LIE SUPERALGEBRAS OF KRICHEVER-NOVIKOV TYPE AND THEIR CENTRAL EXTENSIONS

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Abstract. Classically important examples of Lie superalgebras have been constructed starting from the Witt and Virasoro algebra. In this article we consider Lie superalgebras of Krichever-Novikov type. These algebras are multi-point and higher genus equivalents. The grading in the classical case is replaced by an almost-grading. The almost-grading is determined by a splitting of the set of points were poles are allowed into two disjoint subsets. With respect to a fixed splitting, or equivalently with respect to an almost-grading, it is shown that there is up to rescaling and equivalence a unique non-trivial central extension. It is given explicitly. Furthermore, a complete classification of bounded cocycles (with respect to the almost-grading) is given.

1. Introduction

Krichever–Novikov (KN) type algebras give important examples of infinite dimensional algebras. They are defined via meromorphic objects on compact Riemann surfaces Σ of arbitrary genus with controlled polar behaviour. More precisely, poles are only allowed at a fixed finite set of points denoted by A. The classical examples are the algebras defined by objects on the Riemann sphere (genus zero) with possible poles only at $\{0, \infty\}$. This yields e.g. the well-known Witt algebra, current algebras, and their central extensions the Virasoro, and the affine Kac-Moody algebras. For higher genus, but still only for two points were poles are allowed, they were generalised by Krichever and Novikov [13], [14], [15] in 1986. In 1990 the author [20], [21], [22], [23] extended the approach further to the general multi-point case. This extension was not a straight-forward generalization. The crucial point was to introduce a replacement of the graded algebra structure present in the "classical" case. Krichever and Novikov found that an almost-grading, see Definition 3.1, will be enough to allow constructions in representation theory, like triangular decomposition, highest weight modules, Verma modules and so on. In [22], [23] it was realized that a splitting of A two disjoint non-empty subsets $A = I \cup O$ is crucial for introducing an almost-grading and the corresponding almost-grading was given. In the classical situation there is only one such splitting (up to inversion) hence there is only one almost-grading,

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which is indeed a grading. Similar to the classical situation, a Krichever-Novikov algebra should always be considered as an algebra of meromorphic objects with an almost-grading coming from such a splitting.

I like to point out that already in the genus zero case (i.e. the Riemann sphere case) with more than two points where poles are allowed the algebras will be only almost-graded. In fact, quite a number of interesting new phenomena will show up already there, see [24], [5], [6].

In the context of conformal field theory and string theory superextensions of the classical algebras appeared, see e.g. [7]. Very important examples are the Neveu-Schwarz and the Ramond type superalgebras. Quite soon some physicists also studied superanalogs of the algebra of Krichever-Novikov type, but still only with two points were poles are allowed, e.g. [1], [2], [3], [4], [30]. The multi-point case was also developed by the author. It has not been published yet, but see [27].

Quite recently, these superalgebras of Krichever–Novikov type found again interest in the context of Jordan superalgebras and Lie antialgebras (see Ovsienko [19] for their definitions and Lecomte and Ovsienko[16] for further properties). Starting from Krichever–Novikov type superalgebras interesting explicite infinite dimensional examples of Jordan superalgebras and antialgebras can be constructed. In this respect, see the work of Leidwanger and Morier-Genoud [17], [18], and Kreusch [12].

The goal of this article is to recall the general definition of KN algebras for the multipoint situation and for arbitrary genus. The classical situation will be a special case. In particular, the construction of the Lie superalgebra is recalled. Its almost-graded structure, induced by a fixed splitting $A = I \cup O$ is given. Also the Jordan superalgebra of KN type (with its almost-grading) fits perfectly in this picture. See Remark 3.8.

One of the main results of the paper is the proof that there is, up to rescaling the central element and equivalence, only one non-trivial almost-graded central extension of the Lie superalgebra of KN type with even central element. We stress the fact, that this does not mean that there is essentially only one central extension. In fact, a different splitting of A will yield a different almost-grading and hence an essentially different central extension. Moreover, at least for higher genus, there are central extensions which are not related to any almost-grading. In the classical situation we reprove uniqueness of the non-trivial central extension.

We will give a geometric description for the defining cocycle, see (5.12). For the two-point case the form of cocycle was given by Bryant in [4], correcting some ommission in [1].

A cocycle is bounded from above if its value is zero if the sum of the degrees of the (homogeneous) arguments are higher than a certain bound. Krichever and Novikov introduced the term “local” cocycle, for a cocycle which is bounded from above and from below. Local cocycles are exactly those cocycles which define central extensions which allow that the almost-grading can be extended to them. In the process of proving the uniqueness of local cocycle classes (Theorem 5.5) we give a complete classification of bounded (from above) cocycles. To prove this we show the fact that a bounded cocycle of the Lie superalgebra is already fixed by its restriction to the vector field subalgebra (Proposition 5.4). For the vector field algebra the bounded cocycle were classified by the author [26].

Up to this point we assumed that the central element was an even element. In an additional section we consider the case that the central element is odd. We show that all bounded (from above) cocycles for odd central elements are cohomologically trivial. This means that the corresponding central extension of the Lie superalgebra will split.

We close with some remarks on special examples.

2. The algebras

2.1. **The geometric set-up.** For the whole article let Σ be a compact Riemann surface without any restriction for the genus $g = g(\Sigma)$. Furthermore, let A be a finite subset of Σ. Later we will need a splitting of A into two non-empty disjoint subsets I and O, i.e. $A = I \cup O$. Set $N := \#A$, $K := \#I$, $M := \#O$, with $N = K + M$. More precisely, let

$$I = (P_1, \dots, P_K), \quad \text{and} \quad O = (Q_1, \dots, Q_M) \tag{2.1}$$

be disjoint ordered tuples of distinct points ("marked points", "punctures") on the Riemann surface. In particular, we assume $P_i \neq Q_j$ for every pair (i, j). The points in I are called the *in-points*, the points in O the *out-points*. Sometimes we consider I and O simply as sets.

In the article we sometimes refer to the classical situation. By this we understand $\Sigma = S^2$, the Riemann sphere, or equivalently the projective line over $\mathbb{C}$, $I = \{0\}$ and $O = \{\infty\}$ with respect to the quasi-global coordinate z.

Our objects, algebras, structures, ... will be meromorphic objects defined on Σ which are holomorphic outside the points in A. To introduce them let $\mathcal{K} = \mathcal{K}_\Sigma$ be the canonical line bundle of Σ, resp. the locally free canonically sheaf. The local sections of the bundle are the local holomorphic differentials. If $P \in \Sigma$ is a point and z a local holomorphic coordinate at P then a local holomorphic differential can be written as $f(z)dz$ with a local holomorphic function f defined in a neighbourhood of P. A global holomorphic section can be described locally for a covering by coordinate charts $(U_i, z_i)_{i \in J}$ by a system of local holomorphic functions $(f_i)_{i \in J}$, which are related by the transformation rule induced by the coordinate change map $z_j = z_j(z_i)$ and the condition $f_i dz_i = f_j dz_j$

$$f_j = f_i \cdot \left(\frac{dz_j}{dz_i}\right)^{-1}. \tag{2.2}$$

With respect to a coordinate covering a meromorphic section of $\mathcal{K}$ is given as a collection of local meromorphic functions $(h_i)_{i \in J}$ for which the transformation law (2.2) is true.

In the following λ is either an integer or a half-integer. If λ is an integer then
(1) $\mathcal{K}^\lambda = \mathcal{K}^{\otimes \lambda}$ for $\lambda > 0$,
(2) $\mathcal{K}^0 = \mathcal{O}$, the trivial line bundle, and
(3) $\mathcal{K}^\lambda = (\mathcal{K}^*)^{\otimes(-\lambda)}$ for $\lambda < 0$.

Here as usual $\mathcal{K}^*$ denotes the dual line bundle to the canonical line bundle. The dual line bundle is the holomorphic tangent line bundle, whose local sections are the holomorphic tangent vector fields $f(z)(d/dz)$. If λ is a half-integer, then we first have to fix a "square root" of the canonical line bundle, sometimes called a *theta-characteristics*. This means we fix a line bundle L for which $L^{\otimes 2} = K$.

After such a choice of L is done we set $\mathcal{K}^\lambda = \mathcal{K}^\lambda_L = L^{\otimes 2\lambda}$. In most cases we will drop mentioning L, but we have to keep the choice in mind. Also the structure of the algebras we are about to define will depend on the choice. But the main properties will remain the same.

Remark 2.1. A Riemann surface of genus g has exactly 2^{2g} non-isomorphic square roots of $\mathcal{K}$. For $g = 0$ we have $\mathcal{K} = \mathcal{O}(-2)$ and $L = \mathcal{O}(-1)$, the tautological bundle which is the unique square root. Already for $g = 1$ we have 4 non-isomorphic ones. As in this case $\mathcal{K} = \mathcal{O}$ one solution is $L_0 = \mathcal{O}$. But we have also other bundles L_i, $i = 1, 2, 3$. Note that L_0 has a non-vanishing global holomorphic section, whereas L_1, L_2, L_3 do not have a global holomorphic section. In general, depending on the parity of $\dim \mathrm{H}(\Sigma, L)$, one distinguishes even and odd theta characteristics L. For $g = 1$ the bundle $\mathcal{O}$ is odd, the others are even theta characteristics.

We set

$$\begin{aligned}\mathcal{F}^\lambda := \mathcal{F}^\lambda(A) := \{f \text{ is a global meromorphic section of } K^\lambda \mid \\ \text{such that } f \text{ is holomorphic over } \Sigma \setminus A\}.\end{aligned} \tag{2.3}$$

As in this work the set of A is fixed we do not add it to the notation. Obviously, $\mathcal{F}^\lambda$ is an infinite dimensional $\mathbb{C}$-vector space. Recall that in the case of half-integer λ everything depends on the theta characteristic L.

We call the elements of the space $\mathcal{F}^\lambda$ *meromorphic forms of weight* λ (with respect to the theta characteristic L). In local coordinates z_i we can write such a form as $f_i dz_i^\lambda$, with f_i being a local holomorphic, resp. meromorphic function.

2.2. **Associative Multiplication.** The natural map of the locally free sheaves of rang one

$$\mathcal{K}^\lambda \times \mathcal{K}^\nu \to \mathcal{K}^\lambda \otimes \mathcal{K}^\nu \cong \mathcal{K}^{\lambda+\nu}, \quad (s,t) \mapsto s \otimes t, \tag{2.4}$$

defines a bilinear map

$$\cdot : \mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu}. \tag{2.5}$$

With respect to local trivialisations this corresponds to the multiplication of the local representing meromorphic functions

$$(s\, dz^\lambda, t\, dz^\nu) \mapsto s\, dz^\lambda \cdot t\, dz^\nu = s \cdot t\; dz^{\lambda+\nu}. \tag{2.6}$$

If there is no danger of confusion then we will mostly use the same symbol for the section and for the local representing function.

We set

$$\mathcal{F} := \bigoplus_{\lambda \in \frac{1}{2}\mathbb{Z}} \mathcal{F}^\lambda. \tag{2.7}$$

The following is obvious

Proposition 2.2. *The vector space $\mathcal{F}$ is an associative and commutative graded (over $\frac{1}{2}\mathbb{Z}$) algebra. Moreover, $\mathcal{F}^0$ is a subalgebra.*

We use also $\mathcal{A} := \mathcal{F}^0$. Of course, it is the algebra of meromorphic functions on Σ which are holomorphic outside of A. The spaces $\mathcal{F}^\lambda$ are modules over $\mathcal{A}$.

2.3. **Lie algebra structure.** Next we define a Lie algebraic structure on the space $\mathcal{F}$. The structure is induced by the map

$$\mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu+1}, \qquad (s,t) \mapsto [s,t], \tag{2.8}$$

which is defined in local representatives of the sections by

$$(s\,dz^\lambda, t\,dz^\nu) \mapsto [s\,dz^\lambda, t\,dz^\nu] := \left((-\lambda)s\frac{dt}{dz} + \nu\,t\frac{ds}{dz}\right) dz^{\lambda+\nu+1}, \tag{2.9}$$

and bilinearly extended to $\mathcal{F}$.

Proposition 2.3. *(a) The bilinear map $[.,.]$ defines a Lie algebra structure on $\mathcal{F}$.*
(b) The space $\mathcal{F}$ with respect to $\cdot$ and $[.,.]$ is a Poisson algebra.

Proof. This is done by local calculations. For details see [25], [27]. □

2.4. **The vector field algebra and the Lie derivative.**

Proposition 2.4. *The subspace $\mathcal{L} := \mathcal{F}^{-1}$ is a Lie subalgebra, and the $\mathcal{F}^\lambda$'s are Lie modules over $\mathcal{L}$.*

As forms of weight -1 are vector fields, $\mathcal{L}$ could also be defined as the Lie algebra of those meromorphic vector fields on the Riemann surface Σ which are holomorphic outside of A. The product (2.9) gives the usual Lie bracket of vector fields and the Lie derivative for their actions on forms. We get (again naming the local functions with the same symbol as the section)

$$[e,f]_|(z) = [e(z)\frac{d}{dz}, f(z)\frac{d}{dz}] = \left(e(z)\frac{df}{dz}(z) - f(z)\frac{de}{dz}(z)\right)\frac{d}{dz}\ , \tag{2.10}$$

$$\nabla_e(f)_|(z) = L_e(g)_| = e\,.\,g_| = \left(e(z)\frac{df}{dz}(z) + \lambda f(z)\frac{de}{dz}(z)\right)\frac{d}{dz}\ . \tag{2.11}$$

2.5. **The algebra of differential operators.** In $\mathcal{F}$, considered as Lie algebra, $\mathcal{A} = \mathcal{F}^0$ is an abelian Lie subalgebra and the vector space sum $\mathcal{F}^0 \oplus \mathcal{F}^{-1} = \mathcal{A} \oplus \mathcal{L}$ is also a Lie subalgebra of $\mathcal{F}$. In an equivalent way it can also be constructed as semi-direct sum of $\mathcal{A}$ considered as abelian Lie algebra and $\mathcal{L}$ operating on $\mathcal{A}$ by taking the derivative. This Lie algebra is called the *Lie algebra of differential operators of degree* ≤ 1 and is denoted by $\mathcal{D}^1$. In more direct terms $\mathcal{D}^1 = \mathcal{A} \oplus \mathcal{L}$ as vector space direct sum and endowed with the Lie product

$$[(g,e),(h,f)] = (e\,.\,h - f\,.\,g\,,\,[e,f]). \tag{2.12}$$

The $\mathcal{F}^\lambda$ will be Lie-modules over $\mathcal{D}^1$.

2.6. **Superalgebra of half forms.** Next we consider the associative product

$$\cdot\ \mathcal{F}^{-1/2} \times \mathcal{F}^{-1/2} \to \mathcal{F}^{-1} = \mathcal{L}. \tag{2.13}$$

Introduce the vector space and the product

$$\mathcal{S} := \mathcal{L} \oplus \mathcal{F}^{-1/2}, \quad [(e,\varphi),(f,\psi)] := ([e,f] + \varphi\cdot\psi, e\,.\,\varphi - f\,.\,\psi). \tag{2.14}$$

Usually we will denote the elements of $\mathcal{L}$ by $e, f, \ldots$, and the elements of $\mathcal{F}^{-1/2}$ by $\varphi, \psi, \ldots$.

Definition (2.14) can be reformulated as an extension of $[.,.]$ on $\mathcal{L}$ to a "super-bracket" (denoted by the same symbol) on $\mathcal{S}$ by setting

$$[e,\varphi] := -[\varphi,e] := e\,.\,\varphi = {}_{|}(e\frac{d\varphi}{dz} - \frac{1}{2}\varphi\frac{de}{dz})(dz)^{-1/2} \tag{2.15}$$

and

$$[\varphi,\psi] = \varphi\cdot\psi. \tag{2.16}$$

We call the elements of $\mathcal{L}$ elements of even parity, and the elements of $\mathcal{F}^{-1/2}$ elements of odd parity. For such elements x we denote by $\bar{x}\in\{\bar{0},\bar{1}\}$ their parity.

The sum (2.14) can also be described as $\mathcal{S}=\mathcal{S}_{\bar{0}}\oplus\mathcal{S}_{\bar{1}}$, where $\mathcal{S}_{\bar{i}}$ is the subspace of elements of parity $\bar{i}$.

Proposition 2.5. *The space $\mathcal{S}$ with the above introduced parity and product is a Lie superalgebra.*

Definition 2.6. The algebra $\mathcal{S}$ is the Krichever - Novikov Lie superalgebra.

Before we say a few words on the proof we recall the definition of a Lie superalgebra. Let $\mathcal{S}$ be a vector space which is decomposed into even and odd elements $\mathcal{S}=\mathcal{S}_{\bar{0}}\oplus\mathcal{S}_{\bar{1}}$, i.e. $\mathcal{S}$ is a $\mathbb{Z}/2Z$-graded vector space. Furthermore, let $[.,.]$ be a $\mathbb{Z}/2Z$-graded bilinear map $\mathcal{S}\times\mathcal{S}\to\mathcal{S}$ such that for elements x,y of pure parity

$$[x,y] = -(-1)^{\bar{x}\bar{y}}[y,x]. \tag{2.17}$$

This says that

$$[\mathcal{S}_{\bar{0}},\mathcal{S}_{\bar{0}}]\subseteq\mathcal{S}_{\bar{0}},\qquad [\mathcal{S}_{\bar{0}},\mathcal{S}_{\bar{1}}]\subseteq\mathcal{S}_{\bar{1}},\qquad [\mathcal{S}_{\bar{1}},\mathcal{S}_{\bar{1}}]\subseteq\mathcal{S}_{\bar{0}}, \tag{2.18}$$

and $[x,y]$ is symmetric for x and y odd, otherwise anti-symmetric. Recall that $\mathcal{S}$ is a Lie superalgebra if in addition the super-Jacobi identity (x,y,z of pure parity)

$$(-1)^{\bar{x}\bar{z}}[x,[y,z]] + (-1)^{\bar{y}\bar{x}}[y,[z,x]] + (-1)^{\bar{z}\bar{y}}[z,[x,y]] = 0 \tag{2.19}$$

is valid. As long as the type of the arguments is different from *(even, odd, odd)* all signs can be put to $+1$ and we obtain the form of the usual Jacobi identity. In the remaining case we get

$$[x,[y,z]] + [y,[z,x]] - [z,[x,y]] = 0. \tag{2.20}$$

By the definitions $\mathcal{S}_0$ is a Lie algebra.

Proof. (Proposition 2.5) By (2.14) Equations (2.17) and (2.18) are true. If we consider (2.19) for elements of type *(even, even, even)* then it reduces to the usual Jacobi identity which is of course true for the subalgebra of vector fields $\mathcal{L}$. For *(even, even, odd)* it is true as $\mathcal{F}^{-1/2}$ is a Lie-module over $\mathcal{L}$. For *(even, odd, odd)* we get

$$[e,[\varphi,\psi]] + [\varphi,[\psi,e]] - [\psi,[e,\varphi]] = e\,.\,(\varphi\cdot\psi) - (e\,.\,\psi)\cdot\varphi - (e\,.\,\varphi)\cdot\psi = 0, \tag{2.21}$$

as e acts as derivation on $\mathcal{F}^{-1/2}$. For *(odd, odd, odd)* the super-Jacobi relation writes as

$$[\varphi,[\psi,\chi]] + \quad \text{cyclic permutation} \quad = \quad 0. \tag{2.22}$$

Equivalently

$$-(\psi\cdot\chi)\,.\,\varphi + \quad \text{cyclic permutation} \quad = \quad 0. \tag{2.23}$$

Now (again identifying local representing functions with the element)

$$(\psi \cdot \chi) \,.\, \varphi = \big((\psi \cdot \chi) \cdot \varphi' - 1/2((\psi \cdot \chi)'\varphi)\big)\,(dz)^{-1/2} = \\ \big(\psi\chi\varphi' - 1/2\,\psi'\chi\varphi - 1/2\,\psi\chi'\varphi\big)\,(dz)^{-1/2}. \quad (2.24)$$

Adding up all cyclic permutations yield zero. □

Remark 2.7. The above introduced Lie superalgebra corresponds classically to the Neveu-Schwarz superalgebra. In string theory physicists considered also the Ramond superalgebra as string algebra (in the two-point case). The elements of the Ramond superalgebra do not correspond to sections of the dual theta characteristics. They are only defined on a 2-sheeted branched covering of Σ, see e.g. [1], [3]. Hence, the elements are only multi-valued sections. As we only consider honest sections of half-integer powers of the canonical bundle, we do not deal with the Ramond algebra here.

The choice of the theta characteristics corresponds to choosing a spin structure on Σ. For the relation of the Neveu-Schwarz superalgebra to the geometry of graded Riemann surfaces see Bryant [4].

3. Almost-graded structure

3.1. **Definition of almost-gradedness.** Recall the classical situation. This is the Riemann surface $\mathbb{P}^1(\mathbb{C}) = S^2$, i.e. the Riemann surface of genus zero, and the points where poles are allowed are $\{0, \infty\}$). In this case the algebras introduced in the last chapter are graded algebras. In the higher genus case and even in the genus zero case with more than two points where poles are allowed there is no non-trivial grading anymore. As realized by Krichever and Novikov [13] there is a weaker concept, an almost-grading which to a large extend is a valuable replacement of a honest grading. Such an almost-grading is induced by a splitting of the set A into two non-empty and disjoint sets I and O. The (almost-)grading is fixed by exhibiting certain basis elements in the spaces $\mathcal{F}^\lambda$ as homogeneous.

Definition 3.1. Let $\mathcal{L}$ be a Lie or an associative algebra such that $\mathcal{L} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n$ is a vector space direct sum, then $\mathcal{L}$ is called an *almost-graded* (Lie-) algebra if

(i) $\dim \mathcal{L}_n < \infty$,
(ii) There exist constants $L_1, L_2 \in \mathbb{Z}$ such that

$$\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathcal{L}_h, \qquad \forall n, m \in \mathbb{Z}. \quad (3.1)$$

Elements in $\mathcal{L}_n$ are called *homogeneous* elements of degree n, and $\mathcal{L}_n$ is called homogeneous subspace of degree n.

In a similar manner almost-graded modules over almost-graded algebras are defined. Also of course, we can extend in an obvious way the definition to superalgebras, resp. even to more general algebraic structures. This definition makes complete sense also for more general index sets $\mathbb{J}$. In fact we will consider the index set $\mathbb{J} = (1/2)\mathbb{Z}$ for our superalgebra. Our even elements (with respect to the super-grading) will have integer degree, our odd elements half-integer degree.

3.2. **Separating cycle and Krichever-Novikov duality.** Let C_i be positively oriented (deformed) circles around the points P_i in I and C^*_j positively oriented ones around the points Q_j in O.

A cycle C_S is called a separating cycle if it is smooth, positively oriented of multiplicity one and if it separates the in- from the out-points. It might have multiple components. In the following we will integrate meromorphic differentials on Σ without poles in $\Sigma \setminus A$ over closed curves C. Hence, we might consider C and C' as equivalent if $[C] = [C']$ in $\mathrm{H}(\Sigma \setminus A, \mathbb{Z})$. In this sense we can write for every separating cycle

$$[C_S] = \sum_{i=1}^{K} [C_i] = -\sum_{j=1}^{M} [C^*_j]. \tag{3.2}$$

The minus sign appears due to the opposite orientation. Another way for giving such a C_S is also via level lines of a "proper time evolution", for which I refer to Ref. [22].

Given such a separating cycle C_S (resp. cycle class) we can define a linear map

$$\mathcal{F}^1 \to \mathbb{C}, \qquad \omega \mapsto \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega. \tag{3.3}$$

As explained above the map will not depend on the separating line C_S chosen, as two of such will be homologous and the poles of ω are only located in I and O.

Consequently, the integration of ω over C_S can also be described over the special cycles C_i or equivalently over C^*_j. This integration corresponds to calculating residues

$$\omega \quad \mapsto \quad \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega \quad = \quad \sum_{k=i}^{K} \mathrm{res}_{P_i}(\omega) \quad = \quad -\sum_{l=1}^{M} \mathrm{res}_{Q_l}(\omega). \tag{3.4}$$

Furthermore,

$$\mathcal{F}^\lambda \times \mathcal{F}^{1-\lambda} \to \mathbb{C}, \quad (f,g) \mapsto \langle f, g\rangle := \frac{1}{2\pi \mathrm{i}} \int_{C_S} f \cdot g, \tag{3.5}$$

gives a well-defined pairing, called the *Krichever-Novikov (KN) pairing.*

3.3. **The homogeneous subspaces.** Depending on whether λ is integer or half-integer, we set $\mathbb{J}_\lambda = \mathbb{Z}$ or $\mathbb{J}_\lambda = \mathbb{Z} + 1/2$. For $\mathcal{F}^\lambda$ we introduce for $m \in \mathbb{J}_\lambda$ subspaces $\mathcal{F}^\lambda_m$ of dimension K, where $K = \#I$, by exhibiting certain elements $f^\lambda_{m,p} \in \mathcal{F}^\lambda$, $p = 1, \ldots, K$ which constitute a basis of $\mathcal{F}^\lambda_m$. Recall that the spaces $\mathcal{F}^\lambda$ for $\lambda \in \mathbb{Z} + 1/2$ depend on the square root L (the theta characteristic) of the canonical bundle chosen. The elements of $\mathcal{F}^\lambda_m$ are the elements of degree m.

Let $I = \{P_1, P_2, \ldots, P_K\}$ then the basis element $f^\lambda_{m,p}$ of degree m is of order

$$\mathrm{ord}_{P_i}(f^\lambda_{m,p}) = (n + 1 - \lambda) - \delta^p_i \tag{3.6}$$

at the point $P_i \in I$, $i = 1, \ldots, K$. The prescription at the points in O is made in such a way that the element $f^\lambda_{m,p}$ is essentially uniquely given. Essentially unique means up to multiplication with a constant[1]. After fixing as additional geometric data, a system of

[1] Strictly speaking, there are some special cases where some constants have to be added such that the Krichever-Novikov duality (3.10) below is valid, see [22].

coordinates z_l centered at P_l for $l = 1, \ldots, K$ and requiring that

$$f_{n,p}^{\lambda}(z_p) = z_p^{n-\lambda}(1 + O(z_p))(dz_p)^{\lambda} \tag{3.7}$$

the element $f_{n,p}$ is uniquely fixed. In fact, the element $f_{n,p}^{\lambda}$ only depends on the first jet of the coordinate z_p [28].

Example. Here we will not give the general recipe for the prescription at the points in O, see [22], [23], [27]. Just to give an example which is also an important special case, assume $O = \{Q\}$ is a one-element set. If either the genus $g = 0$, or $g \geq 2$, $\lambda \neq 0,\ 1/2,\ 1$ and the points in A are in generic position then we require

$$\operatorname{ord}_Q(f_{n,p}^{\lambda}) = -K \cdot (n + 1 - \lambda) + (2\lambda - 1)(g - 1). \tag{3.8}$$

In the other cases (e.g. for $g = 1$) there are some modifications at the point in O necessary for finitely many m.

The construction yields [22], [23], [27]

Theorem 3.2. *Set*

$$\mathcal{B}^{\lambda} := \{\, f_{n,p}^{\lambda} \mid n \in \mathbb{J}_{\lambda},\ p = 1, \ldots, K \,\}. \tag{3.9}$$

Then (a) $\mathcal{B}^{\lambda}$ is a basis of the vector space $\mathcal{F}^{\lambda}$.

(b) The introduced basis $\mathcal{B}^{\lambda}$ of $\mathcal{F}^{\lambda}$ and $\mathcal{B}^{1-\lambda}$ of $\mathcal{F}^{1-\lambda}$ are dual to each other with respect to the Krichever-Novikov pairing (3.5), i.e.

$$\langle f_{n,p}^{\lambda}, f_{-m,r}^{1-\lambda} \rangle = \delta_p^r\, \delta_n^m, \quad \forall n, m \in \mathbb{J}_{\lambda}, \quad r, p = 1, \ldots, K. \tag{3.10}$$

From part (b) of the theorem it follows that the Krichever-Novikov pairing is non-degenerate. Moreover, any element $v \in \mathcal{F}^{1-\lambda}$ acts as linear form on $\mathcal{F}^{\lambda}$ via

$$\mathcal{F}^{\lambda} \mapsto \mathbb{C}, \quad w \mapsto \Phi_v(w) := \langle v, w \rangle. \tag{3.11}$$

Via this pairing $\mathcal{F}^{1-\lambda}$ can be considered as subspace of $(\mathcal{F}^{\lambda})^*$. But I like to stress the fact that the identification depends on the splitting of A into I and O as the KN pairing depends on it. The full space $(\mathcal{F}^{\lambda})^*$ can even be described with the help of the pairing. Consider the series

$$\hat{v} := \sum_{m \in \mathbb{Z}} \sum_{p=1}^{K} a_{m,p} f_{m,p}^{1-\lambda} \tag{3.12}$$

as a formal series, then $\Phi_{\hat{v}}$ (as a distribution) is a well-defined element of ${\mathcal{F}^{\lambda}}^*$, as it will be only evaluated for finitely many basis elements in $\mathcal{F}^{\lambda}$. Vice versa, every element of ${\mathcal{F}^{\lambda}}^*$ can be given by a suitable $\hat{v}$. Every $\phi \in (\mathcal{F}^{\lambda})^*$ is uniquely given by the scalars $\phi(f_{m,r}^{\lambda})$. We set

$$\hat{v} := \sum_{m \in \mathbb{Z}} \sum_{p=1}^{K} \phi(f_{-m,p}^{\lambda})\, f_{m,p}^{1-\lambda}. \tag{3.13}$$

Obviously, $\Phi_{\hat{v}} = \phi$. For more information about this "distribution interpretation" see [23], [25].

The dual elements of $\mathcal{L}$ will be given by the formal series (3.12) with basis elements from $\mathcal{F}^2$, the quadratic differentials, and the dual elements of $\mathcal{F}^{-1/2}$ correspondingly from

$\mathcal{F}^{3/2}$. The spaces $\mathcal{F}^2$ and $\mathcal{F}^{3/2}$ themselves can be considered as some kind of restricted duals.

It is quite convenient to use special notations for elements of some important weights:

$$e_{n,p} := f_{n,p}^{-1}, \quad \varphi_{n,p} := f_{n,p}^{-1/2}, \quad \mathcal{A}_{n,p} := f_{n,p}^{0}. \tag{3.14}$$

3.4. **The algebras.**

Proposition 3.3. *There exist constants R_1 and R_2 (depending on the number and splitting of the points in A and of the genus g) independent of $n, m \in \mathbb{J}$ such that for the basis elements*

$$\begin{aligned} f_{n,p}^{\lambda} \cdot f_{m,r}^{\nu} = \quad & f_{n+m,r}^{\lambda+\nu}\delta_p^r \\ & + \sum_{h=n+m+1}^{n+m+R_1}\sum_{s=1}^{K} a_{(n,p)(m,r)}^{(h,s)} f_{h,s}^{\lambda+\nu}, \quad a_{(n,p)(m,r)}^{(h,s)} \in \mathbb{C}, \\ [f_{n,p}^{\lambda}, f_{m,r}^{\nu}] = \quad & (-\lambda m + \nu n)\, f_{n+m,r}^{\lambda+\nu+1}\delta_p^r \\ & + \sum_{h=n+m+1}^{n+m+R_2}\sum_{s=1}^{K} b_{(n,p)(m,r)}^{(h,s)} f_{h,s}^{\lambda+\nu+1}, \quad b_{(n,p)(m,r)}^{(h,s)} \in \mathbb{C}. \end{aligned} \tag{3.15}$$

Proof. For the elements on the l.h.s. of (3.15) we can estimate the maximal pole orders at the points in I and O. Using the KN duality and the prescribed orders of the basis elements we obtain by considering possible pole orders at the points in I the lower bound of the degree, and by considering the pole orders at O the upper bounds R_1 and R_2 for the degree on the r.h.s.. The degree $n+m$ part follows from local calculations at the points in I. See [22], [23], [27] for more details. □

As a direct consequence we obtain

Theorem 3.4. *The algebras $\mathcal{L}$ and $\mathcal{S}$ are almost-graded Lie , resp. Lie superalgebras. The almost-grading depends on the splitting of the set A into I and O. More precisely,*

$$\mathcal{F}^{\lambda} = \bigoplus_{m\in\mathbb{J}_\lambda} \mathcal{F}_m^{\lambda}, \qquad \textit{with} \quad \dim \mathcal{F}_m^{\lambda} = K. \tag{3.16}$$

and there exist R_1, R_2 (independent of n and m) such that

$$[\mathcal{L}_n, \mathcal{L}_m] \subseteq \bigoplus_{h=n+m}^{n+m+R_1} \mathcal{L}_h \ , \qquad [\mathcal{S}_n, \mathcal{S}_m] \subseteq \bigoplus_{h=n+m}^{n+m+R_2} \mathcal{S}_h.$$

The constants R_i depend on the genus of the Riemann surface and the number of points in I and O. In fact they can be explicitly calculated (if needed).

Also from (3.15) we can directly conclude

Proposition 3.5. *For all $m, n \in \mathbb{J}_\lambda$ and $r, p = 1, \ldots, K$ we have*

$$\begin{aligned} [e_{n,p}, e_{m,r}] &= (m-n) \cdot e_{n+m,r}\, \delta_r^p + h.d.t. \\ e_{n,p} \cdot \varphi_{m,r} &= (m - \frac{n}{2}) \cdot \varphi_{n+m,r}\, \delta_r^p + h.d.t. \\ \varphi_{n,p} \cdot \varphi_{m,r} &= e_{n+m,r}\, \delta_r^p + h.d.t. \end{aligned} \tag{3.17}$$

Here h.d.t. denote linear combinations of basis elements of degree between $n+m+1$ and $n+m+R_2$, with the R_2 from Theorem 3.4.

See Section 5.3.2 for an example in the classical case (by ignoring the central extension appearing there for the moment).

Remark 3.6. Note that in certain literature for the classical situation some other normalisation of the structure equation of the superalgebra was given. Instead of the product of the -1/2 forms twice the product was used (corresponding to the anti-commutator). These results can also be obtained by setting $\varphi_{n,p} = \sqrt{2} f_{n,p}^{-1/2}$. The last line of (3.17) will then start with $2e_{n+m,p}\delta_p^r$. This has also consequences for the structure equation of the central extension (5.41).

On the basis of the almost-grading we obtain a triangular decomposition of the algebras

$$\mathcal{L} = \mathcal{L}_{[+]} \oplus \mathcal{L}_{[0]} \oplus \mathcal{L}_{[-]}, \qquad \mathcal{S} = \mathcal{S}_{[+]} \oplus \mathcal{S}_{[0]} \oplus \mathcal{S}_{[-]} \tag{3.18}$$

where e.g.

$$\mathcal{S}_{[+]} := \bigoplus_{m>0} \mathcal{S}_m, \quad \mathcal{S}_{[0]} = \bigoplus_{m=-R_2}^{m=0} \mathcal{S}_m, \quad \mathcal{S}_{[-]} := \bigoplus_{m<-R_2} \mathcal{S}_m. \tag{3.19}$$

By the almost-graded structure the [+] and [−] subspaces are indeed (infinite dimensional) subalgebras. The [0] spaces in general are not subalgebras.

Remark 3.7. In case that O has more than one point there are certain choices, e.g. numbering of the points in O, different rules, etc. involved. Hence, if the choices are made differently the subspaces $\mathcal{F}_n^\lambda$ might depend on them, and consequently also the almost-grading. But as it is shown in the above quoted works the induced filtration

$$\begin{aligned} &\mathcal{F}_{(n)}^\lambda := \bigoplus_{m \geq n} \mathcal{F}_m^\lambda, \\ &\ldots \supseteq \mathcal{F}_{(n-1)}^\lambda \supseteq \mathcal{F}_{(n)}^\lambda \supseteq \mathcal{F}_{(n+1)}^\lambda \ldots \end{aligned} \tag{3.20}$$

has an intrinsic meaning given by

$$\mathcal{F}_{(n)}^\lambda := \{\ f \in \mathcal{F}^\lambda \mid \operatorname{ord}_{P_i}(f) \geq n - \lambda, \forall i = 1, \ldots, K\ \}\,. \tag{3.21}$$

Hence it is independent of these choices.

Remark 3.8. Leidwanger and Morier-Genoux introduced in [17] also a Jordan superalgebra based on the Krichever-Novikov objects, i.e.

$$\mathcal{J} := \mathcal{F}^0 \oplus \mathcal{F}^{-1/2} = \mathcal{J}_{\bar{0}} \oplus \mathcal{J}_{\bar{1}}. \tag{3.22}$$

Recall that $\mathcal{F}^0$ is the associative algebra of meromorphic functions. The (Jordan) product is defined via the algebra structure introduced in Section 2 for the spaces $\mathcal{F}^\lambda$ by

$$\begin{aligned} f \circ g &:= f \cdot g \quad &\in \mathcal{F}^0, \\ f \circ \varphi &:= f \cdot \varphi \quad &\in \mathcal{F}^{-1/2} \\ \varphi \circ \psi &:= [\varphi, \psi] \quad &\in \mathcal{F}^0. \end{aligned} \tag{3.23}$$

By rescaling the second definition with the factor 1/2 one obtains a Lie antialgebra. See [17] for more details and additional results on representations.

Here I only want to add the following. Using the results presented in this section one easily sees that with respect to the almost-grading introduced (depending on a splitting $A = I \cup O$) the Jordan superalgebra becomes indeed an almost-graded algebra

$$\mathcal{J} = \bigoplus_{m \in 1/2\mathbb{Z}} \mathcal{J}_m. \tag{3.24}$$

Hence, it makes sense to call it a Jordan superalgebra of KN type. Calculated for the introduced basis elements we get (using Proposition 3.3)

$$\begin{aligned} A_{n,p} \circ A_{m,r} &= A_{n+m,r}\, \delta_r^p + \text{h.d.t.} \\ A_{n,p} \circ \varphi_{m,r} &= \varphi_{n+m,r}\, \delta_r^p + \text{h.d.t.} \\ \varphi_{n,p} \circ \varphi_{m,r} &= \frac{1}{2}(m-n)\mathcal{A}_{n+m,r}\, \delta_r^p + \text{h.d.t.} \end{aligned} \tag{3.25}$$

4. Central extensions

In this section we recall the results which are needed about central extensions of the vector field algebra in the following discussion of central extensions of the Lie superalgebras. More details can be found in [22], [23], [26].

A central extension of a Lie algebra W is defined on the vector space direct sum $\widehat{W} = \mathbb{C} \oplus W$. If we denote $\hat{x} := (0, x)$ and $t := (1, 0)$ then its Lie structure is given by

$$[\hat{x}, \hat{y}] = \widehat{[x, y]} + \Phi(x, y) \cdot t, \quad [t, \widehat{W}] = 0, \quad x, y \in W. \tag{4.1}$$

$\widehat{W}$ will be a Lie algebra, e.g. fulfill the Jacobi identity, if and only if Φ is antisymmetric and fulfills the Lie algebra 2-cocycle condition

$$0 = d_2\Phi(x, y, z) := \Phi([x, y], z) + \Phi([y, z], x) + \Phi([z, x], y). \tag{4.2}$$

There is the notion of equivalence of central extensions. It turns out that two central extensions are equivalent if and only if the difference of their defining 2-cocycles Φ and Φ' is a coboundary, i.e. there exists a $\phi : W \to \mathbb{C}$ such that

$$\Phi(x, y) - \Phi'(x, y) = d_1\phi(x, y) = \phi([x, y]). \tag{4.3}$$

In this way the second Lie algebra cohomology $\mathrm{H}^2(W, \mathbb{C})$ of W with values in the trivial module $\mathbb{C}$ classifies equivalence classes of central extensions. The class [0] corresponds to the trivial (i.e. split) central extension. Hence, to construct central extensions of our Lie algebras we have to find such Lie algebra 2-cocycles.

We want to generalize the cocycle which defines in the classical case the Virasoro algebra to higher genus and the multi-point situation. We have to geometrize the cocycle. Before we can give this geometric description, we have to introduce the notion of a *projective connection*

Definition 4.1. Let $(U_\alpha, z_\alpha)_{\alpha\in J}$ be a covering of the Riemann surface by holomorphic coordinates, with transition functions $z_\beta = f_{\beta\alpha}(z_\alpha)$. A system of local holomorphic functions $R = (R_\alpha(z_\alpha))$ is called a holomorphic *projective connection* if it transforms as

$$R_\beta(z_\beta)\cdot (f'_{\beta,\alpha})^2 = R_\alpha(z_\alpha) + S(f_{\beta,\alpha}), \qquad \text{with} \quad S(h) = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2, \tag{4.4}$$

the Schwartzian derivative. Here $'$ means differentiation with respect to the coordinate z_α.

It is a classical result [10], [9] that every Riemann surface admits a holomorphic projective connection R. From the definition it follows that the difference of two projective connections is a quadratic differential. In fact starting from one projective projection we will obtain all of them by adding quadratic differentials to it.

Given a smooth differentiable curve C (not necessarily connected) and a fixed holomorphic projective connection R, the following defines for the vector field algebra a two-cocycle

$$\Phi_{C,R}(e,f) := \frac{1}{24\pi \mathrm{i}}\int_C \left(\frac{1}{2}(e'''f - ef''') - R\cdot(e'f - ef')\right) dz\ . \tag{4.5}$$

Only by the term involving the projective connection it will be a well-defined differential, i.e. independent of the chosen coordinates. It is shown in Ref. [23] that it is a cocycle. Another choice of a projective connection will result in a cohomologous one, see also (5.15)

In contrast to the classical situation, for the higher genus and/or multi-point situation there are many essentially different closed curves and also many non-equivalent central extensions defined by the integration.

But we should take into account that we want to extend the almost-grading from our algebras to the centrally extended ones. This means we take $\deg \hat{x} := \deg x$ and assign a degree $deg(t)$ to the central element t, and obtain an almost-grading.

This is possible if and only if our defining cocycle ψ is "local" in the following sense (the name was introduced in the two point case by Krichever and Novikov in Ref. [13]). There exists $M_1, M_2 \in \mathbb{Z}$ such that

$$\forall n,m: \quad \psi(W_n, W_m) \neq 0 \implies M_1 \le n+m \le M_2. \tag{4.6}$$

Here W stands for any of our algebras (including the supercase discussed below). Very important, "local" is defined in terms of the grading, and the grading itself depends on the splitting $A = I \cup O$. Hence what is "local" depends on the splitting too.

We will call a cocycle *bounded* (from above) if there exists $M \in \mathbb{Z}$ such that

$$\forall n,m: \quad \psi(W_n, W_m) \neq 0 \implies n+m \le M. \tag{4.7}$$

Similarly bounded from below can be defined. Locality means bounded from above and below.

Given a cocycle class we call it bounded (resp. local) if and only if it contains a representing cocycle which is bounded (resp. local). Not all cocycles in a bounded class have to be bounded. If we choose as integration path a separating cocycle C_S, or one of the C_i then the above introduced geometric cocycles (4.5) are local, resp. bounded. Recall that in this case integration can be done by calculating residues at the in-points or at the out-points. All these cocycles are cohomologically nontrivial. The following theorem concerns the opposite direction.

Theorem 4.2. [26] *Let $\mathcal{L}$ be the Krichever–Novikov vector field algebra.*
(a) The space of bounded cohomology classes is K-dimensional ($K = \#I$). A basis is given by setting the integration path in (4.5) to C_i, $i = 1, \ldots, K$ the little (deformed) circles around the points $P_i \in I$.
(b) The space of local cohomology classes is one-dimensional. A generator is given by integrating (4.5) over a separating cocycle C_S.
(c) Up to equivalence and rescaling there is only one one-dimensional central extension of the vector field algebra $\mathcal{L}$ which allows an extension of the almost-grading.

5. Central extensions - the supercase

In this section we consider central extensions of our Lie superalgebra $\mathcal{S}$. Such a central extension is given by a bilinear map

$$c : \mathcal{S} \times \mathcal{S} \to \mathbb{C} \tag{5.1}$$

via an expression completely analogous to (4.1). Additional conditions for c follow from the fact that the resulting extension should be again a superalgebra. This implies that for the homogeneous elements $x, y, z \in \mathcal{S}$ ($\mathcal{S}$ might be an arbitrary Lie superalgebra) we have

$$c(x, y) = -(-1)^{\bar{x}\bar{y}} c(x, y). \tag{5.2}$$

The bilinear map c will be symmetric if x and y are odd, otherwise it will be antisymmetric. The super-cocycle condition reads in complete analogy with the super-Jacobi relation as

$$(-1)^{\bar{x}\bar{z}} c(x, [y, z]) + (-1)^{\bar{y}\bar{x}} c(y, [z, x]) + (-1)^{\bar{z}\bar{y}} c(z, [x, y]) \ = \ 0. \tag{5.3}$$

As we will need it anyway, I will write it out for the different type of arguments. For *(even,even,even)*, *(even,even, odd)*, and *(odd,odd,odd)* it will be of the "usual form" of the cocycle condition

$$c(x, [y, z]) + c(y, [z, x]) + c(z, [x, y]) \ = \ 0. \tag{5.4}$$

For *(even,odd,odd)* we obtain

$$c(x, [y, z]) + c(y, [z, x]) - c(z, [x, y]) \ = \ 0. \tag{5.5}$$

Now we have to decide which parity our central element should have. In our context the natural choice is that the central element should be even, as we want to extend the central extension of the vector field algebra to the superalgebra. This implies that our bilinear form c has to be an even form. Consequently,

$$c(x, y) = c(y, x) = 0, \quad \text{for} \ \ \bar{x} = 0, \bar{y} = 1. \tag{5.6}$$

In this case from the super-cocycle conditions only (5.5) for the *(even,odd,odd)* and (5.4) for the *(even,even,even)* case will give relations which are not nontrivially zero. In Section 5.2 we will consider the case that the central element is of odd parity.

Given a linear form $k : \mathcal{S} \to \mathbb{C}$ we assign to it

$$\delta_1 k(x, y) = k([x, y]). \tag{5.7}$$

As in the classical case $\delta_1 k$ will be a super-cocycle. A super-cocycle will be a coboundary if and only if there exists a linear form $k : \mathcal{S} \to \mathbb{C}$ such that $c = \delta_1 k$. As k is a linear form it can be written as $k = k_{\bar{0}} \oplus k_{\bar{1}}$ where $k_{\bar{0}} : \mathcal{S}_{\bar{0}} \to \mathbb{C}$ and $k_{\bar{1}} : \mathcal{S}_{\bar{1}} \to \mathbb{C}$. Again we have the two cases of the parity of the central element. Let c be a coboundary $\delta_1 k$. If the central element is even then c will also be a coboundary of a k with $k_{\bar{1}} = 0$. In other words k is even. In the odd case we have $k_{\bar{0}} = 0$ and k is odd.

After fixing a parity of the central element we consider the quotient spaces

$$\mathrm{H}^2_{\bar{0}}(\mathcal{S}, \mathbb{C}) := \{\text{even cocycles}\}/\{\text{even coboundaries}\}, \tag{5.8}$$

$$\mathrm{H}^2_{\bar{1}}(\mathcal{S}, \mathbb{C}) := \{\text{odd cocycles}\}/\{\text{odd coboundaries}\}. \tag{5.9}$$

These cohomology spaces classify central extensions of $\mathcal{S}$ with even (resp. odd) central elements up to equivalence. Equivalence is defined as in the non-super setting.

For the rest of this section our algebra $\mathcal{S}$ will the Lie superalgebra introduced in Section 2.6. Moreover, for the moment we concentrate on the case of an even central element t. Recall our convention to denote vector fields by $e, f, g, \ldots$ and -1/2-forms by $\varphi, \psi, \chi, \ldots$ From the discussion above we know

$$c(e, \varphi) = 0, \quad e \in \mathcal{L},\ \varphi \in \mathcal{F}^{-1/2}. \tag{5.10}$$

The super-cocycle conditions for the even elements is just the cocycle condition for the Lie subalgebra $\mathcal{L}$. The only other nonvanishing super-cocycle condition is for the *(even,odd,odd)* elements and reads as

$$c(e, [\varphi, \psi]) - c(\varphi, e\,.\,\psi) - c(\psi, e\,.\,\varphi) = 0. \tag{5.11}$$

Here the definition of the product $[e, \psi] := e\,.\,\psi$ was used to rewrite (5.5).

In particular, if we have a cocycle c for the algebra $\mathcal{S}$ we obtain by restriction a cocycle for the algebra $\mathcal{L}$. For the mixing term we know that $c(e, \psi) = 0$. A naive try to put just anything for $c(\varphi, \psi)$ will not work as (5.11) relates the restriction of the cocycle on $\mathcal{L}$ with its values on $\mathcal{F}^{-1/2}$.

Proposition 5.1. *Let C be any closed (differentiable) curve on Σ not meeting the points in A, and let R be any (holomorphic) projective connection, then the bilinear extension of*

$$\begin{aligned}
\Phi_{C,R}(e, f) &:= \frac{1}{24\pi \mathrm{i}} \int_C \left(\frac{1}{2}(e''' f - e f''') - R \cdot (e' f - e f') \right) dz \\
\Phi_{C,R}(\varphi, \psi) &:= -\frac{1}{24\pi \mathrm{i}} \int_C \left(\varphi'' \cdot \psi + \varphi \cdot \psi'' - R \cdot \varphi \cdot \psi \right) dz \\
\Phi_{C,R}(e, \varphi) &:= 0
\end{aligned} \tag{5.12}$$

gives a Lie superalgebra cocycle for $\mathcal{S}$, hence defines a central extension of $\mathcal{S}$

A similar formula was given by Bryant in [4]. By adding the projective connection in the second part of (5.12) he corrected some formula appearing in [1]. He only considered the two-point case and only the integration over a separating cycle. See also [12] for the multi-point case, where still only the integration over a separating cycle is considered.

Proof. First one has to show that the integrands are well-defined differentials. It is exactly this point for which R had to be introduced. This was done for the vector field case in [23], [22]. The case for the second part of (5.12) is completely analogous, and follows from straight-forward calculations.

Next the super-Jacobi identities have to be verified. Again the first one (5.4) was shown in the latter references. For the other one (5.11) we write out the three integrands and sum them up (before we integrate over C). By direct calculations we obtain that the term coming with the projective connection will identically vanish. Hence the rest will be a well-defined meromorphic differential, It will not necessarily vanish identically. We only claim that the sum will vanish after integration over an arbitrary closed curve. Recall that the curve integral over an exact meromorphic differential ω, i.e. a differential which is exact, i.e. can be written locally as $\omega = df$ with f a meromorphic function on Σ, will vanish. In a first step we calculate

$$e'''f = \frac{1}{2}(e'''f - ef''') + 1/2((ef)'' - 3(e'f'))'$$
$$-2(\varphi'\psi') = (\varphi''\psi + \varphi\psi'') - \left((\varphi\psi)'\right)'.$$

Hence, we can replace the corresponding integrands in the cocycle expressions by integrands given by the left hand side. The total integrand of (5.11) can now be written as Bdz with

$$B := e'''(\varphi\psi) - 2\varphi'(e\psi' - 1/2e'\psi)' - 2\psi'(e\varphi' - 1/2e'\varphi)' \tag{5.13}$$

which calculates to

$$B = (e''(\varphi\psi) - 2(e\varphi'\psi'))'. \tag{5.14}$$

Consequently Bdz integrated over any closed curve C will vanish. This shows that the cocycle condition (5.11) is true. □

How will the central extension depend on C and R? Obviously, two cycles lying in the same homology class class of $\Sigma \setminus A$ will define the same cocycle.

Proposition 5.2. *If R and R' are two projective connections then $\Phi_{C,R}$ and $\Phi_{C,R'}$ are cohomologous. Hence the cohomology class will not depend on the choice of R.*

Proof. The difference of two projective connections is a quadratic differential, $\Omega = R' - R$. We calculate

$$\begin{aligned}\Phi_{C,R'}(e,f) - \Phi_{C,R}(e,f) &= \frac{1}{24\pi\mathrm{i}}\int_C \Omega\cdot(ef' - e'f)dz = \frac{1}{24\pi\mathrm{i}}\int_C \Omega\cdot[e,f],\\ \Phi_{C,R'}(\varphi,\psi) - \Phi_{C,R}(\varphi,\psi) &= \frac{1}{24\pi\mathrm{i}}\int_C \Omega\cdot(\varphi\cdot\psi)dz = \frac{1}{24\pi\mathrm{i}}\int_C \Omega\cdot[\varphi,\psi].\end{aligned} \tag{5.15}$$

If we fix the quadratic differential Ω then the map

$$\kappa_C : \mathcal{L} \to \mathbb{C}, \qquad e \mapsto \frac{1}{24\pi\mathrm{i}}\int_C \Omega\cdot e \tag{5.16}$$

is a linear map. We extend this map by zero on $\mathcal{F}^{-1/2}$ and obtain using (5.15) that

$$\Phi_{C,R'} - \Phi_{C,R} = \delta_1\kappa_C. \tag{5.17}$$

Hence both cocycles are cohomologous. □

As in the pure vector field case for the non-classical situation, there will be many inequivalent central extensions given by different cycle classes as integration paths. Recall that classical means $g = 0$ and $N = 2$.

We will need the special integration paths C_i, (C_j^*), the circles around the points $P_i \in I$ $(Q_j \in O)$, introduced in Section 3 and C_S a separating cycle. Recall from (3.2)

$$[C_S] = \sum_{i=1}^{K} [C_i] = -\sum_{j=1}^{M} [C_j^*], \tag{5.18}$$

as homology classes.

Proposition 5.3. *(a) The cocycles $\Phi_{C_i,R}$ are bounded (from above) by zero.*

(b) The cocycle $\Phi_{C_S,R}$ obtained by integrating over a separating cocycle is a local cocycle.

Proof. First, the cocycles evaluated for the vector field subalgebra are bounded resp. local, as shown in [22], [26, Thm. 4.2]. The same argument works for the other part of the cocycle. Just to give the principle idea: We consider the integrand for pairs of elements $(\varphi_{m,p}, \psi_{n,r})$. If $m + n > 0$ it will not have residues at the points P_i. Hence the integration around C_i will yield 0. This shows (a) and the fact that $\Phi_{C_S,R}$ is bounded from above by zero. Integration over C_S can alternatively be done also by summation of integration over the right hand side of expression (5.18). By the definition of the homogeneous elements there is a bound S independent of n, m such that the integrand for pairs of elements for which the sum of their degrees $< S$, do not have poles at the points $Q_j \in O$. Hence the integral will vanish too. This says the cocycle $\Phi_{C_S,R}$ is bounded from below, hence local. □

The question is, will the opposite be also true, meaning that every local or every bounded cocycle will be equivalent to those cocycles defined above, resp. to a certain linear combination of them? As in the vector field algebra case, it will turn out that with respect to a fixed almost-grading the non-trivial almost-graded central extension (with even central element) will be essentially unique, hence given by the a scalar times $\Phi_{C_S,R}$. Also we will make a corresponding statement about bounded cocycles.

The following is the crucial technical result

Proposition 5.4. *Let c be a cocycle for the superalgebra $\mathcal{S}$ which is bounded from above, and which vanishes on the vector field subalgebra $\mathcal{L}$, then c vanishes in total. In other words, every bounded cocycle is uniquely given by its restriction to the vector field subalgebra.*

Before we prove this proposition in Section 5.1 we formulate the main theorem of this article.

Theorem 5.5. *Given the Lie superalgebra $\mathcal{S}$ of Krichever-Novikov type with its induced almost-grading given by the splitting of A into I and O. Then:*

(a) The space of bounded cohomology classes has dimension $K = \#I$. A basis is given by the classes of cocycles (5.12) integrating over the cycles C_i, $i = 1, \ldots, K$.

(b) The space of local cohomology classes is one-dimensional. A generator is given by the class of (5.12) integrating over a separating cocycle C_S.

(c) Up to equivalence and rescaling there is only one non-trivial almost-graded central extension of the Lie superalgebra extending the almost-graded structure on $\mathcal{S}$.

Proof. Let c be a bounded cocycle for $\mathcal{S}$. After restriction to $\mathcal{L}\times\mathcal{L}$ we obtain a bounded cocycle for $\mathcal{L}$. By Theorem 4.2 it is cohomologous to a standard cocycle with a suitable projective connection R

$$\Phi := \sum_{i=1}^{K} a_i \Phi_{C_i,R}, \quad a_i \in \mathbb{C}. \tag{5.19}$$

Let $\kappa : \mathcal{L} \to \mathbb{C}$ be the linear form giving the coboundary, i.e. $c_{|\mathcal{L}} - \Phi = \delta_1(\kappa)$. We extend κ by zero for the elements of $\mathcal{F}^{-1/2}$. Then $\Phi' := c - \delta_1(\kappa)$ is cohomologous to c (as cocycle for $\mathcal{S}$). Moreover, $(c-\delta_1(\kappa))_{|\mathcal{L}} = \Phi$. Next we extend Φ via (5.12) to $\mathcal{S}$ (i.e. taking the same linear combination of integration cycles and the same projective connection) and obtain another cocycle for $\mathcal{S}$, still denoted by Φ. By construction the difference $\Phi' - \Phi$ is zero on $\mathcal{L}$. Hence by Proposition 5.4 it vanishes on $\mathcal{S}$. But Φ is exactly of the form claimed in (a). We get exactly K linearly independent cocycle classes, as they are linearly independent as cocycles for the subalgebra $\mathcal{L}$. Claim (b) follows in a completely analogous manner, now applied to local cocycles. Claim (c) is a direct consequence. □

I like to stress the fact, that it will not be the case that there is only one central extension of the superalgebra $\mathcal{S}$. Only if we fix an almost-grading for $\mathcal{S}$, which means that we fix a splitting of A into I and O, there will be a unique central extension allowing us to extend the almost-grading. For another essentially different splitting (meaning that it is not only the changing of the role of I and O) splitting the almost-grading will be different and we will obtain a different central extension of the algebra. In fact, if the genus g of the Riemann surface is larger than zero, there will be non-equivalent central extensions which are not associated to any almost-grading (neither coming from a local cocycle, nor from a bounded cocycle).

5.1. **Proof of Proposition 5.4.** We start with a bounded cocycle c for $\mathcal{S}$ which vanishes on the subalgebra $\mathcal{L}$ and consider the cocycle condition (5.11). It remains

$$c(\varphi, e\,.\,\psi) + c(\psi, e\,.\,\varphi) = 0, \quad \forall e \in \mathcal{L}, \varphi, \psi \in \mathcal{F}^{-1/2}. \tag{5.20}$$

Our goal is to show that it is identically zero.

For a pair of homogeneous basis elements $(f_{m,p}, g_{n,r})$ of any combination of types we call $l = n+m$ the *level* of the pair. We evaluate the cocycle c at pairs of level l, i.e. $c(f_{m,p}, g_{n-l,r})$. We call these cocycle values, values of level l. We apply the technique developed in [26]. We will consider cocycle values $c(f_{m,p}, g_{n,r})$ on pairs of level $l = n+m$ and will make descending induction over the level. By the boundedness from above, the cocycle values will vanish at all pairs of sufficiently high level. It will turn out that everything will be fixed by the values of the cocycle at level zero. Finally, we will show that the cocycle c also vanishes at level zero. Hence the claim of Proposition 5.4.

For a cocycle c evaluated for pairs of elements of level l we will use the symbol $\equiv$ to denote that the expressions are the same on both sides of an equation involving cocycle values up to values of c at higher level. This has to be understood in the following strong sense:

$$\sum \alpha^{(n,p,r)} c(f_{n,p}, g_{l-n,r}) \equiv 0, \qquad \alpha^{(n,p,r)} \in \mathbb{C} \tag{5.21}$$

denotes a congruence modulo a linear combination of values of c at pairs of basis elements of level $l' > l$. The coefficients of that linear combination, as well as the $\alpha^{(n,p,r)}$, depend only on the structure of the Lie algebra $\mathcal{S}$ and do not depend on c. We will also use the same symbol $\equiv$ for equalities in $\mathcal{S}$ which are true modulo terms of higher degree compared to the terms under consideration.

We consider the triple of basis elements $(\varphi_{m,r}, \varphi_{n,s}, e_{k,p})$ for level $l = n + m + k$. Recall (3.17)

$$[e_{k,p}, \varphi_{n,s}] \equiv (n - \frac{k}{2})\, \delta_p^s\, \varphi_{n+k,p}. \tag{5.22}$$

Hence

$$c(\varphi_{m,r}, [e_{k,p}, \varphi_{n,s}]) \equiv c(\varphi_{m,r}, (n - \frac{k}{2})\, \delta_p^s\, \varphi_{n+k,p}) = (n - \frac{k}{2})\, \delta_p^s\, c(\varphi_{m,r}, \varphi_{n+k,p}). \tag{5.23}$$

If we use this we obtain from (5.20)

$$(n - \frac{k}{2})\, \delta_p^s\, c(\varphi_{m,r}, \varphi_{n+k,p}) + (m - \frac{k}{2})\, \delta_p^r\, c(\varphi_{n,s}, \varphi_{m+k,p}) \equiv 0. \tag{5.24}$$

We set $k = 0$ then (now the level is $n + m$)

$$n\, \delta_p^s\, c(\varphi_{m,r}, \varphi_{n,p}) + m\, \delta_p^r\, c(\varphi_{n,s}, \varphi_{m,p}) \equiv 0. \tag{5.25}$$

If $p = s$ but $p \neq r$ then we obtain

$$n \cdot c(\varphi_{m,r}, \varphi_{n,p}) \equiv 0. \tag{5.26}$$

As $n \in \mathbb{Z} + \frac{1}{2}$, and hence $n \neq 0$ we have

$$c(\varphi_{m,r}, \varphi_{n,p}) \equiv 0, \quad r \neq p. \tag{5.27}$$

Next we consider $r = p = s$ and (5.25) yields

$$n \cdot c(\varphi_{m,s}, \varphi_{n,s}) + m \cdot c(\varphi_{n,s}, \varphi_{m,s}) \equiv 0. \tag{5.28}$$

As c is symmetric on $\mathcal{F}^{-1/2}$ we get

$$(n + m) \cdot c(\varphi_{m,s}, \varphi_{n,s}) \equiv 0. \tag{5.29}$$

This shows that, as long as the level is different from zero, the cocycle is given via universal cocycle values of higher level. By assumption our cocycle c is bounded from above. Hence there exists a level R such that for all levels $> R$ the cocycle values will vanish. We get by induction from (5.27) and (5.29) that the cocycle values will be zero for all levels > 0. Next we will show that it will vanish also at level zero. We go back to (5.24) (for $s = p = r$) and plug in $n = m$ and $k = -2n \in \mathbb{Z}$ and obtain

$$4n \cdot c(\varphi_{n,s}, \varphi_{-n,s}) \equiv 0, \quad \forall n \in \mathbb{Z} + \frac{1}{2}. \tag{5.30}$$

Hence also at level zero everything will be expressed by cocycle values of higher level and consequently will be equal to zero. Continuing with (5.29) we see that also at level < 0 we get that the cocycle will vanish. Hence the claim $\square$.

5.2. **The case of an odd central element.**

In this section we will consider the case where the central element has odd parity. We will show

Theorem 5.6. *Every bounded cocycle yielding a central extension with odd central element is a coboundary.*

Hence,

Corollary 5.7. *There are no non-trivial central extensions of the Lie superalgebra $\mathcal{S}$ with odd central element coming from a bounded cocycle. In other words all such central extensions will split.*

Proof. (Theorem 5.6) In the odd case only the cocycle relations (5.4) for the *(even,even,odd)* and *(odd,odd,odd)* combinations will be non-trivial. We first make a cohomologous change by defining recursively a map

$$\Phi : \mathcal{F}^{-1/2} \to \mathbb{C}, \tag{5.31}$$

which will be extended by zero on $\mathcal{L}$. We consider $c(e_{0,p}, \varphi_{k,r})$. It is of level k. By the boundedness there exists an R such that for $k > R$ all its values will vanish. Recall e.g. (3.17)

$$e_{0,p} \cdot \varphi_{k,p} = k \cdot \varphi_{k,p} + y_{k,p}, \tag{5.32}$$

with $y_{k,p}$ a finite sum of elements of degree $\geq k+1$. Set

$$\Phi(\varphi_{k,p}) := 0, \quad k > R, \ p = 1, \dots, K \tag{5.33}$$

and then recursively for $k = R, R-1, \dots..$

$$\Phi(\varphi_{k,p}) := \frac{1}{k} \left(c(e_{0,p}, \varphi_{k,p}) - \Phi(y_{k,p}) \right), \quad p = 1, \dots, K. \tag{5.34}$$

For the cohomologous cocycle $c' = c - \delta_1 \Phi$ we calculate for $p = 1, \dots, K$

$$c'(e_{0,p}, \varphi_{m,p}) = c(e_{0,p}, \varphi_{m,p}) - \Phi(e_{0,p} \cdot \varphi_{k,p}) = c(e_{0,p}, \varphi_{m,p}) - k\Phi(\varphi_{k,p}) - \Phi(y_{k,p}) = 0. \tag{5.35}$$

Claim: The cocycle c' vanishes identically. For simplicity we will drop the $'$. We consider the cocycle relation for *(even,even,odd)* for the elements $e_{n,p}, e_{m,r}, \varphi_{k,s}$. With the same technique used in the last section we obtain

$$(k - \frac{m}{2}) c(e_{n,p}, \varphi_{k+m,r}) \delta_r^s - (k - \frac{n}{2}) c(e_{m,r}, \varphi_{k+n,s}) \delta_p^s - (m-n) c(e_{m+n,r}, \varphi_{k,s}) \delta_p^r \equiv 0. \tag{5.36}$$

For $n = 0$ this specializes to

$$(k - \frac{m}{2}) c(e_{0,p}, \varphi_{k+m,r}) \delta_r^s - k \cdot c(e_{m,r}, \varphi_{k,s}) \delta_p^s - m \cdot c(e_{m,r}, \varphi_{k,s}) \delta_p^r \equiv 0. \tag{5.37}$$

If we set $s = p \neq r$ then

$$-k \cdot c(e_{m,r}, \varphi_{k,s}) \equiv 0. \tag{5.38}$$

As k is half-integer we get $c(e_{m,r}, \varphi_{k,s}) \equiv 0$. For $r = s = p$ in (5.37) we obtain

$$(k - \frac{m}{2}) c(e_{0,p}, \varphi_{m,p}) - (k+m) c(e_{m,p}, \varphi_{k,p}) \equiv 0. \tag{5.39}$$

As $c(e_{0,p}, \varphi_{m,p}) = 0$ and $k + m \neq 0$ we get $c(e_{m,p}, \varphi_{k,p}) \equiv 0$ for all m and k. Hence all values are determined by values of higher level. By the boundedness it will be zero at all level. □

5.3. **Some special examples.** In this section I like to give reference to some special examples.

5.3.1. *Higher genus, two points.* In this case there is only one almost-grading for $\mathcal{S}$ because there is only one possible splitting. In this case the separating cycle C_S coincides with the cycle C_1. In particular, integration over C_1 already gives a local cocycle. But this does not mean that every bounded cocycle will be local, it only means that the class of bounded cocycles coincides with the class of local cocycles. This case with integration over C_S was considered by Bryant [4]. But he does not prove uniqueness. Also it has to be repeated that for higher genus, there are other non-equivalent cocycles obtained by integration by other non-trivial cycle classes on Σ. See also Zachos [30] for $g = 1$ and two points.

5.3.2. *Genus $g = 0$, two points.* This is the classical situation. By some isomorphism of $\Sigma = S^2$, the Riemann sphere, we can assume that $I = \{0\}$ and $O = \{\infty\}$ with respect to the quasi-global coordinate z. As projective connection $R = 0$ will do. In this situation our algebras are honestly graded and the elements can be given like follows

$$e_n = z^{n+1}\frac{d}{dz}, \quad n \in \mathbb{Z}, \qquad \varphi_m = z^{m+1/2}(dz)^{-1/2}, \quad m \in \mathbb{Z} + 1/2. \tag{5.40}$$

By calculating the cocycle values we obtain the well-known expressions

$$\begin{aligned} [e_n, e_m] &= (m-n)e_{m+n} + \frac{1}{12}(n^3 - n)\,\delta_n^{-m}\,t, \\ [e_n, \varphi_m] &= (m - \frac{n}{2})\,\varphi_{m+n}, \\ [\varphi_n, \varphi_m] &= e_{n+m} - \frac{1}{6}\,(n^2 - \frac{1}{4})\,\delta_n^{-m}\,t. \end{aligned} \tag{5.41}$$

In fact, all higher order terms in the calculations above are now exact not only up to higher order. This means that there is no reference to boundedness needed and the statements are true for all cocycles. This is the same as for the vector field algebra. Note also, that the subspace

$$\langle e_{-1}, e_0, e_{-1}, \varphi_{-1/2}, \varphi_{1/2}\rangle \tag{5.42}$$

is a finite-dimensional sub Lie superalgebra. It consists of the global holomorphic sections of $\mathcal{F}^{-1}$ and $\mathcal{F}^{-1/2}$. Restricted to this subalgebra the cocycle vanishes.

5.3.3. *Genus $g = 0$, more than two points.* Here our algebra will not be graded anymore, but only almost-graded. Different splittings give different separating cycles and non-equivalent central extensions. The $N = 3$ situation was studied by Kreusch [12] and the central extension was calculated independently of this work. The case $N = 3$ is somehow special. If we fix the 3 points then we have 3 essentially different splitting into I and O. Hence we also have 3 non-equivalent different central extensions. If we fix one splitting then by a biholomorphic mapping of S^2 any other splitting can be mapped to this one. Such a mapping induces an automorphism of the algebra $\mathcal{S}$ (and of course also of $\mathcal{L}$). Hence, the obtained central extensions will be isomorphic (but not equivalent).

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