

AN ELEMENTARY PROOF OF THE VANISHING OF THE SECOND COHOMOLOGY OF THE WITT AND VIRASORO ALGEBRA WITH VALUES IN THE ADJOINT MODULE

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ABSTRACT. By elementary and direct calculations the vanishing of the (algebraic) second Lie algebra cohomology of the Witt and the Virasoro algebra with values in the adjoint module is shown. This yields infinitesimal and formal rigidity of these algebras. The first (and up to now only) proof of this important result was given 1989 by Fialowski in an unpublished note. It is based on cumbersome calculations. Compared to the original proof the presented one is quite elegant and considerably simpler.

1. INTRODUCTION

The simplest nontrivial infinite dimensional Lie algebras are the Witt algebra and its central extension the Virasoro algebra. The Witt algebra is related to the Lie algebra of the group of diffeomorphisms of the unit circle. The central extension comes into play as one is typically forced to consider projective actions if one wants to quantize a classical system or wants to regularize a field theory. There is a huge amount of literature about the application of these algebras. Here it is not the place to give even a modest overview about these applications.

The Witt algebra \mathcal{W} is the graded Lie algebra generated as vector space by the elements $\{e_n \mid n \in \mathbb{Z}\}$ with Lie structure

$$[e_n, e_m] = (m - n)e_{n+m}, \quad n, m \in \mathbb{Z}. \quad (1.1)$$

The Virasoro algebra \mathcal{V} is the universal one-dimensional central extension. Detailed definitions and descriptions are given in Section 2. The goal of this article is to give an elementary proof that the second Lie algebra cohomology of both algebras with values in the adjoint modules will vanish. The result will be stated in Theorem 3.1 below. These cohomology spaces are related to deformations of the algebras. In particular, if they vanish the algebras will be infinitesimal and formally rigid, see Corollary 3.2. Note that this does

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not mean that the algebras are analytically or geometrically rigid. Indeed, together with Alice Fialowski we showed in [5] that there exist natural, geometrically defined, nontrivial families of Lie algebras given by Krichever-Novikov type algebras associated to elliptic curves. These families appeared in [13]. In these families the special fiber is the Witt (resp. Virasoro) algebra but all other fibers are non-isomorphic to it. Similar results for the current, resp. affine Lie algebras can be found in [6], [7].

The result on the vanishing of the second Lie algebra cohomology of the Witt algebra should clearly be attributed to Fialowski. There exists an unpublished manuscript [1] by her, dating from 1989, where she does explicit calculations¹. These calculations were quite cumbersome not really appealing for journal publication. In 1990 Fialowski gave the statement of the rigidity of the Witt and Virasoro algebra in [3] without proof. Later, in the above-mentioned joint paper of the author with her [5] a sketch of its proof was presented. It is based on the calculations of the cohomology of the Lie algebra $Vect(S^1)$ of vector fields on S^1 with values in the adjoint module. Hence, this proof is not purely algebraic. Based on results of Tsujishita [14], Reshetnikov [12], and Goncharova [10] we showed that

$$H^*(Vect(S^1), Vect(S^1)) = \{0\}. \quad (1.2)$$

See also the book of Guieu and Roger [11]. We argued that by density arguments the vanishing of the cohomology of the Witt algebra will follow. This is indeed true if one considers continuous cohomology. But here we are dealing with arbitrary cohomology. In a recent attempt to fill the details how to extend the arguments to this setting, we run into troubles. We would have been forced to make considerable cohomological changes to obtain finally certain boundedness properties to guarantee convergence with respect to the topology coming from the Lie algebra of vector fields. Hence, there was an incentive to return to a direct algebraic and elementary proof. Indeed, I found a very elementary, computational, but nevertheless reasonable short and elegant proof which is simpler than Fialowski's original calculations [1].

As the Witt and Virasoro algebra are of fundamental importance inside of mathematics and in the applications, and up to now there is no complete published proof, the presented proof is for sure worth-while to publish. The proof avoids the heavy machinery of Tsujishita, Reshetnikov, and Goncharova.²

Furthermore the presented proof has the advantage that by its algebraic nature it will be valid for every field \mathbb{K} of characteristic zero. Moreover, this is one of the rare occasions where algebraic cohomology is calculated. From the very beginning it was desirable to have a purely algebraic proof of the vanishing of the algebraic cohomology.

In Section 2 we give the definition of both the Witt and Virasoro algebra and make some remarks on their graded structure. The graded structure will play an important role in the article.

In Section 3 after recalling the definition of general Lie algebra cohomology the 2-cohomology of a Lie algebra with values in the adjoint module is considered in more

¹Note added after publication of the current article: The content of the mentioned note of Fialowski is in the meantime accessible on the archive and will appear in Jour. Math. Phys. See [1] for details.

²As Fuks writes in his book [8, p.119] "Theorem 2.3.1 [Goncharova's theorem] still remains one of the most difficult theorems in the cohomology theory of infinite-dimensional Lie algebras."

detail. Some facts about its relation to deformations of the algebra are quoted to allow to judge the importance of this cohomology [7]. The main result about the vanishing of the second cohomology and the rigidity for the Witt and Virasoro algebra is formulated in Theorem 3.1 and Corollary 3.2. The cohomology considered in this article is algebraic cohomology without any restriction on the cocycles.

In Section 4 we use the graded structure of the algebras to decompose their cohomology into graded pieces. It is quite easy then to show that for degree $d \neq 0$, the degree d parts will vanish. Hence, everything is reduced to degree zero. This is due to the fact that the grading is induced by the action of a special element. Here it is the element e_0 .

The vanishing of the degree zero part is more involved. It will be presented for the Witt algebra in Section 5. It is the computational core of the article, but the computations are elementary.

In Section 6 we extend this to the Virasoro algebra \mathcal{V} . We show that the vanishing of the cohomology for \mathcal{W} implies the same for its central extension \mathcal{V} .

As the core of the proof is elementary, I want to keep this spirit throughout the article. Hence, it will be rather self-contained and elementary. We will not use things like long exact cohomology sequences, etc. The proofs are purely algebraic and the vanishing and rigidity theorems are true over every field \mathbb{K} of characteristic zero.

2. THE ALGEBRAS

The *Witt algebra* \mathcal{W} is the Lie algebra generated as vector space over a field \mathbb{K} by the elements $\{e_n \mid n \in \mathbb{Z}\}$ with Lie structure

$$[e_n, e_m] = (m - n)e_{n+m}, \quad n, m \in \mathbb{Z}. \quad (2.1)$$

As in the cohomology computations we have to divide by arbitrary integers we assume for the whole article that $\text{char}(\mathbb{K}) = 0$.

Remark 2.1. In conformal field theory \mathbb{K} will be \mathbb{C} and the Witt algebra can be realized as complexification of the Lie algebra of polynomial vector fields $Vect_{pol}(S^1)$ on the circle S^1 , which is a subalgebra of $Vect(S^1)$, the Lie algebra of all C^∞ vector fields on the circle. In this realization $e_n = \exp(in\varphi) \frac{d}{d\varphi}$. The Lie product is the usual bracket of vector fields.

An alternative realization is given as the algebra of meromorphic vector fields on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ which are holomorphic outside $\{0\}$ and $\{\infty\}$. In this realization $e_n = z^{n+1} \frac{d}{dz}$.

A very important fact is that the Witt algebra is a \mathbb{Z} -graded Lie algebra. We define the degree by setting $\deg(e_n) := n$, then the Lie product between elements of degree n and of degree m is of degree $n + m$ (if nonzero). The homogeneous spaces \mathcal{W}_n of degree n are one-dimensional with basis e_n . Crucial for the following is the additional fact that the eigenspace decomposition of the element e_0 , acting via the adjoint action on \mathcal{W} coincides with the decomposition into homogeneous subspaces. This follows from

$$[e_0, e_n] = n e_n = \deg(e_n) e_n. \quad (2.2)$$

Another property, which will play a role, is that $[\mathcal{W}, \mathcal{W}] = \mathcal{W}$. That means that \mathcal{W} is a perfect Lie algebra. In fact,

$$e_n = \frac{1}{n}[e_0, e_n], \quad n \neq 0, \quad e_0 = \frac{1}{2}[e_{-1}, e_1]. \quad (2.3)$$

The *Virasoro algebra* \mathcal{V} is the universal one-dimensional central extension of \mathcal{W} . As vector space it is the direct sum $\mathcal{V} = \mathbb{K} \oplus \mathcal{W}$. If we set for $x \in \mathcal{W}$, $\hat{x} := (0, x)$, and $t := (1, 0)$ then its basis elements are \hat{e}_n , $n \in \mathbb{Z}$ and t with the Lie product

$$[\hat{e}_n, \hat{e}_m] = (m - n)\hat{e}_{n+m} - \frac{1}{12}(n^3 - m^3)\delta_n^{-m}t, \quad [\hat{e}_n, t] = [t, t] = 0, \quad (2.4)$$

for all $n, m \in \mathbb{Z}$. Here δ_k^l is the Kronecker delta which is equal to 1 if $k = l$, otherwise zero. If we set $\deg(\hat{e}_n) := \deg(e_n) = n$ and $\deg(t) := 0$ then \mathcal{V} becomes a graded algebra. Let ν be the Lie homomorphism mapping the central element t to 0 and \hat{x} to x inducing the following short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathcal{V} \xrightarrow{\nu} \mathcal{W} \longrightarrow 0. \quad (2.5)$$

This sequence does not split, i.e. it is a non-trivial central extension.

In some abuse of notation we identify the element $\hat{x} \in \mathcal{V}$ with $x \in \mathcal{W}$ and after identification we have $\mathcal{V}_n = \mathcal{W}_n$ for $n \neq 0$ and $\mathcal{V}_0 = \langle e_0, t \rangle_{\mathbb{K}}$. Note that the relation (2.2) inducing the eigenspace decomposition for the grading element $\hat{e}_0 = e_0$ remains true.

The expression $\frac{1}{12}(n^3 - m^3)\delta_n^{-m}$ is the defining cocycle for the central extension. This form is given in a standard normalisation – others are possible. It is a Lie algebra two-cocycle of \mathcal{W} with values in the trivial module. The equivalence classes of central extensions are in 1:1 correspondence to the cohomology classes $H^2(\mathcal{W}, \mathbb{K})$, see the next section for more details. It is well-known that $\dim H^2(\mathcal{W}, \mathbb{K}) = 1$, and that the class of the above cocycle is a generator.

3. COHOMOLOGY AND DEFORMATIONS

Let us recall for completeness and further reference the definition of the *Lie algebra cohomology* of a Lie algebra W with values in a Lie module M over W . We denote the Lie module structure by $W \times M \rightarrow M$, $(x, m) \mapsto x \cdot m$. A k -cochain is an alternating k -multilinear map $W \times W \times \cdots \times W \rightarrow M$ (k copies of W). The vector space of k -cochains is denoted by $C^k(W; M)$. Here we will deal exclusively with algebraic cohomology.

Next we have the family of coboundary operators

$$\delta_k : C^k(W; M) \rightarrow C^{k+1}(W; M), \quad k \in \mathbb{N}, \quad \text{with} \quad \delta_{k+1} \circ \delta_k = 0. \quad (3.1)$$

Here we will only consider the second cohomology

$$\begin{aligned} \delta_2 \psi(x, y, z) &:= \psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y) \\ &\quad - x \cdot \psi(y, z) + y \cdot \psi(x, z) - z \cdot \psi(x, y), \\ (\delta_1 \phi)(x, y) &:= \phi([x, y]) - x \cdot \phi(y) + y \cdot \phi(x). \end{aligned} \quad (3.2)$$

A k -cochain ψ is called a *k-cocycle* if it lies in the kernel of the k -coboundary operator δ_k . It is called a *k-coboundary* if it lies in the image of the $(k - 1)$ -coboundary operator.

By $\delta_k \circ \delta_{k-1} = 0$ the vector space quotient of cocycles modulo coboundaries is well-defined. It is called the vector space of *k-Lie algebra cohomology* of W with values in the

module M . It is denoted by $H^k(W; M)$. Two cocycles which are in the same cohomology class are called *cohomologous*.

The trivial module is \mathbb{K} with the Lie action $x \cdot m = 0$, for all $x \in W$ and $m \in \mathbb{K}$. The second cohomology with values in the trivial module classifies equivalence classes of central extensions of W . It is well-known that for the Witt algebra \mathcal{W} we have $\dim H^2(\mathcal{W}; \mathbb{K}) = 1$ and that the class of the cocycle defining \mathcal{V} gives a basis.

The second cohomology of W with values in the adjoint module, $H^2(W; W)$, i.e. with module structure $x \cdot y := [x, y]$, is the cohomology to be studied here. As we will need the formula for explicit calculations later let me specialize the 2-cocycle condition (3.2)

$$\begin{aligned} \delta_2 \psi(x, y, z) := 0 &= \psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y) \\ &\quad - [x, \psi(y, z)] + [y, \psi(x, z)] - [z, \psi(x, y)]. \end{aligned} \quad (3.3)$$

A 2-cocycle will be a coboundary if it lies in the image of the (1-)coboundary operator, i.e. there exists a linear map $\phi : W \rightarrow W$ such that

$$\psi(x, y) = (\delta_1 \phi)(x, y) := \phi([x, y]) - [\phi(x), y] - [x, \phi(y)]. \quad (3.4)$$

The second cohomology $H^2(W, W)$ is related to the deformations of the Lie algebra W .

- (1) $H^2(W, W)$ classifies infinitesimal deformations of W up to equivalence (Gerstenhaber [9]).
- (2) If $\dim H^2(W, W) < \infty$ then there exists a versal formal family for the formal deformations of W whose base is formally embedded into $H^2(W, W)$. This is due to Fialowski [2], and Fialowski and Fuks [4].

A Lie algebra W is called *rigid* if every deformation is locally equivalent to the trivial family. Hence, if $H^2(W, W) = 0$ then W is infinitesimally and formally rigid. See [7] for more information on the connection between cohomology and deformations.

After having recalled the general definitions, I formulate the main result of this article.

Theorem 3.1. *Both the second cohomology of the Witt algebra W and of the Virasoro algebra \mathcal{V} (over a field \mathbb{K} with $\text{char}(\mathbb{K}) = 0$) with values in the adjoint module vanishes, i.e.*

$$H^2(\mathcal{W}; \mathcal{W}) = \{0\}, \quad H^2(\mathcal{V}; \mathcal{V}) = \{0\}. \quad (3.5)$$

Corollary 3.2. *Both \mathcal{W} and \mathcal{V} are formally and infinitesimally rigid.*

I refer to Section 1, the Introduction, for the history of this theorem. Here I only like to repeat that the first proof (at least in the Witt case) was given by Alice Fialowski by very cumbersome unpublished calculations [1]³. The proof which I present in the following is elementary, but much more accessible and elegant as the original proof.

³Note added after publication of the current article: In the meantime they are published, see [1] for details

4. THE DEGREE DECOMPOSITION OF THE COHOMOLOGY

Let W be an arbitrary \mathbb{Z} -graded Lie algebra, i.e.

$$W = \bigoplus_{n \in \mathbb{Z}} W_n. \quad (4.1)$$

Recall that we consider algebraic cohomology, i.e. our 2-cochains $\psi \in C^2(W; W)$ are arbitrary alternating bilinear maps in the usual sense, i.e. for all $v, w \in W$ the cochain $\psi(v, w)$ will be a finite linear combination (depending on v and w) of basis elements in W . We call a k -cochain ψ homogeneous of degree d if there exists a $d \in \mathbb{Z}$ such that for all $i_1, i_2, \dots, i_k \in \mathbb{Z}$ and homogeneous elements $x_{i_l} \in W$, of $\deg(x_{i_l}) = i_l$, for $l = 1, \dots, k$ we have that

$$\psi(x_{i_1}, x_{i_2}, \dots, x_{i_k}) \in W_n, \quad \text{with} \quad n = \sum_{l=1}^k i_l + d. \quad (4.2)$$

The corresponding subspace of degree d homogeneous k -cochains is denoted by $C_{(d)}^k(W; W)$. Every k -cochain can be written as a formal infinite sum

$$\psi = \sum_{d \in \mathbb{Z}} \psi_{(d)}. \quad (4.3)$$

Note that evaluated for a fixed k -tuple of elements only a finite number of the summands will produce values different from zero.

An inspection of the coboundary operators (3.3) and (3.4) for homogeneous elements x, y, z shows

Proposition 4.1. *The coboundary operators δ_k are operators of degree zero, i.e. applied to a k -cocycle of degree d they will produce a $(k+1)$ -cocycle also of degree d .*

In the following we will concentrate on $k = 2$ or $k = 1$. If $\psi = \sum_d \psi_{(d)}$ is a 2-cocycle then $\delta_2 \psi = \sum_d \delta_2 \psi_{(d)} = 0$. By Proposition 4.1 $\delta_2 \psi_{(d)}$ is either zero or of degree d . As we sum over different degrees and the terms cannot cancel if different from zero we obtain that ψ is cocycle if and only if all degree d components $\psi_{(d)}$ will be individually 2-cocycles. Moreover, if $\psi_{(d)}$ is 2-coboundary, i.e. $\psi_{(d)} = \delta_1 \phi$ with a 1-cochain ϕ , then we can find another 1-cochain ϕ' of degree d such that $\psi_{(d)} = \delta_1 \phi'$.

We summarize as follows. Every cohomology class $\alpha \in H^2(W; W)$ can be decomposed as formal sum

$$\alpha = \sum_{d \in \mathbb{Z}} \alpha_{(d)}, \quad \alpha_{(d)} \in H_{(d)}^2(W; W), \quad (4.4)$$

where the latter space consists of classes of cocycles of degree d modulo coboundaries of degree d .

For the rest of this section let W be either \mathcal{W} or \mathcal{V} and $d \neq 0$. We will show that for these algebras the cohomology spaces of degree $d \neq 0$ will vanish. The degree zero case needs a more involved treatment and will be done in the next sections. We start with a cocycle of degree $d \neq 0$ and make first a cohomological change $\psi' = \psi - \delta_1 \phi$ with

$$\phi : W \rightarrow W, \quad x \mapsto \phi(x) = \frac{\psi(x, e_0)}{d}. \quad (4.5)$$

Recall e_0 is the element of either \mathcal{W} or \mathcal{V} which gives the degree decomposition. This implies (note that by definition $\phi(e_0) = 0$)

$$\begin{aligned}\psi'(x, e_0) &= \psi(x, e_0) - (\delta_1 \phi)(x, e_0) = \psi(x, e_0) - \phi([x, e_0]) + [\phi(x), e_0] \\ &= d\phi(x) + \deg(x)\phi(x) - (\deg(x) + d)\phi(x) = 0.\end{aligned}\tag{4.6}$$

We evaluate (3.3) for the cocycle ψ' on the triple (x, y, e_0) and leave out the cocycle values which vanish due to (4.6):

$$\begin{aligned}0 &= \psi'([y, e_0], x) + \psi'([e_0, x], y) - [e_0, \psi'(x, y)] \\ &= (\deg(y) + \deg(x) - (\deg(x) + \deg(y) + d))\psi'(x, y) = -d\psi'(x, y).\end{aligned}\tag{4.7}$$

As $d \neq 0$ we obtain $\psi'(x, y) = 0$ for all $x, y \in W$. We conclude

Proposition 4.2. *The following hold:*

- (a) $H_{(d)}^2(\mathcal{W}; \mathcal{W}) = H_{(d)}^2(\mathcal{V}; \mathcal{V}) = \{0\}$, for $d \neq 0$.
- (b) $H^2(\mathcal{W}; \mathcal{W}) = H_{(0)}^2(\mathcal{W}; \mathcal{W})$, $H^2(\mathcal{V}; \mathcal{V}) = H_{(0)}^2(\mathcal{V}; \mathcal{V})$.

Remark 4.3. In fact the arguments also work for every \mathbb{Z} -graded Lie algebra W for which there exists an element e_0 such that the homogeneous spaces W_n are just the eigenspaces of e_0 under the adjoint action to the eigenvalue n . Such Lie algebras are called internally graded. See also Theorem 1.5.2 in [8].

5. DEGREE ZERO FOR THE WITT ALGEBRA

It remains the degree zero part. In this section we consider only the Witt algebra. Recall that the homogeneous subspaces of degree n are one-dimensional and generated by e_n . Hence, a degree zero cocycle can be written as $\psi(e_i, e_j) = \psi_{i,j}e_{i+j}$ and if it is a coboundary then it can be given as a coboundary of a linear form of degree zero: $\phi(e_i) = \phi_i e_i$. The systems of $\psi_{i,j}$ and ϕ_i for $i, j \in \mathbb{Z}$ fix ψ and ϕ completely. If we evaluate (3.3) for the triple (e_i, e_j, e_k) we get for the coefficients

$$\begin{aligned}0 &= (j-i)\psi_{i+j,k} - (k-i)\psi_{i+k,j} + (k-j)\psi_{j+k,i} \\ &\quad - (j+k-i)\psi_{j,k} + (i+k-j)\psi_{i,k} - (i+j-k)\psi_{i,j}.\end{aligned}\tag{5.1}$$

For the coboundary we obtain

$$(\delta\phi)_{i,j} = (j-i)(\phi_{i+j} - \phi_j - \phi_i).\tag{5.2}$$

Hence, ψ is a coboundary if and only if there exists a system of $\phi_k \in \mathbb{K}$, $k \in \mathbb{Z}$ such that

$$\psi_{i,j} = (j-i)(\phi_{i+j} - \phi_j - \phi_i), \quad \forall i, j \in \mathbb{Z}.\tag{5.3}$$

A degree zero 1-cochain ϕ will be a 1-cocycle (i.e. $\delta_1 \phi = 0$) if and only if

$$\phi_{i+j} - \phi_j - \phi_i = 0.\tag{5.4}$$

This has the solution $\phi_i = i\phi_1, \forall i \in \mathbb{Z}$. Hence, given a ϕ we can always find a ϕ' with $(\phi')_1 = 0$ and $\delta_1 \phi = \delta_1 \phi'$. In the following we will always choose such a ϕ' for our 2-coboundaries.

Step 1: We make a cohomological change

We start with a 2-cocycle ψ given by the system of $\psi_{i,j}$ and will modify it by adding a coboundary $\delta_1\phi$ with suitable ϕ to obtain $\psi' = \psi - \delta_1\phi$. We will determine ϕ inspired by the intended relation (5.3)

$$\psi_{i,1} = (1-i)(\phi_{i+1} - \phi_1 - \phi_i) = (1-i)(\phi_{i+1} - \phi_i). \quad (5.5)$$

Note that we could put $\phi_1 = 0$ by our normalization.

(a) Starting from $\phi_0 := -\psi_{0,1}$ we set in descending order for $i \leq -1$

$$\phi_i := \phi_{i+1} - \frac{1}{1-i} \psi_{i,1}. \quad (5.6)$$

(b) ϕ_2 cannot be fixed by (5.5), instead we use (5.3)

$$\psi_{-1,2} = 3(-\phi_2 - \phi_{-1}), \quad \text{yielding} \quad \phi_2 := -\phi_{-1} - \frac{1}{3}\psi_{-1,2}. \quad (5.7)$$

Then we have $\psi'_{-1,2} = 0$.

(c) We use again (5.5) to calculate recursively in ascending order ϕ_i , $i \geq 3$ by

$$\phi_{i+1} := \phi_i + \frac{1}{1-i} \psi_{i,1}. \quad (5.8)$$

For the cohomologous cocycle ψ' we obtain by construction

$$\psi'_{i,1} = 0, \quad \forall i \in \mathbb{Z}, \quad \text{and} \quad \psi'_{-1,2} = \psi'_{2,-1} = 0. \quad (5.9)$$

Step2: Show that ψ' is identical zero.

To avoid cumbersome notation we will denote the cohomologous cocycle by ψ .

Lemma 5.1. *Let ψ be a 2-cocycle of degree zero such that $\psi_{i,1} = 0, \forall i \in \mathbb{Z}$ and $\psi_{-1,2} = 0$, then ψ will be identical zero.*

Before we proof the lemma we use it to show

Proof. (Witt part of Theorem 3.1.) By Proposition 4.2 it is enough to consider degree zero cohomology. By the cohomological change done in Step 1 every degree zero cocycle ψ is cohomologous to a cocycle fulfilling the conditions of Lemma 5.1. But such a cocycle vanishes by the lemma. Hence, the original cocycle ψ is cohomological trivial. \square

Proof. (Lemma 5.1.) First, we note two special cases of (5.1) which will be useful in the following. For the index triple $(i, -1, k)$ we obtain

$$\begin{aligned} 0 = & -(i+1)\psi_{i-1,k} - (k-i)\psi_{i+k,-1} + (k+1)\psi_{k-1,i} \\ & - (-1+k-i)\psi_{-1,k} + (i+k+1)\psi_{i,k} - (i-1-k)\psi_{i,-1}. \end{aligned} \quad (5.10)$$

For the triple $(i, 1, k)$, and ignoring terms of the type $\psi_{i,1}$ which are zero by assumption, we obtain

$$0 = (1-i)\psi_{i+1,k} + (k-1)\psi_{k+1,i} + (i+k-1)\psi_{i,k}. \quad (5.11)$$

We will consider $\psi_{i,m}$ for certain values of $|m| \leq 2$ and finally make ascending and descending induction on m . We will call the coefficient $\psi_{i,m}$ coefficients of level m (and of level i by antisymmetry). By assumption the cocycle values of level 1 are all zero.

$m = 0$

By the antisymmetry we have $\psi_{1,i} = -\psi_{i,1} = 0$. In (5.11) we consider $k = 0$ this gives

$$0 = (1 - i)(\psi_{i+1,0} - \psi_{i,0}). \quad (5.12)$$

Starting from $\psi_{1,0} = 0$ this implies for $i \leq 0$ that $\psi_{i,0} = 0$ and for $i \geq 3$ that $\psi_{i,0} = \psi_{2,0}$. Next we consider (5.10) for $k = 2, i = 0$ and obtain

$$0 = -\psi_{-1,2} - 2\psi_{2,-1} + 3\psi_{1,0} - \psi_{-1,2} + 3\psi_{0,2} + 3\psi_{0,-1}. \quad (5.13)$$

The $\psi_{-1,2}$ terms cancel and we know already $\psi_{1,0} = \psi_{-1,0} = 0$, hence $\psi_{2,0} = 0$. This implies

$$\psi_{i,0} = 0 \quad \forall i \in \mathbb{Z}. \quad (5.14)$$

$$\boxed{m = -1}$$

In (5.11) we set $k = -1$ and obtain (with $\psi_{0,i} = 0$)

$$-(i - 1)\psi_{i+1,-1} + (i - 2)\psi_{i,-1} = 0. \quad (5.15)$$

Hence,

$$\psi_{i,-1} = \frac{i-1}{i-2}\psi_{i+1,-1}, \quad \text{for } i \neq 2, \quad \psi_{i+1,-1} = \frac{i-2}{i-1}\psi_{i,-1}, \quad \text{for } i \neq 1. \quad (5.16)$$

The first formula implies starting from $\psi_{1,-1} = -\psi_{-1,1} = 0$ that $\psi_{i,-1} = 0$, for all $i \leq 1$. The second formula for $i = 2$ implies $\psi_{3,-1} = 0$ and hence $\psi_{i,-1} = 0$ for $i \geq 3$. But by assumption $\psi_{2,-1} = \psi_{-1,2} = 0$. Hence,

$$\psi_{i,-1} = 0 \quad \forall i \in \mathbb{Z}. \quad (5.17)$$

$$\boxed{m = -2}$$

We plug the value $k = -2$ into (5.11) and get for the terms not yet identified as zero

$$(1 - i)\psi_{i+1,-2} + (i - 3)\psi_{i,-2} = 0. \quad (5.18)$$

This yields

$$\psi_{i+1,-2} = \frac{i-3}{i-1}\psi_{i,-2}, \quad \text{for } i \neq 1, \quad \psi_{i,-2} = \frac{i-1}{i-3}\psi_{i+1,-2}, \quad \text{for } i \neq 3. \quad (5.19)$$

From the first formula we get $\psi_{3,-2} = -\psi_{2,-2}$, $\psi_{4,-2} = 0 \cdot \psi_{3,-2}$, and hence $\psi_{i,-2} = 0$, for all $i \geq 4$.

From the second formula we get starting from $\psi_{1,-2} = 0$ that $\psi_{i,-2} = 0$ for $i \leq 1$.

Altogether, $\psi_{i,-2} = 0$ for $i \neq 2, 3$. The value of $\psi_{2,-2} = -\psi_{3,-2}$ stays undetermined for the moment.

$$\boxed{m = 2}$$

We start from (5.10) for $k = 2$ and recall that terms of levels 0, 1, -1 are zero. This gives

$$-(i + 1)\psi_{i-1,2} + (i + 3)\psi_{i,2} = 0. \quad (5.20)$$

Hence,

$$\psi_{i,2} = \frac{i+1}{i+3}\psi_{i-1,2}, \quad \text{for } i \neq -3, \quad \psi_{i-1,2} = \frac{i+3}{i+1}\psi_{i,2}, \quad \text{for } i \neq -1. \quad (5.21)$$

From the first formula we start from $\psi_{-1,2} = 0$ and get $\psi_{i,2} = 0, \forall i \geq -1$. From the second we get $\psi_{-3,2} = -\psi_{-2,2}$, then $\psi_{-4,2} = 0$ and then altogether $\psi_{i,2} = 0$ for all $i \neq -2, -3$. The value $\psi_{-3,2} = -\psi_{-2,2}$ stays undetermined for the moment.

To find it we consider the index triple $(2, -2, 4)$ in (5.1) and obtain after leaving out terms which are obviously zero

$$0 = -2\psi_{6,-2} - 8\psi_{4,2} + 4\psi_{2,-2}. \quad (5.22)$$

From the level $m = 2$ discussion we get $\psi_{4,2} = 0$, from $m = -2$ we get $\psi_{6,-2} = 0$. This shows that $\psi_{2,-2} = \psi_{3,-2} = \psi_{-3,2} = 0$ and we can conclude

$$\psi_{i,-2} = \psi_{i,2} = 0, \quad \forall i \in \mathbb{Z}. \quad (5.23)$$

$$\boxed{m < -2}$$

We make induction assuming it is true for $m = 2, 1, 0, -1, -2$. We start from (5.10) for k and put the $k - 1$ level element on the l.h.s. By this

$$(k + 1)\psi_{i,k-1} = \text{terms of level } k \text{ and } -1. \quad (5.24)$$

By induction the terms on the r.h.s. are zero. Note that in this region $k < -1$, hence $k + 1 \neq 0$, and $\psi_{i,k-1} = 0$, too.

$$\boxed{m > 2}$$

Again we make induction. Starting from (5.11) we get

$$(k - 1)\psi_{i,k+1} = (1 - i)\psi_{i+1,k} + (i + k - 1)\psi_{i,k}. \quad (5.25)$$

As $k \geq 2$ the value of $1 - k \neq 0$ and we get by induction trivially the statement for $k + 1$.

Altogether we obtain $\psi_{i,k} = 0, \forall i, k \in \mathbb{Z}$. \square

The presented calculation is completely different and much simpler as in [1]

6. EXTENSION TO THE VIRASORO ALGEBRA

Here we show in an elementary way that also for the Virasoro algebra

$$H^2(\mathcal{V}, \mathcal{V}) = 0. \quad (6.1)$$

Hence \mathcal{V} is also infinitesimally and formally rigid. This shows the Virasoro part of Theorem 3.1. It is our intention not to use any higher techniques, but see Remark 6.4 at the end of this section.

First recall that by Proposition 4.2 it is enough to consider degree zero cocycles. We start with a degree zero 2-cocycle $\psi : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ of the Virasoro algebra. If we apply the Lie homomorphism ν we get the bilinear map $\nu \circ \psi$ which we restrict to $\mathcal{W} \times \mathcal{W}$

$$\psi' = \nu \circ \psi| : \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{W}. \quad (6.2)$$

Unfortunately, in general ψ' will not be a 2-cocycle for the Witt algebra. We have to be careful as the Lie product for \mathcal{V} differs from that of \mathcal{W} by multiples of the central element. But we are allowed to make cohomologous changes.

Proposition 6.1. *Given a cocycle $\psi \in C^2(\mathcal{V}, \mathcal{V})$ there exists a cohomologous one $\tilde{\psi} \in C^2(\mathcal{V}, \mathcal{V})$ such that the bilinear map $\psi' = \nu \circ \tilde{\psi} \in C^2(\mathcal{W}, \mathcal{W})$.*

Proof. Let $x, y, z \in \mathcal{W}$. We have $[x, y]_{\mathcal{V}} = [x, y]_{\mathcal{W}} + \alpha(x, y) \cdot t$. We consider (3.3) with the bracket $[\cdot, \cdot]_{\mathcal{V}}$ and rewrite it in terms of $[\cdot, \cdot]_{\mathcal{W}}$ (but drop the index) and α . For the second group of terms we use that ν is a Lie homomorphism and they will not see the central elements. Only in the first group they will play a role. We get

$$\begin{aligned} 0 &= \psi'([x, y], z) + \psi'([y, z], x) + \psi'([z, x], y) \\ &\quad + \alpha([x, y], z) \nu \circ \psi(t, z) + \alpha([y, z], x) \nu \circ \psi(t, x) + \alpha([z, x], y) \nu \circ \psi(t, y) \\ &\quad - [x, \psi'(y, z)] + [y, \psi'(x, z)] - [z, \psi'(x, y)]. \end{aligned} \quad (6.3)$$

We will make a cohomological change for the cocycle ψ in \mathcal{V} such that $\psi(t, z)$ will have only central terms, i.e. $\nu \circ \psi(t, z)$ will vanish. Restricting it to $\mathcal{W} \times \mathcal{W}$ and projecting it by ν will define a 2-cocycle for \mathcal{W} .

As ψ is a degree zero cocycle, we have $\psi(e_1, t) = a e_1$. We set $\phi(t) := a e_0$, and $\phi(e_n) := 0$ for all $n \in \mathbb{Z}$. Let $\tilde{\psi} := \psi - \delta_1 \phi$. We calculate

$$\begin{aligned} \tilde{\psi}(e_1, t) &= \psi(e_1, t) - (\delta_1 \phi)(e_1, t) = a e_1 - \phi([e_1, t]) + [e_1, \phi(t)] + [\phi(e_1), t] \\ &= a e_1 - 0 - a e_1 + 0 = 0. \end{aligned} \quad (6.4)$$

The coefficients a_n (and b , but it will not play any role) are given by

$$\tilde{\psi}(e_n, t) = a_n e_n, \quad n \neq 0, \quad \tilde{\psi}(e_0, t) = a_0 e_n + b \cdot t. \quad (6.5)$$

Note that $a_1 = 0$. If we evaluate (3.3) for the triple (e_n, e_m, t) and leave out terms which are zero due to the fact, that t is central we get

$$0 = \tilde{\psi}([e_n, e_m]_{\mathcal{V}}, t) - [e_n, \tilde{\psi}(e_m, t)]_{\mathcal{V}} + [e_m, \tilde{\psi}(t, e_n)]_{\mathcal{V}}. \quad (6.6)$$

This implies (by evaluating the coefficients at the element e_{m+n})

$$(m - n)(a_{n+m} - a_m - a_n) = 0. \quad (6.7)$$

If we plug in $m = 1$ and use $a_1 = 0$ we get $a_n = 0$ for all $n \leq 1$ and $a_n = a_2$ for all $n \geq 2$. Now plugging in $m = 2, n = -2$ we obtain $a_2 = -a_{-2} = 0$. Hence altogether $a_n = 0$ for all n , and $\nu \circ \tilde{\psi}(w, t) = 0, \forall w \in \mathcal{V}$. \square

Hence, after a cohomologous change we may assume that our restricted and projected ψ' will be a 2-cocycle for the algebra \mathcal{W} with values in the adjoint module \mathcal{W} . In the last section we showed that it is cohomologically trivial, i.e. there exists a $\phi' : \mathcal{W} \rightarrow \mathcal{W}$ such that $\delta_1^{\mathcal{W}} \phi' = \psi' = \nu \circ \psi$. We denote for a moment the corresponding coboundary operators of the two Lie algebras by $\delta^{\mathcal{W}}$ and $\delta^{\mathcal{V}}$. By setting $\phi(t) := 0$ we extend ϕ' to a linear map $\phi : \mathcal{V} \rightarrow \mathcal{V}$. In particular, $\nu \circ \phi = \phi'$ if restricted to \mathcal{W} .

The 2-cocycle $\hat{\psi} = \psi - \delta_1^{\mathcal{V}} \phi$ will be a cohomologous cocycle for \mathcal{V} . If we apply ν then

$$\nu \hat{\psi} = \nu \psi - \nu \delta_1^{\mathcal{V}} \phi = \psi' - \delta_1^{\mathcal{W}} \phi' = 0. \quad (6.8)$$

Hence, $\hat{\psi}$ takes values in the kernel of ν , i.e.

$$\hat{\psi} : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{K} \cdot t, \quad (6.9)$$

with t the central element.

This implies that it is enough to show that every class $\psi : \mathcal{V} \rightarrow \mathcal{V}$ with values in the central ideal $\mathbb{K} \cdot t$ will be a coboundary to show the vanishing of $H^2(\mathcal{V}, \mathcal{V})$. In the following let ψ be of this kind.

As $\psi(x, y) = \psi_{x,y}t$ will be central the 2-cocycle condition (3.3) will reduce to

$$(\delta_2\psi)(x, y, z) = \psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y) = 0. \quad (6.10)$$

The coboundary condition for $\phi : \mathcal{V} \rightarrow \mathbb{K} \cdot t$ reduces to

$$(\delta_1\phi)(x, y) = \phi([x, y]). \quad (6.11)$$

We can reformulate this as that the component function $\psi_{x,y} : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{K}$ is a Lie algebra 2-cocycle for \mathcal{V} with values in the trivial module.

Lemma 6.2. *We have*

$$\psi(x, t) = 0, \quad \forall x \in \mathcal{V}. \quad (6.12)$$

Proof. We evaluate (6.10) for (e_i, e_j, t) . As t is central $[e_j, t] = [t, e_i] = 0$, hence only the first term in (6.10) will survive and we get $(j - i)\psi(e_{i+j}, t) = 0$. Choosing $j = 0$ we obtain $\psi(e_i, t) = 0$ for $i \neq 0$, choosing $i = -1, j = +1$ we obtain also $\psi(e_0, t) = 0$. As $\psi(t, t) = 0$ is automatic and $\{e_n, n \in \mathbb{Z}, t\}$ is a basis of \mathcal{V} this shows the result. \square

Next let $\phi : \mathcal{V} \rightarrow \mathbb{K} \cdot t$ be a linear map. We write for the component functions $\phi(e_i) = \phi_i t$ and $\phi(t) = c \cdot t$. If we evaluate the 1-coboundary operator (6.11) for pairs of basis elements we get

$$\delta_1\phi(e_i, e_j) = \phi([e_i, e_j]) = \left((j - i)\phi_{i+j} - \frac{1}{12}(i^3 - i)\delta_i^{-j}c \right) t, \quad i, j \in \mathbb{Z}, \quad (6.13)$$

and by Lemma 6.2 we get $\delta_1\phi(e_i, t) = 0$, for all $i \in \mathbb{Z}$.

With respect to (6.13) we choose ϕ such that the cohomologous cocycle $\psi' = \psi - \delta_1\phi$ fulfills

$$\psi'(e_i, e_0) = 0, \quad \forall i \in \mathbb{Z}, \quad \text{and} \quad \psi'(e_1, e_{-1}) = \psi'(e_2, e_{-2}) = 0. \quad (6.14)$$

This is obtained by putting

$$\phi_i := -\frac{1}{i}\psi_{e_i, e_0}, \quad i \in \mathbb{Z}, i \neq 0, \quad \phi_0 := -(1/2)\psi_{e_1, e_{-1}}, \quad c := (-2\psi_{e_2, e_{-2}} - 8\phi_0). \quad (6.15)$$

Lemma 6.3. *The cocycle ψ' is identically zero.*

Proof. First we consider the 2-cocycle condition (6.10) for the triple (e_i, e_j, e_0) and obtain using $\psi'(e_i, t) = 0$ that

$$0 = 0 + (-j)\psi'(e_j, e_i) + i\psi'(e_i, e_j) = (i + j)\psi'(e_i, e_j). \quad (6.16)$$

This yields that

$$\psi'(e_i, e_j) = 0 \quad \text{for } j \neq -i. \quad (6.17)$$

Next we show by induction that $\psi'(e_i, e_{-i}) = 0$ for all $i \geq 0$ (and by antisymmetry also for $i \leq 0$). Note that this is true for $i = 0, 1$ and 2 , see (6.14). Consider (6.10) for the triple $(e_n, e_{-(n-1)}, e_{-1})$, $n > 2$. After evaluation of the Lie products we get

$$0 = (-2n + 1)\psi'(e_1, e_{-1}) + (n - 2)\psi'(e_{-n}, e_n) + (-n + 1)\psi'(e_{n-1}, e_{-(n-1)}). \quad (6.18)$$

The first and last term vanishes by induction. As $n > 2$ this implies

$$\psi'(e_n, e_{-n}) = 0. \quad (6.19)$$

As we have always $\psi'(x, t) = 0$ this shows the lemma. \square

Finally we obtain that the ψ we started with was a coboundary. This implies indeed $H^2(\mathcal{V}; \mathcal{V}) = \{0\}$.

Remark 6.4. What has been done in this section can also be interpreted in the framework of long exact cohomology sequences in Lie algebra cohomology. The exact sequence (2.5) of Lie algebras is also a exact sequence of Lie modules over \mathcal{V} . For such sequences we have a long exact sequence in cohomology. If we consider only level two we get

$$\cdots \longrightarrow H^2(\mathcal{V}; \mathbb{K}) \longrightarrow H^2(\mathcal{V}; \mathcal{V}) \xrightarrow{\nu^*} H^2(\mathcal{V}; \mathcal{W}) \longrightarrow \cdots. \quad (6.20)$$

Proposition 6.1 also shows that naturally $H^2(\mathcal{V}; \mathcal{W}) \cong H^2(\mathcal{W}; \mathcal{W})$ (which is not a consequence from the existence of the long exact sequence). The calculations of Lemma 6.2 and Lemma 6.3 showed $H^2(\mathcal{V}; \mathbb{K}) = \{0\}$. From Section 5 we know $H^2(\mathcal{W}; \mathcal{W}) = \{0\}$. Hence, also $H^2(\mathcal{V}; \mathcal{V}) = \{0\}$.

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