositions of morphisms and 2-morphisms we refer to [Pr]. We only recall that the preliminary step for such a construction is the following. For any pair of morphism of orbifold atlases

$$\mathcal{C}\xrightarrow{[\hat{a}]}\mathcal{B}\xleftarrow{[\hat{b}]}\mathcal{A}$$

with $[\hat{b}]$ weak equivalence we choose any reduced orbifold atlas $\mathcal{D}$, any pair of morphisms $[\hat{a}']$, $[\hat{b}']$ with $[\hat{b}']$ weak equivalence and any 2-morphism $[\delta]$ as follows

$$\begin{array}{ccccc} & & \mathcal{D} & & \\ & \swarrow^{[\hat{b}']} & \overset{[\delta]}{\Longrightarrow} & \searrow^{[\hat{a}']} & \\ \mathcal{C} & \xrightarrow[{[\hat{a}]}]{} & \mathcal{B} & \xleftarrow[{[\hat{b}]}]{} & \mathcal{A}. \end{array}$$

Such a choice is always possible by (U3), but it is not unique (neither in general there is a preferred choice). Different choices of "completions" as before will give rise to different but weakly equivalent bicategories. In particular, for any such set of choices, given any pair of morphisms as follows:

$$\mathcal{U}\xleftarrow{[\hat{w}]}\mathcal{U}'\xrightarrow{[\hat{f}]}\mathcal{V}\qquad\qquad\mathcal{V}\xleftarrow{[\hat{v}]}\mathcal{V}'\xrightarrow{[\hat{g}]}\mathcal{W},$$

we use the choice we fixed for the pair $([\hat{f}],[\hat{v}])$ in order to get a new pair of morphisms $([\hat{f}'],[\hat{v}'])$ (defined from a common atlas $\mathcal{U}''$ to $\mathcal{V}'$ and $\mathcal{U}'$ respectively) and we set

$$\Big(\mathcal{V}',[\hat{v}],[\hat{g}]\Big)\circ\Big(\mathcal{U}',[\hat{w}],[\hat{f}]\Big):=\Big(\mathcal{U}'',[\hat{w}]\circ[\hat{v}'],[\hat{g}]\circ[\hat{f}']\Big):\mathcal{U}\longrightarrow\mathcal{W}.$$

- We omit the construction of the composition for 2-morphisms; we only remark that since each 2-morphism is invertible in (**Red Alt**), then it is not difficult to prove that the same property holds in (**REd Orb**).
- The functor H sends each reduced orbifold atlas $\mathcal{U}$ to the same object in (**Red Orb**). For every morphism $[\hat{f}]:\mathcal{U}\to\mathcal{V}$ we have $H([\hat{f}])=(\mathcal{U},\mathrm{id}_{\mathcal{U}},[\hat{f}])$. For every pair of morphisms $[\hat{f}^m]:\mathcal{U}\to\mathcal{V}$ for $m=1,2$ and for every 2-morphism $[\delta]:[\hat{f}^1]\Rightarrow[\hat{f}^2]$ in (**Red Atl**) we have

$$H([\delta])=\Big[\mathcal{U},\mathrm{id}_{\mathcal{U}},\mathrm{id}_{\mathcal{U}},i_{\mathcal{U}},[\delta]\Big].$$

An analogous description holds for the bicategory $(\textbf{PEÉ Gpd})\left[\mathbf{W}^{-1}_{\textbf{PEÉ Gpd}}\right]$ and for the pseudofunctor G.

Theorem 6.9. *There is a weak equivalence of bicategories*

$$L:(\textbf{Red Orb})\longrightarrow(\textbf{PEÉ Gpd})\left[\mathbf{W}^{-1}_{\textbf{PEÉ Gpd}}\right].$$

Proof. Let us consider the functor

$$G\circ F:(\textbf{Red Atl})\longrightarrow(\textbf{PEÉ Gpd})\longrightarrow(\textbf{PEÉ Gpd})\left[\mathbf{W}^{-1}_{\textbf{PEÉ Gpd}}\right].$$

By lemma 6.3 we have that F sends each weak equivalence of reduced orbifold atlases to a weak equivalence of Lie groupoids, therefore $G \circ F$ sends each weak equivalence to an equivalence. Then by the universal property of the pseudofunctor H, there exists a unique pseudofunctor

$$L = (L_0, L_1, L_2) : (\mathbf{Red\,Orb}) \longrightarrow (\mathbf{PE\acute{E}\,Gpd}) \left[\mathbf{W}^{-1}_{\mathbf{PE\acute{E}\,Gpd}}\right]$$

making the following diagram commute

$$\begin{array}{ccc} (\mathbf{Red\,Atl}) & \xrightarrow{\quad F \quad} & (\mathbf{PE\acute{E}\,Gpd}) \\ \Big\downarrow H & \curvearrowright & \Big\downarrow G \\ (\mathbf{Red\,Orb}) & \xrightarrow{\quad L \quad} & (\mathbf{PE\acute{E}\,Gpd}) \left[\mathbf{W}^{-1}_{\mathbf{PE\acute{E}\,Gpd}}\right]. \end{array}$$

In particular, we have that L acts as follows:

- for every reduced orbifold atlas $\mathcal{U}$, $L_0(\mathcal{U}) = F_0(\mathcal{U})$;
- given any 1-morphism $(\mathcal{U}', [\hat{w}], [\hat{f}]) : \mathcal{U} \to \mathcal{V}$ in $(\mathbf{Red\,Orb})$, we have

$$L_1\Big(\mathcal{U}', [\hat{w}], [\hat{f}]\Big) = \Big(F_0(\mathcal{U}'), F_1([\hat{w}]), F_1([\hat{f}])\Big) : F_0(\mathcal{U}) \longrightarrow F_0(\mathcal{V});$$

- for every 2-morphism in $(\mathbf{Red\,Orb})$

$$\Big[\mathcal{U}^3, [\hat{w}^3], [\hat{w}^4], [\delta^1], [\delta^2]\Big] : \Big(\mathcal{U}^1, [\hat{w}^1], [\hat{f}^1]\Big) \Longrightarrow \Big(\mathcal{U}^2, [\hat{w}^2], [\hat{f}^2]\Big)$$

  we have

$$L_2\Big(\Big[\mathcal{U}^3, [\hat{w}^3], [\hat{w}^4], [\delta^1], [\delta^2]\Big]\Big) = \Big[F_0(\mathcal{U}^3), F^1([\hat{w}^3]), F^1([\hat{w}^4]), F^2([\delta^1]), F^2([\delta^2])\Big]$$

  (this is well-defined since F is a 2-functor).

In order to conclude we need to prove that L is a weak equivalence of bicategories. We recall from [St, (1.33)] that a pseudofunctor $L : \mathscr{C} \to \mathscr{D}$ is a *weak equivalence of bicategories* (also known as *biequivalence*) iff the following conditions hold:

(W1) for each object d in $\mathscr{D}$ there exists an object c in $\mathscr{C}$ and an equivalence from $L(c)$ to d in $\mathscr{D}$;

(W2) for each pair of objects c, c' in $\mathscr{C}$, the functor $L(c, c')$ is an equivalence of categories from $\mathscr{C}(c, c')$ to $\mathscr{D}(L(c), L(c'))$.

Let us prove (W1), so let us fix any object of $(\mathbf{PE\acute{E}\,Gpd}) \left[\mathbf{W}^{-1}_{\mathbf{PE\acute{E}\,Gpd}}\right]$; i.e. any proper, effective, étale Lie groupoid $R \overset{s}{\underset{t}{\rightrightarrows}} U$. By lemma 6.6 we get that there exist a reduced orbifolds atlas $\mathcal{U}$ and a weak equivalence of Lie groupoids $(\psi, \Psi) : F_0(\mathcal{U}) \to (R \overset{s}{\underset{t}{\rightrightarrows}} U)$. Then the data

$$F_0(\mathcal{U}) \xleftarrow{\ \mathrm{id}_{F_0(\mathcal{U})}\ } F_0(\mathcal{U}) \xrightarrow{\ (\psi, \Psi)\ } (R \overset{s}{\underset{t}{\rightrightarrows}} U) \tag{6.8}$$

is a morphism from $F_0(\mathcal{U})$ to $R \overset{s}{\underset{t}{\rightrightarrows}} U$ in $(\mathbf{PE\acute{E}\,Gpd}) \left[\mathbf{W}^{-1}_{\mathbf{PE\acute{E}\,Gpd}}\right]$; moreover it is the image via G of the weak equivalence (ψ, Ψ). By proposition 5.3 G sends each weak equivalence to an equivalence. Therefore (6.8) is an equivalence from $F_0(\mathcal{U})$ to $R \overset{s}{\underset{t}{\rightrightarrows}} U$, so (W1) is verified.

Now let us prove (W2); this amounts to proving that given any pair of reduced orbifold atlases $\mathcal{U}, \mathcal{V}$ we have:

(i) for any morphism

$$F_0(\mathcal{U}) \xleftarrow{(\psi,\Psi)} (R \overset{s}{\underset{t}{\rightrightarrows}} U) \xrightarrow{(\phi,\Phi)} F_0(\mathcal{V}) \tag{6.9}$$

in $(\mathbf{PE\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{PE\acute{E}\,Gpd}}\right]$ there is a morphism $(\mathcal{U}', [\hat{w}], [\hat{f}])$ as in (6.6) and there is an invertible 2-morphism from $L_1(\mathcal{U}', [\hat{w}], [\hat{f}])$ to (6.9);

(ii) for every pair of morphisms $(\mathcal{U}^m, [\hat{w}^m], [\hat{f}^m]) : \mathcal{U} \to \mathcal{V}$ for $m = 1, 2$, L_2 induces a bijection between the sets

$$\Big\{\text{2-morphisms from } \Big(\mathcal{U}^1, [\hat{w}^1], [\hat{f}^1]\Big) \text{ to } \Big(\mathcal{U}^2, [\hat{w}^2], [\hat{f}^2]\Big)\Big\} \tag{6.10}$$

and

$$\Big\{\text{2-morphisms from } \Big(F_0(\mathcal{U}^1), F_1([\hat{w}^1]), F_1([\hat{f}^1])\Big) \text{ to } \Big(F_0(\mathcal{U}^2), F_1([\hat{w}^2]), F_1([\hat{f}^2])\Big)\Big\}. \tag{6.11}$$

For any data as in (i), by lemma 6.6 there is an atlas $\mathcal{U}'$ and a weak equivalence $(\theta, \Theta) : F_0(\mathcal{U}') \to (R \overset{s}{\underset{t}{\rightrightarrows}} U)$. Then we have an invertible 2-morphism in $(\mathbf{PE\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{PE\acute{E}\,Gpd}}\right]$:

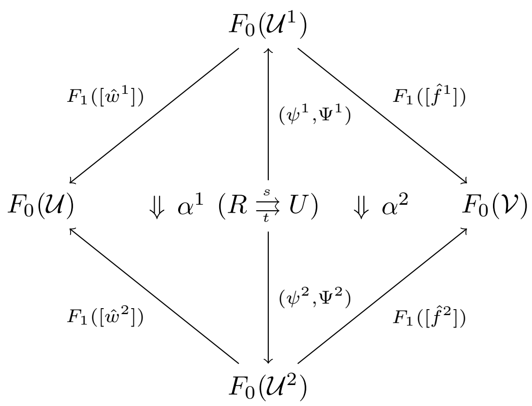
$$\Big[F_0(\mathcal{U}'), (\theta, \Theta), \mathrm{id}_{F_0(\mathcal{U}')}, i_{(\psi\circ\theta,\Psi\circ\Theta)}, i_{(\phi\circ\theta,\Phi\circ\Theta)}\Big] :$$

$$\Big(F_0(\mathcal{U}'), (\psi \circ \theta, \Psi \circ \Theta), (\phi \circ \theta, \Phi \circ \Theta)\Big) \Longrightarrow \Big((R \overset{s}{\underset{t}{\rightrightarrows}} U), (\psi, \Psi), (\phi, \Phi)\Big).$$

By using lemma 4.16 and proposition 6.5 we conclude that there exists a unique pair of morphisms of reduced orbifold atlases as in (6.6) such that

$$\Big(F_0(\mathcal{U}'), (\psi \circ \theta, \Psi \circ \Theta), (\phi \circ \theta, \Phi \circ \Theta)\Big) = L_1\Big(\mathcal{U}', [\hat{w}], [\hat{f}]\Big).$$

This suffices to prove (i). Now let us fix any set of data as in (ii) and let us fix any 2-morphism in (6.11) with representative:



with $R \overset{s}{\underset{t}{\rightrightarrows}} U$ proper, effective, étale Lie groupoid and (ψ^m, Ψ^m) weak equivalences of Lie groupoids for $m = 1, 2$. By lemma 6.6 there exists an orbifold atlas $\mathcal{U}^3$ and a weak equivalence $(\phi, \Phi) : F_0(\mathcal{U}^3) \to (R \overset{s}{\underset{t}{\rightrightarrows}} U)$. Then it is easy to see that

$$\Big[(R \overset{s}{\underset{t}{\rightrightarrows}} U), (\psi^1, \Psi^1), (\psi^2, \Psi^2), \alpha^1, \alpha^2\Big] =$$
$$= \Big[F_0(\mathcal{U}^3), (\psi^1 \circ \phi, \Psi^1 \circ \Phi), (\psi^2 \circ \phi, \Psi^2 \circ \Phi), \alpha^1 * i_{(\phi,\Phi)}, \alpha^2 * i_{(\phi,\Phi)}\Big]. \quad (6.12)$$

Now by using lemmas 4.16 and 4.17 and proposition 6.5 we conclude that (6.12) is the image of a 2-morphism in (6.10). The proof that it is the image of a unique 2-morphism is straightforward. □

7. A weak equivalence between (**Red Orb**) and the 2-category of effective orbifold stacks

As we mentioned in the introduction, a very convenient way to define a 2-category of orbifolds is by exhibiting it as a full 2-subcategory of the 2-category of C^∞-stacks (these are called "differentiable stacks" in several papers, see for example [Pr]). For the definition of the Grothendieck topology used for this definition of stack, we refer to [J2, definition 8.1]. A C^∞-stack is called an *orbifold* (see [J2, definition 9.25]) if it is equivalent to the stack $[R \overset{s}{\underset{t}{\rightrightarrows}} U]$ associated to a proper, étale Lie groupoid $R \overset{s}{\underset{t}{\rightrightarrows}} U$. In particular (see again [J2, definition 9.25]) every orbifold is a separated, locally finitely presented Deligne-Mumford C^∞-stack. An orbifold $\mathcal{X}$ is called *effective* (see [J1, definition 9.4]) if for every point $[x] \in \mathcal{X}_{\mathrm{top}}$ there exists a linear effective action of $G := \mathrm{Iso}_{\mathcal{X}}([x])$ on some $\mathbb{R}^n$, a G-invariant open neighbourhood of 0 in $\mathbb{R}^n$ and a 1-morphism $i : [U/G] \to \mathcal{X}$ which is an equivalence with an open neighbourhood of $[x]$ in $\mathcal{X}$ with $i_{\mathrm{top}}(0) = [x]$ (if $\mathcal{X}$ is not effective, we are in the same setup but the action of each G is not required to be effective). According to [J2] we write (**Orb**) and ($\mathbf{Orb^{eff}}$) for the full 2-subcategories of orbifolds, respectively of effective orbifolds, in the 2-category of C^∞-stacks (or, equivalently, in the 2-category of Deligne-Mumford C^∞-stacks). We recall that by [Pr, Corollary 43] there is a weak equivalence of bicategories

$$\overline{M} : (\mathbf{\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{\acute{E}\,Gpd}}\right] \longrightarrow (\mathbf{C^\infty - Stacks})$$

and that by [J2, theorem 9.26] there is a weak equivalence of bicategories induced by $\overline{M}$:

$$\widetilde{M} : (\mathbf{P\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{P\acute{E}\,Gpd}}\right] \longrightarrow (\mathbf{Orb}).$$

Therefore we get easily that there is also is a weak equivalence of bicategories induced by $\widetilde{M}$:

$$M : (\mathbf{PE\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{PE\acute{E}\,Gpd}}\right] \longrightarrow (\mathbf{Orb^{eff}}).$$

By considering the composition:

$$(\mathbf{Red\,Atl})\left[\mathbf{W}^{-1}_{\mathbf{Red\,Alt}}\right] \xrightarrow[\text{weak equiv.}]{L} (\mathbf{PE\acute{E}\,Gpd})\left[\mathbf{W}^{-1}_{\mathbf{PE\acute{E}\,Gpd}}\right] \xrightarrow[\text{weak equiv.}]{M} (\mathbf{Orb^{eff}})$$

we conclude:

Theorem 7.1. *There is a weak equivalence between the bicategory* (**Red Orb**) *and the* 2*-category* ($\mathbf{Orb^{eff}}$) *of effective orbifolds described as a full* 2*-subcategory of* C^∞*-Deligne-Mumford stacks.*

If we use the axiom of choice, then each weak equivalence of bicategories is an equivalence of bicategories (see [PW, §1]) so under that assumption the weak equivalences of theorems 6.9 and 7.1 are equivalences of bicategories.

8. The homotopy category of reduced orbifolds

Let us fix any pair of reduced orbifold atlases $\mathcal{U}$,$\mathcal{U}'$ that are defined on the same topological space; then it is easy to see that there is an equivalence from $\mathcal{U}$ to $\mathcal{U}'$ in (**Red Orb**) if and only if $\mathcal{U}$ and $\mathcal{U}'$ are equivalent as atlases. Anyway, this description seems somewhat unsatisfactory mainly because one would expect that objects of a bicategory of (reduced) orbifolds should be simply *equivalence classes* of orbifold atlases (on a fixed topological space). With the current description it is not possible to overcome this problem, exactly as it is not possible to overcome a similar problem in the language of Lie groupoids (conversely, in the language of stacks such a problem does not occur).

Anyway, at least we are able to give a satisfactory description for the homotopy category of (**Red Orb**). To be more precise, in this section we will describe a category (**Ho**) with objects given by orbifold structures (i.e. equivalence classes of orbifold atlases) and we will prove that (**Ho**) is equivalent to the homotopy category of (**Red Orb**). So (**Ho**) should be considered as the correct 1-category of reduced orbifolds in the language of differential geometry.

In order to construct (**Ho**), we first prove the following lemmas.

Lemma 8.1. *Let us fix any pair of reduced orbifold atlases $\mathcal{U},\mathcal{V}$ and any pair of weak equivalences $[\hat{w}^m] : \mathcal{U} \to \mathcal{V}$ for $m = 1, 2$. Then the following facts are equivalent:*

(a) the underlying topological maps w^1 and w^2 coincide;
(b) there exists a 2-morphism $[\delta] : [\hat{w}^1] \Rightarrow [\hat{w}^2]$.

Proof. Clearly (b) implies (a) by definition of 2-morphism in (**Red Alt**), so we need only to prove that (a) implies (b). Let

$$\mathcal{U} := \{(\widetilde{U}_i, G_i, \pi_i)\}_{i\in I}, \quad \mathcal{V} := \{(\widetilde{V}_j, H_j, \phi_j)\}_{j\in J}$$

and let

$$\hat{w}^m := (w^m, \overline{w}^m, \{\widetilde{w}^m_i\}_{i\in I}, P^m_w, \nu^m_w) \quad \text{for } m = 1, 2$$

be representatives for $[\hat{w}^1]$ and $[\hat{w}^2]$ respectively. In the proof of lemma 6.3 we have already shown that all the local lifts appearing in a weak equivalence of orbifold atlases are local diffeomorphisms. Therefore for each $i \in I$ there exists an open covering $\{\widetilde{U}^a_i\}_{a\in A(i)}$ of $\widetilde{U}_i$ such that both $\widetilde{w}^1_i$ and $\widetilde{w}^2_i$ are diffeomorphisms if restricted to any $\widetilde{U}^a_i$. Since both $[\hat{w}^1]$ and $[\hat{w}^1]$ are morphisms, for each $i \in I$ we have that

$$\phi_{\overline{w}^1(i)} \circ \widetilde{w}^1_i = w^1 \circ \pi_i, \quad \phi_{\overline{w}^2(i)} \circ \widetilde{w}^2_i = w^2 \circ \pi_i = w^1 \circ \pi_i. \tag{8.1}$$

Then for each $i \in I$ and for each $a \in A(i)$ we set:

$$\delta^a_i := \widetilde{w}^2_i \circ \left(\widetilde{w}^1_i|_{\widetilde{U}^a_i}\right)^{-1}$$

By (8.1) we get that each δ^a_i is a change of charts in $\mathcal{C}h(\mathcal{V}, \overline{w}^1(i), \overline{w}^2(i))$. Then it is easy to see that $\delta := \{(\widetilde{U}^a_i, \delta^a_i)\}_{i\in I, a\in A(i)}$ is a representative of a 2-morphism from $[\hat{w}^1]$ to $[\hat{w}^2]$. □

Lemma 8.2. *Let us fix any pair of reduced orbifold atlases $\mathcal{U}^1$,$\mathcal{U}^2$ on the same topological space X, then $\mathcal{U}^1$ is equivalent to $\mathcal{U}^2$ (see definition 1.7) if and only if there exists a pair of* unit *weak equivalences*

$$\mathcal{U}^1 \xleftarrow{[\hat{w}^1]} \mathcal{U} \xrightarrow{[\hat{w}^2]} \mathcal{U}^2 \tag{8.2}$$

i.e. weak equivalences over the identity of X.

Proof. Let us suppose that $\mathcal{U}^1$ and $\mathcal{U}^2$ are equivalent and let us denote by $\mathcal{U}^{\max}$ the maximal atlas associated to both. We recall that by definition 1.13 there exist natural morphisms over id_X:

$$\mathcal{U}^1 \xhookrightarrow{\iota_{\mathcal{U}^1}} \mathcal{U}^{\max} \xhookleftarrow{\iota_{\mathcal{U}^2}} \mathcal{U}^2.$$

Then by condition (U3) together with [PS, lemma 8.1] we get that there exists an atlas $\mathcal{U}'$, a pair of weak equivalences $[\hat{v}^1]$, $[\hat{v}^2]$ and a 2-morphism $[\delta]$ in (**Red Atl**) as follows:

$$\begin{array}{ccccc} & & \mathcal{U}^{\max} & & \\ & \overset{\iota_{\mathcal{U}^1}}{\nearrow} & & \overset{\iota_{\mathcal{U}^2}}{\nwarrow} & \\ \mathcal{U}^1 & & \overset{[\delta]}{\Longrightarrow} & & \mathcal{U}^2. \\ & \underset{[\hat{v}^2]}{\nwarrow} & & \underset{[\hat{v}^1]}{\nearrow} & \\ & & \mathcal{U}' & & \end{array}$$

Let us denote by X' the topological space where $\mathcal{U}'$ is defined and let

$$\mathcal{U}' := \{(\widetilde{U}_i, G_i, \pi_i)\}_{i\in I}.$$

Since both $\iota_{\mathcal{U}^1}$ and $\iota_{\mathcal{U}^2}$ are defined over the identity of X, then lemma 8.1 implies that $v^1 = v^2 : X' \to X$. Then we can consider the atlas $v^1_*(\mathcal{U}')$ defined on X as

$$v^1_*(\mathcal{U}') := \{(\widetilde{U}_i, G_i, v^1 \circ \pi_i)\}_{i\in I}$$

and the obvious invertible morphism (therefore, equivalence hence weak equivalence)

$$[\hat{\varphi}] : v^*_1(\mathcal{U}') \xrightarrow{\sim} \mathcal{U}' \tag{8.3}$$

over the continuous map $(v^1)^{-1} = (v^2)^{-1}$. Then the morphisms $[\hat{w}^m] := [\hat{v}^m] \circ [\hat{v}]$ for $m = 1, 2$ are the required unit weak equivalences. The converse implication is obvious by definition of unit weak equivalence. □

Remark 8.3. It is obvious that for every homeomorphism $v^1 : X' \to X$ and for every reduced orbifold atlas $\mathcal{U}'$ on X' as before, we have $F_0(\mathcal{U}') = F_0(v^1_*(\mathcal{U}'))$ (see construction 4.8). Therefore F_0 is not injective; actually it is easy to prove that this is the only point where F_0 fails to be injective.

Definition 8.4. Given any pair of morphisms of reduced orbifold atlases $[\hat{f}^m] : \mathcal{U}^m \to \mathcal{V}^m$ for $m = 1, 2$, we say that $[\hat{f}^1]$ is *equivalent* to $[\hat{f}^2]$ if and only if the following conditions hold:

(a) $\mathcal{V}^1$ and $\mathcal{V}^2$ are equivalent orbifold atlases (see definition 1.7); we denote by $\mathcal{V}^{\max}$ the maximal atlas associated to both and by $\iota_{\mathcal{V}^m}$ the "inclusion" of $\mathcal{V}^m$ in $\mathcal{V}^{\max}$ for $m = 1, 2$ (see definition 1.13);

(b) there exists an atlas $\mathcal{U}$, a pair of *unit* weak equivalences $[\hat{w}^1], [\hat{w}^2]$ and a 2-morphism $[\delta]$ as in the following diagram:

$$\begin{array}{ccccccc} & & \mathcal{U}^1 & \xrightarrow{\quad [\hat{f}^1] \quad} & \mathcal{V}^1 & & \\ & {\scriptstyle [\hat{w}^1]} \nearrow & & & & \searrow {\scriptstyle \iota_{\mathcal{V}^1}} & \\ \mathcal{U} & & & \Downarrow [\delta] & & & \mathcal{V}^{\max}. \\ & {\scriptstyle [\hat{w}^2]} \searrow & & & & \nearrow {\scriptstyle \iota_{\mathcal{V}^2}} & \\ & & \mathcal{U}^2 & \xrightarrow[\quad [\hat{f}^2] \quad]{} & \mathcal{V}^2 & & \end{array}$$

In particular, by lemma 8.2 we have $[\mathcal{U}^1] = [\mathcal{U}^2]$; moreover $[\mathcal{V}^1] = [\mathcal{V}^2]$. The previous relation is clearly symmetric and reflexive (by lemma 3.2). Also transitivity is straightforward by using (U2) and (U3), [PS, lemma 8.1] and the construction performed in the proof of lemma 8.2. We denote by $[[\hat{f}^1]] = [[\hat{f}^2]]$ the class of any morphism as before and we say that it is a *morphism from* $[\mathcal{U}^1] = [\mathcal{U}^2]$ *to* $[\mathcal{V}^1] = [\mathcal{V}^2]$.

Remark 8.5. Definition 8.4 is equivalent to the same definition with the additional request that each local lift in $[\hat{w}^1]$ and in $[\hat{w}^2]$ is an embedding. We are not going anyway to use this alternative description.

Now we define a 1-category (**Ho**) as follows. Its objects are classes $[\mathcal{U}]$ of equivalent reduced orbifold atlases on the same topological space, i.e. orbifold structures. By the lemma 8.2, $[\mathcal{U}^1] = [\mathcal{U}^2]$ if and only if there is a pair of unit weak equivalences as in (8.2). Given any pair of orbifold structures $[\mathcal{U}]$, $[\mathcal{V}]$, the morphisms from $[\mathcal{U}]$ to $[\mathcal{V}]$ are all the morphisms given by definition 8.4. In order to have a category, we need also to define a composition, so let us suppose that we have fixed any pair of morphisms

$$[[\hat{f}]] : [\mathcal{U}] \longrightarrow [\mathcal{V}], \quad [[\hat{g}]] : [\mathcal{V}] \longrightarrow [\mathcal{W}].$$

Then we choose any pair of representatives for them

$$[\hat{f}] : \mathcal{U} \longrightarrow \mathcal{V}, \quad [\hat{g}] : \overline{\mathcal{V}} \longrightarrow \mathcal{W} \tag{8.4}$$

and we use (U3) in order to get a reduced orbifold atlas $\overline{\mathcal{U}}$, a pair of morphisms $[\overline{f}]$, $[\hat{w}]$ and a 2-morphism $[\delta]$ in (**Red Atl**) as follows

$$\begin{array}{ccccc} \mathcal{U} & \xrightarrow{[\hat{f}]} & \mathcal{V} & \underset{\iota_{\mathcal{V}}}{\hookrightarrow} & \mathcal{V}^{\max} \\ {\scriptstyle [\hat{w}]} \uparrow & & \searrow [\delta] & & \uparrow {\scriptstyle \iota_{\overline{\mathcal{V}}}} \\ \overline{\mathcal{U}} & & \xrightarrow{\quad [\overline{f}] \quad} & & \overline{\mathcal{V}} \end{array} \tag{8.5}$$

and such that $[\hat{w}]$ is a *unit* weak equivalence. Note that a priori (U3) only gives $[\hat{w}]$ as a weak equivalence; if $[\hat{w}]$ is not a unit weak equivalence, it suffices to compose horizontally $[\delta]$ with the 2-identity over the isomorphism

$$w_*(\overline{\mathcal{U}}) \xrightarrow{\sim} \overline{\mathcal{U}}$$

over w^{-1} (see (8.3)) (we remark that we proved that (U3) holds by using the fact that (U3) holds for Lie groupoids, so we didn't describe how to get a diagram as (8.5) in the general case; however, since $\iota_{\overline{\mathcal{V}}}$ is an "inclusion", then the construction of a diagram (8.5) can be done easily directly). Then we define:

$$[[\hat{g}]] \circ [[\hat{f}]] := \Big[[\hat{g}] \circ [\overline{f}]\Big] : [\mathcal{U}] \longrightarrow [\mathcal{W}]. \tag{8.6}$$

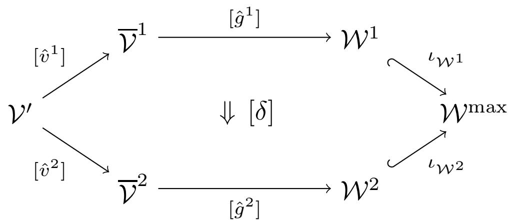
A priori such a definition depends on the choices of representatives (8.4) and on the choice of a diagram as (8.5) (since (U3) does not ensure uniqueness of such a diagram), but actually this is not the case:

Lemma 8.6. *Different choices as before give rise to the same composition in* (**Ho**).

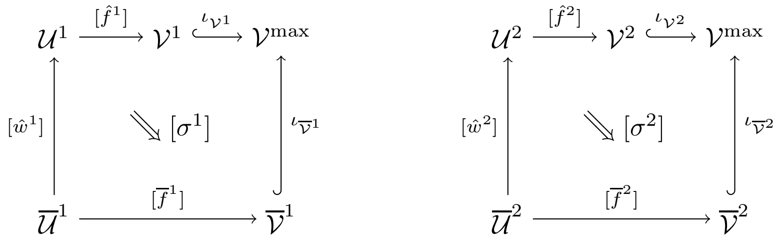
Proof. Let us fix pairs of representatives

$$[\hat{f}^m] : \mathcal{U}^m \longrightarrow \mathcal{V}^m, \quad [\hat{g}^m] : \overline{\mathcal{V}}^m \longrightarrow \mathcal{W}^m \quad \text{for } m = 1, 2$$

for $[[\hat{f}]]$ and $[[\hat{g}]]$ respectively. In particular, we have a diagram as follows:

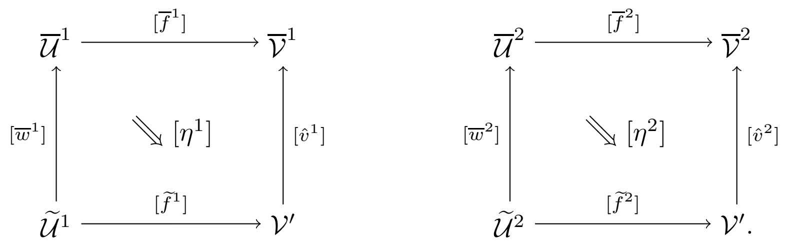


with $[\hat{v}^1]$ and $[\hat{v}^2]$ unit weak equivalences. Now we proceed as before in order to get (non-unique) diagrams as follows



(8.7)

with $[\hat{w}^1]$ and $[\hat{w}^2]$ unit weak equivalences. Now to prove that (8.6) is well-defined is equivalent to proving that the morphisms $[\hat{g}^1] \circ [\overline{f}^1]$ and $[\hat{g}^2] \circ [\overline{f}^2]$ are in the same equivalence class. In order to prove that, we proceed as follows. First of all, we apply (U3) separately to the pair $([\hat{v}^1], [\overline{f}^1])$ and to the pair $([\hat{v}^2], [\overline{f}^2])$, so we get diagrams as follows:



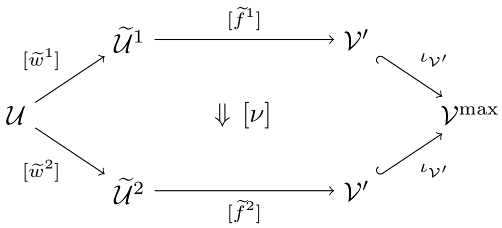
As before, we choose $[\overline{w}^1]$ and $[\overline{w}^2]$ as unit weak equivalences. If we apply lemma 8.1 to the pair of unit weak equivalences $\iota_{\overline{\mathcal{V}}^1} \circ [\hat{v}^1]$ and $\iota_{\mathcal{V}'}$, both defined $\mathcal{V}'$ to $\mathcal{V}^{\max}$, we get that there exists $[\gamma^1] : \iota_{\overline{\mathcal{V}}^1} \circ [\hat{v}^1] \Rightarrow \iota_{\mathcal{V}'}$. Then the existence of the 2-morphism

$$\Big(i_{\iota_{\overline{\mathcal{V}}^1}} * [\eta^1]\Big) \odot \Big([\gamma^1] * i_{[\widetilde{f}^1]}\Big) : \iota_{\overline{\mathcal{V}}^1} \circ [\overline{f}^1] \circ [\overline{w}^1] \Longrightarrow \iota_{\mathcal{V}} \circ [\widetilde{f}^1]$$

proves that $[[\widetilde{f}^1]] = [[\overline{f}^1]]$; analogously we can prove that $[[\widetilde{f}^2]] = [[\overline{f}^2]]$. Then by looking at (8.7) we get that

$$[[\widetilde{f}^1]] = [[\overline{f}^1]] = [[\hat{f}^1]] = [[\hat{f}^2]] = [[\overline{f}^2]] = [[\widetilde{f}^2]].$$

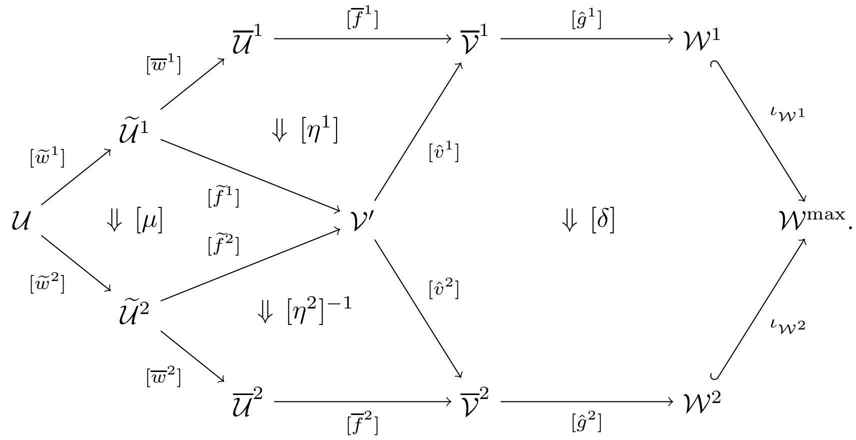
Therefore, there exists a diagram as follows



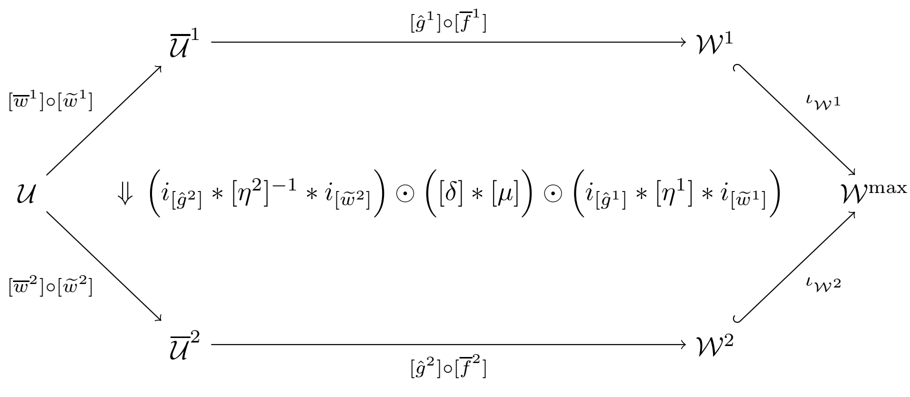
with both $[\widetilde{w}^1]$ and $[\widetilde{w}^1]$ unit weak equivalences. Then by using the notion of 2-morphism in (**Red Alt**), we have that there exists a 2-morphism

$$[\mu] : [\widetilde{f}^1] \circ [\widetilde{w}^1] \Rightarrow [\widetilde{f}^2] \circ [\widetilde{w}^2]$$

such that $[\nu] = i_{\iota_{\mathcal{V}'}} * [\mu]$. Let us consider the following diagram



Then the following diagram



proves the claim. □

The proof that the composition defined in (8.6) is associative is straightforward; the identity in (**Ho**) over any object $[\mathcal{U}]$ is simply given by $[\mathrm{id}_{\mathcal{U}}]$. Then we have proved that (**Ho**) is a category.

Remark 8.7. A straightforward proof shows that the 1-category of C^∞-manifolds is a full subcategory of (**Ho**), i.e. for every pair of manifolds M and N we have a canonical bijection between the set of smooth maps from M to N and the set of morphisms in (**Ho**) from M to N, seen as reduced orbifold structures.

Theorem 8.8. *The category* (**Ho**) *is equivalent to the homotopy category* **Ho**(**Red Orb**) *of* (**Red Orb**).

Proof. We recall that given any 2-category or bicategory $\mathscr{C}$, its homotopy category $\mathbf{Ho}(\mathscr{C})$ is the 1-category whose objects are the same of $\mathscr{C}$ and whose morphisms are classes of equivalence of morphisms of $\mathscr{C}$ connected by an invertible 2-morphism. Therefore, the homotopy category of **Ho**(**Red Orb**) is described as follows:

- its objects are the same objects of (**Red Orb**), i.e. reduced orbifold atlases;
- given any pair of atlases $\mathcal{U}, \mathcal{V}$, a morphism from $\mathcal{U}$ to $\mathcal{V}$ is any class of equivalence of $[\mathcal{U}', [\hat{w}], [\hat{f}]]$ with $\mathcal{U}'$ reduced orbifold atlas, $[\hat{w}] : \mathcal{U}' \to \mathcal{U}$ weak equivalence and $[\hat{f}] : \mathcal{U}' \to \mathcal{V}$ morphism of reduced orbifold atlases. Any two triples $(\mathcal{U}^m, [\hat{w}^m], [\hat{f}^m])$ for $m = 1, 2$ are in the same class of equivalence if and only if there exists data as in diagram (6.7).

Then we define a functor $N : \mathbf{Ho}(\mathbf{Red\,Orb}) \to \mathbf{Ho}$ as follows: for any reduced orbifold atlas $\mathcal{U}$ we set $N(\mathcal{U}) := [\mathcal{U}]$; for any morphism

$$\Big[\mathcal{U}', [\hat{w}], [\hat{f}]\Big] : \mathcal{U} \longrightarrow \mathcal{V}$$

in **Ho**(**Red Orb**) we set

$$N\Big(\Big[\mathcal{U}', [\hat{w}], [\hat{f}]\Big]\Big) := \Big[[\hat{f}] \circ [\hat{\varphi}]\Big] : [\mathcal{U}] \longrightarrow [\mathcal{V}] \tag{8.8}$$

where

$$[\hat{\varphi}] : w_*(\mathcal{U}') \stackrel{\sim}{\longrightarrow} \mathcal{U}'$$

is the isomorphism of orbifold atlases (over w^{-1}) induced by w (see (8.3)). Note that since $[\hat{w}] : \mathcal{U}' \to \mathcal{U}$ is a weak equivalence, then $[\mathcal{U}] = [W_*(\mathcal{U}')]$. We have to prove that N is well-defined on morphisms. So let us suppose that

$$\Big[\mathcal{U}^1, [\hat{w}^1], [\hat{f}^1]\Big] = \Big[\mathcal{U}^2, [\hat{w}^2], [\hat{f}^2]\Big] : \mathcal{U} \longrightarrow \mathcal{V}$$

in **Ho**(**Red Orb**). This implies that there exists data $(\mathcal{U}^3, [\hat{w}^3], [\hat{w}^4], [\delta^1], [\delta^2])$ as in diagram (6.7). By lemma 8.1 we have that necessarily $w_1 \circ w_3 = w_2 \circ w_4$; since all these 4 morphisms are homeomorphisms, we get

$$(w^3)^{-1} \circ (w^1)^{-1} = (w^4)^{-1} \circ (w^2)^{-1}. \tag{8.9}$$

Now we denote by

$$[\hat{\varphi}^m] : w^m_*(\mathcal{U}^m) \stackrel{\sim}{\longrightarrow} \mathcal{U}^m \quad \text{for } m = 1, 2$$

the isomorphisms (over $(w^m)^{-1}$) induced by w^1 and w^2 respectively. Now we use (U3) and [PS, lemma 8.1] separately on the pairs $([\hat{\varphi}^1], [\hat{w}^3])$ and $([\hat{\varphi}^2], [\hat{w}^4])$ and we get (non-unique) diagrams as follows:

$$
\begin{array}{ccc}
w^1_*(\mathcal{U}^1) & \xrightarrow{[\hat{\varphi}^1]} & \mathcal{U}^1 \\
\uparrow{\scriptstyle [\hat{w}^6]} & \Searrow [\sigma^1] & \uparrow{\scriptstyle [\hat{w}^3]} \\
\mathcal{U}^4 & \xrightarrow{[\hat{w}^5]} & \mathcal{U}^3 \\
\mathcal{U}^5 & \xrightarrow{[\hat{w}^7]} & \mathcal{U}^3 \\
\downarrow{\scriptstyle [\hat{w}^8]} & \Searrow [\sigma^2] & \downarrow{\scriptstyle [\hat{w}^4]} \\
w^2_*(\mathcal{U}^2) & \xrightarrow[{[\hat{\varphi}^2]}]{} & \mathcal{U}^2
\end{array}
\tag{8.10}
$$

with $[\hat{w}^m]$ weak equivalences for $m = 5, \cdots, 8$. By using (U3) for the pair $([\hat{w}^5], [\hat{w}^7])$ together with [PS, lemma 8.1] we get that there exists a diagram as follows

$$
\begin{array}{ccccc}
 & & \mathcal{U}^4 & & \\
 & \nearrow{\scriptstyle [\hat{w}^9]} & & \searrow{\scriptstyle [\hat{w}^5]} & \\
\mathcal{U}^6 & & \Downarrow [\sigma^3] & & \mathcal{U}^3 \\
 & \searrow{\scriptstyle [\hat{w}^{10}]} & & \nearrow{\scriptstyle [\hat{w}^7]} & \\
 & & \mathcal{U}^5 & &
\end{array}
\tag{8.11}
$$

with $[\hat{w}^9]$ and $[\hat{w}^{10}]$ weak equivalences. Now using lemma 8.1 together with (8.10) and (8.11) we get that

$$(w^3)^{-1} \circ (w^1)^{-1} \circ w^6 \circ w^9 = w^5 \circ w^9 = w^7 \circ w^{10} = (w^4)^{-1} \circ (w^2)^{-1} \circ w^8 \circ w^{10}.$$

Using (8.9) we conclude that $w^6 \circ w^9 = w^8 \circ w^{10}$. We denote by

$$[\hat{\varphi}] : \mathcal{U}^7 := (w^6 \circ w^9)_*(\mathcal{U}^6) \xrightarrow{\sim} \mathcal{U}^6$$

the isomorphism of orbifold atlases over $(w^6 \circ w^9)^{-1}$ induced by $w^6 \circ w^9$. Then the following diagram proves that (8.8) is well-defined:

$$
\begin{array}{ccccc}
 & w^1_*(\mathcal{U}^1) & \xrightarrow{[\hat{f}^1] \circ [\hat{\varphi}^1]} & \mathcal{V} & \\
 \nearrow{\scriptstyle [\hat{w}^6] \circ [\hat{w}^9] \circ [\hat{\varphi}]} & & & & \searrow{\scriptstyle \iota_{\mathcal{V}}} \\
\mathcal{U}^7 & \multicolumn{3}{c}{\Downarrow\, i_{\iota_{\mathcal{V}}} * \Big(\Big(i_{[\hat{f}^2]} * [\sigma^2] * i_{[\hat{w}^{10}]} \Big) \odot \Big([\delta^2] * [\sigma^3] \Big) \odot \Big(i_{[\hat{f}^1]} * [\sigma^1] * i_{[\hat{w}^9]} \Big) \Big) * i_{[\hat{\varphi}]}} & \mathcal{V}^{\max}. \\
 \searrow{\scriptstyle [\hat{w}^8] \circ [\hat{w}^{10}] \circ [\hat{\varphi}]} & & & & \nearrow{\scriptstyle \iota_{\mathcal{V}}} \\
 & w^2_*(\mathcal{U}^2) & \xrightarrow{[\hat{f}^2] \circ [\hat{\varphi}^2]} & \mathcal{V} &
\end{array}
$$

The proof that N preserves identities and compositions is straightforward. Moreover it is obvious that N is surjective on objects, so in order to conclude that N is an equivalence of categories, it suffices to prove that N is fully faithful. Firstly, we show that N is full. Let us fix any pair $\mathcal{U}, \mathcal{V}$ of reduced orbifold atlases and any morphism from $[\mathcal{U}]$ to $[\mathcal{V}]$ in (**Ho**); let us choose any representative $[\overline{f}] : \overline{\mathcal{U}} \to \overline{\mathcal{V}}$ for such a morphism. Then we use (U3) in order to get a diagram as follows

$$\begin{array}{ccccc} \overline{\mathcal{U}} & \xrightarrow{[\overline{f}]} & \overline{\mathcal{V}} & \xrightarrow{\iota_{\overline{\mathcal{V}}}} & \mathcal{V}^{\max} \\ {\scriptstyle [\hat{w}^1]}\uparrow & & \Searrow [\delta] & & \uparrow{\scriptstyle \iota_{\mathcal{V}}} \\ \mathcal{U}' & & \xrightarrow{\quad [\hat{f}] \quad} & & \mathcal{V} \end{array}$$

with $[\hat{w}^1]$ weak equivalence; as before, without loss of generality we can assume that $[\hat{w}^1]$ is a unit weak equivalence. Since $[\hat{w}^1]$ is a unit weak equivalence, then $\mathcal{U}'$ and $\overline{\mathcal{U}}$ are in the same equivalence class of atlases over X. Moreover, by hypothesis $\overline{\mathcal{U}}$ is in the same class of $\mathcal{U}$. Then by lemma 8.2 there exist an atlas $\mathcal{U}''$ and a pair of unit weak equivalences:

$$\mathcal{U} \xleftarrow{\quad [\hat{w}^3] \quad} \mathcal{U}'' \xrightarrow{\quad [\hat{w}^2] \quad} \mathcal{U}'.$$

The following diagram proves that in (**Ho**) we have $[[\overline{f}]] = [[\hat{f}] \circ [\hat{w}^2]]$:

$$\begin{array}{ccccccc} & \nearrow^{[\hat{w}^1] \circ [\hat{w}^2]} & \overline{\mathcal{U}} & \xrightarrow{\quad [\overline{f}] \quad} & \overline{\mathcal{V}} & \searrow^{\iota_{\overline{\mathcal{V}}}} & \\ \mathcal{U}'' & & & \Downarrow [\delta] * i_{[\hat{w}^2]} & & & \mathcal{V}^{\max}. \\ & \searrow_{\mathrm{id}_{\mathcal{U}''}} & \mathcal{U}'' \xrightarrow[{[\hat{w}^2]}]{} & \mathcal{U}' & \xrightarrow[{[\hat{f}]}]{} \mathcal{V} & \nearrow_{\iota_{\mathcal{V}}} & \end{array}$$

Now let us consider the morphism

$$\Big[\mathcal{U}'', [\hat{w}^3], [\hat{f}] \circ [\hat{w}^2]\Big] : \mathcal{U} \longrightarrow \mathcal{V}$$

in **Ho**(**Red Orb**). Since $[\hat{w}^3]$ is a unit weak equivalence, then the induced isomorphism

$$[\hat{\varphi}^3] : w^3_*(\mathcal{U}'') \xrightarrow{\sim} \mathcal{U}''$$

is the identity. Therefore,

$$N\Big(\Big[\mathcal{U}'', [\hat{w}^3], [\hat{f}] \circ [\hat{w}^2]\Big]\Big) = \Big[[\hat{f}] \circ [\hat{w}^2] \circ [\hat{\varphi}^3]\Big] = \Big[[\hat{f}] \circ [\hat{w}^2]\Big] = [[\overline{f}]].$$

This proves that N is full. Now let us suppose that we have fixed two morphisms in **Ho**(**Red Orb**) with the same source and target:

$$\Big[\mathcal{U}^m, [\hat{w}^m], [\hat{f}^m]\Big] : \mathcal{U} \longrightarrow \mathcal{V} \quad \text{for } m = 1, 2 \tag{8.12}$$

and let us suppose that they are identified by N, i.e. let us suppose that

$$\Big[[\hat{f}^1] \circ [\hat{\varphi}^1]\Big] = \Big[[\hat{f}^2] \circ [\hat{\varphi}^2]\Big], \tag{8.13}$$

where for $m = 1, 2$, $[\hat{\varphi}^m]$ is the usual isomorphism (over $(w^m)^{-1}$) from $w^m_*(\mathcal{U}^m)$ to $\mathcal{U}^m$. Now by (8.13) we have that there exist a reduced orbifold atlas $\mathcal{U}^3$, a pair of unit weak equivalences $[\hat{v}^m] : \mathcal{U}^3 \to w^m_*(\mathcal{U}^m)$ for $m = 1, 2$ and a 2-morphism in (**RedAtl**):

$$[\mu] : \iota_{\mathcal{V}} \circ [\hat{f}^1] \circ [\hat{\varphi}^1] \circ [\hat{v}^1] \Longrightarrow \iota_{\mathcal{V}} \circ [\hat{f}^2] \circ [\hat{\varphi}^2] \circ [\hat{v}^2].$$

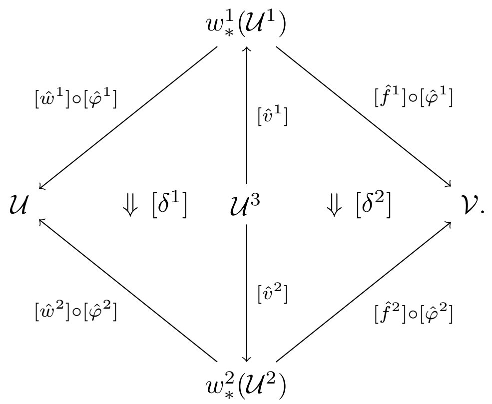
By definition of 2-morphism and of $\iota_{\mathcal{V}}$, we conclude that there exists a 2-morphism

$$[\delta^2] : [\hat{f}^1] \circ [\hat{\varphi}^1] \circ [\hat{v}^1] \Longrightarrow [\hat{f}^2] \circ [\hat{\varphi}^2] \circ [\hat{v}^2]$$

such that $[\mu] = i_{\iota_{\mathcal{V}}} * [\delta^2]$. Now let us consider the morphisms

$$[\hat{w}^m] \circ [\hat{\varphi}^m] \circ [\hat{v}^m] : \mathcal{U}^3 \longrightarrow \mathcal{U} \quad \text{for } m = 1, 2. \tag{8.14}$$

Since for $m = 1, 2$ the morphism $[\hat{v}^m]$ is a unit weak equivalence and since $[\hat{\varphi}^m]$ is defined over $(w^m)^{-1}$, then we get that the maps in (8.14) are both unit weak equivalences. Therefore by lemma 8.1 we conclude that there exists a 2-morphism $[\delta^1]$ from the first unit weak equivalence to the second one. So we have a diagram as follows:



This proves that in **Ho**(**Red Orb**) we have:

$$\Big[w^1_*(\mathcal{U}^1), [\hat{w}^1] \circ [\hat{\varphi}^1], [\hat{f}^1] \circ [\hat{\varphi}^1]\Big] = \Big[w^2_*(\mathcal{U}^2), [\hat{w}^2] \circ [\hat{\varphi}^2], [\hat{f}^2] \circ [\hat{\varphi}^2]\Big].$$

Now it is easy to see that for $m = 1, 2$ we have

$$\Big[w^m_*(\mathcal{U}^m), [\hat{w}^m] \circ [\hat{\varphi}^m], [\hat{f}^m] \circ [\hat{\varphi}^m]\Big] = \Big[\mathcal{U}^m, [\hat{w}^m], [\hat{f}^m]\Big].$$

This proves that any two morphisms as in (8.12) are identified by N if and only if they are already equal. So N is a faithful functor. This suffices to conclude. □

Appendix - Some technical proofs

Here are the proofs of some technical lemmas stated in the previous pages.

Proof of lemma 2.3. Let us fix any $i \in I$; by definition of 2-morphism for $[\sigma]$ and $[\delta]$ we have that $\{\widetilde{U}^a_i\}_{a \in A(i)}$ and $\{\widetilde{U}^b_i\}_{b \in B(i)}$ are both open coverings of $\widetilde{U}_i$, therefore so is the set $\{\widetilde{U}^{a,b}_i\}_{(a,b)}$, so (R1) holds. Also property (R2) is easy to verify. Now let us check property (R3), so let us fix any $i \in I$, any $(a, b) \in A(i) \times B(i)$ and any $\widetilde{x}_i \in \widetilde{U}^{a,b}_i$. Then:

$$\widetilde{f}^3_i(\widetilde{x}_i) = \sigma^b_i \circ \widetilde{f}^2_i(\widetilde{x}_i) = \sigma^b_i \circ \delta^a_i \circ \widetilde{f}^1_i(\widetilde{x}_i) = \theta^{a,b}_i \circ \widetilde{f}^1_i(\widetilde{x}_i),$$

so θ satisfies property (R3). Now let us fix any $i \in I$, let $(a,b),(a',b') \in A(i) \times B(i)$ and let $\widetilde{x}_i \in \widetilde{U}_i^{a,b} \cap \widetilde{U}_i^{a',b'}$. Then we have

$$\begin{aligned}\operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \theta_i^{a,b} &= \operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \sigma_i^b \cdot \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta_i^a = \\ &= \operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \sigma_i^{b'} \cdot \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta_i^{a'} = \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \theta_i^{a',b'},\end{aligned}$$

so θ satisfies (R4). Let us verify also (R5), so let us fix any pair $(i,i') \in I \times I$, $(a,b) \in A(i) \times B(i)$ and $(a',b') \in A(i') \times B(i')$. Let us fix also $\lambda \in \mathcal{C}h(\mathcal{U},i,i')$ and let $\widetilde{x}_i \in \operatorname{dom}\lambda \cap \widetilde{U}_i^{a,b}$ such that $\lambda(\widetilde{x}_i) \in \widetilde{U}_{i'}^{a',b'}$. By property (R5) for δ we get that there exist $(P_f^m,\nu_f^m) \in [P_f^m,\nu_f^m]$ and $\lambda^m \in P_f^m$ for $m=1,2$, such that $\widetilde{x}_i \in \operatorname{dom}\lambda^m$, $\operatorname{germ}_{\widetilde{x}_i}\lambda^m = \operatorname{germ}_{\widetilde{x}_i}\lambda$ for $m=1,2$ and

$$\operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \nu_f^2(\lambda^2) \cdot \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta_i^a = \operatorname{germ}_{\widetilde{f}_{i'}^1(\lambda(\widetilde{x}_i))} \delta_{i'}^{a'} \cdot \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \nu_f^1(\lambda^1). \tag{8.15}$$

Analogously, by property (R5)$'$ for σ we get that there exists $(P_f^3,\nu_f^3) \in [P_f^3,\nu_f^3]$ and $\lambda^3 \in P_f^3$ such that $\widetilde{x}_i \in \operatorname{dom}\lambda^3$, $\operatorname{germ}_{\widetilde{x}_i}\lambda^3 = \operatorname{germ}_{\widetilde{x}_i}\lambda$ and

$$\operatorname{germ}_{\widetilde{f}_i^3(\widetilde{x}_i)} \nu_f^3(\lambda^3) \cdot \operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \sigma_i^b = \operatorname{germ}_{\widetilde{f}_{i'}^2(\lambda(\widetilde{x}_i))} \sigma_{i'}^{b'} \cdot \operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \nu_f^2(\lambda^2). \tag{8.16}$$

Then by multiplying (8.16) by $\operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta_i^a$ on the right we have:

$$\begin{aligned}&\operatorname{germ}_{\widetilde{f}_i^3(\widetilde{x}_i)} \nu_f^3(\lambda^3) \cdot \operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \sigma_i^b \cdot \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta_i^a = \\ &= \operatorname{germ}_{\widetilde{f}_{i'}^2(\lambda(\widetilde{x}_i))} \sigma_{i'}^{b'} \cdot \operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \nu_f^2(\lambda^2) \cdot \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta_i^a \overset{(8.15)}{=} \\ &\overset{(8.15)}{=} \operatorname{germ}_{\widetilde{f}_{i'}^2(\lambda(\widetilde{x}_i))} \sigma_{i'}^{b'} \cdot \operatorname{germ}_{\widetilde{f}_{i'}^1(\lambda(\widetilde{x}_i))} \delta_{i'}^{a'} \cdot \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \nu_f^1(\lambda^1).\end{aligned}$$

Using the definition of θ we can rewrite the previous identity as

$$\operatorname{germ}_{\widetilde{f}_i^3(\widetilde{x}_i)} \nu_f^3(\lambda^3) \cdot \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \theta_i^{a,b} = \operatorname{germ}_{\widetilde{f}_{i'}^1(\lambda(\widetilde{x}_i))} \theta_{i'}^{a',b'} \cdot \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \nu_f^1(\lambda^1).$$

Since by construction $\lambda^1 \in P_f^1$ and $\lambda^3 \in P_f^3$ are such that $\operatorname{germ}_{\widetilde{x}_i}\lambda^1 = \operatorname{germ}_{\widetilde{x}_i}\lambda = \operatorname{germ}_{\widetilde{x}_i}\lambda^3$, then this suffices to prove property (R5) for θ. □

Proof of lemma 2.6. Let us fix any $i \in I$. Then by (R2) for δ we have $\widetilde{f}_i^2(\widetilde{U}_i^a) \subseteq \operatorname{cod}\delta_i^a \subseteq \widetilde{V}_{\overline{f}^2(i)}$; moreover, by (R1) for η we have:

$$\widetilde{V}_{\overline{f}^2(i)} = \bigcup_{c \in C(\overline{f}^2(i))} \widetilde{V}_{\overline{f}^2(i)}^c.$$

Hence:

$$\widetilde{f}_i^2(\widetilde{U}_i^a) \subseteq \bigcup_{c \in C(\overline{f}^2(i))} \operatorname{cod}\delta_i^a \cap \widetilde{V}_{\overline{f}^2(i)}^c.$$

Therefore

$$\widetilde{U}_i^a = \bigcup_{c \in C(\overline{f}^2(i))} \widetilde{U}_i^a \cap \left(\widetilde{f}_i^2\right)^{-1}\left(\operatorname{cod}\delta_i^a \cap \widetilde{V}_{\overline{f}^2(i)}^c\right) = \bigcup_{c \in C(\overline{f}^2(i))} \widetilde{U}_i^{a,c}.$$

Since $\widetilde{U}_i = \cup_{a \in A(i)} \widetilde{U}_i^a$, then we conclude that $\{\widetilde{U}_i^{a,c}\}_{(a,c)}$ is an open covering of $\widetilde{U}_i$, so property (R1) is verified. Now let us fix any $i \in I$, any $(a,c) \in A(i) \times C(\overline{f}^2(i))$ and any $\widetilde{x}_i \in \widetilde{U}_i^{a,c}$. Then

$$\widetilde{f}_i^2(\widetilde{x}_i) \in \widetilde{V}^c_{\overline{f}^2(i)},$$

so by (R2) for η we have:

$$\widetilde{w} := \widetilde{g}^1_{\overline{f}^2(i)} \circ \widetilde{f}_i^2(\widetilde{x}_i) \in \operatorname{dom} \eta^c_{\overline{f}^2(i)}.$$

Now by (R3) for δ and (Q5c) for $\hat{g}^1$ we have

$$\widetilde{w} = \widetilde{g}^1_{\overline{f}^2(i)} \circ \delta_i^a \circ \widetilde{f}_i^1(\widetilde{x}_i) = \nu_g^1(\delta_i^a) \circ \widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i).$$

Therefore

$$\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i) \in \Big(\nu_g^1(\delta_i^a)\Big)^{-1}\Big(\operatorname{cod} \nu_g^1(\delta_i^a) \cap \operatorname{dom} \eta^c_{\overline{f}^2(i)}\Big) = \widetilde{W}_i^{a,c} = \operatorname{dom} \gamma_i^{a,c}. \quad (8.17)$$

Now by (R2) and (R3) for η we have that

$$\eta^c_{\overline{f}^2(i)}(\widetilde{w}) = \widetilde{g}^2_{\overline{f}^2(i)} \circ \widetilde{f}_i^2(\widetilde{x}_i).$$

Therefore we conclude that

$$\widetilde{g}^2_{\overline{f}^2(i)} \circ \widetilde{f}_i^2(\widetilde{x}_i) \in \eta^c_{\overline{f}^2(i)}\Big(\operatorname{cod} \nu_g^1(\delta_i^a) \cap \operatorname{dom} \eta^c_{\overline{f}^2(i)}\Big) = \operatorname{cod} \gamma_i^{a,c}. \quad (8.18)$$

By (8.17) and (8.18) we conclude that property (R2) holds for γ. Now let us prove also (R3), so let us fix any $i \in I$, any $(a,c) \in A(i) \times C(\overline{f}^2(i))$ and any $\widetilde{x}_i \in \widetilde{U}_i^{a,c}$. Then

$$\begin{aligned}\gamma_i^{a,c} \circ \widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i) &= \eta^c_{\overline{f}^2(i)} \circ \nu_g^1(\delta_i^a) \circ \widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i) = \\ &= \eta^c_{\overline{f}^2(i)} \circ \widetilde{g}^1_{\overline{f}^2(i)} \circ \delta_i^a \circ \widetilde{f}_i^1(\widetilde{x}_i) = \widetilde{g}^2_{\overline{f}^2(i)} \circ \widetilde{f}_i^2(\widetilde{x}_i),\end{aligned}$$

so property (R3) is verified. Now let us prove property (R4): let us fix any $i \in I$, any pair $(a,c),(a',c') \in A(i) \times C(\overline{f}^2(i))$ and any $\widetilde{x}_i \in \widetilde{U}_i^{a,c} \cap \widetilde{U}_i^{a',c'}$. Then $\widetilde{x}_i \in \widetilde{U}_i^a \cap \widetilde{U}_i^{a'}$; so by property (R4) for δ we have that

$$\operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta_i^a = \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta_i^{a'}.$$

Therefore, by property (Q5d) for $\hat{g}^1$ we have that:

$$\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i)} \nu_g^1(\delta_i^a) = \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i)} \nu_g^1(\delta_i^{a'}). \quad (8.19)$$

Moreover, by the hypothesis on $\widetilde{x}_i$, we have that $\widetilde{f}_i^2(\widetilde{x}_i) \in \widetilde{V}^c_{\overline{f}^2(i)} \cap \widetilde{V}^{c'}_{\overline{f}^2(i)}$. Therefore, by property (R4) for η we have:

$$\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)} \circ \widetilde{f}_i^2(\widetilde{x}_i)} \eta^c_{\overline{f}^2(i)} = \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)} \circ \widetilde{f}_i^2(\widetilde{x}_i)} \eta^{c'}_{\overline{f}^2(i)}. \quad (8.20)$$

By multiplying (8.20) by (8.19) on the right we get:

$$\begin{aligned}&\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)} \circ \widetilde{f}_i^2(\widetilde{x}_i)} \eta^c_{\overline{f}^2(i)} \cdot \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i)} \nu_g^1(\delta_i^a) = \\ &= \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)} \circ \widetilde{f}_i^2(\widetilde{x}_i)} \eta^{c'}_{\overline{f}^2(i)} \cdot \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i)} \nu_g^1(\delta_i^{a'}).\end{aligned}$$

By definition of γ, this is equivalent to saying:

$$\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i)} \gamma_i^{a,c} = \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i)} \gamma_i^{a',c'},$$

so we have proved property (R4) for γ. Lastly, let us prove also (R5). In order to do that, let us denote by

$$\hat{h}^m := \left(h := g \circ f, \overline{h}^m := \overline{g}^m \circ \overline{f}^m, \left\{\widetilde{h}_i^m := \widetilde{g}^m_{\overline{f}^m(i)} \circ \widetilde{f}_i^m\right\}_{i \in I}, P_h^m, \nu_h^m\right) \text{ for } m = 1, 2$$

two representatives for $[\hat{g}^1] \circ [\hat{f}^1]$ and $[\hat{g}^2] \circ [\hat{f}^2]$ respectively, obtained as in construction 1.14. In particular, we recall that in such a construction we defined set maps $\nu_f^{\text{ind},m} : P_h^m \to P_g$ for $m = 1, 2$, such that $(\nu_f^{\text{ind},m}, P_h^m) \in [P_f^m, \nu_f^m]$ for $m = 1, 2$; we recall also that

$$\nu_h^m := \nu_g^m \circ \nu_f^{\text{ind},m} \quad \text{for} \quad m = 1, 2.$$

Now let us fix any $(i, i') \in I \times I$, any $(a, c) \in A(i) \times C(\overline{f}^2(i))$, any $(a', c') \in A(i') \times C(\overline{f}^2(i'))$, any $\lambda \in \mathcal{C}h(\mathcal{U}, i, i')$ and any $\widetilde{x}_i \in \operatorname{dom} \lambda \cap \widetilde{U}_i^{a,c}$ such that $\lambda(\widetilde{x}_i) \in \widetilde{U}_{i'}^{a',c'}$. By (P1) there exist $\lambda^m \in P_h^m(i, i')$ for $m = 1, 2$ such that $\widetilde{x}_i \in \operatorname{dom} \lambda^m$ and $\operatorname{germ}_{\widetilde{x}_i} \lambda^m = \operatorname{germ}_{\widetilde{x}_i} \lambda$ for $m = 1, 2$. By property (R5)$'$ for δ we get that

$$\operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \nu_f^{\text{ind},2}(\lambda^2) \cdot \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta_i^a = \operatorname{germ}_{\widetilde{f}_{i'}^1(\lambda(\widetilde{x}_i))} \delta_{i'}^{a'} \cdot \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \nu_f^{\text{ind},1}(\lambda^1). \tag{8.21}$$

In the previous expression the terms $\delta_i^a, \delta_{i'}^{a'}$ and $\nu_f^{\text{ind},1}(\lambda^1)$ are all in P_g^1. Since P_g^1 is a good subset of $\mathcal{C}h(\mathcal{V})$, then there exists $\mu \in P_g^1$ such that $\widetilde{f}_i^2(\widetilde{x}_i) \in \operatorname{dom} \mu$ and $\operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \mu = \operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \nu_f^{\text{ind},2}(\lambda^2)$. By substituting in (8.21) and by using property (Q5e) for $\hat{g}^1$ we get:

$$\begin{aligned}
&\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)} \circ \widetilde{f}_i^2(\widetilde{x}_i)} \nu_g^1(\mu) \cdot \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i)} \nu_g^1(\delta_i^a) = \\
&= \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i')} \circ \widetilde{f}_{i'}^1(\lambda(\widetilde{x}_i))} \nu_g^1(\delta_{i'}^{a'}) \cdot \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i)} \nu_g^1 \circ \nu_f^{\text{ind},1}(\lambda^1). \\
&= \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i')} \circ \widetilde{f}_{i'}^1(\lambda(\widetilde{x}_i))} \nu_g^1(\delta_{i'}^{a'}) \cdot \operatorname{germ}_{\widetilde{h}_i^1(\widetilde{x}_i)} \nu_h^1(\lambda^1).
\end{aligned} \tag{8.22}$$

If we multiply (8.22) by $\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i')} \circ \widetilde{f}_{i'}^2(\lambda(\widetilde{x}_i))} \eta^{c'}_{\overline{f}^2(i')}$ on the left we get:

$$\begin{aligned}
&\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i')} \circ \widetilde{f}_{i'}^2(\lambda(\widetilde{x}_i))} \eta^{c'}_{\overline{f}^2(i')} \cdot \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)} \circ \widetilde{f}_i^2(\widetilde{x}_i)} \nu_g^1(\mu) \cdot \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\widetilde{x}_i)} \nu_g^1(\delta_i^a) = \\
&= \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i')} \circ \widetilde{f}_{i'}^2(\lambda(\widetilde{x}_i))} \eta^{c'}_{\overline{f}^2(i')} \cdot \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i')} \circ \widetilde{f}_{i'}^1(\lambda(\widetilde{x}_i))} \nu_g^1(\delta_{i'}^{a'}) \cdot \operatorname{germ}_{\widetilde{h}_i^1(\widetilde{x}_i)} \nu_h^1(\lambda^1).
\end{aligned} \tag{8.23}$$

Now by construction $\mu \in P_g^1, \nu_f^{\text{ind},2}(\lambda^2) \in P_g^2$ and $\operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \mu = \operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \nu_f^{\text{ind},2}(\lambda^2)$. Then by property (R5)$'$ for η we get that

$$\begin{aligned}
&\operatorname{germ}_{\widetilde{g}^2_{\overline{f}^2(i)} \circ \widetilde{f}_i^2(\widetilde{x}_i)} \nu_g^2 \circ \nu_f^{\text{ind},2}(\lambda^2) \cdot \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)} \circ \widetilde{f}_i^2(\widetilde{x}_i)} \eta^c_{\overline{f}^2(i)} = \\
&= \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i')} \circ \mu \circ \widetilde{f}_i^2(\widetilde{x}_i)} \eta^{c'}_{\overline{f}^2(i')} \cdot \operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)} \circ \widetilde{f}_i^2(\widetilde{x}_i)} \nu_g^1(\mu).
\end{aligned} \tag{8.24}$$

We recall that

$$\begin{aligned}
&\operatorname{germ}_{\widetilde{x}_i} \mu \circ \widetilde{f}_i^2 = \operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \nu_f^{\text{ind},2}(\lambda^2) \cdot \operatorname{germ}_{\widetilde{x}_i} \widetilde{f}_i^2 \overset{(Q5c)}{=} \\
&\overset{(Q5c)}{=} \operatorname{germ}_{\lambda^2(\widetilde{x}_i)} \widetilde{f}_{i'}^2 \cdot \operatorname{germ}_{\widetilde{x}_i} \lambda^2 = \operatorname{germ}_{\lambda(\widetilde{x}_i)} \widetilde{f}_{i'}^2 \cdot \operatorname{germ}_{\widetilde{x}_i} \lambda,
\end{aligned}$$

so we can rewrite (8.24) as

$$\begin{aligned}&\operatorname{germ}_{\widetilde{h}^2_i(\widetilde{x}_i)}\,\nu^2_h(\lambda^2)\cdot\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)}\circ\widetilde{f}^2_i(\widetilde{x}_i)}\,\eta^c_{\overline{f}^2(i)}=\\&=\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i')}\circ\widetilde{f}^2_{i'}(\lambda(\widetilde{x}_i))}\,\eta^{c'}_{\overline{f}^2(i')}\cdot\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)}\circ\widetilde{f}^2_i(\widetilde{x}_i)}\,\nu^1_g(\mu).\end{aligned}\tag{8.25}$$

By substituting (8.25) in the left hand side of (8.23) we get:

$$\begin{aligned}&\operatorname{germ}_{\widetilde{h}^2_i(\widetilde{x}_i)}\,\nu^2_h(\lambda^2)\cdot\Big(\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)}\circ\widetilde{f}^2_i(\widetilde{x}_i)}\,\eta^c_{\overline{f}^2(i)}\cdot\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)}\circ\widetilde{f}^1_i(\widetilde{x}_i)}\,\nu^1_g(\delta^a_i)\Big)=\\&=\Big(\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i')}\circ\widetilde{f}^2_{i'}(\lambda(\widetilde{x}_i))}\,\eta^{c'}_{\overline{f}^2(i')}\cdot\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i')}\circ\widetilde{f}^1_{i'}(\lambda(\widetilde{x}_i))}\,\nu^1_g(\delta^{a'}_{i'})\Big)\cdot\operatorname{germ}_{\widetilde{h}^1_i(\widetilde{x}_i)}\,\nu^1_h(\lambda^1).\end{aligned}\tag{8.26}$$

We can rewrite (8.26) as:

$$\operatorname{germ}_{\widetilde{h}^2_i(\widetilde{x}_i)}\,\nu^2_h(\lambda^2)\cdot\operatorname{germ}_{\widetilde{h}^1_i(\widetilde{x}_i)}\,\gamma^{a,c}_i=\operatorname{germ}_{\widetilde{h}^1_{i'}(\lambda(\widetilde{x}_i))}\,\gamma^{a',c'}_{i'}\cdot\operatorname{germ}_{\widetilde{h}^1_i(\widetilde{x}_i)}\,\nu^1_h(\lambda^1),$$

so property (R5) is verified for γ. □

Proof of lemma 3.3. Let us fix representatives (P^m_g,ν^m_g) for $[P^m_g,\nu^m_g]$ for $m=1,2$. As we did in construction 2.5 we can choose representatives

$$\delta=\Big\{\Big(\widetilde{U}^a_i,\delta^a_i\Big)\Big\}_{i\in I,a\in A(i)},\quad\sigma=\Big\{\Big(\widetilde{U}^b_i,\sigma^b_i\Big)\Big\}_{i\in I,b\in B(i)},$$

for $[\delta]$ and $[\sigma]$ respectively, such that

$$\delta^a_i\in P^1_g\quad\forall\,i\in I,\,\forall\,a\in A(i),\quad\sigma^b_i\in P^2_g\quad\forall\,i\in I,\,\forall\,b\in B(i).$$

Let us also fix representatives

$$\eta=\Big\{\Big(\widetilde{V}^c_j,\eta^c_j\Big)\Big\}_{j\in J,c\in C(j)},\quad\mu=\Big\{\Big(\widetilde{V}^d_j,\mu^d_j\Big)\Big\}_{j\in J,d\in D(j)}$$

for $[\eta]$ and $[\mu]$ respectively. Then we have that

$$\begin{aligned}[\sigma]\odot[\delta]&=\Bigg[\Big\{\Big(\widetilde{U}^{a,b}_i,\,\sigma^b_i\circ\delta^a_i|_{\widetilde{V}^{a,b}_i}\Big)\Big\}_{i\in I,(a,b)\in A(i)\times B(i)}\Bigg]:[\hat{f}^1]\Longrightarrow[\hat{f}^3],\\ [\mu]\odot[\eta]&=\Bigg[\Big\{\Big(\widetilde{V}^{c,d}_j,\,\mu^d_j\circ\eta^c_j|_{\widetilde{W}^{c,d}_j}\Big)\Big\}_{j\in J,(c,d)\in C(j)\times D(j)}\Bigg]:[\hat{g}^1]\Longrightarrow[\hat{g}^3],\end{aligned}$$

where

$$\begin{aligned}&\widetilde{U}^{a,b}_i:=\widetilde{U}^a_i\cap\widetilde{U}^b_i,\quad\widetilde{V}^{a,b}_i:=(\delta^a_i)^{-1}\Big(\operatorname{cod}\delta^a_i\cap\operatorname{dom}\sigma^b_i\Big),\\&\widetilde{V}^{c,d}_j:=\widetilde{V}^c_j\cap\widetilde{V}^d_j,\quad\widetilde{W}^{c,d}_j:=\big(\eta^c_j\big)^{-1}\Big(\operatorname{cod}\eta^c_j\cap\operatorname{dom}\mu^d_j\Big).\end{aligned}$$

Since P^1_g is a good subset of $\mathcal{C}h(\mathcal{V})$, then as in construction 2.5 for each $(a,b)\in A(i)\times B(i)$ there exists a set

$$\Big\{\Theta^{a,b,e}_i\Big\}_{e\in E(i,a,b)}\subseteq P^1_g$$

such that

$$\widetilde{V}^{a,b}_i\subseteq\bigcup_{e\in E(i,a,b)}\operatorname{dom}\Theta^{a,b,e}_i$$

and such that for each $\widetilde{x}_i \in \widetilde{U}_i^{a,b}$ there is $e \in E(i,a,b)$ such that $\widetilde{f}_i^1(\widetilde{x}_i) \in \operatorname{dom}\Theta_i^{a,b,e}$ and $\operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \Theta_i^{a,b,e} = \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \sigma_i^b \circ \delta_i^a$. Then we set

$$\widetilde{U}_i^{a,b,e} := \widetilde{U}_i^{a,b} \cap \left(\widetilde{f}_i^1\right)^{-1}\left(\operatorname{dom}\Theta_i^{a,b,e}\right).$$

So we get that a representative for $[\sigma] \odot [\delta]$ is given by

$$\left\{\left(\widetilde{U}_i^{a,b,e}, \Theta_i^{a,b,e}\right)\right\}_{i\in I,(a,b)\in A(i)\times B(i), e\in E(i,a,b)}.$$

Then we conclude that:

$$\Big([\mu]\odot[\eta]\Big) * \Big([\sigma]\odot[\delta]\Big) = \tag{8.27}$$
$$\left[\left\{\left(\widetilde{U}_i^{a,b,c,d,e}, \mu^d_{\overline{f}^3(i)} \circ \eta^c_{\overline{f}^3(i)} \circ \nu_g^1(\Theta_i^{a,b,e})\Big|_{\widetilde{W}_i^{a,b,c,d,e}}\right)\right\}_{i\in I,(a,b,c,d,e)}\right],$$

where for each $i \in I$, (a,b) varies in $A(i)\times B(i)$, $e \in E(i,a,b)$, $(c,d) \in C(\overline{f}^3(i)) \times D(\overline{f}^3(i))$ and

$$\widetilde{U}_i^{a,b,c,d,e} := \widetilde{U}_i^{a,b,e} \cap \left(\widetilde{f}_i^3\right)^{-1}\left(\operatorname{cod}\Theta_i^{a,b,e} \cap \widetilde{V}^{c,d}_{\overline{f}^3(i)}\right),$$
$$\widetilde{W}_i^{a,b,c,d,e} := \left(\nu_g^1(\Theta_i^{a,b,e})\right)^{-1}\left(\operatorname{cod}\nu_g^1(\Theta_i^{a,b,e}) \cap \widetilde{W}^{c,d}_{\overline{f}^3(i)}\right).$$

Since we have chosen each $\delta_i^a \in P_g^1$, then we have:

$$[\eta] * [\delta] = \left[\left\{\left(\widetilde{U}_i^{\overline{a},\overline{c}}, \eta^{\overline{c}}_{\overline{f}^2(i)} \circ \nu_g^1(\delta_i^{\overline{a}})\Big|_{\widetilde{W}_i^{\overline{a},\overline{c}}}\right)\right\}_{i\in I,(\overline{a},\overline{c})\in A(i)\times C(\overline{f}^2(i))}\right],$$

where

$$\widetilde{U}_i^{\overline{a},\overline{c}} := \widetilde{U}_i^{\overline{a}} \cap \left(\widetilde{f}_i^2\right)^{-1}\left(\operatorname{cod}\delta_i^{\overline{a}} \cap \widetilde{V}^{\overline{c}}_{\overline{f}^2(i)}\right),$$
$$\widetilde{W}_i^{\overline{a},\overline{c}} := \left(\nu_g^1(\delta_i^{\overline{a}})\right)^{-1}\left(\operatorname{cod}\nu_g^1(\delta_i^{\overline{a}}) \cap \operatorname{dom}\eta^{\overline{c}}_{\overline{f}^2(i)}\right).$$

Analogously, since we have chosen each $\sigma_i^b \in P_g^2$, we have

$$[\mu] * [\sigma] = \left[\left\{\left(\widetilde{U}_i^{\overline{b},\overline{d}}, \mu^{\overline{d}}_{\overline{f}^3(i)} \circ \nu_g^2(\sigma_i^{\overline{b}})\Big|_{\widetilde{W}_i^{\overline{b},\overline{d}}}\right)\right\}_{i\in I,(\overline{b},\overline{d})\in B(i)\times D(\overline{f}^3(i))}\right],$$

where

$$\widetilde{U}_i^{\overline{b},\overline{d}} := \widetilde{U}_i^{\overline{b}} \cap \left(\widetilde{f}_i^3\right)^{-1}\left(\operatorname{cod}\sigma_i^{\overline{b}} \cap \widetilde{V}^{\overline{d}}_{\overline{f}^3(i)}\right),$$
$$\widetilde{W}_i^{\overline{b},\overline{d}} := \left(\nu_g^2(\sigma_i^{\overline{b}})\right)^{-1}\left(\operatorname{cod}\nu_g^2(\sigma_i^{\overline{b}}) \cap \operatorname{dom}\mu^{\overline{d}}_{\overline{f}^3(i)}\right).$$

Then we have:

$$\Big([\mu]*[\sigma]\Big) \odot \Big([\eta]*[\delta]\Big) = \tag{8.28}$$
$$= \left[\left\{\left(\widetilde{U}_i^{\overline{a},\overline{b},\overline{c},\overline{d}}, \mu^{\overline{d}}_{\overline{f}^3(i)} \circ \nu_g^2(\sigma_i^{\overline{b}}) \circ \eta^{\overline{c}}_{\overline{f}^2(i)} \circ \nu_g^1(\delta_i^{\overline{a}})\Big|_{\widetilde{W}_i^{\overline{a},\overline{b},\overline{c},\overline{d}}}\right)\right\}_{i\in I,(\overline{a},\overline{b},\overline{c},\overline{d})}\right],$$

where for each $i\in I$, $(\overline{a},\overline{b},\overline{c},\overline{d})$ varies in $A(i)\times B(i)\times C(\overline{f}^2(i))\times D(\overline{f}^3(i))$ and

$$\widetilde{U}_i^{\overline{a},\overline{b},\overline{c},\overline{d}}:=\widetilde{U}_i^{\overline{a},\overline{c}}\cap\widetilde{U}_i^{\overline{b},\overline{d}},$$

$$\widetilde{W}_i^{\overline{a},\overline{b},\overline{c},\overline{d}}:=\left(\eta^{\overline{c}}_{\overline{f}^2(i)}\circ\nu_g^1(\delta_i^{\overline{a}})|_{\widetilde{W}_i^{\overline{a},\overline{c}}}\right)^{-1}$$

$$\left(\left(\operatorname{cod}\eta^{\overline{c}}_{\overline{f}^2(i)}\circ\nu_g^1(\delta_i^{\overline{a}})|_{\widetilde{W}_i^{\overline{a},\overline{b}}}\right)\cap\left(\operatorname{dom}\mu^{\overline{d}}_{\overline{f}^3(i)}\circ\nu_g^2(\sigma_i^{\overline{b}})|_{\widetilde{W}_i^{\overline{b},\overline{d}}}\right)\right).$$

Now we claim that (8.27) and (8.28) are the same 2-morphism. In order to prove that, let us fix any $i\in I$, any (a,b,c,d,e) as in (8.27), any $(\overline{a},\overline{b},\overline{c},\overline{d})$ as in (8.28) and any point $\widetilde{x}_i\in\widetilde{U}_i^{a,b,c,d,e}\cap\widetilde{U}_i^{\overline{a},\overline{b},\overline{c},\overline{d}}$. Since P_g^1 is a good subset of $\mathcal{C}h(\mathcal{V})$, then there exist $\hat{\sigma}_i^b$ in P_g^1 such that $\widetilde{f}_i^2(\widetilde{x}_i)\in\operatorname{dom}\hat{\sigma}_i^b$ and $\operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)}\hat{\sigma}_i^b=\operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)}\sigma_i^b$. Then we have:

$$\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)}\circ\widetilde{f}_i^1(\widetilde{x}_i)}\left(\mu^d_{\overline{f}^3(i)}\circ\eta^c_{\overline{f}^3(i)}\circ\nu_g^1(\Theta_i^{a,b,e})\right)=$$

$$=\operatorname{germ}_{\widetilde{g}^2_{\overline{f}^3(i)}\circ\widetilde{f}_i^3(\widetilde{x}_i)}\mu^d_{\overline{f}^3(i)}\cdot\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^3(i)}\circ\widetilde{f}_i^3(\widetilde{x}_i)}\eta^c_{\overline{f}^3(i)}\cdot\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)}\circ\widetilde{f}_i^1(\widetilde{x}_i)}\nu_g^1(\Theta_i^{a,b,e})\stackrel{(Q5e)}{=}$$

$$\stackrel{(Q5e)}{=}\operatorname{germ}_{\widetilde{g}^2_{\overline{f}^3(i)}\circ\widetilde{f}_i^3(\widetilde{x}_i)}\mu^d_{\overline{f}^3(i)}\cdot\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^3(i)}\circ\widetilde{f}_i^3(\widetilde{x}_i)}\eta^c_{\overline{f}^3(i)}\cdot$$

$$\cdot\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)}\circ\widetilde{f}_i^2(\widetilde{x}_i)}\nu_g^1(\hat{\sigma}_i^b)\cdot\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)}\circ\widetilde{f}_i^1(\widetilde{x}_i)}\nu_g^1(\delta_i^a)\stackrel{(R5)'}{=}$$

$$\stackrel{(R5)'}{=}\operatorname{germ}_{\widetilde{g}^2_{\overline{f}^3(i)}\circ\widetilde{f}_i^3(\widetilde{x}_i)}\mu^d_{\overline{f}^3(i)}\cdot\operatorname{germ}_{\widetilde{g}^2_{\overline{f}^2(i)}\circ\widetilde{f}_i^2(\widetilde{x}_i)}\nu_g^2(\sigma_i^b)\cdot$$

$$\cdot\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)}\circ\widetilde{f}_i^2(\widetilde{x}_i)}\eta^c_{\overline{f}^2(i)}\cdot\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)}\circ\widetilde{f}_i^1(\widetilde{x}_i)}\nu_g^1(\delta_i^a)=$$

$$=\operatorname{germ}_{\widetilde{g}^2_{\overline{f}^3(i)}\circ\widetilde{f}_i^3(\widetilde{x}_i)}\mu^{\overline{d}}_{\overline{f}^3(i)}\cdot\operatorname{germ}_{\widetilde{g}^2_{\overline{f}^2(i)}\circ\widetilde{f}_i^2(\widetilde{x}_i)}\nu_g^2(\sigma_i^{\overline{b}})\cdot$$

$$\cdot\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^2(i)}\circ\widetilde{f}_i^2(\widetilde{x}_i)}\eta^{\overline{c}}_{\overline{f}^2(i)}\cdot\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)}\circ\widetilde{f}_i^1(\widetilde{x}_i)}\nu_g^1(\delta_i^{\overline{a}})=$$

$$=\operatorname{germ}_{\widetilde{g}^1_{\overline{f}^1(i)}\circ\widetilde{f}_i^1(\widetilde{x}_i)}\left(\mu^{\overline{d}}_{\overline{f}^3(i)}\circ\nu_g^2(\sigma_i^{\overline{b}})\circ\eta^{\overline{c}}_{\overline{f}^2(i)}\circ\nu_g^1(\delta_i^{\overline{a}})\right).$$

So by definition 1.19 we conclude that (8.27) and (8.28) are the same 2-morphism. □

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Mathematics Research Unit
University of Luxembourg
6, rue Richard Coudenhove-Kalergi
L-1359 Luxembourg
E-mail address: matteo.tommasini2@gmail.com