Equivalent Representations of a Set Function with Applications to Game Theory and Multicriteria Decision Making

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Abstract. This paper introduces four alternative representations of a set function: the Möbius transformation, the co-Möbius transformation, and the interactions between elements of any subset of a given set as extensions of Shapley and Banzhaf values. The links between the five equivalent representations of a set function are emphasized in this presentation.

Keywords: set functions, pseudo-Boolean functions, Möbius inversion formula, Shapley and Banzhaf values, interaction indices, game theory, multicriteria decision making.

1 Introduction

Real valued set functions, which are not necessarily additive, are extensively used in decision theory. This paper mostly concentrates on some alternative representations of set functions and on their usefulness in game theory and in multicriteria decision making.

Consider a real valued set function $v : 2^N \to \mathbb{R}$, where N is a discrete set of n elements, $n \in \mathbb{N}_0$. In cooperative game theory, v is a *game* that assigns to each coalition S of players a real number $v(S)$ representing the *worth* or the *power* of S. One also defines the *unanimity game* for $T \subseteq N$, as the game v_T such that $v_T(S) = 1$ if and only if $S \supseteq T$, and 0 otherwise. In multicriteria decision making, when N represents a set of criteria, v is a *weight function* and the number $v(S)$ represents the *weight* related to the combination S of criteria.

There exist several equivalent ways to define v. The first one is to give for any subset S the number $v(S)$. The second one is to observe that v can be expressed in a unique way as:

$$v(S) = \sum_{T\subseteq S} a(T), \quad S \subseteq N. \tag{1}$$

In game theory, the real coefficients $\{a(T)\}_{T\subseteq N}$ are called the *dividends* of the coalitions in game v, see [11, 15]. In combinatorics, a viewed as a set function on N is called the *Möbius transform* of v (see e.g. Rota [17]), which is given by

$$a(S) = \sum_{T\subseteq S} (-1)^{s-t} v(T), \quad S \subseteq N, \tag{2}$$

where $s = |S|$ and $t = |T|$.

The set function a is a *representation of* v since there is a bijection between the set of games and the set of dividends, i.e. defining one of the two allows to compute the other without ambiguity. More formally, a set function $w : 2^N \to \mathbb{R}$ is a representation of v if there exists an invertible transform $\mathcal{T}$ such that

$$w = \mathcal{T}(v) \quad \text{and} \quad v = \mathcal{T}^{-1}(w).$$

In addition to the Möbius representation of v, we introduce the following definitions:

– The *dual representation* of v, denoted v^*, is defined by

$$v^*(S) := v(N) - v(N \setminus S), \quad S \subseteq N.$$

– The *co-Möbius representation* of v, denoted b, is defined by

$$b(S) := \sum_{T\supseteq N\setminus S} (-1)^{n-t} v(T) = \sum_{T\subseteq S} (-1)^{t}\, v(N \setminus T), \quad S \subseteq N. \tag{3}$$

– The *Banzhaf interaction index* related to v, denoted I_{B}, is defined by

$$I_{\mathrm{B}}(S) := \frac{1}{2^{n-s}} \sum_{T\subseteq N\setminus S} \sum_{L\subseteq S} (-1)^{s-l} v(L \cup T), \quad S \subseteq N. \tag{4}$$

– The *Shapley interaction index* related to v, denoted I_{Sh}, is defined by

$$I_{\mathrm{Sh}}(S) := \sum_{T\subseteq N\setminus S} \frac{(n-t-s)!\,t!}{(n-s+1)!} \sum_{L\subseteq S} (-1)^{s-l} v(L \cup T), \quad S \subseteq N. \tag{5}$$

In evidence theory (Shafer [19]), v corresponds to the belief function, v^* is called the plausibility function, a corresponds to the mass or basic probability assignment and b is called the commonality function.

The interaction indices I_{B} and I_{Sh} have been introduced respectively by Roubens [18] and Grabisch [7] to model interaction among players or criteria. Actually, the problem of modelling interaction remains a difficult question, often overlooked in practical applications. Although everybody agrees that interaction phenomena do exist in real situations, the lack of suitable tool to model them frequently causes the practitioner to assume that his criteria are independent and exhaustive. This comes primarily from the absence of a precise definition of interaction.

However, the problem has recently been addressed under the viewpoint of cooperative game theory and multicriteria decision making, and an approach which seems suitable has been pointed out. The origin of the idea is due to Murofushi and Soneda [13], who propose an interaction index among a pair of criteria, based on multiattribute utility theory. Later, Grabisch [7] generalized this index to any subset S, thus giving rise to a new representation of set functions. This representation is called the Shapley interaction index (5). Viewed as a set function, it coincides on singletons with the Shapley value

$$\phi_{\mathrm{Sh}}(i)=\sum_{T\subseteq N\setminus\{i\}}\frac{(n-t-1)!\,t!}{n!}\,[v(T\cup\{i\})-v(T)],\quad i\in N,$$

which is a fundamental concept in game theory [20] expressing a power index. There is in fact another common way of defining a power index, due to Banzhaf [2] (see also Dubey and Shapley [4]). The so-called Banzhaf value, defined as

$$\phi_{\mathrm{B}}(i)=\frac{1}{2^{n-1}}\sum_{T\subseteq N\setminus\{i\}}[v(T\cup\{i\})-v(T)],\quad i\in N,$$

can be viewed as an alternative to the Shapley value, and Roubens [18] developed a parallel notion of interaction index, based on the Banzhaf value: the Banzhaf interaction index (4).

It should be noted that the interaction indices I_{B} and I_{Sh} have been axiomatically characterized by Grabisch and Roubens [8].

If $v(\emptyset)=0$ then v^* is a representation of v since $(v^*)^*=v$. Moreover, for any v, it is already known that the Möbius transform is invertible and thus is a representation of v. The main aim of this paper is to show that b is also a representation of v, as well as the interaction indices I_{B} and I_{Sh}. All these representations are linear, that is, such that $\mathcal{T}$ is a linear operator. We also give all the conversion formulas between v, a, b, I_{B} and I_{Sh}.

On this issue, Roubens [18] has expressed I_{B} in terms of the dividends:

$$I_{\mathrm{B}}(S)=\sum_{T\supseteq S}(\frac{1}{2})^{t-s}\,a(T),\quad S\subseteq N, \tag{6}$$

while Grabisch [7] has shown that:

$$I_{\mathrm{Sh}}(S)=\sum_{T\supseteq S}\frac{1}{t-s+1}\,a(T),\quad S\subseteq N. \tag{7}$$

The paper is organized as follows. In Section 2, we introduce the concept of pseudo-Boolean functions as substitutes for set functions. In Sections 3 and 4, we introduce the multilinear and Lovász extensions. In Section 5, we present some conversion formulas which can be obtained from the multilinear extension.

In Section 6, we consider the algebraic aspect of the transformations by investigating the matricial relations between the representations. Two types of transformations are pointed out: fractal and cardinality transformations. In Section 7, we apply our results to a certain problem of approximations of pseudo-Boolean functions.

2 The use of pseudo-Boolean functions

For any subset $S \subseteq N$, e_S is the *characteristic vector* (or *incidence vector*) of S, i.e. the vector of $\{0,1\}^n$ whose i-th component is 1 if and only if $i \in S$. Geometrically, the characteristic vectors are the 2^n vertices of the hypercube $[0,1]^n$.

Any real valued set function $v : 2^N \to \mathbb{R}$ can be assimilated unambiguously with a pseudo-Boolean function, that is a function $f : \{0,1\}^n \to \mathbb{R}$. The correspondence is straightforward: we have

$$f(x) = \sum_{T \subseteq N} v(T) \prod_{i \in T} x_i \prod_{i \notin T} (1 - x_i), \quad x \in \{0,1\}^n,$$

and $v(S) = f(e_S)$ for all $S \subseteq N$. We shall henceforth make this identification.

Hammer and Rudeanu [10] showed that any pseudo-Boolean function has a unique expression as a multilinear polynomial in n variables:

$$f(x) = \sum_{T \subseteq N} a(T) \prod_{i \in T} x_i, \quad x \in \{0,1\}^n, \tag{8}$$

where the coefficients $a(T)$ are nothing else than the dividends (2). Moreover, equation (8) is the decomposition of the set function v into unanimity games: indeed, $\prod_{i \in T} x_i$ corresponds to the unanimity game v_T and we have, for all $S \subseteq N$,

$$v(S) = f(e_S) = \sum_{T \subseteq N} a(T) \prod_{i \in T} (e_S)_i = \sum_{T \subseteq N} a(T)\, v_T(S).$$

Thus, any game v has a canonical representation in terms of unanimity games that determine a linear basis for v. Note that Gilboa and Schmeidler [5] and Pap [16] extended this unanimity-basis representation to general (infinite) spaces of players.

Let us introduce the concept of derivatives of pseudo-Boolean functions, which will be useful as we continue, see e.g. [9].

Definition 2.1 *Given $S = \{i_1, \ldots, i_s\} \subseteq N$, the s-th order derivative of a pseudo-Boolean function $f : \{0,1\}^n \to \mathbb{R}$ with respect to $x_{i_1}, \ldots, x_{i_s}$ is the function $\Delta_S\, f : \{0,1\}^n \to \mathbb{R}$ defined inductively as*

$$\Delta_S\, f(x) = \Delta_{i_1}(\Delta_{S \setminus \{i_1\}}\, f)(x),$$

where $\Delta_i f(x)$ $(i \in N)$ *is the (first) derivative defined by*

$$\Delta_i f(x) := f(x \,|\, x_i = 1) - f(x \,|\, x_i = 0), \quad x \in \{0,1\}^n,$$

and, as usual, $\Delta_\emptyset f(x) = f(x)$ *for all* $x \in \{0,1\}^n$*. For all* $S \subseteq N$*,* $\Delta_S f(x)$ *will be called the* S*-derivative of* $f(x)$*.*

Thus defined, $\Delta_S f(x)$ depends only on the variables x_i for $i \notin S$, but we still regard it as a function on $\{0,1\}^n$. For instance, we have, for all $T \subseteq N$,

$$(\Delta_i f)(e_T) = \begin{cases} f(e_T) - f(e_{T\setminus\{i\}}), & \text{if } i \in T, \\ f(e_{T\cup\{i\}}) - f(e_T), & \text{if } i \notin T. \end{cases}$$

If f is given under the form (8) then we can easily see that:

$$\Delta_S f(x) = \sum_{T \supseteq S} a(T) \prod_{i \in T\setminus S} x_i, \quad \forall x \in \{0,1\}^n, \ \forall S \subseteq N. \tag{9}$$

Hence we have

$$(\Delta_S f)(e_T) = \sum_{L \subseteq T} a(L \cup S), \quad \forall S \subseteq N, \ \forall T \subseteq N \setminus S. \tag{10}$$

Moreover, we can see that (use induction over $|S|$):

$$(\Delta_S f)(e_T) = \sum_{L \subseteq S} (-1)^{s-l} v(L \cup T), \quad \forall S \subseteq N, \ \forall T \subseteq N \setminus S. \tag{11}$$

In particular, for $S = \{i\}$, we obtain the marginal contribution of player i to the coalition $T \subseteq N \setminus \{i\}$:

$$(\Delta_i f)(e_T) = v(T \cup \{i\}) - v(T).$$

For $S = \{i, j\}$, we obtain the marginal contribution of j in the presence of i minus the marginal contribution of j in the absence of i:

$$(\Delta_{\{i,j\}} f)(e_T) = v(T \cup \{i,j\}) - v(T \cup \{i\}) - v(T \cup \{j\}) + v(T).$$

This difference represents the marginal interaction between i and j, conditioned to the presence of elements of the coalition $T \subseteq N \setminus \{i,j\}$.

More generally, $(\Delta_S f)(e_T)$ represents the marginal interaction between the elements of a coalition $S \subseteq N$ in the presence of elements of the coalition $T \subseteq N \setminus S$.

By combining (11) and (3), we obtain

$$b(S) = (\Delta_S f)(e_{N\setminus S}) = (\Delta_S f)(e_N), \quad S \subseteq N, \tag{12}$$

and by using (9),

$$b(S) = \sum_{T \supseteq S} a(T), \quad S \subseteq N. \tag{13}$$

Moreover, it should be noted that, from (11), equations (4) and (5) become:

$$I_{\mathrm{B}}(S)=\frac{1}{2^{n-s}}\sum_{T\subseteq N\setminus S}(\Delta_S\, f)(e_T),\quad S\subseteq N, \tag{14}$$

$$I_{\mathrm{Sh}}(S)=\frac{1}{n-s+1}\sum_{T\subseteq N\setminus S}\binom{n-s}{t}^{-1}(\Delta_S\, f)(e_T),\quad S\subseteq N, \tag{15}$$

thus showing that I_{B} and I_{Sh} are of the form (see [8]):

$$I(S)=\sum_{T\subseteq N\setminus S}p_T^S\,(\Delta_S\, f)(e_T),\quad S\subseteq N. \tag{16}$$

This points out the following probabilistic interpretation of I_{B} and I_{Sh}: let us suppose that any coalition $S\subseteq N$ joins a coalition $T\subseteq N\setminus S$ at random with a probability p_T^S. Then the interaction index (16) can be thought of as the mathematical expectation of the marginal interaction $(\Delta_S\, f)(e_T)$. Depending on the given randomization scheme, this interaction index takes a well defined form (see also [22]):

- if the coalition S is equally likely to join any coalition $T\subseteq N\setminus S$, its probability to join is $p_T^S=\frac{1}{2^{n-s}}$ and we get I_{B}.
- if the coalition S is equally likely to join any coalition $T\subseteq N\setminus S$ of size t $(0\leq t\leq n-s)$ and that all coalitions of size t are equally likely, its probability to join is $p_T^S=\frac{1}{n-s+1}\binom{n-s}{t}^{-1}$ and we get I_{Sh}.

The following result shows that equations (14) and (15) can be rewritten in another form.

Proposition 2.1 *We have*

$$I_{\mathrm{B}}(S)=\frac{1}{2^{n}}\sum_{x\in\{0,1\}^n}(\Delta_S\, f)(x),\quad S\subseteq N, \tag{17}$$

$$I_{\mathrm{Sh}}(S)=\frac{1}{n+1}\sum_{x\in\{0,1\}^n}\binom{n}{\sum_i x_i}^{-1}(\Delta_S\, f)(x),\quad S\subseteq N. \tag{18}$$

Proof. Given $S\subseteq N$, we simply have

$$\begin{aligned}\frac{1}{2^{n}}\sum_{x\in\{0,1\}^n}(\Delta_S\, f)(x)&=\frac{1}{2^{n}}\sum_{T\supseteq S}a(T)\sum_{x\in\{0,1\}^n}\prod_{i\in T\setminus S}x_i\quad\text{(by (9))}\\&=\frac{1}{2^{n}}\sum_{T\supseteq S}a(T)\sum_{K\subseteq N\setminus(T\setminus S)}1\\&=\sum_{T\supseteq S}(\frac{1}{2})^{t-s}a(T)\\&=I_{\mathrm{B}}(S)\quad\text{(by (6))},\end{aligned}$$

and

$$
\begin{aligned}
&\frac{1}{n+1}\sum_{x\in\{0,1\}^n}\binom{n}{\sum_i x_i}^{-1}(\Delta_S\, f)(x)\\
&=\frac{1}{n+1}\sum_{T\supseteq S}a(T)\sum_{x\in\{0,1\}^n}\binom{n}{\sum_i x_i}^{-1}\prod_{i\in T\setminus S}x_i \quad \text{(by (9))}\\
&=\frac{1}{n+1}\sum_{T\supseteq S}a(T)\sum_{K\subseteq N\setminus(T\setminus S)}\binom{n}{k}^{-1}\\
&=\frac{1}{n+1}\sum_{T\supseteq S}a(T)\sum_{k=0}^{n-t+s}\binom{n-t+s}{k}\binom{n}{k}^{-1}\\
&=\sum_{T\supseteq S}\frac{1}{t-s+1}\,a(T)\\
&=I_{\mathrm{Sh}}(S)\quad \text{(by (7))},
\end{aligned}
$$

which proves the result. □

Before going on, let us make some observations.

1. It should be noted that equations (6) and (7) can be easily obtained from (14) and (15) respectively by using the following formula:

$$
\sum_{T\subseteq N\setminus S}p_t^s\,(\Delta_S\, f)(e_T)=\sum_{T\supseteq S}\Big[\sum_{k=0}^{n-t}\binom{n-t}{k}p^s_{k+t-s}\Big]a(T),\quad S\subseteq N.
$$

   The proof of this formula is simple: setting $L':=L\cup S$, we have, from (10),

$$
\begin{aligned}
\sum_{T\subseteq N\setminus S}p_t^s\,(\Delta_S\, f)(e_T)&=\sum_{T\subseteq N\setminus S}p_t^s\sum_{L\subseteq T}a(L\cup S)\\
&=\sum_{L'\supseteq S}\Big[\sum_{T:L'\setminus S\subseteq T\subseteq N\setminus S}p_t^s\Big]a(L')\\
&=\sum_{L'\supseteq S}\Big[\sum_{t=l'-s}^{n-s}\binom{n-l'}{t-l'+s}p_t^s\Big]a(L')\\
&=\sum_{L'\supseteq S}\Big[\sum_{k=0}^{n-l'}\binom{n-l'}{k}p^s_{k+l'-s}\Big]a(L').
\end{aligned}
$$

2. In their definitions, Grabisch and Roubens [8] introduced the notation

$$
\delta_S v(T):=(\Delta_S f)(e_T),\quad S\subseteq T\subseteq N.
$$

   Using this concept of derivative, we immediately have

$$
(\Delta_S f)(e_T)=\delta_S v(T\cup S),\quad \forall S,T\subseteq N,
$$

and (16) becomes

$$I(S) = \sum_{T \subseteq N \setminus S} p_T^S \, \delta_S v(T \cup S), \quad S \subseteq N.$$

3 Multilinear extension of pseudo-Boolean functions

From any pseudo-Boolean function $f : \{0,1\}^n \to \mathbb{R}$, we can define a variety of extensions $\bar{f} : [0,1]^n \to \mathbb{R}$ which interpolate f at the 2^n vertices of $[0,1]^n$, that is $\bar{f}(e_S) = f(e_S) = v(S)$ for all $S \subseteq N$.

The S-derivative of any extension $\bar{f}$ is defined inductively in the same way as for f. In particular, we have

$$\Delta_S \bar{f}(x) = \Delta_S f(x), \quad \forall x \in \{0,1\}^n, \ \forall S \subseteq N.$$

Let us introduce the notation $\underline{x} := (x, \ldots, x) \in [0,1]^n$ for all $x \in [0,1]$. By (10) and (12), we immediately have

$$\begin{aligned} a(S) &= (\Delta_S \bar{f})(\underline{0}), \quad S \subseteq N \\ b(S) &= (\Delta_S \bar{f})(\underline{1}), \quad S \subseteq N \end{aligned}$$

for any extension $\bar{f}$ of f.

The polynomial expression (8) was used in game theory in 1972 by Owen [14] as the *multilinear extension* of a game.

Definition 3.1 *If the pseudo-Boolean function f has the unique multilinear expression (8) then the* multilinear extension *of f (MLE) is the function g : $[0,1]^n \to \mathbb{R}$ defined by*

$$g(x) := \sum_{T \subseteq N} f(e_T) \prod_{i \in T} x_i \prod_{i \notin T} (1 - x_i) = \sum_{T \subseteq N} a(T) \prod_{i \in T} x_i, \quad x \in [0,1]^n. \tag{19}$$

It has been proved by Owen [15] that g is the only multilinear function (i.e. linear in each of the variables x_i) on $[0,1]^n$ that coincides with f on $\{0,1\}^n$. More precisely, g corresponds to the classical linear interpolation (with respect to each of the n variables) of f.

It is easy to see that:

$$\Delta_S \, g(x) = \sum_{T \supseteq S} a(T) \prod_{i \in T \setminus S} x_i, \quad \forall x \in [0,1]^n, \ \forall S \subseteq N, \tag{20}$$

and, by (9), we can observe that $\Delta_S \, g(x)$ is the MLE of $\Delta_S \, f(x)$.

From (6) and (20), we can readily see that the Banzhaf interaction index related to S is obtained by integrating the S-derivative of the MLE of game v over the hypercube. Formally, this result can be stated as follows.

Proposition 3.1 *We have*

$$I_{\mathrm{B}}(S)=\int_{[0,1]^n}(\Delta_S\, g)(x)\,dx,\quad S\subseteq N. \tag{21}$$

This result can be interpreted by analogy with (17): $I_{\mathrm{B}}(S)$ is the average value of $\Delta_S\, f$ over $\{0,1\}^n$, but also the average value of its MLE over $[0,1]^n$.

From (20), we immediately have:

$$(\Delta_S\, g)(\underline{x})=\sum_{T\supseteq S}a(T)\,x^{t-s},\quad \forall x\in[0,1],\ \ \forall S\subseteq N. \tag{22}$$

Consequently, we have, using (13), (6), and (7):

$$a(S)=(\Delta_S\, g)(\underline{0}),\quad S\subseteq N \tag{23}$$
$$b(S)=(\Delta_S\, g)(\underline{1}),\quad S\subseteq N \tag{24}$$
$$I_{\mathrm{B}}(S)=(\Delta_S\, g)(\underline{1/2}),\quad S\subseteq N \tag{25}$$
$$I_{\mathrm{Sh}}(S)=\int_0^1(\Delta_S\, g)(\underline{x})\,dx,\quad S\subseteq N. \tag{26}$$

We see that the Banzhaf interaction index related to S is the value of the S-derivative of the MLE of game v on the center of the hypercube $[0,1]^n$, while the Shapley interaction index related to S is obtained by integrating the S-derivative of the MLE of game v along the main diagonal of the hypercube. This latter result has been proved by Owen [15] when $|S|=1$.

4 Lovász extension of pseudo-Boolean functions

Let Π_n denote the family of all permutations π of N. The *Lovász extension* $\hat{f}:[0,1]^n\to\mathbb{R}$ of any pseudo-Boolean function f is defined on each n-simplex

$$\mathcal{B}_\pi=\{x\in[0,1]^n\,|\,x_{\pi(1)}\leq\cdots\leq x_{\pi(n)}\},\quad \pi\in\Pi_n,$$

as the unique affine function which interpolates f at the $n+1$ vertices of $\mathcal{B}_\pi$, see Lovász [12, §3] and Singer [21, §2].

When f is given under the form (8), the Lovász extension of f can take the form of a min-polynomial as follows:

$$\hat{f}(x)=\sum_{T\subseteq N}a(T)\bigwedge_{i\in T}x_i\quad x\in[0,1]^n, \tag{27}$$

where $\wedge$ denotes the *min* operation; indeed, the function (27) agrees with f at all the vertices of $[0,1]^n$, and identifies with an affine function on each simplex $\mathcal{B}_\pi$.

It is easy to see that:

$$\Delta_S\,\hat{f}(x)=\sum_{T\supseteq S}a(T)\bigwedge_{i\in T\setminus S}x_i,\quad \forall x\in[0,1]^n,\ \ \forall S\subseteq N, \tag{28}$$

and, by (9), we can observe that $\Delta_S\,\hat{f}$ is the Lovász extension of $\Delta_S\, f$.

The following lemma will be very useful as we continue.

Lemma 4.1 *There holds*

$$\int_{[0,1]^n} \bigwedge_{i\in S} x_i \, dx = \frac{1}{s+1}, \quad S \subseteq N. \tag{29}$$

Proof. Observe first that we can assume $S = N$. Next, we have

$$\begin{aligned}
\int_{[0,1]^n} \bigwedge_{i\in N} x_i \, dx &= \sum_{\pi\in\Pi_n} \int_{\mathcal{B}_\pi} x_{\pi(1)} \, dx \\
&= \sum_{\pi\in\Pi_n} \int_0^1 \int_0^{x_{\pi(n)}} \cdots \int_0^{x_{\pi(2)}} x_{\pi(1)} \, dx_{\pi(1)} \cdots dx_{\pi(n)} \\
&= \sum_{\pi\in\Pi_n} \frac{1}{(n+1)!} = \frac{1}{n+1},
\end{aligned}$$

and the lemma is proved. □

From (7), (28) and (29), we can readily see that the Shapley interaction index related to S is obtained by integrating the S-derivative of the Lovász extension of game v over the hypercube. This result, which is to be compared with (21), can be stated as follows.

Proposition 4.1 *We have*

$$I_{\mathrm{Sh}}(S) = \int_{[0,1]^n} \Delta_S \, \hat{f}(x) \, dx, \quad S \subseteq N.$$

From (28), we immediately have

$$(\Delta_S \, \hat{f})(\underline{x}) = a(S) + x \sum_{\substack{T\supseteq S \\ T\neq S}} a(T), \quad \forall x \in [0,1], \;\; \forall S \subseteq N.$$

Consequently, we have, using (13):

$$\begin{aligned}
a(S) &= (\Delta_S \, \hat{f})(\underline{0}), \quad S \subseteq N \\
b(S) &= (\Delta_S \, \hat{f})(\underline{1}), \quad S \subseteq N \\
\frac{a(S)+b(S)}{2} &= (\Delta_S \, \hat{f})(\underline{1/2}), \quad S \subseteq N \\
\frac{a(S)+b(S)}{2} &= \int_0^1 (\Delta_S \, \hat{f})(\underline{x}) \, dx, \quad S \subseteq N
\end{aligned}$$

5 Some conversion formulas derived from the MLE

It is easy to see that, for any function g of the form (19), the operator Δ_S identifies with the classical S-derivative, that is,

$$\Delta_S \, g(x) = \frac{\partial^s \, g(x)}{\partial x_{i_1} \cdots \partial x_{i_s}} \quad \text{where } S = \{i_1, \ldots, i_s\}.$$

The Taylor formula for functions of several variables then can be applied to g. This leads to the equality:

$$g(x) = \sum_{T \subseteq N} \prod_{i \in T} (x_i - y_i)\, \Delta_T\, g(y), \quad x, y \in [0,1]^n. \tag{30}$$

Replacing x by e_S and y by $\underline{y}$ provides:

$$v(S) = \sum_{T \subseteq N} \prod_{i \in T} ((e_S)_i - y)\, (\Delta_T\, g)(\underline{y}), \quad \forall y \in [0,1],\ \ \forall S \subseteq N. \tag{31}$$

On the basis of (23)–(25), we can obtain the conversions from a, b, I_{B} to v by replacing y respectively by 0, 1 and 1/2 in (31). The corresponding formulas can be found in Tables 3 and 4 (Section 6).

By successive derivations of (30), we obtain:

$$\Delta_S\, g(x) = \sum_{T \supseteq S} \prod_{i \in T \setminus S} (x_i - y_i)\, \Delta_T g(y), \quad \forall x, y \in [0,1]^n,\ \ \forall S \subseteq N.$$

In particular, we have:

$$(\Delta_S\, g)(\underline{x}) = \sum_{T \supseteq S} (x-y)^{t-s}\, (\Delta_T g)(\underline{y}), \quad \forall x, y \in [0,1],\ \ \forall S \subseteq N. \tag{32}$$

We can get all the conversions between a, b, and I_{B} by replacing x and y by 0, 1, and 1/2 in (32). The corresponding formulas are written in Tables 3 and 4.

Combining (26) and (32), we immediately have:

$$\begin{aligned} I_{\mathrm{Sh}}(S) &= \sum_{T \supseteq S} \left[\int_0^1 (x-y)^{t-s}\, dx \right] (\Delta_T g)(\underline{y}) \\ &= \sum_{T \supseteq S} \frac{(1-y)^{t-s+1} - (-y)^{t-s+1}}{t-s+1}\, (\Delta_T g)(\underline{y}),\ \forall y \in [0,1],\ \ \forall S \subseteq N. \end{aligned} \tag{33}$$

We then obtain the conversions from a, b, I_{B} to I_{Sh} by replacing y successively by 0, 1, and 1/2 in (33), see Tables 3 and 4.

The conversions from I_{Sh} to a, b, I_{B}, are a little bit more delicate. Let $\{B_n\}_{n \in \mathbb{N}}$ be the sequence of Bernoulli numbers defined recursively by

$$\begin{cases} B_0 = 1, \\ \displaystyle\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0, \quad n \in \mathbb{N}_0. \end{cases}$$

The first elements of the sequence are:

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, \ldots.$$

The Bernoulli polynomials are then defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k\, x^{n-k}, \quad \forall n \in \mathbb{N},\ \forall x \in \mathbb{R}.$$

It is well known that these polynomials fulfil the following properties (see e.g. [1]):

$$B_n(0) = B_n, \quad \forall n \in \mathbb{N} \tag{34}$$

$$B_n(1) = (-1)^n\, B_n, \quad \forall n \in \mathbb{N} \tag{35}$$

$$B_n(1/2) = (\frac{1}{2^{n-1}} - 1)\, B_n, \quad \forall n \in \mathbb{N} \tag{36}$$

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x)\, y^{n-k}, \quad \forall n \in \mathbb{N},\ \forall x, y \in \mathbb{R} \tag{37}$$

$$\int_0^1 B_n(x)\, dx = 0, \quad \forall n \in \mathbb{N}_0. \tag{38}$$

We have the following lemma.

Lemma 5.1 *For all $S, K \subseteq N$ such that $S \subseteq K$, we have:*

$$\sum_{T : S \subseteq T \subseteq K} B_{t-s}(x)\, \frac{1}{k-t+1} = x^{k-s}, \quad x \in [0,1]. \tag{39}$$

Proof. We have

$$\begin{aligned}
\sum_{T : S \subseteq T \subseteq K} B_{t-s}(x)\, \frac{1}{k-t+1} &= \sum_{t=s}^{k} \binom{k-s}{t-s} B_{t-s}(x)\, \frac{1}{k-t+1} \\
&= \sum_{u=0}^{k-s} \binom{k-s}{u} B_u(x)\, \frac{1}{k-s-u+1} \\
&= \int_0^1 \sum_{u=0}^{k-s} \binom{k-s}{u} B_u(x)\, y^{k-s-u}\, dy \\
&= \int_0^1 \sum_{u=0}^{k-s} \binom{k-s}{u} B_u(y)\, x^{k-s-u}\, dy \quad \text{(by (37))} \\
&= x^{k-s} \quad \text{(by (38))},
\end{aligned}$$

which proves the result. □

We then have the following result.

Proposition 5.1 *We have*

$$(\Delta_S\, g)(\underline{x}) = \sum_{T \supseteq S} B_{t-s}(x)\, I_{\mathrm{Sh}}(T), \quad \forall x \in [0,1],\ \forall S \subseteq N. \tag{40}$$

Proof. We have

$$\begin{aligned}
\sum_{T\supseteq S} B_{t-s}(x)\, I_{\mathrm{Sh}}(T) &= \sum_{T\supseteq S} B_{t-s}(x) \sum_{K\supseteq T} \frac{1}{k-t+1}\, a(K) \quad \text{(by (7))}\\
&= \sum_{K\supseteq S} a(K) \sum_{T: S\subseteq T\subseteq K} B_{t-s}(x)\, \frac{1}{k-t+1}\\
&= \sum_{K\supseteq S} a(K)\, x^{k-s} \quad \text{(by (39))}\\
&= (\varDelta_S\, g)(\underline{x}) \quad \text{(by (22))},
\end{aligned}$$

and the result is proved. □

We then obtain the conversions from I_{Sh} to a, b, I_{B} by replacing x successively by 0, 1, and 1/2 in (40), and by using (34)–(36).

6 Fractal and cardinality transformations

In this section, we give all the conversion formulas between the five representations v, a, b, I_{B}, I_{Sh} of a game v. All these representations are linear, that is, such that the transform $\mathcal{T}$ is a linear operator which can be written in a matrix form.

Any pair (x, y) extracted from the set $\{v, a, b, I_{\mathrm{B}}, I_{\mathrm{Sh}}\}$ can produce a matricial relation

$$y = T \circ x$$

where $x, y : 2^N \to \mathbb{R}$ and where T is a *transformation matrix* of dimension $2^n \times 2^n$ if the 2^n elements of 2^N are ordered according to some sequence.

Let us consider the following total ordering of the elements of 2^N,

$$\mathcal{O} : \emptyset, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{4\}, \ldots, N.$$

This order is obtained as follows. We consider the natural sequence of integers from 0 to $2^n - 1$, that is $0, 1, 2, \ldots, i, \ldots, 2^n - 1$, and its binary notation $[0]_2, [1]_2, \ldots, [i]_2, \ldots, [2^n - 1]_2$, which is (with n digits) $000\cdots00$, $000\cdots01$, $000\cdots10, \ldots,$ $111\cdots11$. To any number $[i]_2$ in binary notation corresponds a unique subset $I \subseteq N$ such that $j \in I$ if and only if the $(n+1-j)$-th digit in $[i]_2$ is 1.

We obtain the vectors of $\mathbb{R}^{2^n}$:

$$\begin{aligned}
x^t_{(n)} &= (\, x(\emptyset) \quad x(\{1\}) \quad x(\{2\}) \quad x(\{1,2\}) \quad \ldots \quad x(N)\,)\\
y^t_{(n)} &= (\, y(\emptyset) \quad y(\{1\}) \quad y(\{2\}) \quad y(\{1,2\}) \quad \ldots \quad y(N)\,)
\end{aligned}$$

(here the superscript t represents the transposition operation) and we determine the matricial relation

$$y_{(n)} = T_{(n)} \circ x_{(n)}$$

with

$$
\begin{array}{cc}
 & \begin{array}{cccc} \emptyset & \{1\} & \cdots & N \end{array} \\
T_{(n)} = \begin{array}{c} \emptyset \\ \{1\} \\ \vdots \\ N \end{array} & \begin{pmatrix} T(\emptyset,\emptyset) & T(\emptyset,\{1\}) & \cdots & T(\emptyset,N) \\ T(\{1\},\emptyset) & T(\{1\},\{1\}) & \cdots & T(\{1\},N) \\ \vdots & \vdots & & \vdots \\ T(N,\emptyset) & T(N,\{1\}) & \cdots & T(N,N) \end{pmatrix}.
\end{array}
$$

Three particular transformations will be considered:

(i) the *fractal* transformation linked to a "fractal matrix" $T = F$ defined with the help of one "basic fractal matrix" $F_{(1)}$ which is supposed to be invertible.

$$
\begin{aligned}
F_{(1)} &:= \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}, \quad f_i \in \mathbb{R}, \ \ i = 1,2,3,4 \\
F_{(1)}^{-1} &:= \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \\
F_{(k)} &:= \begin{pmatrix} f_1\, F_{(k-1)} & f_2\, F_{(k-1)} \\ f_3\, F_{(k-1)} & f_4\, F_{(k-1)} \end{pmatrix}, \quad k = 2,\ldots,n.
\end{aligned}
$$

It can be shown that the inverse matrix is also fractal. In general we have:

$$
F_{(k)}^{-1} = \begin{pmatrix} g_1\, F_{(k-1)}^{-1} & g_2\, F_{(k-1)}^{-1} \\ g_3\, F_{(k-1)}^{-1} & g_4\, F_{(k-1)}^{-1} \end{pmatrix}, \quad k = 2,\ldots,n.
$$

(ii) the *upper-cardinality* transformation linked to an "upper-cardinality matrix" $T = C$ based on asequence of real numbers $(c_0, c_1, \ldots, c_k, \ldots, c_n)$, $c_0 = 1$, and

$$
\begin{aligned}
C_{(1)} &:= \begin{pmatrix} c_0 & c_1 \\ 0 & c_0 \end{pmatrix}, \quad C_{(1)}^l := \begin{pmatrix} c_{l-1} & c_l \\ 0 & c_{l-1} \end{pmatrix}, \ l = 1,\ldots,n \\
C_{(2)} &:= \begin{pmatrix} C_{(1)}^1 & C_{(1)}^2 \\ 0 & C_{(1)}^1 \end{pmatrix}, \quad C_{(2)}^l := \begin{pmatrix} C_{(1)}^l & C_{(1)}^{l+1} \\ 0 & C_{(1)}^l \end{pmatrix}, \ l = 1,\ldots,n-1 \\
C_{(k)} &:= \begin{pmatrix} C_{(k-1)}^1 & C_{(k-1)}^2 \\ 0 & C_{(k-1)}^1 \end{pmatrix}, \ k = 2,\ldots,n.
\end{aligned}
$$

Using the sequence $\mathcal{O}$ to order the rows and the columns of $C_{(n)}$, one obtains (blanks replace zeroes):

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} \emptyset & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & \cdots \end{array} \\
C_{(n)} = \begin{array}{l} \emptyset \\ \{1\} \\ \{2\} \\ \{1,2\} \\ \{3\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \\ \vdots \end{array} & \begin{pmatrix}
c_0 & c_1 & c_1 & c_2 & c_1 & c_2 & c_2 & c_3 & \\
 & c_0 & & c_1 & & c_1 & & c_2 & \\
 & & c_0 & c_1 & & & c_1 & c_2 & \\
 & & & c_0 & & & & c_1 & \\
 & & & & c_0 & c_1 & c_1 & c_2 & \\
 & & & & & c_0 & & c_1 & \\
 & & & & & & c_0 & c_1 & \\
 & & & & & & & c_0 & \\
 & & & & & & & &
\end{pmatrix}.
\end{array}
$$

(iii) the *lower-cardinality* transformation linked to a "lower-cardinality matrix" $T = C^t$ based on a sequence of real numbers $(c_0, c_1, \ldots, c_k, \ldots, c_n)$, $c_0 = 1$, and

$$C^t_{(1)} := \begin{pmatrix} c_0 & 0 \\ c_1 & c_0 \end{pmatrix}, \ldots$$

$$C^t_{(k)} := \begin{pmatrix} C^{1\,t}_{(k-1)} & 0 \\ C^{2\,t}_{(k-1)} & C^{1\,t}_{(k-1)} \end{pmatrix}, \; k = 2, \ldots, n.$$

Both fractal and cardinality transformations correspond to a two-place real valued set function Φ. We introduce the product of two such transformations Φ and Ψ to define:

$$(\Phi \circ \Psi)(A, B) := \sum_{C \subseteq N} \Phi(A, C)\, \Psi(C, B), \quad A, B \subseteq N.$$

In the case of the upper-cardinality transformation (see Denneberg and Grabisch [3])

$$\Phi(A, B) = \Phi(\emptyset, B \setminus A) = \begin{cases} c_{|B \setminus A|}, & \text{if } A \subseteq B, \\ 0, & \text{otherwise,} \end{cases}$$

and this definition justifies the terminology used.

If we are concerned with a lower-cardinality transformation,

$$\Phi(A, B) = \Phi(A \setminus B, \emptyset) = \begin{cases} c_{|A \setminus B|}, & \text{if } B \subseteq A, \\ 0, & \text{otherwise.} \end{cases}$$

Let us now consider the families:

$\mathcal{G}_F := \{F : 2^N \times 2^N \to \mathbb{R} \,|\, F \text{ is built on a basic invertible fractal matrix } F_{(1)}\}$
$\mathcal{G}_{\overline{C}} := \{C : 2^N \times 2^N \to \mathbb{R} \,|\, C \text{ is determined by a sequence } (c_k)\}$
$\mathcal{G}_{\underline{C}} := \{C^t : 2^N \times 2^N \to \mathbb{R} \,|\, C^t \text{ is determined by a sequence } (c_k)\}$

The three families form a multiplicative group for the composition law $(\circ)$ with neutral element

$$I(A, B) := \begin{cases} 1, & \text{if } A = B, \\ 0, & \text{else.} \end{cases}$$

The families $\mathcal{G}_{\overline{C}}$ and $\mathcal{G}_{\underline{C}}$ form an Abelian group (i.e. commutative) but the property of commutativity is generally not satisfied for $\mathcal{G}_F$.

In the case of the upper-cardinality transformation, $y_{(n)} = C_{(n)}\, x_{(n)}$ can be rewritten as

$$y(S) = \sum_{T \supseteq S} c_{t-s}\, x(T), \quad S \subseteq N. \tag{41}$$

Moreover, if C^1 and C^2 represent two upper-cardinality transformations, the sequence (c_k) related to $C^1 \circ C^2$ corresponds to (see [3])

$$c_k = \sum_{l=0}^{k} \binom{k}{l} c^1_{k-l}\, c^2_l = \sum_{l=0}^{k} \binom{k}{l} c^2_{k-l}\, c^1_l, \quad k = 0, \ldots, n. \tag{42}$$

The inverse C^{-1} of the upper-cardinality transformation C is obtained with $c_0^{-1} = 1$ and

$$c_k^{-1} = -\sum_{l=0}^{k-1} \binom{k}{l} c_{k-l}\, c_l^{-1}, \quad k = 1, \ldots, n. \tag{43}$$

It is obvious that the lower-cardinality transformation $y_{(n)} = C^t_{(n)}\, x_{(n)}$ can be rewritten as

$$y(S) = \sum_{T \subseteq S} c_{s-t}\, x(T), \quad S \subseteq N.$$

If $C^{1\,t}$ and $C^{2\,t}$ represent two lower-cardinality transformations, the sequence (c_k) related to $C^{1\,t} \circ C^{2\,t}$ corresponds to the formula (42) and the inverse $(C^t)^{-1}$ of the lower-cardinality transformation C^t is obtained with (43).

If a fractal transformation F is considered, $y_{(n)} = F_{(n)}\, x_{(n)}$ can be rewritten as

$$y(S) = \sum_{T \subseteq N} F(S,T)\, x(T), \quad S \subseteq N.$$

We know that F^{-1} is also a fractal transformation and we can easily check that

$$F^{-1}_{(n)}(S,T) = \frac{(-1)^{t-s}}{(\det F_{(1)})^n}\, F_{(n)}(N \setminus T, N \setminus S), \quad \forall S, T \subseteq N. \tag{44}$$

Moreover, the composition of two fractal transformations F^1 and F^2 corresponds to a fractal transformation with basic fractal matrix $F_{(1)} = F^1_{(1)} \circ F^2_{(1)}$.

It should be noted that any fractal transformation with a basic fractal matrix:

$$F_{(1)} = \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix}$$

is an upper (lower)-cardinality transformation with the sequence $c_k = \rho^k$. The converse is also true.

From classical results in combinatorics [17], all conversion formulas between v, a and b are well known. We can observe that all the transformations between v, a, b and I_{B} are fractal. For instance, the Möbius representation (2) can be rewritten under the fractal form

$$a_{(n)} = M_{(n)} \circ v_{(n)}$$

with the use of the basic fractal matrix:

$$M_{(1)} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{45}$$

We see that transformation M also corresponds to a lower-cardinality transformation with $c_k = (-1)^k$ and we immediately obtain that

$$v_{(n)} = M^{-1}_{(n)} \circ a_{(n)}$$

where M^{-1} corresponds to the basic fractal matrix:

$$M^{-1}_{(1)} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

or the lower-cardinality transformation with sequence $c_k = 1$, which gives (1).

More generally, one can easily see that the generating conversion formula (31) corresponds, for any fixed $y \in [0, 1]$, to a fractal transformation whose basic fractal matrix is

$$F_{(1)} = \begin{pmatrix} 1 & -y \\ 1 & 1-y \end{pmatrix}.$$

By (44), the formula (31) can immediately be inverted into

$$(\Delta_S\, g)(\underline{y}) = \sum_{T \subseteq N} (-1)^{t-s} \prod_{i \in N \setminus S} ((e_{N\setminus T})_i - y)\, v(T), \quad y \in [0, 1]. \tag{46}$$

Replacing y respectively by 0, 1 and 1/2 in (46), we obtain the conversions from v to a, b and I_{B}, see Table 3.

The generating conversion formula (32) corresponds, for any fixed $x, y \in [0, 1]$, to a fractal transformation with basic fractal matrix:

$$F_{(1)} = \begin{pmatrix} 1 & x-y \\ 0 & 1 \end{pmatrix}.$$

Observe that this transformation also corresponds to an upper-cardinality transformation with sequence $c_k = (x-y)^k$.

We have just shown that all the transformations between v, a, b and I_{B} are fractal. The corresponding basic fractal matrices are summarized in Table 1.

Due to (41), it is clear that the generating conversion formula (33) corresponds, for any fixed $y \in [0, 1]$, to an upper-cardinality transformation with sequence

$$c_k = \int_0^1 (x-y)^k\, dx = \frac{(1-y)^{k+1} - (-y)^{k+1}}{k+1},$$

whereas the inverse transformation (40) corresponds to an upper-cardinality transformation with sequence $c_k^{-1} = B_k(y)$. Thus, all the transformations between a, b, I_{B} and I_{Sh} are upper-cardinality transformations. The corresponding sequences are summarized in Table 2.

Now, let us turn to the two remaining cases: the transformations from v to I_{Sh} and the converse, which are neither fractal, nor upper-cardinal. From (5), we obtain, by setting $T' := T \cup L$ (which implies $L = T' \cap S$ and $T = T' \setminus S$):

$$I_{\mathrm{Sh}}(S) = \sum_{T' \subseteq N} \frac{|N \setminus (S \cup T')|!\, |T' \setminus S|!}{(n-s+1)!} (-1)^{|S \setminus T'|}\, v(T'), \quad S \subseteq N,$$

which can also be written under the form

$$I_{\mathrm{Sh}}(S) = \sum_{T\subseteq N} \frac{(-1)^{|S\setminus T|}}{(n-s+1)\binom{n-s}{|T\setminus S|}}\, v(T), \quad S\subseteq N. \tag{47}$$

With matricial notation, this identity is written:

$$I_{S\,(n)} = H_{(n)} \circ a_{(n)} = H_{(n)} \circ M_{(n)} \circ v_{(n)} \tag{48}$$

where $H_{(n)}$ is an upper-cardinality matrix based on the sequence $h_k = \frac{1}{k+1}$, and $M_{(n)}$ is the fractal matrix generated by (45).

The inverse formula can be found in [3, 6]: for all $S \subseteq N$, we have, using adequate correspondence formulas,

$$\begin{aligned} v(S) &= \sum_{K\subseteq S} a(K) = \sum_{K\subseteq S}\sum_{T\supseteq K} B_{t-k}\, I_{\mathrm{Sh}}(T) = \sum_{T\subseteq N} I_{\mathrm{Sh}}(T) \sum_{K\subseteq T\cap S} B_{t-k} \\ &= \sum_{T\subseteq N} I_{\mathrm{Sh}}(T) \sum_{k=0}^{|T\cap S|} \binom{|T\cap S|}{k} B_{t-k}, \end{aligned}$$

that is

$$v(S) = \sum_{T\subseteq N} \beta^{|T|}_{|T\cap S|}\, I_{\mathrm{Sh}}(T), \quad S\subseteq N,$$

with

$$\beta^l_k := \sum_{j=0}^{k} \binom{k}{j} B_{l-j}.$$

The first values of β^l_k are:

$k\backslash l$	0	1	2	3	4
0	1	−1/2	1/6	0	−1/30
1		1/2	−1/3	1/6	−1/30
2			1/6	−1/6	2/15
3				0	−1/30
4					−1/30

Some properties of the β^l_k are shown in [3, 6]. This inverse formula corresponds to

$$v_{(n)} = M^{-1}_{(n)} \circ H^{-1}_{(n)} \circ I_{S\,(n)}.$$

Although these transformations between I_{Sh} and v are neither fractal nor cardinal, their associated matrices have nevertheless a remarkable structure, and we call them *Pascal matrices*:

(i) a *direct Pascal matrix* P based on a sequence of real numbers $(p_0, p_1, \ldots, p_k, \ldots, p_n)$, such that:

$$P_{(1)} := \begin{pmatrix} p_0 & p_1 \\ p_0 & p_0 + p_1 \end{pmatrix}, \quad P^l_{(1)} := \begin{pmatrix} p_{l-1} & p_l \\ p_{l-1} & p_{l-1} + p_l \end{pmatrix},\ l = 1, \ldots, n$$
$$P_{(2)} := \begin{pmatrix} P^1_{(1)} & P^2_{(1)} \\ P^1_{(1)} & P^1_{(1)} + P^2_{(1)} \end{pmatrix}, \quad P^l_{(2)} := \begin{pmatrix} P^l_{(1)} & P^{l+1}_{(1)} \\ P^l_{(1)} & P^l_{(1)} + P^{l+1}_{(1)} \end{pmatrix},\ l = 1, \ldots, n-1$$
$$P_{(k)} := \begin{pmatrix} P^1_{(k-1)} & P^2_{(k-1)} \\ P^1_{(k-1)} & P^1_{(k-1)} + P^2_{(k-1)} \end{pmatrix},\ k = 2, \ldots, n.$$

(ii) an *inverse Pascal matrix* Q based on a sequence of real numbers $(q_0, q_1, \ldots, q_k, \ldots, q_n)$, such that:

$$Q_{(1)} := \begin{pmatrix} q_0 - q_1 & q_1 \\ -q_0 & q_0 \end{pmatrix}, \quad Q^l_{(1)} := \begin{pmatrix} q_{l-1} - q_l & q_l \\ -q_{l-1} & q_{l-1} \end{pmatrix},\ l = 1, \ldots, n$$
$$Q_{(2)} := \begin{pmatrix} Q^1_{(1)} - Q^2_{(1)} & Q^2_{(1)} \\ -Q^1_{(1)} & Q^1_{(1)} \end{pmatrix}, \quad Q^l_{(2)} := \begin{pmatrix} Q^l_{(1)} - Q^{l+1}_{(1)} & Q^{l+1}_{(1)} \\ -Q^l_{(1)} & Q^l_{(1)} \end{pmatrix},\ l = 1, \ldots, n-1$$
$$Q_{(k)} := \begin{pmatrix} Q^1_{(k-1)} - Q^2_{(k-1)} & Q^2_{(k-1)} \\ -Q^1_{(k-1)} & Q^1_{(k-1)} \end{pmatrix},\ k = 2, \ldots, n.$$

The name "Pascal matrix" comes from the fact that, as in the Pascal triangle, elements are obtained by the sum of two preceding elements. Direct Pascal matrices are constructed from the upper left-hand corner, while inverse Pascal matrices start from the lower right-hand corner. An example of each kind is shown below ($n = 2$), where for $P_{(2)}$ the sequence of Bernoulli numbers have been chosen (thus retrieving the β^l_k coefficients and all their properties shown in [3, 6]), and for $Q_{(2)}$ the sequence $h_k = \frac{1}{k+1}$, $k = 0, 1, 2$, defined above (see (48)) (thus retrieving the coefficients of (47)):

$$P_{(2)} = M^{-1}_{(2)} \circ H^{-1}_{(2)} = \begin{pmatrix} 1 & -1/2 & -1/2 & 1/6 \\ 1 & 1/2 & -1/2 & -1/3 \\ 1 & -1/2 & 1/2 & -1/3 \\ 1 & 1/2 & 1/2 & 1/6 \end{pmatrix}$$
$$Q_{(2)} = H_{(2)} \circ M_{(2)} = \begin{pmatrix} 1/3 & 1/6 & 1/6 & 1/3 \\ -1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Any Pascal matrix can be written as the product of an upper-cardinal matrix and either the Möbius matrix M or its inverse:

$$P_{(n),\,p_0,\ldots,p_n} = M^{-1}_{(n)} \circ C_{(n),\,p_0,\ldots,p_n}$$
$$Q_{(n),\,q_0,\ldots,q_n} = C_{(n),\,q_0,\ldots,q_n} \circ M_{(n)}$$

(the generating sequence is written in subscript), as it can be easily verified. The set of (direct or inverse) Pascal matrices does not form a group since the product of two such matrices is no more a Pascal matrix. However, since the inverse of an upper-cardinality transformation is again upper-cardinal, it follows that the inverse of a direct (resp. inverse) Pascal matrix is an inverse (resp. direct) Pascal matrix.

Before closing this section, we present the explicit transformation formulas between v, a, b, I_{B} and I_{Sh}. They are gathered in Tables 3 and 4.

7 Approximations of pseudo-Boolean functions

Hammer and Holzman [9] investigated the approximation of a pseudo-Boolean function by a multilinear polynomial of (at most) a specified degree. According to them, fixing $k \in \mathbb{N}$ with $k \leq n$, the best k-th approximation of f is the multilinear polynomial $f^{(k)} : \{0,1\}^n \to \mathbb{R}$ of degree $\leq k$ defined by

$$f^{(k)}(x) = \sum_{\substack{T \subseteq N \\ t \leq k}} a^{(k)}(T) \prod_{i \in T} x_i$$

which minimizes

$$\sum_{x \in \{0,1\}^n} [f(x) - f^{(k)}(x)]^2$$

among all multilinear polynomials of degree $\leq k$. They proved that the best k-th approximation $f^{(k)}$ is given by the unique solution $\{a^{(k)}(T) \,|\, T \subseteq N, t \leq k\}$ of the triangular linear system:

$$\frac{1}{2^n} \sum_{x \in \{0,1\}^n} \Delta_S f^{(k)}(x) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \Delta_S f(x), \quad \forall S \subseteq N, \;\; s \leq k. \tag{49}$$

They also solved this system for $k = 1$ and $k = 2$. In this final section, we intend to solve the system for any $k \leq n$.

Let $I_{\mathrm{B}}^{(k)}$ be the Banzhaf interaction index related to $f^{(k)}$. By (17), the system (49) can be written as

$$I_{\mathrm{B}}^{(k)}(S) = I_{\mathrm{B}}(S), \quad \forall S \subseteq N, \;\; s \leq k. \tag{50}$$

This shows that the approximation problem amounts to finding a multilinear polynomial of degree $\leq k$ that has the same Banzhaf interaction indices as f for subsets of at most k elements.

By (6), the system (50) becomes

$$\sum_{\substack{T \supseteq S \\ t \leq k}} (\frac{1}{2})^{t-s} a^{(k)}(T) = I_{\mathrm{B}}(S), \quad \forall S \subseteq N, \;\; s \leq k.$$

In particular, we have $a^{(k)}(S) = I_{\mathrm{B}}(S)$ for all $S \subseteq N$ such that $s = k$. Hammer and Holzman [9, §3] obtained this result for $k = 1$: $a^{(1)}(\{i\}) = \phi_{\mathrm{B}}(i)$ for all $i \in N$.

By (6), we observe immediately that $I_{\mathrm{B}}^{(k)}(S)=0$ for all $S\subseteq N$ such that $s>k$. Hence, by using the conversion formula from I_{B} to a, we have

$$a^{(k)}(S)=\sum_{\substack{J\supseteq S\\ j\leq k}}(-\frac{1}{2})^{j-s}I_{\mathrm{B}}^{(k)}(J),\quad \forall S\subseteq N,\;\; s\leq k.$$

The system (50) then becomes

$$a^{(k)}(S)=\sum_{\substack{J\supseteq S\\ j\leq k}}(-\frac{1}{2})^{j-s}I_{\mathrm{B}}(J),\quad \forall S\subseteq N,\;\; s\leq k,$$

and by (6), we have, for all $S\subseteq N$ with $s\leq k$,

$$\begin{aligned}
a^{(k)}(S) &= \sum_{\substack{J\supseteq S\\ j\leq k}}(-\frac{1}{2})^{j-s}\sum_{T\supseteq J}(\frac{1}{2})^{t-j}a(T)\\
&= \sum_{T\supseteq S}(\frac{1}{2})^{t-s}a(T)\sum_{\substack{J:S\subseteq J\subseteq T\\ j\leq k}}(-1)^{j-s}\\
&= \sum_{T\supseteq S}(\frac{1}{2})^{t-s}a(T)\sum_{j=s}^{\min(k,t)}\binom{t-s}{j-s}(-1)^{j-s}.
\end{aligned}$$

However, we have

$$\sum_{j=s}^{\min(k,t)}\binom{t-s}{j-s}(-1)^{j-s}=\begin{cases}(1-1)^{t-s}, & \text{if } t\leq k,\\ (-1)^{k-s}\binom{t-s-1}{k-s}, & \text{if } t>k \text{ (use induction over } k\geq s).\end{cases}$$

Therefore, we obtain an explicit formula for $a^{(k)}(S)$:

Proposition 7.1 *The coefficients of the best k-th approximation of f are given by*

$$a^{(k)}(S)=a(S)+(-1)^{k-s}\sum_{\substack{T\supseteq S\\ t>k}}\binom{t-s-1}{k-s}(\frac{1}{2})^{t-s}\,a(T),\quad \forall S\subseteq N,\;\; s\leq k.$$

Some particular cases are shown in Table 5. We thus retrieve the solutions obtained by Hammer and Holzman for $k=1$ and $k=2$.

8 Conclusions

In this paper, we have analyzed the mathematical structure of the transformations from some linear representation of a game (or weight function) to another one, where a representation is any bijective transform of a game. Interactions indices, as well as dividends, are examples of linear representations. It was shown that the underlying matrices have remarkable properties.

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	v	a	b	I_{B}
v	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1/2 \\ 1 & 1/2 \end{pmatrix}$
a	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix}$
b	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$
I_{B}	$\begin{pmatrix} 1/2 & 1/2 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Table 1. Basic fractal matrices for the equivalent representations $(v, a, b, I_{\mathrm{B}})$.

	a	b	I_{B}	I_{Sh}
a	$\begin{cases} c_0 = 1 \\ c_{k>0} = 0 \end{cases}$	$c_k = (-1)^k$	$c_k = (-\frac{1}{2})^k$	$c_k = B_k$
b	$c_k = 1$	$\begin{cases} c_0 = 1 \\ c_{k>0} = 0 \end{cases}$	$c_k = (\frac{1}{2})^k$	$c_k = (-1)^k B_k$
I_{B}	$c_k = (\frac{1}{2})^k$	$c_k = (-\frac{1}{2})^k$	$\begin{cases} c_0 = 1 \\ c_{k>0} = 0 \end{cases}$	$c_k = (\frac{1}{2^{k-1}} - 1) B_k$
I_{Sh}	$c_k = \frac{1}{k+1}$	$c_k = \frac{(-1)^k}{k+1}$	$c_k = \frac{1+(-1)^k}{(k+1)2^{k+1}}$	$\begin{cases} c_0 = 1 \\ c_{k>0} = 0 \end{cases}$

Table 2. Cardinality sequences for the equivalent representations $(a, b, I_{\mathrm{B}}, I_{\mathrm{Sh}})$.

	v	a	b
$v(S) =$	$v(S)$	$\sum_{T \subseteq S} a(T)$	$\sum_{T \subseteq N \setminus S} (-1)^t b(T)$
$a(S) =$	$\sum_{T \subseteq S} (-1)^{s-t} v(T)$	$a(S)$	$\sum_{T \supseteq S} (-1)^{t-s} b(T)$
$b(S) =$	$\sum_{T \supseteq N \setminus S} (-1)^{n-t} v(T)$	$\sum_{T \supseteq S} a(T)$	$b(S)$
$I_{\mathrm{B}}(S) =$	$(\frac{1}{2})^{n-s} \sum_{T \subseteq N} (-1)^{\lvert S \setminus T \rvert} v(T)$	$\sum_{T \supseteq S} (\frac{1}{2})^{t-s} a(T)$	$\sum_{T \supseteq S} (-\frac{1}{2})^{t-s} b(T)$
$I_{\mathrm{Sh}}(S) =$	$\sum_{T \subseteq N} \frac{(-1)^{\lvert S \setminus T \rvert}}{(n-s+1)\binom{n-s}{\lvert T \setminus S \rvert}} v(T)$	$\sum_{T \supseteq S} \frac{1}{t-s+1} a(T)$	$\sum_{T \supseteq S} \frac{(-1)^{t-s}}{t-s+1} b(T)$

Table 3. Alternative representations in terms of v, a, b

	I_{B}	I_{Sh}
$v(S) =$	$\sum_{T \subseteq N} (\frac{1}{2})^t (-1)^{\lvert T \setminus S \rvert} I_{\mathrm{B}}(T)$	$\sum_{T \subseteq N} \left[\sum_{k=0}^{\lvert T \cap S \rvert} \binom{\lvert T \cap S \rvert}{k} B_{t-k} \right] I_{\mathrm{Sh}}(T)$
$a(S) =$	$\sum_{T \supseteq S} (-\frac{1}{2})^{t-s} I_{\mathrm{B}}(T)$	$\sum_{T \supseteq S} B_{t-s}\, I_{\mathrm{Sh}}(T)$
$b(S) =$	$\sum_{T \supseteq S} (\frac{1}{2})^{t-s} I_{\mathrm{B}}(T)$	$\sum_{T \supseteq S} (-1)^{t-s} B_{t-s}\, I_{\mathrm{Sh}}(T)$
$I_{\mathrm{B}}(S) =$	$I_{\mathrm{B}}(S)$	$\sum_{T \supseteq S} (\frac{1}{2^{t-s-1}} - 1) B_{t-s}\, I_{\mathrm{Sh}}(T)$
$I_{\mathrm{Sh}}(S) =$	$\sum_{T \supseteq S} \frac{1 + (-1)^{t-s}}{(t-s+1)\, 2^{t-s+1}}\, I_{\mathrm{B}}(T)$	$I_{\mathrm{Sh}}(S)$

Table 4. Alternative representations in terms of I_{B} and I_{Sh}

$$a_\emptyset^{(0)} = \sum_{T \subseteq N} \frac{1}{2^t}\, a_T$$

$$a_\emptyset^{(1)} = \sum_{T \subseteq N} \frac{-(t-1)}{2^t}\, a_T$$

$$a_i^{(1)} = \sum_{T \ni i} \frac{1}{2^{t-1}}\, a_T, \qquad i \in N$$

$$a_\emptyset^{(2)} = \sum_{T \subseteq N} \frac{(t-1)(t-2)}{2^{t+1}}\, a_T$$

$$a_i^{(2)} = \sum_{T \ni i} \frac{-(t-2)}{2^{t-1}}\, a_T, \qquad i \in N$$

$$a_{ij}^{(2)} = \sum_{T \ni i,j} \frac{1}{2^{t-2}}\, a_T, \qquad \{i,j\} \subseteq N$$

$$a_S^{(n-1)} = a_S - (-\frac{1}{2})^{n-s}\, a_N, \qquad S \subseteq N, \;\; s \leq n-1$$

Table 5. Coefficients of the best k-th approximation for some values of k