

# SOME NATURALLY DEFINED STAR PRODUCTS FOR KÄHLER MANIFOLDS

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- ▶ One mathematical aspect of quantization is the **passage** from the **commutative world** to the **non-commutative world**.
- ▶ one way **a deformation quantization** (also called **star product**)
- ▶ can only be done on the level of **formal power series** over the algebra of functions
- ▶ was pinned down in a mathematically satisfactory manner by **Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer**.

# OUTLINE

- ▶ give an overview of some **naturally defined star products** in the case that our “phase-space manifold” is a (compact) **Kähler manifold**
- ▶ here we have additional **complex structure** and search for star products **respecting** it
- ▶ yield star products of **separation of variables type** (**Karabegov**) resp. **Wick or anti-Wick type** (**Bordemann and Waldmann**)
- ▶ both constructions are quite different, but there is a **1:1 correspondence** (**Neumaier**)
- ▶ still quite a lot of them

- ▶ single out certain naturally given ones.
- ▶ restrict to quantizable Kähler manifolds
- ▶ Berezin-Toeplitz star product, Berezin transform, Berezin star product
- ▶ a side result: star product of geometric quantization
- ▶ all of them are equivalent star product, but not the same
- ▶ give Deligne-Fedosov class and Karabegov forms
- ▶ give the equivalence transformations

# GEOMETRIC SET-UP

- ▶  $(M, \omega)$  a pseudo-Kähler manifold.  
 $M$  a complex manifold, and  $\omega$ , a non-degenerate closed  $(1, 1)$ -form
- ▶ if  $\omega$  is a positive form then  $(M, \omega)$  is a honest Kähler manifold
- ▶  $C^\infty(M)$  the algebra of complex-valued differentiable functions with associative product given by point-wise multiplication
- ▶ define the Poisson bracket

$$\{f, g\} := \omega(X_f, X_g) \quad \omega(X_f, \cdot) = df(\cdot)$$

- ▶  $C^\infty(M)$  becomes a Poisson algebra.

# STAR PRODUCT

*star product* for  $M$  is an **associative product**  $\star$  on  
 $\mathcal{A} := C^\infty(M)[[\nu]]$ , such

1.  $f \star g = f \cdot g \pmod{\nu}$ ,
2.  $(f \star g - g \star f) / \nu = -i\{f, g\} \pmod{\nu}$ .

Also

$$f \star g = \sum_{k=0}^{\infty} \nu^k C_k(f, g), \quad C_k(f, g) \in C^\infty(M),$$

**differential** (or **local**) if  $C_k(, )$  are bidifferential operators.

Usually:  $1 \star f = f \star 1 = f$ .

## Equivalence of star products

$\star$  and  $\star'$  (the same Poisson structure) are *equivalent* means there exists a formal series of linear operators

$$B = \sum_{i=0}^{\infty} B_i \nu^i, \quad B_i : C^\infty(M) \rightarrow C^\infty(M),$$

with  $B_0 = id$  and  $B(f) \star' B(g) = B(f \star g)$ .

to every equivalence class of a differential star product one assigns its **Deligne-Fedosov class**

$$cl(\star) \in \frac{1}{i} \left( \frac{1}{\nu} [\omega] + H_{dR}^2(M, \mathbb{C})[[\nu]] \right).$$

gives a 1:1 correspondence

**Existence:** by DeWilde-Lecomte, Omori-Maeda-Yoshioka, Fedosov, ...., Kontsevich.

## SEPARATION OF VARIABLES TYPE

- ▶ **pseudo-Kähler** case: we look for star products adapted to the complex structure
- ▶ **separation of variables type** (Karabegov)
- ▶ **Wick and anti-Wick type** (Bordemann - Waldmann)
- ▶ **Karabegov convention:** of separation of variables type if in  $C_k(.,.)$  for  $k \geq 1$  the first argument differentiated in anti-holomorphic and the second argument in holomorphic directions.
- ▶ we call this convention **separation of variables (anti-Wick) type** and call a star product of **separation of variables (Wick) type** if the role of the variables is switched
- ▶ we **need** both conventions

# KARABEGOV CONSTRUCTION (SKETCH OF A SKETCH)

- ▶  $(M, \omega_{-1})$  the **pseudo-Kähler** manifold
- ▶ a **formal deformation** of the form  $(1/\nu)\omega_{-1}$  is a formal form

$$\widehat{\omega} = (1/\nu)\omega_{-1} + \omega_0 + \nu\omega_1 + \dots$$

$\omega_r$ ,  $r \geq 0$ , closed  $(1,1)$ -forms on  $M$ .

- ▶ **Karabegov:** to every such  $\widehat{\omega}$  there exists a star product  $\star$
- ▶ and **vice-versa**
- ▶ **Karabegov form** of the star product  $\star$  is  $kf(\star) := \widehat{\omega}$ ,
- ▶ the star product  $\star_K$  with classifying Karabegov form  $(1/\nu)\omega_{-1}$  is Karabegov's **standard star product**.

- ▶ Formal Berezin transform
- ▶ antiholomorphic functions  $a$  and holomorphic functions  $b$  on  $U \subset M$  we have the relation

$$a \star b = I_{\star}(b \star a) = I_{\star}(b \cdot a),$$

- ▶ can be written as

$$I_{\star} = \sum_{i=0}^{\infty} I_i \nu^i, \quad I_i : C^{\infty}(M) \rightarrow C^{\infty}(M), \quad I_0 = id, \quad I_1 = \Delta.$$

- ▶ the formal Berezin transform  $I_{\star}$  **determines** the  $\star$  uniquely.

- ▶ Start with  $\star$  separation of variables type (anti-Wick)  $(M, \omega_{-1})$
- ▶ dual star-product  $\tilde{\star}$  on  $M$

$$f \tilde{\star} g = I^{-1}(I(g) \star I(f)).$$

(anti-Wick) on  $(M, -\omega_{-1})$ .

- ▶ given  $\star$  , the opposite star product is obtained

$$f \star^{op} g = g \star f$$

for  $(M, -\omega_{-1})$  and switches anti-Wick with Wick type.

- ▶ opposite of the dual  $\star' = \tilde{\star}^{op}$ ,

$$f \star' g = g \tilde{\star} f = I^{-1}(I(f) \star I(g)).$$

on  $(M, \omega_{-1})$ , of Wick type

- ▶ opposite of the dual  $\star' = \tilde{\star}^{op}$ ,

$$f \star' g = g \tilde{\star} f = I^{-1}(I(f) \star I(g)).$$

on  $(M, \omega_{-1})$ , of **Wick type**

- ▶ the **formal Berezin transform**  $I_*$  establishes an equivalence of the star products

$$(\mathcal{A}, \star) \text{ and } (\mathcal{A}, \star')$$

# CLASSIFYING FORMS

★ star product of anti-Wick type with Karabegov form  $kf(\star) = \widehat{\omega}$   
Deligne-Fedosov class calculates as

$$cl(\star) = \frac{1}{i}([\widehat{\omega}] - \frac{\delta}{2}).$$

[..] denotes the de-Rham class of the forms and  $\delta$  is the canonical class of the manifold i.e.  $\delta := c_1(K_M)$ .

for the opposite star product  $cl(\star^{op}) = -cl(\star)$

standard star product  $\star_K$  (with Karabegov form  $\widehat{\omega} = (1/\nu)\omega_{-1}$ )

$$cl(\star_K) = \frac{1}{i}(\frac{1}{\nu}[\omega_{-1}] - \frac{\delta}{2}).$$

- ▶ For the Karabegov form to be in 1:1 correspondence, we need to fix a convention: **Wick** or **anti-Wick** for reference
- ▶ here we refer to the **anti-Wick** type product
- ▶ if  $\star$  is of Wick type we set

$$kf(\star) := kf(\star^{op}),$$

- ▶ Recall  $\star^{op}$  is a star product of **(anti-Wick)** type for the pseudo-Kähler manifold  $(M, -\omega)$

## OTHER GENERAL CONSTRUCTIONS

- ▶ **Bordemann and Waldmann:** modification of Fedosov's geometric existence proof.
- ▶ fibre-wise Wick product.
- ▶ by a **modified Fedosov connection** a star product  $\star_{BW}$  of Wick type is obtained.
- ▶ **Karabegov form** is  $-(1/\nu)\omega$
- ▶ Deligne class class

$$cl(\star_{BW}) = -cl(\star_{BW}^{op}) = \frac{1}{i} \left( \frac{1}{\nu} [\omega] + \frac{\delta}{2} \right).$$

Neumaier: by adding a formal closed  $(1, 1)$  form as parameter each star product of separation of variables type can be obtained by the Bordemann-Waldmann construction

Reshetikhin and Takhtajan:

formal Laplace expansions of formal integrals related to the star product.

coefficients of the star product can be expressed (roughly) by Feynman diagrams

# BEREZIN-TOEPLITZ STAR PRODUCT

- ▶ compact and quantizable Kähler manifold  $(M, \omega)$ ,
- ▶ quantum line bundle  $(L, h, \nabla)$ ,  $L$  is a holomorphic line bundle over  $M$ ,  $h$  a hermitian metric on  $L$ ,  $\nabla$  a compatible connection
- ▶ recall  $(M, \omega)$  is quantizable, if there exists such  $(L, h, \nabla)$ , with

$$\text{curv}_{(L, \nabla)} = -i \omega$$

- ▶ consider all positive tensor powers  $(L^m, h^{(m)}, \nabla^{(m)})$ ,

scalar product

$$\langle \varphi, \psi \rangle := \int_M h^{(m)}(\varphi, \psi) \Omega, \quad \Omega := \frac{1}{n!} \underbrace{\omega \wedge \omega \cdots \wedge \omega}_n$$

$$\Pi^{(m)} : L^2(M, L^m) \longrightarrow \Gamma_{hol}(M, L^m)$$

Take  $f \in C^\infty(M)$ , and  $s \in \Gamma_{hol}(M, L^m)$

$$s \quad \mapsto \quad T_f^{(m)}(s) := \Pi^{(m)}(f \cdot s)$$

defines

$$T_f^{(m)} : \quad \Gamma_{hol}(M, L^m) \rightarrow \Gamma_{hol}(M, L^m)$$

the Toeplitz operator of level  $m$ .

## Berezin-Toeplitz operator quantization

$$f \mapsto \left( T_f^{(m)} \right)_{m \in \mathbb{N}_0}.$$

has the correct **semi-classical behavior**

**Theorem** (Bordemann, Meinrenken, and Schl.)

(a)

$$\lim_{m \rightarrow \infty} \| T_f^{(m)} \| = \| f \|_\infty$$

(b)

$$\| m i [T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)} \| = O(1/m)$$

(c)

$$\| T_f^{(m)} T_g^{(m)} - T_{f \cdot g}^{(m)} \| = O(1/m)$$

Theorem (BMS, Schl., Karabegov and Schl.)

$\exists$  a unique differential star product

$$f \star_{BT} g = \sum \nu^k C_k(f, g)$$

such that

$$T_f^{(m)} T_g^{(m)} \sim \sum_{k=0}^{\infty} \left( \frac{1}{m} \right)^k T_{C_k(f, g)}^{(m)}$$

Further properties: is of separation of variables type (Wick type)

classifying Deligne-Fedosov class  $\frac{1}{i} \left( \frac{1}{\nu} [\omega] - \frac{\delta}{2} \right)$  and Karabegov form  $\frac{-1}{\nu} \omega + \omega_{can}$

possible: auxiliary hermitian line (or even vector) bundle can be added, meta-plectic correction.

# GEOMETRIC QUANTIZATION

Further result: The Toeplitz map

$$T_{(m)} : C^\infty(M) \rightarrow \text{End}(\Gamma_{hol}(M, L^m))$$

is **surjective**

implies that the operator  $Q_f^{(m)}$  of **geometric quantization** (with holomorphic polarization) can be written as **Toeplitz operator** of a function  $f_m$  (maybe different for every  $m$ )

indeed **Tuynman relation**:

$$Q_f^{(m)} = i T_{f - \frac{1}{2m} \Delta f}^{(m)}$$

- ▶ star product of geometric quantization
- ▶ set  $B(f) := (id - \nu \frac{\Delta}{2})f$

$$f \star_{GQ} g = B^{-1}(B(f) \star_{BT} B(g))$$

defines an equivalent star product

- ▶ can also be given by the asymptotic expansion of product of geometric quantisation operators
- ▶ it is not of separation of variable type
- ▶ but equivalent to  $\star_{BT}$ .

## Where is the Berezin star product ??

- ▶ It is an important star product: Berezin, Cahen-Gutt-Rawnsley, etc.
- ▶ The original definition is limited in applicability.
- ▶ We will give a definition for quantizable Kähler manifold.
- ▶ Clue: define it as the opposite of the dual of  $\star_{BT}$ .
- ▶  $f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g))$
- ▶ Problem: How to determine  $I$ ?
- ▶ describe the formal  $I$  by asymptotic expansion of some geometrically defined  $I^{(m)}$

## COHERENT STATES - BEREZIN SYMBOLS

- ▶ assume the bundle  $L$  is **very ample** (i.e. has enough global sections)
- ▶ pass to its **dual**  $(U, k) := (L^*, h^{-1})$  with dual metric  $k$
- ▶ inside of the total space  $U$ , consider the **circle bundle**

$$Q := \{\lambda \in U \mid k(\lambda, \lambda) = 1\},$$

- ▶  $\tau : Q \rightarrow M$  (or  $\tau : U \rightarrow M$ ) the **projection**,

coherent vectors/states in the sense of  
Berezin-Rawnsley-Cahen-Gutt:

$$x \in M \mapsto \alpha = \tau^{-1}(x) \in U \setminus 0 \mapsto e_{\alpha}^{(m)} \in \Gamma_{hol}(M, L^m)$$

defined by

$$\langle e_{\alpha}^{(m)}, s \rangle = \alpha^{\otimes m}(s(\tau(\alpha)))$$

As

$$e_{c\alpha}^{(m)} = \bar{c}^m \cdot e_{\alpha}^{(m)}, \quad c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}.$$

we obtain:

$$x \in M \mapsto e_x^{(m)} := [e_{\alpha}^{(m)}] \in \mathbb{P}(\Gamma_{hol}(M, L^m))$$

# APPLICATIONS

- ▶ Bergman projectors  $\Pi^{(m)}$ , Bergman kernels, ....
- ▶ Covariant Berezin symbol  $\sigma^{(m)}(A)$  (of level  $m$ ) of an operator  $A \in \text{End}(\Gamma_{hol}(M, L^{(m)}))$

$$\sigma^{(m)}(A) : M \rightarrow \mathbb{C}, \quad x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e_\alpha^{(m)}, Ae_\alpha^{(m)} \rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle},$$

$$\alpha \in \tau^{-1}(x) \neq 0$$

# IMPORTANCE OF THE COVARIANT SYMBOL

- ▶ Construction of the **Berezin star product**, only for limited classes of manifolds (see Berezin, Cahen-Gutt-Rawnsley)
- ▶  $\mathcal{A}^{(m)} \leq C^\infty(M)$ , of level  $m$  covariant symbols.
- ▶ symbol map is **injective** (follows from Toeplitz map surjective)
- ▶ for  $\sigma^{(m)}(A)$  and  $\sigma^{(m)}(B)$  the operators  $A$  and  $B$  are uniquely fixed
- ▶

$$\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B)$$

- ▶  $\star_{(m)}$  on  $\mathcal{A}^{(m)}$  is an associative and noncommutative product
- ▶ **Crucial problem**, how to obtain from  $\star_{(m)}$  a star product for all functions (or symbols) independent from the level  $m$  ?

# BEREZIN TRANSFORM

$$I^{(m)} : C^\infty(M) \rightarrow C^\infty(M), \quad f \mapsto I^{(m)}(f) := \sigma^{(m)}(T_f^{(m)})$$

**Theorem:** (Karabegov - Schl.)

$I^{(m)}(f)$  has a complete **asymptotic expansion** as  $m \rightarrow \infty$

$$I^{(m)}(f)(x) \sim \sum_{i=0}^{\infty} I_i(f)(x) \frac{1}{m^i},$$

$$I_i : C^\infty(M) \rightarrow C^\infty(M), \quad I_0(f) = f, \quad I_1(f) = \Delta f.$$

- $\Delta$  is the **Laplacian** with respect to the metric given by the Kähler form  $\omega$

## BEREZIN STAR PRODUCT

- ▶ from asymptotic expansion of the Berezin transform get formal expression

$$I = \sum_{i=0}^{\infty} I_i \nu^i, \quad I_i : C^{\infty}(M) \rightarrow C^{\infty}(M)$$

- ▶ set  $f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g))$
- ▶  $\star_B$  is called the Berezin star product
- ▶  $I$  gives the equivalence to  $\star_{BT}$  ( $I_0 = id$ ). Hence, the same Deligne-Fedosov classes
- ▶ if the covariant symbol works star product works, it will coincide with the star product  $\star_B$ .

- ▶ separation of variables type (but now of anti-Wick type).
- ▶ Karabegov form is  $\frac{1}{\nu}\omega + \mathbb{F}(i\partial\bar{\partial}\log u_m)$
- ▶  $u_m$  is the Bergman kernel  $\mathcal{B}_m(\alpha, \beta) = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle$  evaluated along the diagonal
- ▶  $\mathbb{F}$  means: take asymptotic expansion in  $1/m$  as formal series in  $\nu$
- ▶  $I = I_{\star_B}$ , the geometric Berezin transform equals the formal Berezin transform of Karabegov for  $\star_B$
- ▶ both star products  $\star_B$  and  $\star_{BT}$  are dual and opposite to each other

# SUMMARY OF NATURALLY DEFINED STAR PRODUCT

	name	Karabegov form	Deligne Fedosov class
$\star_{BT}$	Berezin-Toeplitz	$\frac{-1}{\nu}\omega + \omega_{can}$ (Wick)	$\frac{1}{i}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$ .
$\star_B$	Berezin	$\frac{1}{\nu}\omega + \mathbb{F}(i\partial\bar{\partial}\log u_m)$ (anti-Wick)	$\frac{1}{i}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$ .
$\star_{GQ}$	geometric quantization	(—)	$\frac{1}{i}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$ .
$\star_K$	standard product	$(1/\nu)\omega$ (anti-Wick)	$\frac{1}{i}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$ .
$\star_{BW}$	Bordemann- Waldmann	$-(1/\nu)\omega$ (Wick)	$\frac{1}{i}(\frac{1}{\nu}[\omega] + \frac{\delta}{2})$ .

$u_m$  Bergman kernel evaluated along the diagonal in  $Q \times Q$

$\delta$  the canonical class of the manifold  $M$



- ▶ Berezin transform is not only the equivalence relating  $\star_{BT}$  with  $\star_B$
- ▶ also it (resp. the Karabegov form) can be used to calculate the coefficients of these naturally defined star products,
- ▶ either directly
- ▶ or with the help of the certain type of graphs (see the very interesting work of Gammelgaard and Hua Xu).

$\tau(\alpha) = x, \tau(\beta) = y$  with  $\alpha, \beta \in Q$

integral representation

$$\begin{aligned} \left( I^{(m)}(f) \right)(x) &= \frac{1}{\mathcal{B}_m(\alpha, \alpha)} \int_Q \mathcal{B}_m(\alpha, \beta) \mathcal{B}_m(\beta, \alpha) \tau^* f(\beta) \mu(\beta) \\ &= \frac{1}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle} \int_M \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle \cdot \langle e_\beta^{(m)}, e_\alpha^{(m)} \rangle f(y) \Omega(y) . \end{aligned}$$

Note that:

$$u_m(x) := \mathcal{B}_m(\alpha, \alpha) = \langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle,$$

$$v_m(x, y) := \mathcal{B}_m(\alpha, \beta) \cdot \mathcal{B}_m(\beta, \alpha) = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle \cdot \langle e_\beta^{(m)}, e_\alpha^{(m)} \rangle$$

are well-defined on  $M$  and on  $M \times M$  respectively.