

SOME NATURALLY DEFINED STAR PRODUCTS FOR KÄHLER MANIFOLDS

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- ▶ One mathematical aspect of quantization is the **passage** from the **commutative world** to the **non-commutative world**.
- ▶ one way **a deformation quantization** (also called **star product**)
- ▶ can only be done on the level of **formal power series** over the algebra of functions
- ▶ was pinned down in a mathematically satisfactory manner by **Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer**.

- ▶ give an overview of some **naturally defined star products** in the case that our “phase-space manifold” is a (compact) **Kähler manifold**
- ▶ here we have additional **complex structure** and search for star products **respecting** it
- ▶ yield star products of **separation of variables type** (**Karabegov**) resp. **Wick or anti-Wick type** (**Bordemann and Waldmann**)
- ▶ both constructions are quite different, but there is a **1:1 correspondence** (**Neumaier**)
- ▶ still quite a lot of them

- ▶ **single out** certain **naturally** given ones.
- ▶ restrict to **quantizable** Kähler manifolds
- ▶ **Berezin-Toeplitz** star product, **Berezin** transform, **Berezin star product**
- ▶ a side result: star product of **geometric quantization**
- ▶ all of them are **equivalent star product**, but not the same
- ▶ give **Deligne-Fedosov class** and **Karabegov forms**
- ▶ give the **equivalence transformations**

- ▶ (M, ω) a pseudo-Kähler manifold.
 M a complex manifold, and ω , a non-degenerate closed $(1, 1)$ -form
- ▶ if ω is a positive form then (M, ω) is a honest Kähler manifold
- ▶ $C^\infty(M)$ the algebra of complex-valued differentiable functions with associative product given by point-wise multiplication
- ▶ define the Poisson bracket

$$\{f, g\} := \omega(X_f, X_g) \quad \omega(X_f, \cdot) = df(\cdot)$$

- ▶ $C^\infty(M)$ becomes a Poisson algebra.

STAR PRODUCT

star product for M is an **associative product** \star on

$\mathcal{A} := C^\infty(M)[[\nu]]$, such

1. $f \star g = f \cdot g \mod \nu$,
2. $(f \star g - g \star f) / \nu = -i\{f, g\} \mod \nu$.

Also

$$f \star g = \sum_{k=0}^{\infty} \nu^k C_k(f, g), \quad C_k(f, g) \in C^\infty(M),$$

differential (or **local**) if $C_k(,)$ are bidifferential operators.

Usually: $1 \star f = f \star 1 = f$.

Equivalence of star products

\star and \star' (the same Poisson structure) are *equivalent* means there exists

a formal series of linear operators

$$B = \sum_{i=0}^{\infty} B_i \nu^i, \quad B_i : C^\infty(M) \rightarrow C^\infty(M),$$

with $B_0 = id$ and $B(f) \star' B(g) = B(f \star g)$.

to every equivalence class of a differential star product one assigns its **Deligne-Fedosov class**

$$cl(\star) \in \frac{1}{i} \left(\frac{1}{\nu} [\omega] + H_{dR}^2(M, \mathbb{C})[[\nu]] \right).$$

gives a 1:1 correspondence

Existence: by DeWilde-Lecomte, Omori-Maeda-Yoshioka, Fedosov, ..., Kontsevich.

SEPARATION OF VARIABLES TYPE

- ▶ **pseudo-Kähler** case: we look for star products adapted to the complex structure
- ▶ **separation of variables type** (Karabegov)
- ▶ **Wick and anti-Wick type** (Bordemann - Waldmann)
- ▶ **Karabegov convention**: of separation of variables type if in $C_k(.,.)$ for $k \geq 1$ the first argument differentiated in anti-holomorphic and the second argument in holomorphic directions.
- ▶ we call this convention **separation of variables (anti-Wick) type** and call a star product of **separation of variables (Wick) type** if the role of the variables is switched
- ▶ we **need** both conventions

KARABEGOV CONSTRUCTION (SKETCH OF A SKETCH)

- ▶ (M, ω_{-1}) the pseudo-Kähler manifold
- ▶ a formal deformation of the form $(1/\nu)\omega_{-1}$ is a formal form

$$\hat{\omega} = (1/\nu)\omega_{-1} + \omega_0 + \nu\omega_1 + \dots$$

$\omega_r, r \geq 0$, closed $(1,1)$ -forms on M .

- ▶ **Karabegov**: to every such $\hat{\omega}$ there exists a star product \star
- ▶ and vice-versa
- ▶ **Karabegov form** of the star product \star is $kf(\star) := \hat{\omega}$,
- ▶ the star product \star_K with classifying Karabegov form $(1/\nu)\omega_{-1}$ is Karabegov's standard star product.

- **Formal Berezin transform**
- **antiholomorphic** functions a and **holomorphic** functions b on $U \subset M$ we have the relation

$$a \star b = I_{\star}(b \star a) = I_{\star}(b \cdot a),$$

- can be written as

$$I_{\star} = \sum_{i=0}^{\infty} I_i \nu^i, \quad I_i : C^{\infty}(M) \rightarrow C^{\infty}(M), \quad I_0 = id, \quad I_1 = \Delta.$$

- the formal Berezin transform I_{\star} **determines** the \star uniquely.

- ▶ Start with \star separation of variables type (anti-Wick)
(M, ω_{-1})
- ▶ dual star-product $\tilde{\star}$ on M

$$f \tilde{\star} g = I^{-1}(I(g) \star I(f)).$$

(anti-Wick) on $(M, -\omega_{-1})$.

- ▶ given \star , the opposite star product is obtained

$$f \star^{op} g = g \star f$$

for $(M, -\omega_{-1})$ and switches anti-Wick with Wick type.

- ▶ opposite of the dual $\star' = \tilde{\star}^{op}$,

$$f \star' g = g \tilde{\star} f = I^{-1}(I(f) \star I(g)).$$

on (M, ω_{-1}) , of Wick type

- ▶ opposite of the dual $\star' = \tilde{\star}^{op}$,

$$f \star' g = g \tilde{\star} f = I^{-1}(I(f) \star I(g)).$$

on (M, ω_{-1}) , of **Wick type**

- ▶ the formal Berezin transform I_{\star} establishes an equivalence of the star products

$$(\mathcal{A}, \star) \text{ and } (\mathcal{A}, \star')$$

CLASSIFYING FORMS

\star star product of anti-Wick type with Karabegov form $kf(\star) = \widehat{\omega}$
Deligne-Fedosov class calculates as

$$cl(\star) = \frac{1}{i}([\widehat{\omega}] - \frac{\delta}{2}).$$

[..] denotes the de-Rham class of the forms and δ is the
canonical class of the manifold i.e. $\delta := c_1(K_M)$.

for the opposite star product $cl(\star^{op}) = -cl(\star)$

standard star product \star_K (with Karabegov form $\widehat{\omega} = (1/\nu)\omega_{-1}$)

$$cl(\star_K) = \frac{1}{i}(\frac{1}{\nu}[\omega_{-1}] - \frac{\delta}{2}).$$

- ▶ For the Karabegov form to be in 1:1 correspondence, we need to fix a convention: **Wick** or **anti-Wick** for reference
- ▶ here we refer to the **anti-Wick** type product
- ▶ if \star is of Wick type we set

$$kf(\star) := kf(\star^{op}),$$

- ▶ Recall \star^{op} is a star product of **(anti-Wick)** type for the pseudo-Kähler manifold $(M, -\omega)$

OTHER GENERAL CONSTRUCTIONS

- ▶ **Bordemann and Waldmann:** modification of Fedosov's geometric existence proof.
- ▶ fibre-wise Wick product.
- ▶ by a **modified Fedosov connection** a star product \star_{BW} of Wick type is obtained.
- ▶ **Karabegov form** is $-(1/\nu)\omega$
- ▶ Deligne class class

$$cl(\star_{BW}) = -cl(\star_{BW}^{op}) = \frac{1}{i}(\frac{1}{\nu}[\omega] + \frac{\delta}{2}).$$

Neumaier: by adding a formal closed $(1, 1)$ form as parameter each star product of separation of variables type can be obtained by the Bordemann-Waldmann construction

Reshetikhin and Takhtajan:

formal Laplace expansions of formal integrals related to the star product.

coefficients of the star product can be expressed (roughly) by Feynman diagrams

BEREZIN-TOEPLITZ STAR PRODUCT

- ▶ compact and quantizable Kähler manifold (M, ω) ,
- ▶ quantum line bundle (L, h, ∇) , L is a holomorphic line bundle over M , h a hermitian metric on L , ∇ a compatible connection
- ▶ recall (M, ω) is quantizable, if there exists such (L, h, ∇) , with

$$\text{curv}_{(L, \nabla)} = -i \omega$$

- ▶ consider all positive tensor powers $(L^m, h^{(m)}, \nabla^{(m)})$,

scalar product

$$\langle \varphi, \psi \rangle := \int_M h^{(m)}(\varphi, \psi) \Omega, \quad \Omega := \frac{1}{n!} \underbrace{\omega \wedge \omega \cdots \wedge \omega}_n$$

$$\Pi^{(m)} : L^2(M, L^m) \longrightarrow \Gamma_{hol}(M, L^m)$$

Take $f \in C^\infty(M)$, and $s \in \Gamma_{hol}(M, L^m)$

$$s \mapsto T_f^{(m)}(s) := \Pi^{(m)}(f \cdot s)$$

defines

$$T_f^{(m)} : \Gamma_{hol}(M, L^m) \rightarrow \Gamma_{hol}(M, L^m)$$

the Toeplitz operator of level m .

Berezin-Toeplitz operator quantization

$$f \mapsto \left(T_f^{(m)} \right)_{m \in \mathbb{N}_0}.$$

has the correct **semi-classical behavior**

Theorem (Bordemann, Meinrenken, and Schl.)

(a)

$$\lim_{m \rightarrow \infty} \|T_f^{(m)}\| = \|f\|_{\infty}$$

(b)

$$\|m [T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}\| = O(1/m)$$

(c)

$$\|T_f^{(m)} T_g^{(m)} - T_{f \cdot g}^{(m)}\| = O(1/m)$$

Theorem (BMS, Schl., Karabegov and Schl.)

\exists a **unique differential star product**

$$f \star_{BT} g = \sum \nu^k C_k(f, g)$$

such that

$$T_f^{(m)} T_g^{(m)} \sim \sum_{k=0}^{\infty} \left(\frac{1}{m} \right)^k T_{C_k(f,g)}^{(m)}$$

Further properties: is of **separation of variables type** (Wick type)

classifying **Deligne-Fedosov class** $\frac{1}{i}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$ and **Karabegov form** $\frac{-1}{\nu}\omega + \omega_{can}$

possible: auxiliary hermitian line (or even vector) bundle can be added, **meta-plectic correction**.

Further result: The Toeplitz map

$$T_{(m)} : C^\infty(M) \rightarrow \text{End}(\Gamma_{\text{hol}}(M, L^m))$$

is **surjective**

implies that the operator $Q_f^{(m)}$ of **geometric quantization** (with holomorphic polarization) can be written as **Toeplitz operator** of a function f_m (maybe different for every m)

indeed **Tuynman relation**:

$$Q_f^{(m)} = i T_{f - \frac{1}{2m} \Delta f}^{(m)}$$

- ▶ star product of geometric quantization
- ▶ set $B(f) := (id - \nu \frac{\Delta}{2})f$

$$f \star_{GQ} g = B^{-1}(B(f) \star_{BT} B(g))$$

defines an **equivalent star product**

- ▶ can also be given by the **asymptotic expansion** of product of geometric quantisation operators
- ▶ it is **not** of separation of variable type
- ▶ but **equivalent** to \star_{BT} .

Where is the Berezin star product ??

- ▶ It is an important star product: Berezin, Cahen-Gutt-Rawnsley, etc.
- ▶ The original definition is limited in applicability.
- ▶ We will give a definition for quantizable Kähler manifold.
- ▶ **Clue:** define it as the opposite of the dual of \star_{BT} .
- ▶ $f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g))$
- ▶ **Problem:** How to determine I ?
- ▶ describe the formal I by asymptotic expansion of some geometrically defined $I^{(m)}$

- ▶ assume the bundle L is **very ample** (i.e. **has enough global sections**)
- ▶ pass to its **dual** $(U, k) := (L^*, h^{-1})$ with dual metric k
- ▶ inside of the total space U , consider the **circle bundle**

$$Q := \{\lambda \in U \mid k(\lambda, \lambda) = 1\},$$

- ▶ $\tau : Q \rightarrow M$ (or $\tau : U \rightarrow M$) the **projection**,

coherent vectors/states in the sense of
Berezin-Rawnsley-Cahen-Gutt:

$$x \in M \mapsto \alpha = \tau^{-1}(x) \in U \setminus 0 \mapsto e_{\alpha}^{(m)} \in \Gamma_{hol}(M, L^m)$$

defined by

$$\langle e_{\alpha}^{(m)}, s \rangle = \alpha^{\otimes m}(s(\tau(\alpha)))$$

As

$$e_{c\alpha}^{(m)} = \bar{c}^m \cdot e_{\alpha}^{(m)}, \quad c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}.$$

we obtain:

$$x \in M \mapsto e_x^{(m)} := [e_{\alpha}^{(m)}] \in \mathbb{P}(\Gamma_{hol}(M, L^m))$$

- ▶ Bergman projectors $\Pi^{(m)}$, Bergman kernels,
- ▶ Covariant Berezin symbol $\sigma^{(m)}(A)$ (of level m) of an operator $A \in \text{End}(\Gamma_{hol}(M, L^{(m)}))$

$$\sigma^{(m)}(A) : M \rightarrow \mathbb{C}, \quad x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e_\alpha^{(m)}, A e_\alpha^{(m)} \rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle},$$

$$\alpha \in \tau^{-1}(x) \neq 0$$

IMPORTANCE OF THE COVARIANT SYMBOL

- ▶ Construction of the **Berezin star product**, **only for limited classes of manifolds** (see Berezin, Cahen-Gutt-Rawnsley)
- ▶ $\mathcal{A}^{(m)} \leq C^\infty(M)$, of level m covariant symbols.
- ▶ symbol map is **injective** (follows from Toeplitz map surjective)
- ▶ for $\sigma^{(m)}(A)$ and $\sigma^{(m)}(B)$ the operators A and B are uniquely fixed



$$\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B)$$

- ▶ $\star_{(m)}$ on $\mathcal{A}^{(m)}$ is an associative and noncommutative product
- ▶ **Crucial problem**, how to obtain from $\star_{(m)}$ a star product for all functions (or symbols) independent from the level m ?

BEREZIN TRANSFORM

$$I^{(m)} : C^\infty(M) \rightarrow C^\infty(M), \quad f \mapsto I^{(m)}(f) := \sigma^{(m)}(T_f^{(m)})$$

Theorem: (Karabegov - Schl.)

$I^{(m)}(f)$ has a complete **asymptotic expansion** as $m \rightarrow \infty$

$$I^{(m)}(f)(x) \sim \sum_{i=0}^{\infty} l_i(f)(x) \frac{1}{m^i},$$

$$l_i : C^\infty(M) \rightarrow C^\infty(M), \quad l_0(f) = f, \quad l_1(f) = \Delta f.$$

- Δ is the **Laplacian** with respect to the metric given by the Kähler form ω

BEREZIN STAR PRODUCT

- ▶ from asymptotic expansion of the Berezin transform get formal expression

$$I = \sum_{i=0}^{\infty} I_i \nu^i, \quad I_i : C^\infty(M) \rightarrow C^\infty(M)$$

- ▶ set $f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g))$
- ▶ \star_B is called the Berezin star product
- ▶ I gives the equivalence to \star_{BT} ($I_0 = id$). Hence, the same Deligne-Fedosov classes
- ▶ if the covariant symbol works star product works, it will coincide with the star product \star_B .

- ▶ separation of variables type (but now of **anti-Wick type**).
- ▶ **Karabegov form** is $\frac{1}{\nu}\omega + \mathbb{F}(i\partial\bar{\partial}\log u_m)$
- ▶ u_m is the **Bergman kernel** $\mathcal{B}_m(\alpha, \beta) = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle$ evaluated along the diagonal
- ▶ \mathbb{F} means: take asymptotic expansion in $1/m$ as **formal series** in ν
- ▶ $I = I_{\star_B}$, the **geometric Berezin** transform equals the **formal Berezin transform** of Karabegov for \star_B
- ▶ both star products \star_B and \star_{BT} are **dual and opposite** to each other

SUMMARY OF NATURALLY DEFINED STAR PRODUCT

	name	Karabegov form	Deligne Fedosov class
\star_{BT}	Berezin-Toeplitz	$\frac{-1}{\nu}\omega + \omega_{can}$ (Wick)	$\frac{1}{i}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$.
\star_B	Berezin	$\frac{1}{\nu}\omega + \mathbb{F}(i \partial \bar{\partial} \log u_m)$ (anti-Wick)	$\frac{1}{i}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$.
\star_{GQ}	geometric quantization	$(-)$	$\frac{1}{i}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$.
\star_K	standard product	$(1/\nu)\omega$ (anti-Wick)	$\frac{1}{i}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$.
\star_{BW}	Bordemann-Waldmann	$-(1/\nu)\omega$ (Wick)	$\frac{1}{i}(\frac{1}{\nu}[\omega] + \frac{\delta}{2})$.

u_m **Bergman kernel** evaluated along the diagonal in $Q \times Q$

δ the **canonical class** of the manifold M

- ▶ **Berezin transform** is not only the equivalence relating \star_{BT} with \star_B
- ▶ also it (resp. the Karabegov form) can be used to **calculate the coefficients** of these naturally defined star products,
- ▶ either **directly**
- ▶ or with the help of the **certain type of graphs** (see the very interesting work of **Gammelgaard** and **Hua Xu**).

$\tau(\alpha) = x, \tau(\beta) = y$ with $\alpha, \beta \in Q$

integral representation

$$\begin{aligned} \left(I^{(m)}(f) \right)(x) &= \frac{1}{\mathcal{B}_m(\alpha, \alpha)} \int_Q \mathcal{B}_m(\alpha, \beta) \mathcal{B}_m(\beta, \alpha) \tau^* f(\beta) \mu(\beta) \\ &= \frac{1}{\langle \mathbf{e}_\alpha^{(m)}, \mathbf{e}_\alpha^{(m)} \rangle} \int_M \langle \mathbf{e}_\alpha^{(m)}, \mathbf{e}_\beta^{(m)} \rangle \cdot \langle \mathbf{e}_\beta^{(m)}, \mathbf{e}_\alpha^{(m)} \rangle f(y) \Omega(y) . \end{aligned}$$

Note that:

$$u_m(x) := \mathcal{B}_m(\alpha, \alpha) = \langle \mathbf{e}_\alpha^{(m)}, \mathbf{e}_\alpha^{(m)} \rangle,$$

$$v_m(x, y) := \mathcal{B}_m(\alpha, \beta) \cdot \mathcal{B}_m(\beta, \alpha) = \langle \mathbf{e}_\alpha^{(m)}, \mathbf{e}_\beta^{(m)} \rangle \cdot \langle \mathbf{e}_\beta^{(m)}, \mathbf{e}_\alpha^{(m)} \rangle$$

are well-defined on M and on $M \times M$ respectively.