Counting non isomorphic maximal independent sets in the cycle graph

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Plan

The n-cycle graph

Définition

- Let $G = (V, E)$ be a simple undirected graph, with vertex set V and edge set E.
- The n-cycle C_n is the graph consisting of a cycle with n vertices.
- The set of vertices of C_n are labelled either clockwise or counterclockwise. The index set of the vertices is assumed to be the cyclic group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with addition modulo n.

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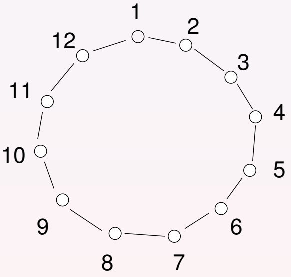

12–cycle

- An *independent set* of G is a subset X of V such that no two vertices in X are adjacent.
- An independent set is *maximal*, denoted MIS, if it is not a proper subset of any other independent set.

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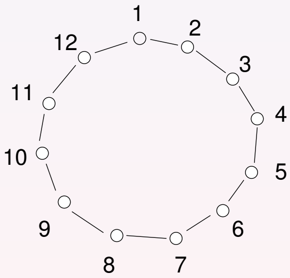

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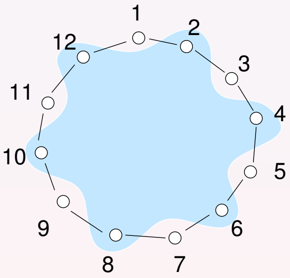

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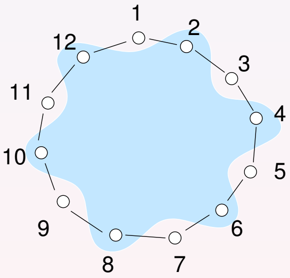

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Counting the MIS of C_n

- It was shown by Füredi (1987) that the total number of MIS's of C_n is given by the nth term $p(n)$ of the Perrin sequence (Sloane's A001608),
- $p(n)$ is defined recursively as $p(1) = 0$, $p(2) = 2$, $p(3) = 3$, and

$$p(n) = p(n-2) + p(n-3), \qquad (n \geqslant 4).$$

Hence the 12-cycle admits $p(12) = 29$ different MISs.

- As a corollary we note that it is possible to generate the MISs of C_n in a univocal way from both the MISs of C_{n-3} and the MISs of C_{n-2}.

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The automorphisms group of the n-cyle

- An isomorphism from a graph G to itself is called an automorphism of G. The set of automorphisms of a digraph G is a group called the automorphism group of G and denoted Aut(G).
- It is well known that the group Aut(C_n) is isomorphic to the *dihedral group* of order $2n$,

$$D_{2n} = \{1, \sigma, \ldots, \sigma^{n-1}, \tau, \sigma\tau, \ldots, \sigma^{n-1}\tau\},$$

  where σ is the rotation and τ is the reflection with the properties $\sigma^n = 1$, $\tau^2 = 1$, and $\tau\sigma = \sigma^{-1}\tau$.
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Non-isomorphic MISs of the n-cycle

- The action of the automorphism group Aut(C_n) of C_n on the family of MIS's of C_n gives rise to a partition of this family into orbits, each containing *isomorphic* (i.e., identical up to a rotation and a reflection) MIS's.
- For instance, it is easy to verify that the 29 MISs of C_{12} can be grouped into four orbits : an orbit with two isomorphic MIS's of size 6, an orbit with three isomorphic MIS's of size 4 and two orbits with each one 12 isomorphic MIS's of size 5.

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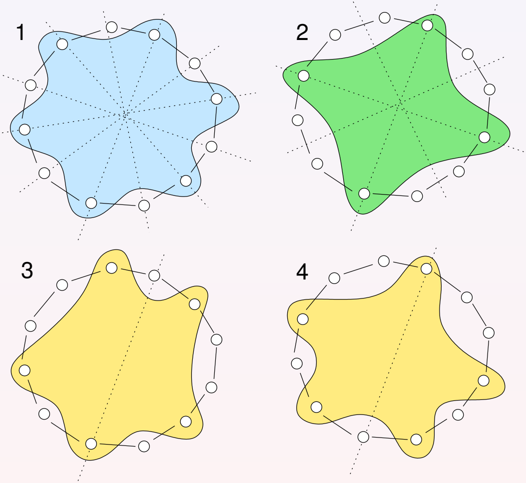

The four non isomorphic MISs of the 12−cycle

Some properties of Aut(C_n)

- $\mathbf{X}_n \subseteq 2^{V_n}$, the set of MISs of C_n, is a family of subsets of V_n with the property that if $X \in \mathbf{X}_n$ then $g(X) \in \mathbf{X}_n$ for all $g \in D_{2n}$.
- For any $X \in \mathbf{X}_n$, the *orbit* and the *stabilizer* of X under the action of D_{2n} are respectively defined as

$$\begin{aligned} \mathrm{Orb}(X) &= \{g(X) : g \in D_{2n}\}, \\ \mathrm{Stab}(X) &= \{g \in D_{2n} : g(X) = X\}. \end{aligned}$$

- Stab(X) is either a cyclic or a dihedral subgroup of D_{2n} and by the orbit-stabilizer theorem, we have

$$|\mathrm{Orb}(X)| \times |\mathrm{Stab}(X)| = |D_{2n}| = 2n,$$

which implies that both $|\mathrm{Orb}(X)|$ and $|\mathrm{Stab}(X)|$ divide $2n$.

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MIS orbit decomposition

- Let $\mathcal{O}_n = \mathbf{X}_n / D_{2n}$ denote the set of orbits of $\mathbf{X}_n$ under the action of D_{2n}. $X, X' \in \mathbf{X}_n$ are *isomorphic*, and we write $X \sim X'$, if $\mathrm{Orb}(X) = \mathrm{Orb}(X')$.
- For any divisor $d \geqslant 1$ of $2n$ (we write $d|2n$), denote by $\mathcal{O}_n^d$ the set of orbits of $\mathbf{X}_n$ of cardinality $\frac{2n}{d}$. $\mathcal{O}_n^d$ is also the set of orbits whose elements have a stabilizer of cardinality d.
- Thus :

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MIS orbit decomposition (continue)

- The previous formulation leads to

$$\mathrm{orb}(n) = \sum_{d|2n} \mathrm{orb}_d(n).$$

- Since the orbits partition the set $\mathbf{X}_n$ we immediately have

$$|\mathbf{X}_n| = \sum_{d|2n} \frac{2n}{d} \mathrm{orb}_d(n).$$

Proposition

For any $d|2n$, we have

$$\mathrm{orb}_d(n) = \begin{cases} \mathrm{orb}_1(n/d), & \text{if } d \text{ is odd}, \\ \mathrm{orb}_2(2n/d), & \text{if } d \text{ is even}. \end{cases}$$

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The orbit decomposition of the MISs of C_n

			Size of the stabilizer																													
n	**#MIS**	**orb(n)**	**...**	**9**	**7**	**5**	**3**	**1**	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**	**32**	**34**	**36**	**38**	**40**	**42**	**44**	**46**	**48**
1	(0)																															
2	2	1							1																							
3	3	1							1																							
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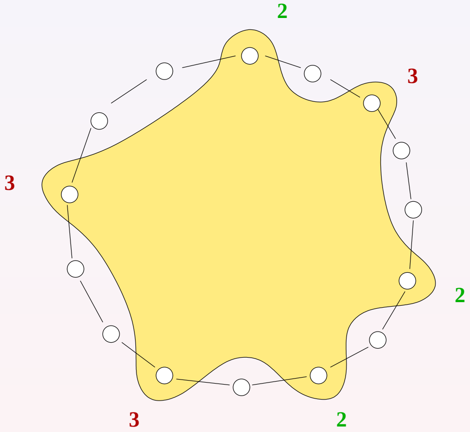
10	17	3							1	1			1																			
11	22	2							2																							
12	29	4							2			1		1																		
13	39	3							3																							
14	51	5							3	1					1																	
15	68	5						1	2		1		1																			
16	90	7							5	1						1																
17	119	6						1	5																							
18	158	10						1	6	1				1			1															
19	209	9						2	7																							
20	277	14						2	9	1		1						1														
21	367	14						5	7		1				1																	
22	486	20						4	13	2									1													
23	644	20						8	12																							
24	853	30						9	16	2	1					1				1												
25	1130	31						15	15				1																			
26	1497	44						16	24	3											1											
27	1983	48						27	19		1						1															
28	2627	67						30	32	3		1										1										
29	3480	74						46	28																							
30	4610	104						54	44	2	1			1				1					1									
31	6107	117						80	37																							
32	8090	161						96	58	5		1												1								
33	10717	188						139	46		2								1													
34	14197	254						167	81	5															1							
35	18807	302						237	63				1		1																	
36	24914	407						292	104	6	2	1								1						1						
37	33004	489						403	86																							
38	43721	654						501	145	7																	1					
39	57918	801						687	110		3										1											
40	76725	1064						862	189	9		1	1			1												1				
41	101639	1315						1164	151																							
42	134643	1742						1472	257	7	3			1								1							1			
43	178364	2174						1974	200																							
44	236282	2867						2512	339	13		2																		1		
45	313007	3613					1	3347	260		2		1				1						1									
46	414646	4747						4274	460	12																					1	



Irregular MIS of C_{15}

A very useful result

The Padovan sequence $(q(n))_{n\in\mathbb{N}}$ (shifted Sloane's A000931), is defined as $q(1) = 0$, $q(2) = 1$, $q(3) = 1$, and

$$q(n) = q(n-2) + q(n-3) \qquad (n \geqslant 4),$$

Proposition

- orb(n) *is the number of cyclic compositions of n in which each term is either 2 or 3, where a clockwise writing is not distinguished from its counterclockwise counterpart.*
- *$q(n)$ is the number of (linear) compositions of n in which each term is either 2 or 3.*

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A very useful result (continue)

Let $(r(n))_{n\in\mathbb{N}}$ be the sequence defined by

$$r(n) = \begin{cases} q(k), & \text{if } n = 2k-1, \\ q(k+2), & \text{if } n = 2k. \end{cases}$$

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- *$r(n)$ is the number of cyclic and palindromic compositions of n in which each term is either 2 or 3.*
- *$r(n)$ is also the number of orbits of $\mathbf{X}_n$ whose elements have a stabilizer not included in $\langle\sigma\rangle$.*

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$$r(n) = \begin{cases} q(k), & \text{if } n = 2k-1, \\ q(k+2), & \text{if } n = 2k. \end{cases}$$

Proposition

- *$r(n)$ is the number of cyclic and palindromic compositions of n in which each term is either 2 or 3.*
- *$r(n)$ is also the number of orbits of $\mathbf{X}_n$ whose elements have a stabilizer not included in $\langle\sigma\rangle$.*

Some useful definitions

- The *Dirichlet convolution product* of two sequences $(f(n))_{n\in\mathbb{N}}$ and $(g(n))_{n\in\mathbb{N}}$ is the sequence $((f*g)(n))_{n\in\mathbb{N}}$ defined as

$$(f*g)(n) = \sum_{d|n} f(d)\, g(\frac{n}{d}).$$

- Consider the following integer sequences :

$$\begin{aligned}
\mu(n) &= \mathrm{A008683}(n) \qquad (\textit{Möbius function}),\\
\mathrm{A113788}(n) &= \frac{1}{n}(p*\mu)(n),\\
\mathbf{1}(n) &= 1,\\
e_k(n) &= \begin{cases} 1, & \text{if } n = k,\\ 0, & \text{else,}\end{cases} \qquad (k\in\mathbb{N}).
\end{aligned}$$

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Our main result

Theorem

There holds

$$\begin{aligned} \mathrm{orb} &= r + \mathrm{orb}_1 * \mathbf{1}, \\ 2\mathrm{orb}_1 &= \mathrm{A113788} - r * \mu, \\ \mathrm{orb}_2 &= r * \mu + \mathrm{orb}_1 * e_2. \end{aligned}$$

Summary

$r(n)$ is the number of non-isomorphic MIS's of C_n having at least one symmetry axis, where two MIS's are isomorphic if they are identical up to a rotation and a reflection. It is also the number of cyclic and palindromic compositions of n in which each term is either 2 or 3.

$$r(n) = \begin{cases} q(k), & \text{if } n = 2k - 1, \\ q(k+2), & \text{if } n = 2k. \end{cases}$$

Summary

For any $d|2n$, $\text{orb}_d(n)$ gives the number of non-isomorphic MIS's of C_n having $\frac{2n}{d}$ isomorphic representatives. This sequence can always be expressed from one of the sequences $\text{orb}_1(n)$ and $\text{orb}_2(n)$.

$$\begin{aligned} \text{orb}_1(n) &= \frac{1}{2}\left(\text{A113788}(n) - (r * \mu)(n)\right) \\ \text{orb}_2(n) &= (r * \mu)(n) + (\text{orb}_1 * e_2)(n). \end{aligned}$$

Summary

orb(n) gives the number of non-isomorphic MIS's of C_n. It is also the number of cyclic compositions of n in which each term is either 2 or 3, where a clockwise writing is not distinguished from its counterclockwise counterpart.

$$\mathrm{orb}(n) = r(n) + (\mathrm{orb}_1 * \mathbf{1})(n) = \frac{1}{2}\Big(r(n) + \frac{1}{n}(p * \phi)(n)\Big).$$

where $\phi = \mathrm{A000010}$ is the *Euler totient function*.

Summary

$(\mathrm{orb}_1 * \mathbf{1})(n)$ is the number of non-isomorphic MIS's of C_n having no symmetry axis. It is also the number of cyclic and non-palindromic compositions of n in which each term is either 2 or 3, where a clockwise writing is not distinguished from its counterclockwise counterpart.

$$(\mathrm{orb}_1 * \mathbf{1})(n) = \mathrm{orb}(n) - r(n).$$

n	$p(n)$	$q(n)$	$r(n)$	$\mathrm{orb}(n)$	$\mathrm{orb}_1(n)$	$\mathrm{orb}_2(n)$	$\mathrm{orb}^{\sigma}(n)$	$\mathrm{orb}_1^{\sigma}(n)$
1	0	0	0	0	0	0	0	0
2	2	1	1	1	0	1	1	1
3	3	1	1	1	0	1	1	1
4	2	1	1	1	0	0	1	0
5	5	2	1	1	0	1	1	1
6	5	2	2	2	0	0	2	0
7	7	3	1	1	0	1	1	1
8	10	4	2	2	0	1	2	1
9	12	5	2	2	0	1	2	1
10	17	7	3	3	0	1	3	1
11	22	9	2	2	0	2	2	2
12	29	12	4	4	0	2	4	2
13	39	16	3	3	0	3	3	3
14	51	21	5	5	0	3	5	3
15	68	28	4	5	1	2	6	4
16	90	37	7	7	0	5	7	5
17	119	49	5	6	1	5	7	7
18	158	65	9	10	1	6	11	8
19	209	86	7	9	2	7	11	11
20	277	114	12	14	2	9	16	13

TAB.: First 20 values of the main sequences

n	$p(n)$	$q(n)$	$r(n)$	$\mathrm{orb}(n)$	$\mathrm{orb}_1(n)$	$\mathrm{orb}_2(n)$	$\mathrm{orb}^\sigma(n)$	$\mathrm{orb}_1^\sigma(n)$
21	367	151	9	14	5	7	19	17
22	486	200	16	20	4	13	24	21
23	644	265	12	20	8	12	28	28
24	853	351	21	30	9	16	39	34
25	1130	465	16	31	15	15	46	45
26	1497	616	28	44	16	24	60	56
27	1983	816	21	48	27	19	75	73
28	2627	1081	37	67	30	32	97	92
29	3480	1432	28	74	46	28	120	120
30	4610	1897	49	104	54	44	159	151
31	6107	2513	37	117	80	37	197	197
32	8090	3329	65	161	96	58	257	250
33	10717	4410	49	188	139	46	327	324
34	14197	5842	86	254	167	81	422	414
35	18807	7739	65	302	237	63	539	537
36	24914	10252	114	407	292	104	700	687
37	33004	13581	86	489	403	86	892	892
38	43721	17991	151	654	501	145	1157	1145
39	57918	23833	114	801	687	110	1488	1484
40	76725	31572	200	1064	862	189	1928	1911

TAB.: 21 - 40 values of the main sequences