Disaggregation of bipolar-valued outranking relations and application to the inference of model parameters

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Some questions beforehand

What data is **underlying** a bipolar-valued outranking relation?

How do these data **look like**?

Can we **help** the decision maker to determine the parameters of the model?

Structure of the presentation

- Introduction
- Models for the bipolar-valued outranking relation
- Disaggregation of bipolar-valued outranking relation
- On the rank of a bipolar-valued outranking relation
- Illustrative examples
- Usefulness in MCDA: inference of model parameters

Introductive considerations

Recall a few facts . . .

- X is a finite set of n **alternatives**
- N is a finite set of p **criteria**
- $g_i(x)$ is the **performance** of alternative x on criterion i
- $w_i \in [0,1]$, rational, is the **weight** associated with criterion i of N, s.t. $\sum_{i \in N} w_i = 1$
- q_i, p_i, wv_i and v_i are **thresholds** associated with each criterion i to model local or overall *at least as good as* preferences

Recall a few facts . . .

- $xSy \equiv$ "x **outranks** y"
- **Classically**: An outranking situation xSy between two alternatives x and y of X is assumed to hold if there is a **sufficient majority** of criteria which supports an "*at least as good as*" preferential statement and there is no criterion which raises **a veto** against it
- $\widetilde{S}(x,y) \in [-1,1]$ is the **credibility of the validation** of the statement xSy
- $\widetilde{S}$ is called the **bipolar-valued** outranking relation

Goal

Primary objective

Disaggregate the bipolar-valued outranking relation to determine how the underlying data looks like

In other words:
Given $\widetilde{S}(x,y)$ $\forall x \neq y \in X$ and thresholds q_i, p_i, wv_i, v_i $\forall i \in N$, determine the ***performances*** *of alternatives $g_i(x)$ $\forall x \in X, \forall i \in N$, and the* ***weights*** *w_i $\forall i \in N$*

3 different models:

- $\mathcal{M}_1$: Model with a single preference threshold
- $\mathcal{M}_2$: Model with two preference thresholds
- $\mathcal{M}_3$: Model with two preference and two veto thresholds

Goal

Secondary objective

Infer model parameters based on **a priori** knowledge provided by the decision maker

In other words:
Given the ***performances*** *$g_i(x)$ $\forall x \in X$ $\forall i \in N$ and some* ***a priori*** *info from the decision maker, determine the values of the thresholds and the weights*

Usefulness in Multiple Criteria Decision Analysis (MCDA):
Help to elicit the decision maker's **preferences** via questions on his domain of expertise

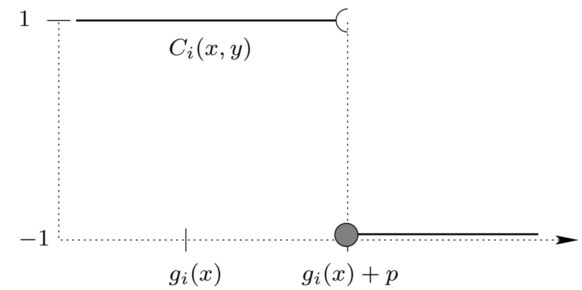
Different models for the outranking relation

$\mathcal{M}_1$: Model with a single preference threshold

A local "*at least as good as*" situation between two alternatives x and y of X, for each criterion i of N is represented by the function $C_i : X \times X \rightarrow \{0,1\}$ defined by:

$$C_i(x,y) = \begin{cases} 1 & \text{if} \quad g_i(y) < g_i(x) + p\,; \\ -1 & \text{otherwise}\,, \end{cases}$$

where $p \in]0,1[$ is a constant **preference threshold** associated with all the preference dimensions

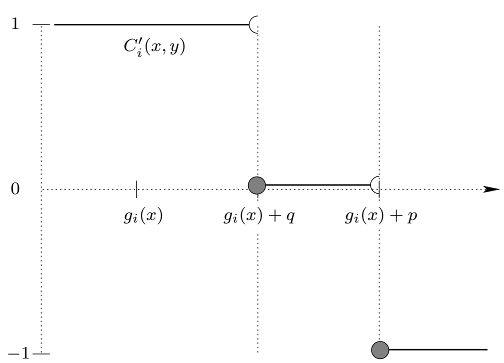


$\mathcal{M}_2$: Model with two preference thresholds

A local "*at least as good as*" situation between two alternatives x and y of X, for each criterion i of N is represented by the function $C_i' : X \times X \rightarrow \{-1,0,1\}$ s.t.:

$$C_i'(x,y) = \begin{cases} 1 & \text{if} \quad g_i(y) < g_i(x) + q\,; \\ -1 & \text{if} \quad g_i(y) \geq g_i(x) + p\,; \\ 0 & \text{otherwise}\,, \end{cases}$$

where $q \in]0,p[$ is a constant **weak preference** threshold associated with all the preference dimensions.



$\mathcal{M}_1$ & $\mathcal{M}_2$

Bipolar-valued outranking relation

$$\widetilde{S}'(x,y) = \sum_{i \in N} w_i C_i'(x,y) \quad \forall x \neq y \in X$$

Recall:
$\widetilde{S}'(x,y) \in [-1,1]$ represents the credibility of the validation of the outranking situation xSy

Meaning of $\widetilde{S}'$:

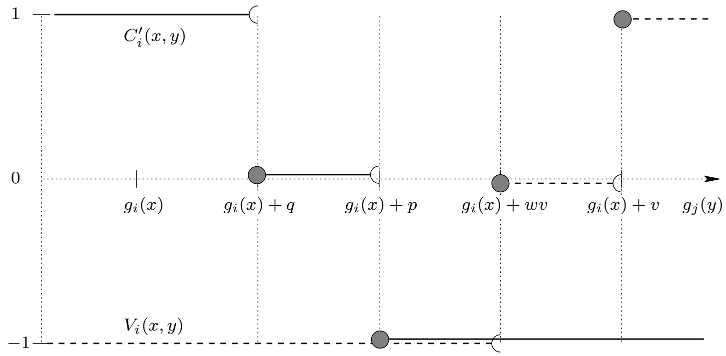
- $\widetilde{S}(x,y) = +1$ means that statement xSy is **clearly validated**.
- $\widetilde{S}(x,y) = -1$ means that statement xSy is **clearly not validated**.
- $\widetilde{S}(x,y) > 0$ means that statement xSy is **more validated than not validated**.
- $\widetilde{S}(x,y) < 0$ means that statement xSy is **more not validated than validated**.
- $\widetilde{S}(x,y) = 0$ means that statement xSy is **indeterminate**.

$\mathcal{M}_3$: Model with two preference and two veto thresholds

A *local veto* situation for each criterion i of N is characterised by a veto function $V_i : X \times X \to \{-1, 0, 1\}$ s.t.:

$$V_i(x,y) = \begin{cases} 1 & \text{if} \quad g_i(y) \geq g_i(x) + v\,; \\ -1 & \text{if} \quad g_i(y) < g_i(x) + wv\,; \\ 0 & \text{otherwise}\,, \end{cases}$$

where $wv \in]p, 1[$ (resp. $v \in]wv, 1[$) is a constant **weak veto threshold** (resp. **veto threshold**) associated with all the preference dimensions



$\mathcal{M}_3$

Bipolar-valued outranking relation

$$\widetilde{S}''(x,y) = \min\Big\{\sum_{i\in N} w_i C_i'(x,y), -V_1(x,y), \ldots, -V_n(x,y)\Big\}.$$

Note:
The min operator tranlsates the **conjunction** between the overall concordance and the negated local veto indexes for each criterion

Disaggregation of the outranking relation

How?

Objective

Disaggregate the bipolar-valued outranking relation to determine how the underlying data looks like

How?

By **mathematical programming**!

$\Rightarrow$ **Given** $\widetilde{S}$, **determine** $g_i(x)$ $(\forall i \in N, \forall x \in X)$ and w_i $(\forall i \in N)$

Disaggregation of $\mathcal{M}_1$ by mathematical programming

Minimise the number of active criteria

P1:

Variables:

$g_i(x) \in [0,1]$ $\quad \forall i \in N, \forall x \in X$

$w_i \in [0,1]$ $\quad \forall i \in N$

$W_i \in \{0,1\}$ $\quad \forall i \in N$

$C_i(x,y) \in \{-1,1\}$ $\quad \forall i \in N, \forall x \neq y \in X$

Parameters:

$p \in]0,1[$

$\widetilde{S}(x,y) \in [-1,1]$ $\quad \forall x \neq y \in X$

$\delta \in]0,p[$

Objective function:

min $\sum_{i=1}^{n} W_i$

Constraints:

s.t. $\sum_{i=1}^{n} w_i = 1$

$w_i \leq W_i$ $\quad \forall i \in N$

$\sum_{i=1}^{n} w_i C_i(a,b) = \widetilde{S}(x,y)$ $\quad \forall x \neq y \in X$

$(-1+p)(C_i(x,y)-1)+\delta \leq g_i(x) - g_i(y) + p$ $\quad \forall x \neq y \in X, \forall i \in N$

$g_i(x) - g_i(y) + p \leq (1+p)(C_i(x,y)+1)$ $\quad \forall x \neq y \in X, \forall i \in N$

Disaggregation of $\mathcal{M}_1$ by mathematical programming

Minimise the number of active criteria

If no solution exists:

- The selected maximal number n of criteria is too **small**
- The model with a constant preference threshold ($\mathcal{M}_1$) is **too poor** to represent the given $\widetilde{S}$
- p is chosen **inappropriately** and does not allow the $g_i(x)$ to take enough distinct values in $[0,1]$
- ...

Disaggregation of $\mathcal{M}_1$ by mathematical programming

Minimise the number of active criteria

OK, but what if there are some slight **errors** in the given $\widetilde{S}$?

Disaggregation of $\mathcal{M}_1$ by mathematical programming

Minimise the maximal gap between the given and the calculated $\widetilde{S}$

MIP1bis:

Variables:

$\varepsilon \geq 0$

$g_i(x) \in [0,1]$ $\quad \forall i \in N, \forall x \in X$

$w_i \in [0,1]$ $\quad \forall i \in N$

$C_i(x,y) \in \{0,1\}$ $\quad \forall i \in N, \forall x \neq y \in X$

$w_i' \in [-1,1]$ $\quad \forall i \in N$

Parameters:

$p \in]0,1[$

$\widetilde{S}(x,y) \in [0,1]$ $\quad \forall x \neq y \in X$

$\delta \in]0,p[$

Objective function:

min ε

Constraints:

s.t. $\sum_{i=1}^{n} w_i = 1$

...

$\sum_{i=1}^{n} w_i'(x,y) \leq \widetilde{S}(x,y) + \varepsilon$ $\quad \forall x \neq y \in X$

$\sum_{i=1}^{n} w_i'(x,y) \geq \widetilde{S}(x,y) - \varepsilon$ $\quad \forall x \neq y \in X$

...

Disaggregation of $\mathcal{M}_1$ by mathematical programming

Minimise the maximal gap between the given and the calculated $\widetilde{S}$

Motivations:

- By construction, $\widetilde{S}(x, y)$ is **rational** in $[-1, 1]$
- If the decimal expansion of a rational number $r \in [-1, 1]$ is **periodic**, then r is hardly representable as a float
- Consequently, the value stored for $\widetilde{S}(x, y)$ might be an **approximation**
- In such a case, **P1** might have **no solution**

Discussion:

- If $\varepsilon = 0$, then there exist $g_i(x)$ $(\forall i \in N, \forall x \in X)$ and associated weights w_i $(\forall i \in N)$ generating $\widetilde{S}$ via $\mathcal{M}_1$
- Else there exists no solution to the problem via the selected representation, and the output of **MIP1bis** is an approximation of $\widetilde{S}$ by $\mathcal{M}_1$

Disaggregation of $\mathcal{M}_2$ and $\mathcal{M}_3$

Similar as $\mathcal{M}_1$ via mixed integer programs by minimising ε

MIP3:

Variables:

$\varepsilon \geq 0$	
$g_i(x) \in [0, 1]$	$\forall i \in N, \forall x \in X$
$w_i \in]0, 1]$	$\forall i \in N$
$\alpha_i(x, y) \in \{0, 1\}$	$\forall i \in N, \forall x \neq y \in X$
$\beta_i(x, y) \in \{0, 1\}$	$\forall i \in N, \forall x \neq y \in X$
$\alpha'_i(x, y) \in \{0, 1\}$	$\forall i \in N, \forall x \neq y \in X$
$\beta'_i(x, y) \in \{0, 1\}$	$\forall i \in N, \forall x \neq y \in X$
...	

Parameters:

$q \in]0, p[$	
$p \in]q, 1[$	
$wv \in]p, 1[$	
$v \in]wv, 1[$	
$\widetilde{S}''(x, y) \in [0, 1]$	$\forall x \neq y \in X$
$\delta \in]0, q[$	

Objective function:

min $\quad \varepsilon$

Constraints:

s.t. $\quad \sum_{i=1}^{n} w_i = 1$

...

On the rank of the outranking relation

On the rank of a bipolar-valued outranking relation

Definition

The **rank** of a bipolar-valued outranking relation is given by the minimal number of criteria necessary to construct it via the selected model.

Practical determination:

- **MIP1**: the objective function gives the rank of $\widetilde{S}$
- **MIP1bis**, **MIP2**, **MIP3**:
  - $n := 0$;
  - do {
    - $n++$;
    - solve the optimisation problem;
  - } while $\varepsilon > 0$;
  - **rank** $= n$;

Note: The algorithm might never stop, if $\widetilde{S}$ cannot be constructed by the chosen model

Illustrative examples

Illustration

MIP1 & MIP1bis ($p = 0.1$, $\delta = 0.001$, $n = 5$):

$\widetilde{S}_1$	a	b	c
a	·	0.258	-0.186
b	0.334	·	0.556
c	0.778	0.036	·

	g_1	g_2	g_3	g_4
a	1.000	0.100	0.000	0.000
b	0.900	0.000	1.000	0.000
c	0.000	0.200	0.100	0.099
w_i	0.111	0.222	0.371	0.296

MIP1: there exists an optimal solution for 4 criteria

MIP1bis:

- for $n \geq 4$: optimal solution with $\varepsilon = 0$
- for $n < 4$: optimal solutions with $\varepsilon > 0$

$\Rightarrow$ **rank($\widetilde{S}_1$) = 4 under $\mathcal{M}_1$**

Illustration

MIP2 & MIP3 ($q = 0.1$, $p = 0.2$, $wv = 0.6$ and $v = 0.8$, $\delta = 0.001$, $n = 5$):

$\widetilde{S}_2$	a	b	c
a	·	0.258	-0.186
b	0.334	·	0.556
c	-1.000	0.036	·

$\widetilde{S}_2^{\mathbf{MIP2}}$	a	b	c	g_1	g_2
a	·	0.407	0.407	0.280	0.000
b	0.296	·	1.000	0.090	1.000
c	-0.407	0.407	·	0.000	0.200
w_i				0.704	0.296

MIP2: for $n = 4$: opt. sol. with $\varepsilon = 0.593$

MIP3:

- for $n \geq 4$: optimal solution with $\varepsilon = 0$
- for $n < 4$: optimal solution with $\varepsilon > 0$

MIP3	g_1	g_2	g_3	g_4
a	0.000	0.000	0.000	1.000
b	0.400	0.100	0.090	0.590
c	0.200	0.290	0.000	0.000
w_i	0.149	0.444	0.074	0.333

$\Rightarrow$ **rank($\widetilde{S}_2$) = 4 under $\mathcal{M}_3$**

Note: Veto between c and a on criterion 4 ($\widetilde{S}(c,a) = -1$)

On the inference of model parameters

Usefulness in MCDA: inference of model parameters

In real-world decision problems involving multiple criteria:

- Performances $g_i(x)$ ($\forall i \in N$, $\forall x \in X$) are **known**
- Weights and thresholds are usually **unknown**

Objective

Show how these parameters can be determined from **a priori** knowledge provided by the decision maker

A priori information

In our context, the **a priori** preferences of the decision maker could take the form of:

- a partial weak order over the credibilities of the validation of outrankings;
- a partial weak order over the importances of some criteria;
- quantitative intuitions about some credibilities of the validation of outrankings;
- quantitative intuitions about the importance of some criteria;
- quantitative intuitions about some thresholds;
- subsets of criteria important enough for the validation of an outranking situation;
- subsets of criteria not important enough for the validation of an outranking situation;
- etc.

A priori information: constraints

- the validation of wSx is strictly more credible than that of ySz can be translated as $\widetilde{S}(w,x) - \widetilde{S}(y,z) \geq \delta$;
- the validation of wSx is similar to that of ySz can be translated as $-\delta \leq \widetilde{S}(w,x) - \widetilde{S}(y,z) \leq \delta$;
- the importance of criterion i is strictly higher than that of j can be translated as $w_i - w_j \geq \delta$;
- the importance of criterion i is similar to that of j can be translated as $-\delta \leq w_i - w_j \leq \delta$;

where $w, x, y, z \in X$, $i, j \in N$ and δ is a non negative separation parameter.

A priori information: constraints

- a quantitative intuition about the credibility of the validation of xSy can be translated as $\eta_{(x,y)} \leq \widetilde{S}(x,y) \leq \theta_{(x,y)}$, where $\eta_{(x,y)} \leq \theta_{(x,y)} \in [-1,1]$ are to be fixed by the DM;
- a quantitative intuition about the importance of criterion i can be translated as $\eta_{w_i} \leq w_i \leq \theta_{w_i}$, where $\eta_{w_i} \leq \theta_{w_i} \in]0,1]$ are to be fixed by the DM;
- a quantitative intuition about the preference threshold p_i of criterion i can be translated as $\eta_{p_i} \leq p_i \leq \theta_{p_i}$, where $\eta_{p_i} \leq \theta_{p_i} \in [0,1]$ are to be fixed by the DM;
- the fact that the subset $M \subset N$ of criteria is sufficient (resp. not sufficient) to validate an outranking statement can be translated as $\sum_{i \in M} w_i \geq \eta_M$ (resp. $\sum_{i \in M} w_i \leq -\eta_M$), where $\eta_M \in]0,1]$ is a parameter of *concordant coallition* which is to be fixed by the DM.

MIP3-MCDA:

Variables:

$\varepsilon \geq 0$	
$w_i \in]0, 1]$	$\forall i \in N$
$q_i \in]0, p[$	$\forall i \in N$
$p_i \in]q, 1[$	$\forall i \in N$
$wv_i \in]p, 1[$	$\forall i \in N$
$v_i \in]wv, 1[$	$\forall i \in N$
$\widetilde{S}''(x, y) \in [0, 1]$	$\forall x \neq y \in X$
...	

Parameters:

$g_i(x) \in [0, 1]$	$\forall i \in N, \forall x \in X$
$\delta \in]0, q[$	

Objective function:

min ε

MIP3 (some of them linearised)

...

Constraints of a priori information (informal):

$\widetilde{S}(w, x) - \widetilde{S}(y, z) \geq \delta$	for some pairs of alternatives
$-\delta \leq \widetilde{S}(w, x) - \widetilde{S}(y, z) \leq \delta$	for some pairs of alternatives
$w_i - w_j \geq \delta$	for some pairs of weights
$-\delta \leq w_i - w_j \leq \delta$	for some pairs of weights
$\eta_{(x,y)} \leq \widetilde{S}(x, y) \leq \theta_{(x,y)}$	for some pairs of alternatives
$\eta_{w_i} \leq w_i \leq \theta_{w_i}$	for some weights
$\eta_{p_i} \leq p_i \leq \theta_{p_i}$	for some thresholds and some weights
$\sum_{i \in M} w_i \geq \eta_M$	for some subsets M of weights
$\sum_{i \in M} w_i \leq -\eta_M$	for some subsets M of weights

Illustration

Starting point:

	g_1	g_2	g_3	g_4
a	0.000	0.000	0.000	1.000
b	0.400	0.100	0.090	0.590
c	0.200	0.290	0.000	0.000

Unknown:

- $w_i \quad \forall i \in N$
- $q_i, p_i, wv_i, w_i \quad \forall i \in N$

A priori preferences:

$\widetilde{S}_3$	a	b	c
a	$\cdot$	$\in]0, 0.5]$	$\in [-0.5, 0[$
b	$\in]0, 0.5]$	$\cdot$	$\in]0.5, 1]$
c	$= -1$	$\in [-0.1, 0.1]$	$\cdot$

Illustration

Output of MIP3-MCDA:

$\widetilde{S}_3$	a	b	c
a	$\cdot$	0.500	-0.010
b	0.500	$\cdot$	1.000
c	-1.000	0.000	$\cdot$

Table: $\widetilde{S}_3$

	g_1	g_2	g_3	g_4
w_i	0.120	0.380	0.250	0.250
q_i	0.970	0.270	0.000	0.000
p_i	0.980	0.280	0.090	0.410
wv_i	0.990	0.290	0.990	0.590
v_i	1.000	0.300	1.000	0.600

Table: Model parameters for $\widetilde{S}_3$ via $\mathcal{M}_3$

Note: $\widetilde{S}_3(c, a) = -1$ (resp $\widetilde{S}_3(c, b) = 0$) results from a veto (resp. weak veto) situation on criterion 4.

A few words on the implementation

On the implementation

- Implemented in the **GNU MathProg** programming language
- Simple examples of this presentation have been solved on a standard desktop computer with **Glpsol**
- Harder examples are solved with **ILOG CPLEX 9.1** on a HP rx4640-8 server with four Itanium 2 processors
- **Very** time consuming!

Introduction Models Disaggregation Rank Illustration Inference Implementation

That's all folks