Travaux mathématiques

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Special Issue
based on the Commemorative Colloquium
dedicated to Nikolai Neumaier Mulhouse,
France, Juin 2011

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Editors
Martin Bordemann
Kurusch Ebrahimi-Fard
Abdenacer Makhlouf
Martin Schlichenmaier
Stefan Waldmann

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Commemorative Colloquium
dedicated to

Nikolai Neumaier

June 16 – 18, 2011

LMIA, Université de Haute-Alsace
Mulhouse, France

Edited by

Martin Bordemann
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Commemorative Colloquium Nikolai Neumaier
LMIA, Université de Haute-Alsace, June 16 – 18, 2011
Mulhouse, France

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Preface

The following volume constitutes the Proceedings of the "Nikolai Neumaier Commemorative Colloquium 2011" which took place at the Université de Haute Alsace in Mulhouse, France, from 16th to 18th of June 2011. It was organised by the Mathematics Departments of Mulhouse and of the University of Luxembourg. The contributions are related to Nikolai's work in Mathematical Physics where each author was lead by the principle to write something about a subject Nikolai would have liked to listen to.

Nikolai Neumaier died on the 20th of March 2010 at the age of only 38 having finally lost his courageous fight against the cancer he was suffering from for two years. On this sad occasion, the Physics Department of Freiburg University – where Nikolai had been working until his death – had organised a Gedächtniskolloquium on the 13th of December 2010. Thereafter, several of his former colleagues and coworkers had the idea to come together to a larger conference to commemorate Nikolai and try to build on the fruitful research of this brilliant young mathematical physicist. We hope that these proceedings contribute to fulfill this task.

The Editors

Martin Bordemann
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July 2012

Nikolai Neumaier 1971 – 2010

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Nikolai Neumaier

1 Introduction

It is a sad task to remind of the scientific work of one of our most valuable colleague, coworker and truly outstanding friend: Nikolai Neumaier. His life and his scientific career ended after a long but in the end hopeless fight in 2010: though struggling with cancer for almost two years, Nikolai kept working till the very last days of his only too short life.

In this note we try to collect the main results and achievements of Nikolai's scientific oeuvre, knowing well that a personality like his is much too rich to be recalled or described in such an insufficient way. All his research centered around the mathematical description of quantization using the notions of star products and deformation theory. Instead of commenting on his papers separately, we try to group them under certain topics: quantization of cotangent bundles, classification results, quantization and symmetries, as well as applications of deformation quantization to various topics in mathematical physics.

When Nikolai joined the little deformation quantization group around Martin Bordemann and Stefan Waldmann within the research division chaired by Hartmann Römer at the Physics department in Freiburg University in 1997 to work on his Master thesis, he already had a reputation as an excellent physics student who also was mathematically well-trained, in particular in differential geometry. He also seemed to have liked the unique atmosphere and 'style' in that group: in those days those unfortunate mathematical physicists having stranded in a physics department constantly had to defend their 'funny' approaches as still being 'physical' while on the same time feeling of course superior to their true physics colleagues: for the former 'understanding' meant to establish deep truths about physics in utmost mathematical precision, whereas the latter were despised since they were apparently content with parroting their teachers' half-wisdoms. Of course this was a desperate battle doomed to be lost –there is presently no more mathematical physics in Freiburg–, but on the other hand highly entertaining and enjoyable, in particular for people having a refined sense of humour like Nikolai. Very quickly Nikolai had developed his own characteristic style of research: in his problem solving mode he would silently listen with great attention, and in case the problem seemed interesting, murmuring "Ich glaube, das muss man einfach mal tun" he would dive away and resurface, say two weeks later with the problem completely solved. Nikolai always needed a silent retreat where he could concentrate on his work: for a long time he did everything to keep his office on the top of the ten storey physics building within a rather draughty, in the wintertime almost unheatable penthouse. Nikolai's theory making mode functioned similarly: carefully studying interesting literature on which he –rather occasionally– had very precise and never trivial questions, he would finally emerge from his retreat with an interesting problem to look at. He found both subjects of his PhD-thesis in his own very autonomous way.

Beside his research Nikolai will always be remembered as a talented university teacher: already during is PhD thesis he supervised exercise groups at the physics department of the University of Freiburg. His group was known to be challenging and sometimes certainly more difficult than others, Nevertheless, it was always overcrowded since the students liked his clear way of explaining, his patience with everybody as long as he was willing to learn, and his supportive help whenever needed. Later on in his career, his own lectures were well accepted by students and not few profited essentially from his clever choice of topics. It was certainly one of the very characteristics of Nikolai that he intended to make his point clear without compromises: in his lectures, he had the very ambition to present every proof in such a detail that students really were able to understand. A cheap excuse like "this is easy to see" was never heard from him. His teaching had an enormous importance for him, in particular during his illness: Only a few weeks before his death, when he was not able to give his lecture personally, he still prepared notes for the course such that Svea Beiser could substitute him and give the lecture at his place.

Above all, and everybody knowing Nikolai will agree, he was a very honourable person; modest and sometimes almost shy, but generous and helpful in any possible way.

2 Quantization of Cotangent Bundles

After starting as a diploma student in the group of Hartmann Römer in Freiburg, Nikolai was given the task to investigate the relation of a standard-ordered quantization based on a chosen connection on a configuration space manifold Q with the still to be found Weyl-ordered analog. This was part of a project about the deformation quantization of cotangent bundles using a connection and a smooth volume density as geometric ingredient. Very typical for him, he disappeared for a couple of days and came back with the complete solution. He computed the adjoint of the standard-ordered quantized symbol with respect to a not necessarily covariantly constant integration density explicitly in terms of his operator N, being the exponential of a d'Alembert-like operator on the cotangent bundle T^*Q associated to the canonical, maximally indefinite metric originating from the choice of the connection on Q, see eqn (20) in [3] and eqn (1.8) of [2]. From there on it was clear that Nikolai was not just another student to be supervised, but that we have found a collaborator on equal footing. In the course of his diploma thesis [13] this resulted in two papers on star products on cotangent bundles [3, 4].

Throughout his career, Nikolai came back to cotangent bundles once in a while and transferred many ideas and techniques to other more general situations. Started as a follow-up project we joined forces with Markus Pflaum in [2], where various generalizations including magnetic terms on cotangent bundles (in order to incorporate the Aharonov-Bohm effect and magnetic monopoles) as well as analytic aspects were investigated. One main theme of this paper was to view a (global) symbol calculus for differential operators as a representation of a star product algebra and thus the study of the representation theory of the star product algebras appeared to be of crucial importance. Nikolai came back to this point of view at several other occasions.

In the work [10], the interplay of quantization on cotangent bundles on the one hand and phase space reduction of cotangent bundles with respect to symmetries of the configuration space on the other hand were investigated. The result was a very explicit construction of a star product on the cotangent bundle of the quotient (with respect to a symplectic form involving magnetic terms in the case of a non-vanishing value of the momentum map) supporting the meta-theorem of "quantization commutes with reduction".

Also the paper [19] on the deformation quantization of Lie algebroids can be seen as originating from Nikolai's early investigations on cotangent bundles: here the essence of the Fedosov construction used in[3, 4] is distilled in such a way that it becomes applicable to the dual of a Lie algebroid as a generalization of a cotangent bundle. Even though the Poisson structure is far from being symplectic, the homogeneity arguments used in the case of cotangent bundles can be transferred to this situation.

3 Classification Results

Being part of his PhD thesis [14], Nikolai used the Fedosov construction of star products on symplectic manifolds and adapted it to various more specific situations. In particular, the question was raised how the different classification schemes of symplectic star products by formal series in the second de Rham cohomology fit together. Transferring ideas of Deligne (1995) and in particular the ν-Euler derivations from Gutt and Rawnsley (1999) to the Fedosov framework, Nikolai provided an elegant proof for a fact stated by Deligne, namely that the Deligne class (now called the characteristic class $c(\star)$) of a symplectic star-product coincides with the Fedosov class appearing in the Fedosov construction, up to a trivial and convention-depending rescaling [15]. Moreover, he showed how certain additional requirements important in physics, like the existence of a *-involution, can be understood in terms of corresponding properties of the characteristic class $c(\star)$.

On Kähler manifolds there were at that time two essentially different constructions: the local & glueing construction by Karabegov and the Fedosov-inspired construction using the Kähler connection. In both constructions a characteristic class was implementable, and Nikolai showed how the two resulting parametrizations by formal series of closed two-forms of type $(1,1)$ match together [16].

In a very remarkable paper [17] deformation quantizations of the convolution algebra associated to a proper étale Lie groupoid were studied. This is one of the main tools to understand the deformation quantization of symplectic spaces with not too bad singularities like symplectic orbifolds. Among many other things, a major result is the classification of trace functionals for this quantization in terms of cyclic (co)homology computations.

Still within the main theme of classification results one can also locate the work [18]. Here the setting was again the realm of Kähler manifolds on which particular Morita equivalence bimodules were constructed. The bimodule structure has the property of "separation of variables". Moreover, the relation between the canonical Wick, Weyl, and anti-Wick star product on a Kähler manifold are revealed.

4 Symmetries and Quantization

Symmetries in form of Lie group actions or Lie algebra actions attracted Nikolai's imagination from the very first steps, still in his diploma thesis. There he related the general quantization of a cotangent bundle with connection for the case of a Lie group $Q = G$ with Gutt's construction of a star product on T^*G, showing that the two star products coincide once the $\frac{1}{2}$-commutator connection is chosen. Apart from his thesis these results have appeared in [3].

Later on, together with Michael Müller-Bahns he investigated invariant star products on symplectic manifolds with invariant connection by means of the Fedosov construction. In [12, 11] they found several existence and classification results on quantum momentum maps. The ideas also influenced the work on reduction [10].

Being already seriously ill, Nikolai kept working on his last project, the classification of invariant star products on symplectic manifolds up to equivariant Morita equivalence. The resulting paper will appear now posthumous [8].

Encoding the gauge symmetries in gauge field theory, principal bundles are the core ingredient. For many reasons one is interested in finding analogs of principal bundles in a more noncommutative setting. In [5] the deformation quantization of principal bundles over a Poisson manifold with star product was defined to be a (right) module deformation of the functions on the total space, invariant under the principal action of the structure group. More generally, a deformation quantization of a surjective submersion was defined as a (right) module deformation. Then the main result is that such a deformation always exists and is unique up to equivalence. The main step in the proof is an explicit computation of the relevant Hochschild cohomologies: they simply vanish, making the deformation problem trivially solvable.

5 Applications to Mathematical Physics

Even though this section title does not seem to be very specific, it should emphasize that it always was Nikolai's main concern not just to produce fine and interesting mathematical results, but also mathematics relevant for physics. In this sense he was a true mathematical physicist. His main motivation for his work came from the quest of a good mathematical description of the quantization problem. In this spirit several papers can be seen as applications of the techniques from deformation quantization to more physical questions.

The papers [6, 7] focus on the idea that the spacetime we are living in is not a smooth manifold at all length scales. Instead, moving to smaller scales it becomes "noncommutative" and thus a much more complicated object. Inspired by noncommutative geometry, many people have tried to formulate a noncommutative spacetime using, among other techniques, star products. In [6, 7] a C^*-algebraic approach to locally noncommutative spacetimes is proposed and investigated. As a follow-up project in [9] the deformation of states was studied: for a C^*-algebraic deformation quantization a la Rieffel it is shown that a state, i.e. a positive linear functional, of the undeformed algebra can be deformed into a state for all the deformed algebras, depending continuously on the deformation parameter $\hbar$. From a physical point of view this is a crucial observation as only this way a classical limit is consistent: every classical state is the classical limit of (non-unique) quantum states.

Finally, in [1] open systems are studied by techniques of deformation quantization. Here an open system is understood as arising from a Cartesian product of the system and the "bath" with a coupled dynamics. The important question was then how the positivity of states and the complete positivity of the time evolution can be restored by deformation. It turns out that there is an affirmative answer.

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Polynomial separating overgroups

by Didier ARNAL

Abstract

We first prove that the moment map for a unitary representation of a Lie group G, defined by N. J. Wildberger is a geometric moment map, coming from a strongly Hamiltonian action of G on a real Fréchet symplectic manifold. Then we define a Fréchet Lie group $\widetilde{G}$, containing G and extensions of each irreducible unitary G-representation into an affine $\widetilde{G}$-action, whose moment map characterizes the unitary representation.

Then we look for construction of overgroups G^+, *i. e.* Lie groups containing G and extensions of the generic coadjoint orbits, resp. unitary representations of G to corresponding objects for G^+, by using a quadratic function. We consider here the cases G nilpotent, for which this is possible if $\dim G \leq 7$, and the case $G = SL(n, \mathbb{R})$, where such a construction exists only if $n = 2$.

1 Introduction

This lecture is the presentation of a common work, with Mohamed Selmi and Amel Zergane from the Sousse University (Tunisia).

N. J. Wildberger introduced the moment map for a unitary representation $(\mathcal{H}, \pi)$ of a Lie group G (see [17, 18]). He proved that, for compact Lie group and irreducible π, the range of this map characterizes the representation π (see also [14]). Then he studied the nilpotent case for which, if π is irreducible, the range is the closed convex hull of the coadjoint orbit $\mathcal{O}$, associated to π. But they are example of distinct orbits having same convex hull.

In a series of papers ([4, 9, 3, 1]), L. Abdelmoula, A. Baklouti, J. Ludwig, M. Selmi, and myself try to characterize the representation π through an extension of the moment map to the whole universal enveloping algebra $\mathfrak{A}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G.

In the last period, with Mohamed Selmi and Amel Zergane, (see [5, 6, 7, 19]) we prefer to extend the moment map to a larger group G^+, containing G. We call such a group an overgroup for G. We expect to keep in this extension a geometric interpretation of the moment map. But we hope also to define these extensions by using mapping as regular as possible, that means polynomial mapping and, if it is possible, quadratic mappings.

Here, we first prove that the Wildberger moment map is a geometric moment map, coming from a strongly Hamiltonian action of G on a Fréchet symplectic manifold, namely $(\mathbb{P}\mathcal{H}^\infty)_\mathbb{R}$. This needs good definitions for Fréchet differential calculus and Fréchet manifolds. Then we can define Fréchet Lie groups, and build what we call the universal overgroup for G, which is a Fréchet Lie group $\widetilde{G}$, containing G and extensions of the irreducible unitary G-representations π into affine actions $\widetilde{\pi}$, such that the moment map of $\widetilde{\pi}$ characterizes π.

In a second part of the lecture, we look for the construction of Lie overgroups G^+, semi direct product $G^+ = G \rtimes V$, where V is a finite dimensional module and extensions of the generic coadjoint orbits, resp. generic unitary representations of G to corresponding objects for G^+, by using a polynomial or even a quadratic function $\varphi : \mathfrak{g}^\star \longrightarrow (\mathfrak{g}^+)^\star$. We want to present here the main results in this direction, thus we will restrict ourselves to the two cases G nilpotent, connected and simply connected, and $G = SL(n,\mathbb{R})$.

In the first case, we present different constructions for such quadratic overgroups, holding for different classes of nilpotent groups G. Then we prove that if G is nilpotent and $\dim G \leq 7$, it is possible to build such a group G^+, and extensions, which characterize the generic coadjoint orbits $\mathcal{O}$ in $\mathfrak{g}^\star$ by the convex hull of the corresponding coadjoint orbit $\mathcal{O}^+ = \varphi(\mathcal{O})$ in $(\mathfrak{g}^+)^\star$.

In the case $G = SL(n,\mathbb{R})$, since it is possible to describe explicitely finite dimensional modules, we prove this program is possible if and only if $n = 2$. Moreover, there is such a construction with a polynomial mapping φ, with $\deg(\varphi) = n$, thus there is a cubic overgroup for $n \leq 3$, but not for $n = 4$.

2 Moment for a representation

2.1 Linear and projective actions

Let G be a Lie group and $(\mathcal{H},\pi)$ a unitary representation of G. Let us first suppose that $\mathcal{H}$ is a finite dimensional vector space. Then the underlying real space $\mathcal{H}_\mathbb{R}$ is a symplectic vector space for the form:

$$\omega^{\mathcal{H}}(w_1,w_2) = \mathfrak{Im}(w_1|w_2).$$

Of course, this form $\omega^{\mathcal{H}}$ is invariant under the linear action $(g,v) \mapsto \pi(g)v$.

Similarly, the real manifold $(\mathbb{P}\mathcal{H})_\mathbb{R}$ (*i. e.* the complex projective space considered as a real manifold) is a symplectic manifold:

First it is a smooth manifold, equipped with the following local charts φ_v. Let v be any non vanishing vector in $\mathcal{H}$, put

$$U_v = \{[w] \in \mathbb{P}\mathcal{H}, \text{ such that } (v|w) \neq 0\},$$

and

$$\varphi_v : U_v \longrightarrow (v^\perp)_\mathbb{R}, \qquad \varphi_v : [w] \mapsto \|v\|^2 \frac{w}{(v|w)} - v.$$

These maps are smooth, bijective and $\varphi_v^{-1}(u) = [v+u]$. The family of all these charts form an atlas for the manifold $(\mathbb{P}\mathcal{H})_\mathbb{R}$, moreover a direct computation shows that the formulas:

$$\omega_{[v]}^{\mathbb{P}\mathcal{H}}(W_1, W_2) = \mathfrak{Im}\frac{(d\varphi_v(W_1)|d\varphi_v(W_2))}{\|v\|^2}$$

defines a smooth 2-form on the manifold $(\mathbb{P}\mathcal{H})_\mathbb{R}$, and this form is closed and everywhere non degenerated.

Now, G acts on $(\mathbb{P}\mathcal{H})_\mathbb{R}$ by $(g,[v]) \mapsto [\pi(g)v]$, this action is smooth, and $\omega^{\mathbb{P}\mathcal{H}}$ is invariant.

These two actions, the linear and the projective one, are strongly Hamiltonian: for each X in the Lie algebra $\mathfrak{g}$ of G, there is a smooth function J_X on $\mathcal{H}_\mathbb{R}$ (resp. on $(\mathbb{P}\mathcal{H})_\mathbb{R}$) such that, for any smooth function f,

$$\{J_X, f\} = \frac{d}{dt}\Big|_{t=0} f(\pi(\exp -tX)\cdot),$$

moreover these functions J_X can be chosen in such a manner that, for any X and Y,

$$\{J_X, J_Y\} = J_{[X,Y]}.$$

The corresponding moment maps ψ_π and Ψ_π are the following mappings, $\psi_\pi : \mathcal{H}_\mathbb{R} \longrightarrow \mathfrak{g}^\star$ (resp. $\Psi_\pi : (\mathbb{P}\mathcal{H})_\mathbb{R} \longrightarrow \mathfrak{g}^\star$):

$$J_X(v) = \langle \psi_\pi(v), X\rangle = \frac{1}{2i}(\pi'(X)v|v),$$

$$\Big(\text{resp. } J_X([v]) = \langle \Psi_\pi([v]), X\rangle = \frac{1}{2i}\frac{(\pi'(X)v|v)}{\|v\|^2}\Big).$$

2.2 Moment set for π

In any case, *i. e.* $\dim\mathcal{H}$ finite or infinite, N. J. Wildberger ([17, 18]) defined

Definition 2.1. The moment set $\mathcal{I}_\pi$ for π is the closure in $\mathfrak{g}^\star$ of the set:

$$\{\Psi_\pi([v]), \quad v \in \mathcal{H}^\infty \setminus \{0\}\}.$$

Remark this map is something like a dequantization procedure. Indeed, suppose for instance G is an exponential Lie group, then there is an one-to-one, onto map from the space $\mathfrak{g}^\star/G$ of coadjoint orbit in $\mathfrak{g}^\star$ and the unitary dual $\widehat{G}$ of the group G ([10]). This map can be considered as a quantization of each coadjoint orbit $\mathcal{O}$ in $\mathfrak{g}^\star$ ([13]).

Here, we consider a map in the "opposite" direction, from $\widehat{G}$ to the family $\mathcal{C}$ of closed subsets in $\mathfrak{g}^\star$. Unfortunately, the map $\pi \mapsto \mathcal{I}_\pi$ is not directly the inverse of the map $\mathcal{O} \mapsto \pi$.

Anyway, let us recall some known facts about this map:

- If G is compact, the map $\pi \mapsto \mathcal{I}_\pi$ is one-to-one: the moment set $\mathcal{I}_\pi$ characterizes the representation π ([14, 17]),
- If G is solvable, $\mathcal{I}_\pi$ is convex, generally, $\pi \mapsto \mathcal{I}_\pi$ is not injective ([4]),
- Very generally, for irreducible π, $\mathcal{I}_\pi = \overline{\mathrm{Conv}}(\mathcal{O})$, the closed convex hull of a coadjoint orbit $\mathcal{O}$, but, even if G is nilpotent, $\mathcal{O} \mapsto \mathrm{Conv}(\mathcal{O})$ is not one-to-one ([18]),
- If we extend the map Ψ_π to the universal enveloping algebra $\mathfrak{A}(\mathfrak{g})$ of $\mathfrak{g}$, these extensions define an injective mapping from $\widehat{G}$ to $(\mathfrak{A}(\mathfrak{g}))^\star$ ([1]).

2.3 Schedule of the lecture

Today, we look for two different goals.

1. We want to view the Wildberger maps ψ_π and Ψ_π as true geometric moment maps. This needs use of infinite dimensional manifolds, on Fréchet spaces. Therefore, we look for Fréchet differential calculus and Fréchet manifolds as defined by R. S. Hamilton in [12].

   Then, with this notion of Fréchet manifolds, we can build a Fréchet Lie-group, semi-direct product $\widetilde{G} = G \rtimes V$ of G with a real Fréchet vector space, and extend each unitary irreducible representation π of G to an affine action $\widetilde{\pi} = \Phi(\pi)$ of $\widetilde{G}$, which is Hamiltonian and such that the moment $\mathcal{I}_{\widetilde{G}}$ for this action does characterize π: we say that $(\widetilde{G}, \Phi)$ is a $\widehat{G}$-separating overgroup of G.

   This construction is very general, but uses a very "large" infinite dimensional Fréchet Lie group $\widetilde{G}$, we call this group, the universal overgroup for G.

2. Of course, we prefer to work with (finite dimensional) Lie groups, so we will restrict ourselves to Lie overgroups, semi-direct products $G^+ = G \rtimes V$ and look to the existence of a polynomial mapping $\phi : \mathfrak{g}^\star \longrightarrow V^\star$ such that, if $\varphi(\ell) = (\ell, \phi(\ell))$, we get, for any generic ℓ in $\mathfrak{g}^\star$:

   $$\varphi(G{\cdot}\ell) = G^+{\cdot}\varphi(\ell), \quad \text{and} \quad \mathrm{Conv}(\varphi(G{\cdot}\ell)) = \mathrm{Conv}(\varphi(G{\cdot}\ell')) \Longrightarrow G{\cdot}\ell = G{\cdot}\ell'.$$

   If such a ϕ exists, we say that (G^+, φ) is a polynomial $\mathfrak{g}^\star/G$-separating overgroup for G. If we get separation for a large subset $\mathfrak{g}^\star_{gen}/G$ of $\mathfrak{g}^\star/G$, we say that (G^+, φ) is a polynomial $\mathfrak{g}^\star_{gen}/G$-separating overgroup for G.

   If it is possible to choose ϕ, with $\deg(\phi) \leq 2$, we just say that G admits a quadratic overgroup, or (G^+, φ) is a quadratic overgroup for G.

Existence of such overgroups seems to be a very restrictive condition for the structure of the group G, but we shall present here the nilpotent case, for which the quadratic condition is in fact not too restrictive and the $SL(n, \mathbb{R})$ case, for which the quadratic condition is very strict.

3 Fréchet differential geometry

3.1 Fréchet differential calculus

Let U be an open subset in a real Fréchet vector space V and W another real Fréchet space. Following R. S. Hamilton ([12]), we say that a continuous function $f : U \longrightarrow W$ is derivable in the direction $h \in V$ if the following limit exists:

$$\lim_{t\to 0} \frac{1}{t}\left(f(u+th) - f(u)\right) = Df(u)(h) = \langle \nabla f|_u, h\rangle.$$

We shall say that f is C^1 if $(u,h) \mapsto Df(u)(h)$ is defined and continuous from $U \times V$ into W. Similarly, f is C^2 if, for any h_1, $u \mapsto Df(u)(h_1)$ is continuous, derivable in any direction h_2, and $(u,h_1,h_2) \mapsto (D^2f)(u)(h_1,h_2)$ is continuous from $U \times V \times V$ into W, and so one ...

The Schwarz lemma, the chain rule hold, but there is no local inverse function theorem as the following example (see [12]) shows:

Let V be the space $C^\infty(\mathbb{R})_1$ of smooth real functions u on $\mathbb{R}$, periodic, with period 1 (with its natural Fréchet topology). Let $F : \mathbb{R} \times V \longrightarrow V$ be the function defined by:

$$(t,u) \mapsto F(t,u)(x) = \int_0^1 u(x+ts)\, ds.$$

We can verify that F is a C^∞ map, we have $F(0,0) = 0$ and $F(0,u) = u$. Then the partial derivative $\frac{DF}{Du}|_{u=0}$ is the invertible map id_V. But fix n, and put, for any k, $u_k(x) = \sin(2kn\pi x)$. We get for any k

$$F(\frac{1}{n}, u_k) = 0,$$

and $u \mapsto F(\frac{1}{n}, u)$ is not one-to-one. If n varies, there is no open subset containing 0 such that the equation $F(t,u) = 0$ has an unique solution.

3.2 Fréchet symplectic manifolds

A Fréchet manifold is a Hausdorff space with local charts $\varphi_i : U_i \longrightarrow E$ (where E is a Fréchet space), such that

Each φ_i is a homeomorphism from U_i onto an open subset in E, and,

For any i and j, $\varphi_j \circ \varphi_i^{-1}$ is a homeomorphism and a C^∞ map between two open subsets in E.

We define C^∞ functions, vector fields and forms on a Fréchet manifold as for a finite dimensional manifold: the smoothness of these quantities can be tested in each local chart.

Let us now come to the case $V = (\mathcal{H}^\infty)_\mathbb{R}$, the real space of C^∞ vectors for a unitary representation $(\mathcal{H}, \pi)$ of the Lie group G. Put

$$\omega^V(w_1, w_2) = \mathfrak{Im}(w_1|w_2).$$

It is a non degenerated, bilinear, antisymmetric form, constant, thus C^∞. Consider now the flat-map:

$$\flat : V \longrightarrow V^\star, \qquad \langle u^\flat, v\rangle = \mathfrak{Im}(u|v).$$

Denote $V^\flat$ the range of this map. On $V^\flat$, we put the topology coming from V through the bijective map $\flat$. If $\dim \mathcal{H}$ is infinite, then $V^\flat \subsetneq V^\star$.

Take for instance for G the Heisenberg group, and $(\mathcal{H}, \pi)$ the usual unitary irreducible representation of G, onto $\mathcal{H} = L^2(\mathbb{R})$. Then V is the Schwartz space $\mathcal{S}(\mathbb{R})$, and

$$V^\flat = \mathcal{S}(\mathbb{R}) \neq V^\star = \mathcal{S}'(\mathbb{R}).$$

Therefore, we define Hamiltonian functions as:

Definition 3.1. A C^∞ function on V is a Hamiltonian function, if ∇f is C^∞ from V into $V^\flat$.

Suppose $V = \mathcal{S}(\mathbb{R})$, as above and f is the linear function $f : u \mapsto u(0)$, f being linear and continuous is C^∞, but it is not a Hamiltonian function, since $\nabla f = \delta_0$ is the Dirac distribution.

Denote now $\mathbb{P}V$ the set $(\mathbb{P}\mathcal{H}^\infty)_\mathbb{R}$. With the local charts (U_v, φ_v) defined as above, it is a C^∞ manifold, we put

$$\omega^{\mathbb{P}V}_{[v]}(W_1, W_2) = \frac{\mathfrak{Im}(d\varphi_v(W_1)|d\varphi_v(W_2))}{\|v\|^2}.$$

This formula defines a well defined, C^∞ 2-form on $\mathbb{P}V$. This 2-form is non degenerated at any point and it is closed:

We just say that $\mathbb{P}V$ is a symplectic Fréchet manifold.

We say that a C^∞ function on $\mathbb{P}V$ is Hamiltonian if it is Hamiltonian in each local chart.

3.3 Linear and projective actions

Consider now the linear and projective actions of G on the symplectic Fréchet manifolds V and $\mathbb{P}V$.

Proposition 3.2. *([8])*

These two actions preserves the corresponding forms, moreover they are strongly Hamiltonian with the Wildberger Hamiltonian functions:

$$v \mapsto \langle \psi_\pi(v), X\rangle, \qquad [v] \mapsto \langle \Psi_\pi([v]), X\rangle.$$

Indeed, a direct computation shows these functions are well defined, C^∞ and Hamiltonian functions, they generate vector fields $\flat^{-1}(\nabla\psi_\pi)$, respectively $\flat^{-1}(\nabla\Psi_\pi)$ which are the infinitesimal generators for the linear, respectively projective actions.

3.4 Fréchet Lie group, universal overgroup

Let us come to the notion of Fréchet Lie group. It is a smooth manifold and a group, such that the maps $(g_1, g_2) \mapsto g_1g_2$ and $g \mapsto g^{-1}$ are C^∞.

For instance, if $(\mathcal{H}, \pi)$ is a unitary representation of G, if $V = (\mathcal{H}^\infty)_\mathbb{R}$, thus $G \rtimes V$ is a Fréchet Lie group, with $\mathfrak{g} \rtimes V$ as Lie algebra. The product and the Lie bracket are:

$$(g_1, v_1)(g_2, v_2) = (g_1g_2, v_1 + \pi(g_1)v_2),$$

and

$$[(X_1, v_1), (X_2, v_2)] = ([X_1, X_2], \pi'(X_1)v_2 - \pi'(X_2)v_1)\,.$$

Consider now the space $\mathcal{H} = \oplus_{\pi\in\widehat{G}}\mathcal{H}_\pi$ and $V = (\mathcal{H}^\infty)_\mathbb{R}$, denote $\widetilde{G}$ the group $G \rtimes V$.

For any π_0 in $\widehat{G}$, we note p_{π_0} the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_{\pi_0}$. We finally extend the linear action π_0 of G into an affine action of $\widetilde{G}$ on $V_{\pi_0} = (\mathcal{H}^\infty_{\pi_0})_\mathbb{R}$ by putting:

$$\widetilde{\pi}_0(g, u)v = \pi_0(g)v + p_{\pi_0}(u) \qquad\qquad (u \in V, \quad v \in V_{\pi_0}).$$

This affine action is Hamiltonian, but not strongly Hamiltonian, its moment map (vanishing at the identity) is

$$\psi_{\widetilde{\pi}_0}(v)(X, u) = \frac{1}{2}\mathfrak{Im}(\pi_0'(X)v|v) + \mathfrak{Im}(p_{\pi_0}(u)|v).$$

Denote $\mathcal{I}_{\widetilde{\pi}_0}$ the corresponding moment set and q the canonical projection from $\widetilde{\mathfrak{g}}^\star$ onto $\mathfrak{g}^\star$. Then

$$q(\mathcal{I}_{\widetilde{\pi}_0}) = \mathcal{C}(\mathcal{I}_{\pi_0}) = \text{ Cone with base } \mathcal{I}_{\pi_0}.$$

Finally, $\mathcal{I}_{\widetilde{\pi}_0}$ characterizes the representation π_0:

$$\mathcal{I}_{\widetilde{\pi}_1} = \mathcal{I}_{\widetilde{\pi}_2} \implies \pi_1 = \pi_2.$$

So we get

Theorem 3.3. *([8])*

The Fréchet Lie group $\widetilde{G}$ and the extensions $\Phi : \pi \mapsto \widetilde{\pi}$ of the linear actions π to affine actions $\widetilde{\pi}$ define a universal overgroup for G, i. e. a Fréchet Lie group, which is $\widehat{G}$-separating.

4 Polynomial Lie overgroups

Recall we are looking for Lie groups of the form $G^+ = G \rtimes V$ where $\dim V$ is a finite dimensional vector space, and polynomial mapping $\phi : \mathfrak{g}^\star \longrightarrow V^\star$ (thus the polynomial map $\varphi(\ell) = (\ell, \phi(\ell))$ from $\mathfrak{g}^\star$ into $(\mathfrak{g}^+)^\star$), such that, for any generic ℓ_i in $\mathfrak{g}^\star$,

$$\varphi(G \cdot \ell_1) = G^+ \cdot \varphi(\ell_1),$$

and

$$\mathrm{Conv}(G^+ \cdot \varphi(\ell_1)) = \mathrm{Conv}(G^+ \cdot \varphi(\ell_2)) \quad \Longrightarrow \quad G \cdot \ell_1 = G \cdot \ell_2,$$

we say that (G^+, φ) is a polynomial $\mathfrak{g}^\star_{gen}/G$-separating overgroup for G.

Let us consider the two 'extremal' cases G nilpotent and $G = SL(n, \mathbb{R})$.

4.1 The nilpotent case

Suppose now G is nilpotent, connected and simply connected. Let us first define generic points ℓ in $\mathfrak{g}^\star$. Fix a Jordan-Hölder basis $(e_1, \dots, e_n)$ in $\mathfrak{g}^\star$, with respect to the coadjoint action, for any subset K of $\{1, \dots, n\}$, denote V_K the vector space generated by the e_k, for k in K.

For each ℓ in $\mathfrak{g}^\star$, there is a subset $J \subset \{1, \dots, n\}$, called the set of jump indices for the orbit $G \cdot \ell$, they are the direction where the orbit growths. More precisely (see [16]), if $J' = \{1, \dots, n\} \setminus J$,

1. The restriction, to the orbit $G \cdot \ell$, of the projection onto V_J, parallel to $V_{J'}$ is a global diffeomorphism: $G \cdot \ell \simeq V_J$,

2. The intersection $G \cdot \ell \cap V_{J'}$ is a singleton: $G \cdot \ell \cap V_{J'} = \{\lambda(\ell)\}$.

Consider now the subset $\mathfrak{g}^\star_{gen}$ of the points ℓ in $\mathfrak{g}^\star$, such that J is minimal for the lexicographic ordering. This set is an invariant, dense, Zariski open subset in $\mathfrak{g}^\star$. On this set, the function λ is rational, and take the form:

$$\lambda(\ell) = \sum_{k \notin J} \frac{P_k(\ell)}{Q_k(\ell)} e_k.$$

The functions P_k are in fact invariant polynomial functions and they generate the field $R(\mathfrak{g})$ of rational invariant functions on $\mathfrak{g}^\star$:

$$R(\mathfrak{g}) = \mathbb{R}(P_k).$$

Thanks to this well known construction (see for instance [16]), we re-find here a result of Baklouti, Ludwig and Selmi in [9]:

Proposition 4.1. *Put $\phi(\ell) = \sum_{k \notin J} P_k(\ell) e_k$, then $(G \times V_{J'}, \varphi)$ (the action of G on $V_{J'}$ is trivial) is a polynomial $\mathfrak{g}^\star_{gen}/G$-separating overgroup for G.*

Since $\widehat{G}$ is homeomorphic to the space of orbits in $\mathfrak{g}^\star$, we can write this at the level of representations:

Define $\Phi : (\widehat{G})_{gen} \longrightarrow \widehat{G^+}$ as the map associating to each generic, irreducible unitary representation π of G (π is associated to a generic orbit) its extension to G^+ defined by $\pi'(e_k) = iP_k(\ell)$. Then (G^+, Φ) is a $(\widehat{G})_{gen}$-separating overgroup for G.

Especially, if the maximum of the degree of the P_k is at most 2, G admits a quadratic overgroup.

On the other hand, there is a different way to build quadratic overgroup for G. Let us present here this method:

1. We say that G is *special* if there is an abelian ideal $\mathfrak{a}$ in $\mathfrak{g}$, with codim $\mathfrak{a} = 1/2\ \#J$.

2. If G is special, then its generic coadjoint orbits are fibre bundles over the G-orbit in $\mathfrak{a}^\star$, with $\mathfrak{a}^\perp$ as fiber. Thus, we can rebuild the orbit, starting with the G-orbit in $\mathfrak{a}^\star$.

3. Now the map $\theta : \ell|_\mathfrak{a} \mapsto (\ell|_\mathfrak{a})^2$ from $\mathfrak{a}^\star$ into $S^2(\mathfrak{a})$ is strictly convex, that means, if p is the natural projection from $\mathfrak{a}^\star \oplus S^2(\mathfrak{a})$ onto $\mathfrak{a}^\star$ and $\vartheta(f) = (f, \theta(f))$, for any subset A in $\mathfrak{a}^\star$ (see [6]),
$$p\left(\overline{\text{Conv}}\ (\vartheta(A)) \cap \vartheta(\mathfrak{a}^\star)\right) = \overline{A}.$$

4. Consider $G^+ = G \rtimes S^2(\mathfrak{a}^\star)$, define $\varphi : \mathfrak{g}^\star \longrightarrow (\mathfrak{g}^+)^\star$ by $\varphi(\ell) = (\ell, \theta(\ell|_\mathfrak{a}))$, a direct computation shows that, for any generic ℓ, $\varphi(G \cdot \ell) = G^+ \cdot \varphi(\ell)$.

Proposition 4.2. *([6, 7])*

A special nilpotent Lie group G admits $(G^+ = G \rtimes S^2(\mathfrak{a}), \varphi)$ as quadratic overgroup. More precisely, this overgroup is $\mathfrak{g}^\star_{gen}/G$-separating.

Moreover, for any generic representation $\pi \in \widehat{G}_{gen}$, associated to the coadjoint orbit $\mathcal{O}$, if $\Phi(\pi)$ is the representation of G^+ associated to the orbit $\varphi(\mathcal{O})$, then $(G^+ = G \rtimes S^2(\mathfrak{a}), \Phi)$ is $\widehat{G}_{gen}$-separating.

Considering the classification of small dimensional nilpotent Lie algebras (see [11, 15]), we see that all the nilpotent Lie groups, with $\dim G \leq 6$, except one called $G_{6,20}$ either are special or verify $\max\deg P_k \leq 2$, thus admit a quadratic overgroup.

Finally, we generalize the spacial case, by considering the case of two ideals in $\mathfrak{g}$: $\mathfrak{a} \subset \mathfrak{b} \subset \mathfrak{g}$, with $\mathfrak{b}$ special, $\mathfrak{a}$ being the abelian ideal given in the '$\mathfrak{b}$ special' definition. We moreover suppose that, for generic ℓ, $\mathfrak{b} + \mathfrak{g}(\ell) = \mathfrak{g}$. With these hypothesis, the G orbits are diffeomorphic to the B-orbits in $\mathfrak{b}^\star$, moreover, if $\lambda_\mathfrak{g}$ and $\lambda_\mathfrak{b}$ are the functions λ defined above, but for the Lie algebras $\mathfrak{g}$ and $\mathfrak{b}$, then any convex combination of points in $G \cdot \lambda_\mathfrak{g}(\ell)$, which are such that $\ell|_\mathfrak{b} = \lambda_\mathfrak{b}(\ell|_\mathfrak{b})$ takes exactly the value $\lambda_\mathfrak{g}(\ell)$. Therefore,

Proposition 4.3. *([2])*

For a nilpotent Lie group G satisfying the above conditions, $(G^+ = G \rtimes S^2(\mathfrak{a}), \varphi)$, where $\varphi(\ell) = (\ell, (\ell|_{\mathfrak{a}})^2)$, is a quadratic overgroup. More precisely, this overgroup is a $\mathfrak{g}^\star_{gen}/G$-separating group.

Remark that we cannot separate the closure of the convex hull of the orbits, thus we cannot separate the irreducible unitary representation by their moment sets.

However, it is possible to verify, case by case, that any nilpotent Lie group G, with $\dim G \leq 7$ satisfies the hypothesis of one of our proposition, thus admits a quadratic overgroup.

Last remark: there is a 12-dimensional example of a Lie group, whose invariants are not generated by quadratic functions, which is not special, and not in the last class of groups. This example admits a quadratic overgroup.

Indeed, it is probably impossible to prove that a given nilpotent Lie group has no quadratic overgroup, just because there is no classification of finite dimensional modules for a nilpotent Lie algebra. For this reason, we consider now a totally different setting, the case $G = SL(n, \mathbb{R})$.

4.2 The $SL(n, \mathbb{R})$ case

Recall well known facts about the $SL(n, \mathbb{R})$-coadjoint orbits:

1. Thanks to the Killing form, we identify adjoint and coadjoint actions.

2. The set of generic ℓ is the set of $n \times n$ matrices, with n distinct eigenvalues. This set is dense and open in $\mathfrak{sl}(n, \mathbb{R})$.

3. If $n \geq 3$, then any generic orbit satisfies $\overline{\mathrm{Conv}}(\mathcal{O}) = \mathfrak{sl}(n, \mathbb{R})$.

4. The invariant polynomial functions on $\mathfrak{sl}(n, \mathbb{R})$ are polynomials in the functions t_k, for $k = 2, \dots, n$:
$$t_k(\ell) = Tr(\ell^k).$$

5. These functions separate almost the generic orbits, the only case where there is no separation, is the case n even, and orbits of matrices ℓ having only non real eigenvalues. In this case, there are exactly 2 orbits $G \cdot \ell_1$, and $G \cdot \ell_2$ on which the invariant functions take the same values: ℓ_1 and ℓ_2 have the same spectrum, but are conjugated through a matrix with determinant -1.

Thus we can say:

Proposition 4.4. *$SL(n, \mathbb{R})$ admits an almost $\mathfrak{sl}(n, \mathbb{R})^\star_{gen}$-separating polynomial overgroup, with degree n, namely $(G^+ = SL(n, \mathbb{R}) \times \mathbb{R}^{n-1}, \varphi)$, with*

$$\varphi(\ell) = (\ell, t_2(\ell), \dots, t_n(\ell)).$$

Now, suppose there is a quadratic overgroup for $SL(n,\mathbb{R})$, then, using semi simplicity of finite dimensional real $\mathfrak{sl}(n,\mathbb{R})$-modules, A. Zergane can prove there is such a quadratic overgroup of the form

$$G^+ = SL(n,\mathbb{R}) \rtimes (\mathfrak{sl}(n,\mathbb{R}) + S^2(\mathfrak{sl}(n,\mathbb{R}))), \qquad \phi(\ell) = b_1(\ell) + b_2(\ell,\ell),$$

where b_1 and b_2 are intertwining operators respectively for the modules $\mathfrak{sl}(n,\mathbb{R})^\star$ and $S^2(\mathfrak{sl}(n,\mathbb{R}))^\star$.

Let us now compute all these intertwining operators: if $n > 3$, there are the trace operators defined as:

$$\langle P_0(\ell), X\rangle = Tr(\ell X),$$

$$\langle P_1(\ell.\ell'), X.X'\rangle = Tr(\ell X\ell' X') + Tr(\ell X'\ell' X),$$

$$\langle P_2(\ell.\ell'), X.X'\rangle = Tr(\ell X)Tr(\ell' X') + Tr(\ell X')Tr(\ell' X),$$

$$\langle P_3(\ell.\ell'), X.X'\rangle = Tr(\ell\ell' XX') + Tr(\ell\ell' X'X) + Tr(\ell'\ell XX') + Tr(\ell'\ell X'X),$$

$$\langle P_4(\ell.\ell'), X.X'\rangle = Tr(\ell\ell')Tr(XX').$$

If $n = 3$, P_3 is a linear combination of P_1, P_2, P_4.

Finally, looking to the condition $G^+ \cdot \varphi(\ell) = \varphi(SL(n,\mathbb{R})\cdot\ell)$, A. Zergane proves the only possibilities are in fact non separating:

Proposition 4.5. *([19])*
If $n \geq 3$, $SL(n,\mathbb{R})$ does not admit any quadratic overgroup.

If $n = 2$, we found a quadratic overgroup. Similarly, $SL(3,\mathbb{R})$ admits a cubic overgroup and we can prove, with the same method, that $SL(4,\mathbb{R})$ does not admit any cubic overgroup.

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Atiyah Classes and Equivariant Connections on Homogeneous Spaces

by Martin Bordemann

Abstract

We shall give a 'pedagogical' review of G-invariant connections on not necessarily reductive homogeneous spaces, its obstructions -due to Nguyen-van Hai, 1965- leading to the Atiyah classes recently dealt with in the literature, and applications to multi-differential operators on homogeneous spaces.

Introduction

In recent years, the interest in not necessarily reductive homogeneous spaces seems to have increased: among other things, the reason is a new attack on an old problem by Michel Duflo where algebras of invariant differential operators are compared to the algebras of their symbols, see e.g. [14], [10], [11]. Moreover, in the preprint [7], Calaque, Căldăraru, and Tu observed that for any Lie algebra inclusion $\mathfrak{h} \subset \mathfrak{g}$ the $\mathfrak{h}$-module $\mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}$ is isomorphic (as a filtered module) to the $\mathfrak{h}$-module $\mathsf{S}(\mathfrak{g}/\mathfrak{h})$ iff a certain cohomology class of rank 1 in the Chevalley-Eilenberg cohomology of $\mathfrak{h}$ vanishes which they called the *Atiyah class* of the Lie algebra inclusion. For a particular case, where $\mathfrak{g}$ is the Lie algebra of all vector fields on a coisotropic submanifold C of a symplectic manifold, and $\mathfrak{h}$ is the subalgebra of all vector fields on C along the canonical foliation the author has observed that this class –which had been defined by P.Molino in 1971, see [29], [30]– was related to obstructions of the representability of a star-product on the ambient symplectic manifold on the space of smooth functions on C, see the preprint [5] and the proceeding [6]. Similar results to [7] have been extended to Lie algebroids, see [12] and [8].

The aim of this proceeding is to relate the above observations to a classical subject in the theory of homogeneous spaces, namely the question of whether a homogeneous space admits invariant connections in a G-equivariant principal bundle over the space. This had already been done in a work by H.-C. Wang, [36] in 1958, but where the cohomological nature of the existence of these connections had not explicitly been mentioned: in those days, the main focus seemed to have been the study of compact or more generally reductive homogeneous spaces for which invariant connections always exist. In 1965 Nguyen-van Hai [31] formulated the cohomological obstruction now called the Atiyah-class for the case of a linear connection which has been rediscovered in [7]. With the coadjoint orbits, a lot of examples of nonreductive homogeneous spaces have been studied. For instance, the work of Pikulin and Tevelev [33] deals with the question of invariant symplectic connections on nilpotent coadjoint orbits of reductive groups.

The main idea of the Atiyah class is very simple: let

$$\{0\} \to (\mathsf{A}, d_A) \to (\mathsf{B}, d_B) \to (\mathsf{C}, d_C) \to \{0\}$$

be an exact sequence of nonnegatively graded cocomplexes (i.e. the degree of the differentials is $+1$). According to classical homological algebra there is the associated long exact cohomology sequence

$$\{0\} \to H^0(\mathsf{A}) \to H^0(\mathsf{B}) \to H^0(\mathsf{C}) \stackrel{\text{connecting hom}}{\longrightarrow} H^1(\mathsf{A}) \to H^1(\mathsf{B}) \to \cdots$$

It is immediate that a class $[\gamma]$ in $H^0(\mathsf{C})$ lifts to a class in $H^0(\mathsf{B})$ iff its image $c_{[\gamma]}$ under the connecting homomorphism in $H^1(\mathsf{A})$, which we may call the *Atiyah class with respect to* $[\gamma]$, vanishes, see also Atiyah's original work [2]. For the important particular case where the cocomplexes are either the smooth Lie group cohomology complexes of a Lie group H or the Chevalley-Eilenberg complexes of a Lie algebra $\mathfrak{h}$ with values in a short exact sequence of H-modules (resp. $\mathfrak{h}$-modules)

$$\{0\} \to P \to Q \to R \to \{0\}$$

the above problem is the *lifting of invariants* in the quotient module R to Q. For the particular Atiyah classes in the literature, the Lie algebra $\mathfrak{h}$ is a subalgebra of a bigger Lie algebra $\mathfrak{g}$, and $\mathfrak{u}$ is a second Lie algebra. We suppose that there is a either a Lie group U having Lie algebra $\mathfrak{u}$ and a Lie group homomorphism $\chi : H \to U$, or just a morphism of Lie algebras $\dot{\chi} : \mathfrak{h} \to \mathfrak{u}$. In the first case $\mathfrak{u}$ is a H-module (via $h\zeta = Ad_{\chi(h)}(\zeta)$), and in the second case $\mathfrak{u}$ is an $\mathfrak{h}$-module via $\zeta \mapsto [\dot{\chi}(\eta), \zeta]$ for all $\eta \in \mathfrak{h}$. Then the above short exact sequence of three H-modules (resp. $\mathfrak{h}$-modules) P, Q, and R specializes to the following:

$$\{0\} \to \mathbf{Hom}_{\mathbb{K}}\big(\mathfrak{g}/\mathfrak{h}, \mathfrak{u}\big) \to \mathbf{Hom}_{\mathbb{K}}\big(\mathfrak{g}, \mathfrak{u}\big) \to \mathbf{Hom}_{\mathbb{K}}\big(\mathfrak{h}, \mathfrak{u}\big) \to \{0\}.$$

The class $[\gamma]$ in the zeroth cohomology group corresponding to $\mathbf{Hom}_{\mathbb{K}}\big(\mathfrak{h}, \mathfrak{u}\big)$ is an invariant in $\mathbf{Hom}_{\mathbb{K}}\big(\mathfrak{h}, \mathfrak{u}\big)$, namely $T_e\chi$ (in the group case) or $\dot{\chi}$ (in the Lie algebra case). In most of the literature there is the following important particular case $U = GL(\mathfrak{g}/\mathfrak{h})$ or $\mathfrak{u} = \mathfrak{gl}(\mathfrak{g}/\mathfrak{h}) = \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$, and $\chi(h) = Ad'_h$ (the induced adjoint representation in $\mathfrak{g}/\mathfrak{h}$ which is isomorphic to the tangent space of the homogeneous space G/H at the distinguished point $o = \pi(e)$ which is also called the linear isotropy representation, see [22, p.187]) or $ad'(\eta)$ (the analogous representation for the subalgebra $\mathfrak{h}$).

We shall give a review of these things which is meant -alas- pedagogical, and I am convinced that almost all the material presented here is more or less well-known, but some of it, as for instance the description of multi-differential operators or jet prolongations on homogeneous spaces is less easy to find in the literature, at least for me. In order to do this we shall formulate the underlying geometric concepts around not necessarily reductive homogeneous spaces in a mild categorical form. This is useful for the general programme relating homogeneous structures over a homogeneous space, such as vector bundles, fibre bundles, principal bundles, groupoids, etc. which mostly form a suitable category, to a very often small(er) category of the typical fibres which almost always can be expressed as an equivalence of categories. Moreover, to do at least a tiny bit of hopefully original work we shall generalize the G-invariant connections in a G-invariant principal bundle to those ones where the left G-action on the total space does no longer commute with the right action of the structure group U but is twisted by an automorphic action ϑ of G on U. The only example I found where this may be relevant is the treatment of most of the coadjoint orbits of the Poincaré group in geometric quantization: as time reversal is demanded to be antisymplectic by physicists, the connection (and hence its curvature form which is symplectic) is not fully invariant under the Poincaré group, but may change signs, whence the curvature form differs from the Kirillov-Kostant-Souriau form by a sign on some of the connected components of the orbit: this can be described by introducing the above automorphic action. I shall also describe how –on an infinitesimal (Lie algebra) level– coadjoint orbits can be generalized in the direction that given the Lie algebra $\mathfrak{g}$, given the Lie algebra of the structure group $\mathfrak{u}$, and given a linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ the subalgebra $\mathfrak{h}$ can be defined as an isotropy subalgebra of p in a certain way, see Proposition 2.9 in order to get infinitesimal G-ϑ-equivariant connections.

The second aim of these proceedings is to give a differential geometric interpretation –which I had announced a year ago– of the above-mentioned result [7] in terms of G-invariant symbol calculus on the homogeneous space because vanishing Atiyah class means to have a G-invariant linear connection in the tangent bundle of the homogeneous space.

At last I shall mention the result –that all experts in deformation quantization such as Nikolai will find completely unsurprising– that a coadjoint orbit having vanishing Atiyah class (for the tangent bundle) admits a G-invariant star-product which is clear by a result by B. Fedosov.

Notation: All manifolds are assumed to be smooth, Hausdorff and second countable. Following [23] we shall denote the category of all smooth manifolds whose morphism sets are smooth maps by $\mathcal{M}f$. In a cartesian product $M = M_1 \times \cdots \times M_n$ of sets let $\mathrm{pr}_i : M \to M_i$ denote the canonical projection on the ith factor. The symbol $\mathbb{K}$ will denote either the field of all real numbers or the field of all complex numbers. For a vector field X on a manifold M and a diffeomorphism $\Phi : M \to M$ let $\Phi^* X$ denote the pull-back $x \mapsto (T_x\Phi)^{-1}\big(X(\Phi(x))\big)$.

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1 Some Preliminaries

In the first subsection 1.1 we just recall –in some homeopathical quantities of categorical language, see e.g. [27], [20]– well-known facts on fibered manifolds, principal bundles, associated bundles, and connections dealt with for instance in [21], [22], and [23] in order to fix notation. The second subsection 1.2 treats the notion of multi-differential operators on associated bundles which seems to be a bit less well-known.

1.1 Fibered Manifolds, Principal Bundles, their Associated Bundles, and Connections

Recall that a *fibered manifold* is a triple (E, τ, M) where E (the *total space*) and M (the *base*) are smooth manifolds and $\tau : E \to M$ (the *projection*) is a smooth surjective submersion. A morphism between two fibered manifolds (E, τ, M) and (E', τ', M') is a pair of smooth maps $\Phi : E \to E'$ and $\phi : M \to M'$ (the base map) such that the obvious diagram

$$\begin{array}{ccc} E & \stackrel{\Phi}{\to} & E' \\ \tau \downarrow & & \downarrow \tau' \\ M & \stackrel{\phi}{\to} & M' \end{array} \tag{1.1}$$

commutes. The class of all fibered manifolds with their morphism sets hence forms a category denoted by $\mathcal{FM}$ by [23]. Returning to fibered manifolds, recall that –by the implicit function theorem– around any y_0 of the total space E of a fibered manifold (E, τ, M) there is a chart $\big(\mathcal{V}, \psi\big)$ and a chart $\big(\mathcal{U}, \varphi\big)$ around $\tau(y_0) \in M$ such that $\psi(\mathcal{V}) \subset \mathcal{U}$ and the local representative $\varphi \circ \tau|_{\mathcal{V}} \circ \psi^{-1}$ of τ is of the simple form $(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) \mapsto (x^1, \ldots, x^m)$. As a consequence, each fibre $\tau^{-1}(\{x\}) \subset E$ over $x \in M$ is a closed submanifold of E. Moreover, it follows that for any fibered manifold the differentiable structure of the base is uniquely determined by the differentiable structure on the total space and the condition that

the projection be a surjective submersion. Furthermore there is the well-known criterion of *passage to the quotient*: for any fibered manifold (E, τ, M) and any set-theoretic map $f : M \to M'$ such that $f \circ \tau : E \to M'$ is smooth it follows that f is smooth. In particular the base map of a morphism is uniquely induced by the map between the total spaces. Recall that a smooth *section* of a fibered manifold is a smooth map $\sigma : M \to E$ with $\tau \circ \sigma = \mathrm{id}_M$, and we shall denote the set of all smooth sections by $\Gamma^\infty(M, E)$. By the above-mentioned local form of τ the following well-known fact is clear that any fibered manifold (E, τ, M) admits *families of local sections*, $(\mathcal{U}_\kappa, \sigma_\kappa)_{\kappa\in\mathfrak{S}}$, i.e. there is an open cover $(\mathcal{U}_\kappa)_{\kappa\in\mathfrak{S}}$ of the base M and smooth maps $\sigma_\kappa : \mathcal{U}_\kappa \to \tau^{-1}(\mathcal{U}_\kappa)$ such that $\tau \circ \sigma_\kappa = \mathrm{id}_{\mathcal{U}_\kappa}$.
For a given manifold M we shall also need the subcategory $\mathcal{FM}_M$ of $\mathcal{FM}$ consisting of all *fibered manifolds over* M, i.e. where the objects all have the same base M, and all the base maps are equal to the identity map of M. We shall treat many more categories of more particular fibered manifolds, and for each such category $\mathcal{C}$ there will be an 'over M'-version denoted by $\mathcal{C}_M$ where all bases are equal to M and all base maps are equal to the identity map on M. Let $\iota_M : \mathcal{FM}_M \to \mathcal{FM}$ the inclusion functor. Since for any smooth section s of the fibered manifold (E, τ, M) and any morphism $(\Phi, \mathrm{id}_M) : (E, \tau, M) \to (E', \tau', M)$ of fibered manifolds over M the map $\Phi \circ s$ is a smooth section of (E', τ', M) it follows that the assignment $(E, \tau, M) \to \Gamma^\infty(M, E)$ and $\Phi \to \Gamma^\infty(M, \Phi) : (s \mapsto \Phi \circ s)$ defines a covariant functor $\Gamma^\infty(M, \ )$ from $\mathcal{FM}_M$ to Set.

Particular cases of fibered manifolds are the well-known *fibre bundles* which are given by quadruples (E, τ, M, S) such that (E, τ, M) is a fibered manifold, S is a manifold (the *typical fibre*), and such that there is an open cover $(\mathcal{U}_\kappa)_{\kappa\in\mathfrak{S}}$ of the base M and smooth isomorphisms $f_\kappa : (\tau^{-1}(\mathcal{U}_\kappa), \tau|_{\tau^{-1}(\mathcal{U}_\kappa)}, \mathcal{U}_\kappa) \to (\mathcal{U}_\kappa \times S, \mathrm{pr}_1, \mathcal{U}_\kappa)$ of fibered manifolds over $\mathcal{U}_\kappa$ (the *local trivialisations*). Hence each f_κ has the general form $f_\kappa(y) = \big(\tau(y), f_\kappa^{(2)}(y)\big)$ with a smooth map $f_\kappa^{(2)} : \tau^{-1}(\mathcal{U}_\kappa) \to S$. Note that for any chosen point $z_0 \in S$ the maps $\sigma_\kappa : \mathcal{U}_\kappa \to \tau^{-1}(\mathcal{U}_\kappa)$ given by $\sigma_\kappa(x) = f_\kappa^{-1}(x, z_0)$ are local sections. The class of all fibre bundles equipped with the morphism sets of its underlying fibered manifolds forms a category denoted by $\mathcal{FB}$ in [23]. For a given manifold M let $\mathcal{FB}_M$ denote the subcategory of all *fibre bundles over* M where all objects have the same base M, and all the base maps are equal to the identity map of M. In case the typical fibre is a finite-dimensional vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ there is the well-known (non full) subcategory $\mathbb{K}\mathcal{VB}$ of $\mathcal{FB}$ of all *vector bundles*: here every fibre carries the structure of a $\mathbb{K}$-vector space, and the morphisms have to be fibrewise $\mathbb{K}$-linear.
In particular for differential operators we shall encounter the following slightly more general situation: let $E := (E_n, \tau_n, M)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{K}$-vector bundles over M and for each $n \in \mathbb{N}$ let $i_n : E_n \to E_{n+1}$ be an injective vector bundle morphisms over M (projecting on the identity map on M). One is tempted to

form the "inductive limit $\lim_{n\to\infty} E_n$" which in general would no longer lead to a vector bundle over M with finite-dimensional fibres so it does not belong to the original category. However it is quite practical to consider such situations, e.g. the symmetric power of the tangent bundle, $\mathsf{S}(TM)$, which of course only symbolizes the sequence $\big(\oplus_{k=0}^n \mathsf{S}^k(TM)\big)_{n\in\mathbb{N}}$ of 'true' finite-dimensional vector bundles. We shall call these sequences *filtered vector bundles over M*, and agree upon that $\Gamma^\infty(M,E) := \lim_{n\to\infty}\Gamma^\infty(M,E_n)$ where the inductive limit is taken in the category of $\mathbb{K}$-vector spaces with respect to the linear maps of the section spaces induced by the $(i_n)_{n\in\mathbb{N}}$. Moreover a morphism of filtered vector bundles over M, $\Phi : E = (E_n,\tau_n,M,i_n)_{n\in\mathbb{N}} \to E' = (E'_n,\tau'_n,M,i'_n)_{n\in\mathbb{N}}$ is a sequence of vector bundle morphisms over M i.e. for each nonnegative integer n: $\Phi_n : E_n \to E'_n$ intertwining the maps i_n and i'_n, i.e. $\Phi_{n+1}\circ i_n = i'_n\circ\Phi_n$ for each $n\in\mathbb{N}$.

Let G be a Lie group where we write $e = e_G$ for its unit element and $(g_1,g_2)\mapsto g_1g_2$ for the multiplication. Let **Lie** $\mathcal{G}$ denote the category of all Lie groups where morphisms are smooth morphisms of Lie groups. Let $(\mathfrak{g},[\ ,\ ])$ denote its Lie algebra. Recall that a *left G-space* (resp. *right G-space*) is a smooth manifold M equipped with a smooth *left G-action* $G\times M\to M$, mostly written $(g,x)\mapsto gx$, (resp. smooth *right G-action* $M\times G\to M$ mostly written $(x,g)\mapsto xg$) satisfying $g_1(g_2x) = (g_1g_2)x$ and $ex = x$ (resp. $(xg_1)g_2 = x(g_1g_2)$ and $xe = x$) for all $g_1,g_2\in G$ and all $x\in M$. For each $x\in M$ let $\mathcal{O}^G_x$ denote the *G-orbit through x*, i.e. $\mathcal{O}^G_x = \{gx\in M \mid g\in G\}$ (resp. $\mathcal{O}^G_x = \{xg\in M \mid g\in G\}$ for right G-actions) which is of course well-known to be an immersed submanifold of M. Recall that an action is called *transitive* on M iff there is only one orbit. Recall that it is called *free* iff for all $g\in G$: if there is $x\in M$ such that $gx = x$ (resp. $xg = x$) then $g = e$. Moreover, for each $\xi\in\mathfrak{g}$ we shall denote the *fundamental vector field* of the left G-action (resp. right G-action) by $\xi_M(x) := \frac{d}{dt}\big(exp(t\xi)x\big)|_{t=0}$ (resp. $\xi^*(x) = \frac{d}{dt}\big(x(exp(t\xi))\big)|_{t=0}$) for all $x\in M$. Recall the Lie bracket rules $[\xi_M,\eta_M] = -[\xi,\eta]_M$ (resp. $[\xi^*,\eta^*] = [\xi,\eta]^*$) for all $\xi,\eta\in\mathfrak{g}$. Let G' be another Lie group, and M' a left G'-space (resp. a right G'-space). A pair (ϕ,θ) is called a morphism from the G-space M to the G'-space M' iff $\phi : M\to M'$ is a smooth map, and $\theta : G\to G'$ is a smooth morphism of Lie groups, such that the following intertwining property holds: $\phi(gx) = \theta(g)\phi(x)$ (resp. $\phi(xg) = \phi(x)\theta(g)$) for all $g\in G$ and $x\in M$ (we also say that ϕ is *G-θ equivariant*). Again the class of all left (resp. right) G-spaces with varying G forms a category. For fixed G we shall denote the subcategory of all left (resp. right) G-spaces whose morphisms all have $\theta = \mathrm{id}_G$ (so-called maps intertwining the G-action) by $G\cdot\mathcal{M}f$ (resp. $\mathcal{M}f\cdot G$). Finally note the functors $I : G\cdot\mathcal{M}f\to\mathcal{M}f\cdot G$ and $I : \mathcal{M}f\cdot G\to G\cdot\mathcal{M}f$ which replace actions by the action of the inverse, i.e. for a left G-space M one defines a right action by $yg := g^{-1}y$ for all $y\in M$ and $g\in G$. In this work we shall call left (resp. right) G-module a $\mathbb{K}$-vector space V ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) on which G acts from the left (resp. from the right) in a $\mathbb{K}$-linear way. As usual, V^G denotes the subspace of all fixed vectors (i.e. those $v\in V$ such that $gv = v$ for all $g\in G$).

Recall that for a fixed Lie group U a *principal bundle* over a manifold M with structure group U (or a principal U-bundle) is a fibre bundle (P, τ, M, U) equipped with a free right U-action $P \times U \to E$ such that for each $y \in P$ the fibre through y, $\tau^{-1}\big(\{\tau(y)\}\big)$, coincides with the U-orbit $\{yu \in P \mid u \in G\}$ passing through y, and all the local trivializations $f_\kappa : \tau^{-1}(\mathcal{U}_\kappa) \to \mathcal{U}_\kappa \times U$ are U-equivariant in the sense that $f_\kappa^{(2)}(yu) = f_\kappa^{(2)}(y)u$ for all $y \in \tau^{-1}(\mathcal{U}_\kappa)$ and $u \in U$, see e.g. [21, p.50]. Note that any family of local sections $(\mathcal{U}_\kappa, \sigma_\kappa)_{\kappa \in \mathfrak{S}}$ of P ('local frames') gives rise to local trivializations via $f_\kappa^{-1} : \mathcal{U}_\kappa \times U \to \tau^{-1}(\mathcal{U}_\kappa)$ given by $f_\kappa^{-1}(x, u) = \sigma_\kappa(x)u$. It is not hard to see that the right U-action is always proper, see e.g. [32] for a definition. Conversely, by the slice theorem (see [32]) it follows that each right U-space P whose action is free and proper gives rise to a principal U-bundle $(P, \tau, P/U, U)$ over the quotient space $M = P/U$ of U-orbits where τ is the canonical projection. Note also that a fibered manifold (P, τ, M) admitting a smooth free right U-action on the total space P such that the fibres coincide with the U-orbits is automatically a principal U-bundle, see e.g. [23, p.87, Lemma 10.3]. Principal bundles (with varying U) form a category denoted by $\mathcal{PB}$ in [23] for which a morphism from a principal bundle (P, τ, M, U) to a principal bundle (P', τ', M', U') is a triple (Φ, θ, ϕ) where $(\Phi, \theta) : P \to P'$ is a morphism from the right U-space E to the right U'-space E' and $\phi : M \to M'$ is the map induced by Φ. Note that the forgetful functor $\mathcal{PB} \to \mathcal{FB}$ is not full, i.e. the above morphisms between principal bundles are more specific than just fibre-preserving maps. We shall denote by $\mathcal{PB}(U)$ the subcategory of all those principal bundles having fixed structure group U and morphisms as in $\mathcal{M}f \cdot U$, i.e. smooth maps $\Phi : P \to P'$ between total spaces intertwining the U-action, i.e. $\Phi(yu) = \Phi(y)u$ for all $y \in P$ and $u \in U$. We denote the subcategory of all principal fibre bundles over a fixed M having fixed structure group U by $\mathcal{PB}(U)_M$.

One particular case which is very important for us is the principal H-bundle $(G, \pi, G/H)$ where G is a Lie group, $H \subset G$ is a closed subgroup, and G/H is the quotient space of the right H-action on G given by right multiplication in the group G where $\pi : G \to G/H$ denotes the canonical projection. G/H is called a *homogeneous space* and is known to be a left G-space by means of the induced left multiplication $\ell : G \times G/H \to G/H$ given by $\ell(g', gH) = \ell_{g'}(gH) := (g'g)H$ in the group. This left G action is transitive.

Very frequently, we shall encounter *Lie algebra versions* of the preceding notions: a pair of (not necessarily finite-dimensional) Lie algebras $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{h} \subset \mathfrak{g}$ is subalgebra can be called a *Lie algebra inclusion* (see e.g. [7]) or an *infinitesimal homogeneous space*.

Recall the important notion of an *associated bundle to a principal fibre bundle*: let (P, τ, M, U) be a principal U-bundle and S a left U-space. The right action $(P \times S) \times U \to P \times S$ given by $(y, z)u = (yu, u^{-1}z)$ is free and proper because the right U-action on P is free and proper. Then the quotient $P_U[S] = P[S] := (P \times S)/U$ is known to be a well-defined manifold where the canonical projection $P \times S \to P[S]$

is a smooth submersion. We shall not very often use the classical notation $P\varkappa_U S$ used for instance in [21] because it may be confused with fibered products. For computations we shall denote the equivalence class of the pair $(y, z) \in P \times S$ by $[y, z] \in P[S]$. The projection $\tau_{P[S]} : P[S] \to M$ given by $\tau_{P[S]}([y, z]) = \tau(y)$ is a well-defined smooth surjective submersion, and the quadruple $(P[S], \tau_{P[S]}, M, S)$ is a fibre bundle over M with typical fibre S, called the associated bundle to P: note that a family of local trivializations $(\mathcal{U}_\kappa, f_\kappa)_{\kappa\in\mathfrak{S}}$ for $P[S]$ can be obtained by a family of local sections $(\mathcal{U}_\kappa, \sigma_\kappa)_{\kappa\in\mathfrak{S}}$ of (P, τ, M, U) by setting $f_\kappa^{-1}(x, z) = [\sigma_\kappa(x), z]$. Moreover, note that any morphism $\phi : S \to S'$ of left U-spaces in $U\cdot\mathcal{M}f$ (i.e. of the particular form (ϕ, id_U)) gives rise to a well-defined morphism $P[\phi] : P[S] \to P[S']$ of fibre bundles over M given by $P[\phi]([y, z]) = [y, \phi(z)]$. Hence the assignment $S \to P[S]$, $\phi \to P[\phi]$ defines a covariant functor, the *associated bundle functor* $P[\ ]$ from $U \cdot \mathcal{M}f$ to $\mathcal{FB}_M$.

We also need to recall the well-known description of *sections of associated bundles as U-equivariant maps*, see e.g. [21, p.115]: let S be a left U-space where the left U-action is denoted by l. Note first that for each y in the total space P of a principal U-bundle (P, τ, M, U) the smooth map $\Phi_y : S \to E[S]$ defined by $z \mapsto [y, z]$ is a diffeomorphism onto the fibre $\tau_{P[S]}^{-1}\big(\{\tau(y)\}\big)$ over $\tau(y)$, and clearly for each $u \in U$ $\Phi_{yu} = \Phi_y \circ l_u$. For any section $\sigma \in \Gamma^\infty(M, P[S])$ let $\hat{\sigma} : P \to S$ denote the map $y \mapsto \Phi_y^{-1}\Big(\sigma\big(\tau(y)\big)\Big)$ which is clearly well-defined, smooth and U-equivariant, i.e. $\hat{\sigma}(yu) = u^{-1}\hat{\sigma}(y)$. The map $\hat{\sigma}$ is called the *frame form of the section* σ, see e.g. [23, p.95]. Let $\mathcal{C}^\infty(P, S)^U$ denote the space of all U-equivariant smooth maps $P \to S$ which can be written as $\mathbf{Hom}_{\mathcal{M}f\cdot U}\big(F(P), I(S)\big)$ in categorical terms (where $F : \mathcal{PB}(U) \to \mathcal{M}f \cdot U$ denotes the forgetful functor). Conversely, let $f \in \mathcal{C}^\infty(P, S)^U$ and set $\check{f} : M \to P[S]$ as $\check{f}\big(\tau(y)\big) = [y, f(y)]$ which is clearly a well-defined smooth section of the fibre bundle $P[S]$ over M. The two maps $\hat{(\ )}$ and $\check{(\ )}$ are inverses and constitute a natural isomorphism of the covariant functors $S \to \Gamma^\infty(M, P[S])$ and $S \to \mathcal{C}^\infty(P, S)^U$ from $U \cdot \mathcal{M}f$ to Set.

Thirdly, recall for any smooth homomorphism of Lie groups $\varphi : U \to U'$ the associated bundle $P_U[U']$ where U acts on the left on U' via $u.u' := \varphi(u)u'$ carries a right U'-action defined by $[p, u']u'_1 := [p, u'u'_1]$ which is well-defined and free, and the fibres of $P_U[U']$ are clearly in bijection with the right U'-orbits. According to [23, p.87, Lemma 10.3], $P_U[U']$ is a principal U'-bundle over M. There is a natural morphism of principal bundles over M defined by

$$\Phi : P \to P_U[U'] : p \mapsto [p, e_{U'}] \tag{1.2}$$

where the Lie group homomorphism $U \to U'$ is given by φ. Next, let U' act smoothly on the left of a smooth manifold S, and form the associated bundle $\big(P_U[U']\big)_{U'}[S]$. Since U also acts smoothly on the left on S via the action of U' and the homomorphism φ we can form the associated bundle $P_U[S]$ over M. It is not hard to see that the map (for all $p \in P$, $u' \in U'$, and $z \in S$)

$$\Phi_S : [p, z] \mapsto \big[[p, e_{U'}], z\big] \quad \text{with inverse} \quad \big[[p, u'], z\big] \mapsto [p, u'z] \tag{1.3}$$

is a well-defined smooth isomorphism $\Phi_S : P_U[S] \to \big(P_U[U']\big)_{U'}[S]$ of fibre bundles over M.

Finally, recall the notion of a *connection* in a principal fibre bundle (P, τ, M, U): let $(\mathfrak{u}, [\ ,\ ])$ denote the Lie algebra of the structure group U. We shall denote the right U-action on P by r. A connection 1-form α is a $\mathfrak{u}$-valued 1-form on the total space P, i.e. a smooth section in the vector bundle $\mathrm{Hom}(TP, \mathfrak{u})$ over P, such that

$$\forall\, y \in P,\ \forall\, u \in U:\ \ \big(r_u^*\alpha\big)_y = \alpha_{yu} \circ T_y r_u \ = \ Ad_{u^{-1}} \circ \alpha_y, \tag{1.4}$$

$$\forall\, y \in P,\ \forall\, \zeta \in \mathfrak{u}:\ \ \alpha_y(\zeta_y^*) \ = \ \zeta. \tag{1.5}$$

Let $(\Phi, \theta, \phi) : (P, \tau, M, U) \to (P', \tau', M', U')$ be a morphism between two principal fibre bundles, let α be a connection 1-form on (P, τ, M, U) and let α' be a connection 1-form on (P', τ', M', U'). Then both the pull-back $\Phi^*\alpha'$ and the form $T_e\theta \circ \alpha$ are $\mathfrak{u}'$-valued 1-forms over P. It is therefore reasonable to introduce the following *category of principal fibre bundles with connection*, written $\mathcal{PBC}$ whose objects are quintuples (P, τ, M, U, α) where (P, τ, M, U) is a principal fibre bundle equipped with a connection 1-form α, and where the set of morphisms is defined as follows

$$\begin{aligned}&\mathbf{Hom}_{\mathcal{PBC}}\big((P, \tau, M, U, \alpha), (P', \tau', M', U', \alpha')\big) := \\ &\big\{(\Phi, \theta, \phi) \in \mathbf{Hom}_{\mathcal{PB}}\big((P, \tau, M, U), (P', \tau', M', U')\big) \mid\ T_e\theta \circ \alpha = \Phi^*\alpha'\big\}.\end{aligned} \tag{1.6}$$

Again we can fix the structure group U to have the category $\mathcal{PBC}(U)$ where in the above definition of morphisms we set of course $\theta = \mathrm{id}_U$, and its 'over M'-version $\mathcal{PBC}(U)_M$. Recall that any principal bundle over a manifold M admits a connection, which can be seen by a partition of unity argument, see e.g. [21, p.67-68].

Moreover, recall that any connection in a principal U-bundle gives rise to a U-invariant splitting of the tangent bundle TP of the total space into the subbundle of vertical subspaces $\{\zeta^*(p) \mid \zeta \in \mathfrak{u}\}$, $p \in P$, and the bundle of horizontal subspaces $H_p := \{v_p \in T_pP \mid \alpha_p(v_p) = 0\}$. By means of this, one can introduce a *horizontal lift* of any vector field X on the base M to a vector field X^h on P which is uniquely determined by the condition that for each $p \in P$ the value $X^h(p)$ lies in H_p and $T_p\tau(X^h(p)) = X(\tau(p))$. A horizontal lift is always U-invariant: $r_u^*X^h = X^h$. Let $E = P[V]$ an associated vector bundle with H-module V. For any smooth section ψ of E let $\hat{\psi} \in \Gamma^\infty(P, V)^U$ be the corresponding U-equivariant map $P \to V$, and let X be a vector field on M. It is well-known that the following formula defines a *covariant derivative* $\nabla_X\psi$, i.e. a smooth section of E such that

$$\widehat{\nabla_X\psi} := X^h\big(\hat{\psi}\big), \tag{1.7}$$

see e.g. [21, p.116, Prop.1.3]. Clearly everything is well-defined since X^h is H-invariant. It follows that $(X, \psi) \mapsto \nabla_X\psi$ defines a bidifferential operator such that $\nabla_{fX}\psi = f\nabla_X\psi$ and $\nabla_X(f\psi) = X\big(f\big)\psi + f\nabla_X\psi$ for all vector fields X on

M, smooth real-valued functions f on M and smooth sections ψ of E: this is precisely the classical definition of a connection in a vector bundle, see e.g. [21, p.116, Prop.1.2].

1.2 Multidifferential Operators in Associated Vector Bundles

Let M be a manifold of dimension m, and let (E_1, τ_1, M),...,(E_k, τ_k, M), (F, τ_F, M) be $\mathbb{K}$-vector bundles over M of fibre dimension $p_1, \ldots, p_k, q$, respectively. There is an open cover $(\mathcal{U}_\kappa)_{\kappa\in\mathfrak{S}}$ of M trivializing all the $k+1$ vector bundles and serving as the family of domains for an atlas of M. For each integer $1 \leq j \leq k$ let $f_1^{(j)}, \ldots, f_{p_j}^{(j)}$ and $g_1, \ldots, g_q$ be local sections of E_j and F, respectively, forming a base of the free module of all local sections $\Gamma^\infty(\mathcal{U}_\kappa, E_j)$ and $\Gamma^\infty(\mathcal{U}_\kappa, F)$, respectively, over the ring $\mathcal{C}^\infty(\mathcal{U}_\kappa, \mathbb{K})$. Hence for each integer $1 \leq j \leq k$ any smooth section $\psi_{(j)}$ of $E_{(j)}$ and ψ of F is locally a linear combination $\psi_{(j)}|_{\mathcal{U}_\kappa} = \sum_{a_j=1}^{p_j} \psi_{(j)}^{a_j} f_{a_j}^{(j)}$ and $\psi|_{\mathcal{U}_\kappa} = \sum_{b=1}^{q} \psi^b g_b$, respectively, with smooth locally defined coefficient functions $\psi_{(j)}^{a_j}, \psi^b : \mathcal{U}_\kappa \to \mathbb{K}$. Furthermore, for any multi-index $I = (n_1, \ldots, n_m) \in \mathbb{N}^{\times m}$ let $|I| := n_1 + \cdots + n_m$ and let ∂_I be short for the partial derivative

$$\partial_I := \frac{\partial^{|I|}}{(\partial x^1)^{n_1} \cdots (\partial x^m)^{n_m}}.$$

Recall that a general *k-differential operator D of maximal order N* in the above bundles is a k-multilinear map $D : \Gamma^\infty(M, E_1) \times \cdots \times \Gamma^\infty(M, E_k) \to \Gamma^\infty(M, F)$ taking the following local form: there is a nonnegative integer N and for each collection of multi-indices $I_1, \ldots, I_k$ with $|I_j| \leq N$ (for all $1 \leq j \leq k$) and collection of positive integers $a_1, \ldots, a_k, b$ with $1 \leq a_j \leq p_j$ for all $1 \leq j \leq k$ and $1 \leq b \leq q$ there is a smooth $\mathbb{K}$-valued function $D_{\kappa a_1 \cdots a_k}^{bI_1 \cdots I_k}$ defined on $\mathcal{U}_\kappa$ such that for all $x \in \mathcal{U}_\kappa$ we get

$$(1.8)\quad D\big(\psi_{(1)}, \ldots, \psi_{(k)}\big)(x) = \sum_{a_1=1}^{p_1} \cdots \sum_{a_k=1}^{p_k} \sum_{b=1}^{q} \sum_{\substack{I_1,\ldots,I_k \\ |I_1|,\ldots,|I_k|\leq N}} D_{\kappa a_1 \cdots a_k}^{bI_1 \cdots I_k}(x) \big(\partial_{I_1} \psi_{(1)}^{a_1}\big)(x) \cdots \big(\partial_{I_k} \psi_{(k)}^{a_k}\big)(x) g_b(x).$$

The integer N, the maximal order, is of course required to be a global property of D and does not depend on the chart. The $\mathbb{K}$-vector space of all such k-differential operators of maximal order N will be denoted by the symbol $\mathbf{Diff}_M^{(N)}(E_1, \ldots, E_k; F)$. Let $\mathbf{Diff}_M(E_1, \ldots, E_k; F)$ the union of all the $\mathbf{Diff}_M^{(N)}(E_1, \ldots, E_k; F)$ in the $\mathbb{K}$-vector space of all k-multilinear maps. Recall that it is also a left module for the ring $\mathcal{C}^\infty(M, \mathbb{K})$ by multiplying smooth functions with the values of D. Recall the following $\mathbb{K}$-bilinear operadic composition $D' \circ_{j'} D$ of $D \in \mathbf{Diff}_M(E_1, \ldots, E_k; F)$ and $D' \in \mathbf{Diff}_M(F_1, \ldots, F_l; G)$ where $F_1, \ldots, F_l, G$ are also vector bundles over

M and there is an integer $1 \leq j_0 \leq l$ such that $F = F_{j_0}$: let $\psi_{(j)} \in \Gamma^\infty(M, E_j)$, $1 \leq j \leq k$, and $\chi_{(j')} \in \Gamma^\infty(M, F_{j'})$, $1 \leq j' \leq l$, $j' \neq j_0$ then

$$(D' \circ_{j_0} D)\big(\chi_{(1)}, \ldots, \chi_{(j_0-1)}, \psi_{(1)}, \ldots, \psi_{(k)}, \chi_{(j_0+1)}, \ldots, \chi_{(l)}\big) := D'\Big(\chi_{(1)}, \ldots, \chi_{(j_0-1)}, D\big(\psi_{(1)}, \ldots, \psi_{(k)}\big), \chi_{(j_0+1)}, \ldots, \chi_{(l)}\Big) \tag{1.9}$$

is in $\mathbf{Diff}_M(F_1, \ldots, F_{j_0-1}, E_1, \ldots, E_k, F_{j_0+1}, \ldots, F_l; G)$, i.e. a $(k+l-1)$-differential operator. In the case $k = l = 1$ the above composition $\circ_1$ is course ordinary composition of linear maps.

Multidifferential operators can be seen as smooth sections of certain filtered vector bundles: we shall recall the notions of *jet bundles*, see the book [23], Section 12, for details. Let M and N be two manifolds having dimensions m and n, respectively, and let r be a nonnegative integer. Recall that two smooth curves $\gamma_1, \gamma_2 : \mathbb{R} \to M$ have rth order contact if in some (and a posteriori in any) chart φ the difference $\varphi \circ \gamma_1 - \varphi \circ \gamma_2$ vanishes to rth order at 0. Let $\phi, \psi : M \to N$ be two smooth maps. Recall that they are said to determine the same r-jet at $x \in M$ if for any smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = x$ the two smooth curves $\phi \circ \gamma$ and $\psi \circ \gamma$ have rth order contact at 0. Let ${}_xJ^r(M, N)$ denote the quotient of $\mathcal{C}^\infty(M, N)$ by the equivalence relation that $\phi \sim \psi$ iff ϕ and ψ determine the same r-jet at x, and let $J^r(M, N)$ be the disjoint union $\bigcup_{x \in M} {}_xJ^r(M, N)$. For each $\phi \in \mathcal{C}^\infty(M, N)$ let $j^r_x(\phi)$ denote its r-jet at x, i.e. the equivalence class of ϕ in ${}_xJ^r(M, N)$. In the particular case $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ the equivalence class j^r_x can be identified with the Taylor series of ϕ at x up to order r. There is an obvious surjective projection $\alpha : J^r(M, N) \to M$ by mapping $j^r_x(\phi)$ to x. Furthermore, the map $j^r_x(\phi) \to \phi(x)$ is a well-defined surjective projection $\beta : J^r(M, N) \to N$. Now each set $J^r(M, N)$ can be given a canonical differentiable structure of a smooth manifold of dimension $m + \binom{m+r}{r} n$ such that by means of the above projection $\pi^r_0 : J^r(M, N) \to M \times N : X \mapsto (\alpha(X), \beta(X))$ is a smooth fibre bundle over $M \times N$. For any $x' \in N$ let denote by $J^r(M, N)_{x'}$ the submanifold $\beta^{-1}(\{x'\})$ of $J^r(M, N)$. Note also that for two nonnegative integers r, s with $r \geq s$ there is a canonical surjective submersion $\pi^r_s : J^r(M, N) \to J^s(M, N)$ defined by $\pi^r_s\big(j^r_x(\phi)\big) = j^s_x(\phi)$. An important issue is the fact that for three manifolds M, N, P, each $x \in M$, and any two smooth maps $\phi : M \to P$ and $\psi : N \to P$ each r-jet $j^r_x(\psi \circ \phi)$ only depends on the r-jets $j^r_{\phi(x)}(\psi)$ and $j^r_x(\phi)$ which defines a composition $J^r(N, P) \times J^r(M, N) \to J^r(M, P)$ which is associative in the appropriate sense. Moreover, let (E, τ, M) be a $\mathbb{K}$-vector bundle over M. Define its *rth jet prolongation* to be the subset $J^rE := \{j^r_x(\phi) \in J^r(M, E) \mid x \in M \text{ and } \phi \in \Gamma^\infty(M, E)\}$ which is a smooth submanifold of $J^r(M, E)$. Defining linear combinations of r-jets of sections as the r-jet of the linear combination of sections endows the fibre bundle $(J^rE, \alpha|_{J^rE}, M)$ with the structure of a smooth vector bundle over M. In the same way it is shown that for the particular case of the trivial bundle $E = M \times \mathbb{K}$ every rth jet prolongation $J^r(M \times \mathbb{K}) = J^r(M, \mathbb{K})$ is a smooth $\mathbb{K}$-vector bundle over M whose fibres are associative commutative unital $\mathbb{K}$-algebras of dimension $\binom{m+r}{r}$ which are all isomorphic to the quotient of the free symmetric algebra $\mathsf{S}(\mathbb{K}^m)$ modulo the ideal $\oplus_{k=r+1}^{\infty}\mathsf{S}^k(\mathbb{K}^m)$. Furthermore, note that the multiplication of smooth sections by $\mathbb{K}$-valued smooth functions endows each J^rE with the structure of a (fibrewise) $J^r(M,\mathbb{K})$-module. There is the following canonical isomorphism of $\mathcal{C}^\infty(M,\mathbb{K})$-modules:

$$\text{(1.10)}\quad \Gamma^\infty\big(M, \mathbf{Hom}(J^rE_1\otimes\cdots\otimes J^rE_k, F)\big) \;\cong\; \mathbf{Diff}_M^{(r)}(E_1,\ldots,E_k;F)$$

defined by

$$\text{(1.11)}\qquad F \;\mapsto\; \Big((\psi_1,\ldots,\psi_k)\mapsto \Big(x\mapsto F_x\big(j_x^r(\psi_1)\otimes\cdots\otimes j_x^r(\psi_k)\big)\Big)\Big).$$

upon using the projections ${\pi_s^r}_j : J^rE_j \to J^sE_j$ for $r\geq s$ it is easy to see that the sequence of $\mathbb{K}$-vector bundles $\Big(\mathbf{Hom}(J^rE_1\otimes\cdots\otimes J^rE_k, F)\Big)_{r\in\mathbb{N}}$ is a filtered vector bundle with $i_r = ({\pi_r^{r+1}}_1\otimes\cdots\otimes {\pi_r^{r+1}}_k)^*$.

Let us consider now the well-known particular case where $M=G$ where G is a Lie group having Lie algebra $(\mathfrak{g},[\ ,\ ])$. Recall that for any $\xi\in\mathfrak{g}$ the fundamental field of the right multiplication $R: G\times G\to G$ where $R_g(g_0) := g_0g$ is given by the well-known *left-invariant vector field* denoted by $\xi^+(g) := T_eL_g(\xi)$ where $L: G\times G\to G$ is the canonical left multiplication $L_g(g_0) = gg_0$. Fix a base $e_1,\ldots,e_n$ of $\mathfrak{g}$. It is well-known that the tangent vectors $e_1^+(g),\ldots,e_n^+(g)$ form a vector space basis of T_gG for each $g\in G$. In the above formula (1.8) for the particular case $k=1$ and $E=F=G\times\mathbb{K}$ we can hence replace the iterated partial coordinate derivatives ∂_I by linear combinations with smooth coefficients of iterations of Lie derivatives with respect to left invariant vector fields.
These iterations correspond to algebraic iterations of Lie algebra elements described by the well-known *universal envelopping algebra* $\mathsf{U}(\mathfrak{g})$ *of the Lie algebra* $\mathfrak{g}$ (which we only need to consider over the field of real numbers, but which is of course much more general): $\mathsf{U}(\mathfrak{g})$ is defined to the quotient of the the free $\mathbb{R}$-algebra $\mathsf{T}(\mathfrak{g}) = \mathbb{R}1\oplus\bigoplus_{i=1}^{\infty}\mathsf{T}^i(\mathfrak{g})$ generated by the real vector space $\mathsf{T}^1(\mathfrak{g})=\mathfrak{g}$ modulo the two-sided ideal I of $\mathsf{T}(\mathfrak{g})$ spanned by all elements of the form $a(\xi\eta-\eta\xi-[\xi,\eta])b$ with $a,b\in\mathsf{T}(\mathfrak{g})$ and $\xi,\eta\in\mathfrak{g}$. It is well-known that $\mathsf{U}(\mathfrak{g})$ has the structure of a real Hopf algebra: the counit map $\epsilon:\mathsf{U}(\mathfrak{g})\to\mathbb{R}$ sends $\lambda 1\in\mathsf{U}(\mathfrak{g})$ to $\lambda\in\mathbb{R}$ and the image of $\bigoplus_{i=1}^{\infty}\mathsf{T}^i(\mathfrak{g})$ to zero, the comultiplication $\Delta:\mathsf{U}(\mathfrak{g})\to\mathsf{U}(\mathfrak{g})\otimes\mathsf{U}(\mathfrak{g})$ is defined to be as $\Delta(\xi)=\xi\otimes 1+1\otimes\xi$ for all $\xi\in\mathfrak{g}$ and uniquely extends to a morphism of unital associative algebras $\mathsf{U}(\mathfrak{g})\to\mathsf{U}(\mathfrak{g})\otimes\mathsf{U}(\mathfrak{g})$, and the antipode $S:\mathsf{U}(\mathfrak{g})\to\mathsf{U}(\mathfrak{g})$ is defined by $S(\xi_1\cdots\xi_N) := (-1)^N x_Nx_{N-1}\cdots x_2x_1$. The assignment $\mathfrak{g}\to\mathsf{U}(\mathfrak{g})$ is the left adjoint of the forgetful functor of the category of all real associative algebras to the category of all real Lie algebras, or in other words $\mathsf{U}(\mathfrak{g})$ is universal in the sense that each Lie algebra map $\phi:\mathfrak{g}\to A^-$ -where A^- is an associative algebra A seen as a Lie algebra with the commutator– uniquely lifts to a map of associative algebras $\Phi:\mathsf{U}(\mathfrak{g})\to A$ such that $\Phi\circ i_{\mathfrak{g}}=\phi$ where $i_{\mathfrak{g}}:\mathfrak{g}\to\mathsf{U}(\mathfrak{g})$ is the map induced by the natural injection $\mathfrak{g}\to\mathsf{T}(\mathfrak{g})$.

The Poincaré-Birkhoff-Witt-Theorem states that $\mathsf{U}(\mathfrak{g})$ is isomorphic to the vector space of the free commutative algebra generated by the vector space $\mathfrak{g}$, $\mathsf{S}(\mathfrak{g})$, as a cocommutative counital coalgebra. We shall write the comultiplication of an element $\mathsf{u} \in \mathsf{U}(\mathfrak{g})$ in Sweedler's notation $\Delta(\mathsf{u}) = \sum_{(\mathsf{u})} \mathsf{u}^{(1)} \otimes \mathsf{u}^{(2)}$. Note also that $\mathsf{U}(\mathfrak{g})$ is a filtered vector space, i.e. $\mathsf{U}(\mathfrak{g}) = \bigcup_{n\in\mathbb{N}} \mathsf{U}(\mathfrak{g})_n$ where each $\mathsf{U}(\mathfrak{g})_n$ is equal to $\oplus_{i=0}^{n} \mathsf{T}(\mathfrak{g})$ modulo I. Moreover the algebra $\mathsf{U}(\mathfrak{g})$ is filtered in the sense that $\mathsf{U}(\mathfrak{g})_n \mathsf{U}(\mathfrak{g})_p \subset \mathsf{U}(\mathfrak{g})_{n+p}$ for all nonnegative integers n, p, and that each $\mathsf{U}(\mathfrak{g})_n$ is a sub-coalgebra of the coalgebra $\mathsf{U}(\mathfrak{g})$.
Since the $\mathbb{R}$-linear map $\mathfrak{g} \to \mathbf{Diff}_G(G\times\mathbb{K}; G\times\mathbb{K})$ sending $\xi \in \mathfrak{g}$ to the Lie derivative with respect to the left invariant vector field ξ^+ is a morphism of Lie algebras we get a unique algebra map $\mathsf{U}(\mathfrak{g}) \to \mathbf{Diff}_G(G\times\mathbb{K}; G\times\mathbb{K})$ induced by the former and denoted by $\mathsf{u} \mapsto \mathsf{u}^+$. Consider now the $\mathbb{K}$-vector space $\mathcal{C}^\infty(G,\mathbb{K}) \otimes \mathsf{U}(\mathfrak{g})$ (here: $\otimes = \otimes_{\mathbb{R}}$) with unit $1 \otimes 1$ and multiplication given by (for all $\varphi, \psi \in \mathcal{C}^\infty(G,\mathbb{K})$ and for all $\mathsf{u}, \mathsf{v} \in \mathsf{U}(\mathfrak{g})$)

$$(\varphi \otimes \mathsf{u})(\psi \otimes \mathsf{v}) := \sum_{(\mathsf{u})} \varphi(\mathsf{u}^{(1)+}(\psi)) \otimes \mathsf{u}^{(2)}\mathsf{v}. \tag{1.12}$$

Moreover consider the following linear map $\mathcal{C}^\infty(G,\mathbb{K}) \otimes \mathsf{U}(\mathfrak{g}) \to \mathbf{Diff}_G(G\times\mathbb{K}; G\times\mathbb{K})$ defined by

$$(\varphi \otimes \mathsf{u}) \mapsto \Big(\psi \mapsto \varphi\big(\mathsf{u}^+(\psi)\big)\Big). \tag{1.13}$$

The following Proposition is well-known, see e.g. [17] for a star-product version, and not hard to check using the preceding facts.

Proposition 1.1. *The $\mathbb{K}$-vector space $\mathcal{C}^\infty(G,\mathbb{K}) \otimes \mathsf{U}(\mathfrak{g})$ equipped with its unit and multiplication (1.12) is an associative unital $\mathbb{K}$-algebra which is isomorphic as an associative unital $\mathbb{K}$-algebra to $\mathbf{Diff}_G(G\times\mathbb{K}; G\times\mathbb{K})$ equipped with the composition $\circ = \circ_1$ by means of the map (1.13).*

Let (P, τ, M, U) be a principal U-bundle, and let $V_1, \ldots, V_k, W$ be finite-dimensional vector spaces over $\mathbb{K}$ of dimension $p_1, \ldots, p_k, q$, respectively. Suppose that these vector spaces are left U-modules, i.e. U acts linearly on the left on each of these vector spaces where we denote the smooth linear action by $\rho_j : U \to GL(V_j)$ and $\rho : U \to GL(W)$, respectively, and by $\dot\rho_j : \mathfrak{u} \to \mathfrak{gl}(V_j)$ and $\dot\rho : \mathfrak{u} \to \mathfrak{gl}(W)$, respectively, the induced map of Lie algebras, i.e. $\dot\rho_j(\zeta) = \frac{d}{dt}\big(\rho_j(\exp(t\zeta))\big)\big|_{t=0}$. Let $E_1 := P[V_1], \ldots, E_k := P[V_k], F := P[W]$ the corresponding associated bundles over M which are of course $\mathbb{K}$-vector bundles. We should like to relate the space of k-differential operators on M, $\mathbf{Diff}_M(E_1, \ldots, E_k; F)$ to the corresponding space of k-differential operators on P, $\mathbf{Diff}_P(P\times V_1, \ldots, P\times V_k; P\times W)$: in the latter case, the bundles are trivial, hence the spaces of k-differential operators are easier to compute. It is clear that we can identify $\mathcal{C}^\infty(P, V)$ with $\Gamma^\infty(P, P\times V)$ for every finite-dimensional $\mathbb{K}$-vector space V. Recall also the natural isomorphism

$\Gamma^\infty(M,P[V]) \to \mathcal{C}^\infty(P,V)^U : \psi \mapsto \hat{\psi}$ for every finite-dimensional left U-module V which has been mentioned before. Each space of smooth functions $\mathcal{C}^\infty(P,V_j)$ $(1 \le j \le k)$ and $\mathcal{C}^\infty(P,W)$ is a U-module in the obvious way: for all $u \in U$ and $\psi'_{(j)} \in \mathcal{C}^\infty(P,V_j)$ one sets $u\psi'_{(j)} = \rho_j(u)\circ\psi'_{(j)}\circ r_u$ and likewise for $\psi' \in \mathcal{C}^\infty(P,W)$. There is an induced linear U-action on the space of k-differential operators, defined as usual for all $u \in U$ and $D' \in \mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)$:

$$(1.14)\qquad (uD')\big(\psi'_{(1)},\ldots,\psi'_{(1)}\big) := \rho(u)\Big(D'\big(u^{-1}\psi'_{(1)},\ldots,u^{-1}\psi'_{(1)}\big)\Big).$$

Clearly, this action preserves operadic composition, i.e. for all $u \in U$ we have $u\big(D'_1 \circ_{j_0} D'_2\big) = (uD'_1)\circ_{j_0}(uD'_2)$. Moreover, for each integer $1 \le j \le k$ and each $\zeta \in \mathfrak{u}$ let $\zeta^* + \dot{\rho}_j(\zeta)$ denote the differential operator in $\mathbf{Diff}_P(P\times V_j;P\times V_j)$ given by the sum of the Lie derivative of the fundamental field (applied to the 'arguments' of a smooth function $\psi_{(j)} \in \mathcal{C}^\infty(P,V_j)$) and the linear map $\dot{\rho}_j(\zeta)$ (applied to the values of $\psi_{(j)}$). For each integer $1 \le j \le k$ let $\mathbf{K}_j$ be the subspace of all those k-differential operators in $\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)$ which is spanned by all elements of the form

$$(1.15)\quad D\circ_j\big(\zeta^*+\dot{\rho}_j(\zeta)\big)\ \text{ where }\ D\in\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)\text{ and }\zeta\in\mathfrak{u}.$$

Since $u\big(\zeta^* + \dot{\rho}_j(\zeta)\big) = \big(Ad(u)(\zeta)\big)^* + \dot{\rho}_j\big(Ad(u)(\zeta)\big)$ for all $u \in U$ and $\zeta \in \mathfrak{u}$ it follows that each $\mathbf{K}_j$ is a U-submodule of the U-module of the above k-differential operators. Consider now the natural restriction map of $D' \in \mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)$ to the U-equivariant sections $\hat{\psi}_{(1)} : P \to V_1$,..., $\hat{\psi}_{(k)} : P \to V_k$ which come from smooth sections $\psi_{(1)} \in \Gamma^\infty(M,P[V_1])$,..., $\psi_{(k)} \in \Gamma^\infty(M,P[V_k])$, so we define

$$\begin{aligned}(1.16)\qquad \mathrm{res} : &\ \mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)\\ \to&\ \mathbf{Hom}_{\mathbb{K}}\big(\Gamma^\infty(M,P[V_1])\otimes\cdots\otimes\Gamma^\infty(M,P[V_1]);\mathcal{C}^\infty(P,W)\big)\\ D' \mapsto&\ \Big(\psi_{(1)}\otimes\cdots\otimes\psi_{(k)} \mapsto D'\big(\hat{\psi}_{(1)},\ldots,\hat{\psi}_{(k)}\big)\Big)\end{aligned}$$

Clearly $u\hat{\psi}_{(j)} = \hat{\psi}_{(j)}$ for all $u \in U$ and $1 \le j \le k$, hence each operator $\zeta^* + \dot{\rho}_j(\zeta)$ vanishes on each $\hat{\psi}_{(j)}$, hence

$$\mathrm{res}(\mathbf{K}_1 + \cdots + \mathbf{K}_k) = \{0\}$$

and the restriction map res passes to the quotient of the space all k-differential operators modulo $\mathbf{K}_1 + \cdots + \mathbf{K}_k$. There is the following

Proposition 1.2. *With the above notations:*
The following two vector spaces are isomorphic, viz

$$\frac{\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)^U}{\big(\mathbf{K}_1+\cdots+\mathbf{K}_k\big)^U} \cong \left(\frac{\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)}{\mathbf{K}_1+\cdots+\mathbf{K}_k}\right)^U$$

where the canonical injection of the left hand side into the right hand side is an isomorphism, and the restriction map maps the right hand side isomorphically to the space of all k-differential operators on M whence

$$\left(\frac{\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)}{\mathbf{K}_1+\cdots+\mathbf{K}_k}\right)^U\cong \mathbf{Diff}_M(E_1,\ldots,E_k;F).$$

Proof: Since we have to work locally we have to prepare the grounds by introducing some notation: for each integer $1\le j\le k$ let $v_1^{(j)},\ldots,v_{p_j}^{(j)}$ be a base of V_j, and let $w_1,\ldots,w_q$ be a base of W. Hence any smooth map $\psi'_{(j)}:P\to V_j$ and $\psi':P\to W$ is a linear combination $\sum_{a_j=1}^{p_j}\psi'^{a_j}_{(j)}v^{(j)}_{a_j}$ and $\sum_{b=1}^{q}\psi'^b w_b$ with smooth $\mathbb{K}$-valued coefficient functions. Next, choose a family of local sections $(\mathcal{U}_\kappa,\sigma_\kappa)_{\kappa\in\mathfrak{S}}$ of the principal bundle (P,τ,M,U) such that each $\mathcal{U}_\kappa$ is the domain of a chart of the manifold M. Hence for any smooth section $\psi_{(j)}$ of the bundle E_j and ψ of the bundle F the associated equivariant maps $\hat{\psi}_{(j)}:P\to V_j$ and $\hat{\psi}:P\to W$ are linear combinations with smooth coefficients of the above bases, and we therefore can compute the following particular local expressions for all $x\in\mathcal{U}_\kappa$:

$$\begin{aligned}\psi_{(j)}(x) &= [\sigma_\kappa(x),\hat{\psi}_{(j)}\big(\sigma_\kappa(x)\big)]=\sum_{a_j=1}^{p_j}\hat{\psi}^{a_j}_{(j)}\big(\sigma_\kappa(x)\big)[\sigma_\kappa(x),v^{(j)}_{a_j}]=:\sum_{a_j=1}^{p_j}\psi^{a_j}_{(j)}(x)f^{(j)}_{a_j}(x),\\ \psi(x) &= [\sigma_\kappa(x),\hat{\psi}\big(\sigma_\kappa(x)\big)]=\sum_{b=1}^{q}\hat{\psi}^{b}\big(\sigma_\kappa(x)\big)[\sigma_\kappa(x),w_b]=:\sum_{b=1}^{q}\psi^{b}(x)g_b(x).\end{aligned}$$

Hence we get the bijection for all $x\in\mathcal{U}_\kappa$ and $u\in U$

$$\psi^{a_j}_{(j)}(x):=\hat{\psi}^{a_j}_{(j)}\big(\sigma_\kappa(x)\big)\ \text{ and }\ \hat{\psi}^{a_j}_{(j)}\big(\sigma_\kappa(x)u\big):=\sum_{a'_j=1}^{p_j}\rho_j(u^{-1})^{a_j}_{a'_j}\psi^{a'_j}_{(j)}(x)\tag{1.17}$$

and likewise for ψ. Denote the coordinate vector field $\partial/(\partial x^\mu)$ in $\mathcal{U}_\kappa$ by ∂_μ for each integer $1\le\mu\le m=\dim(M)$. Then, in $\tau^{-1}(U_\kappa)\subset P$ define the following U-invariant horizontal lifts ∂^h_μ for all $x\in\mathcal{U}_\kappa$ and $u\in U$:

$$\partial^h_\mu\big(\sigma_\kappa(x)u\big):=T_{\sigma_\kappa(x)}r_u\big(T_x\sigma_\kappa(\partial_\mu)\big).$$

Clearly, these horizontal lifts are U-invariant, commute, are τ-related with the ∂_μ, and commute with all the fundamental fields ζ^*, $\zeta\in\mathfrak{u}$. For each multi-index $I\in\mathbb{N}^m$ denote by ∂^h_I the iteration $(\partial^h_1)^{n_1}\cdots(\partial^h_m)^{n_m}$. Let $e_1,\ldots,e_n$ be a vector space base of the Lie algebra $\mathfrak{u}$. For each multi-index $J=(n'_1,\ldots,n'_n)\in\mathbb{N}^{\times n}$ let e_J denote the product $e_1^{n'_1}\cdots e_n^{n'_n}$ in the universal envelopping algebra $\mathsf{U}(\mathfrak{u})$ of $\mathfrak{u}$. For each $\mathsf{u}\in\mathsf{U}(\mathfrak{u})$ let $\mathsf{u}\mapsto\mathsf{u}^*$ denote the algebra map of $\mathsf{U}(\mathfrak{u})$ into the differential operators on $\mathcal{C}^\infty(P,\mathbb{K})$ induced by the Lie algebra map $\zeta\mapsto\zeta^*$ from the Lie algebra $\mathfrak{u}$ to the fundamental fields being part of the differential operators. Since obviously the vector fields $\partial^h_1,\ldots,\partial^h_m$, $e^*_1\ldots,e^*_n$ are a local base of all the vector fields in $\tau^{-1}(\mathcal{U}_\kappa)$ any differential operator D' in $\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)$ takes the following local form for all $y\in\tau^{-1}(\mathcal{U}_\kappa)$

$$(1.18)\qquad D'\big(\psi'_{(1)},\ldots,\psi'_{(k)}\big)(y)= \sum_{a_1=1}^{p_1}\cdots\sum_{a_k=1}^{p_k}\sum_{b=1}^{q}\sum_{\substack{I_1,\ldots,I_k\\ |I_1|,\ldots,|I_k|\leq N}}\sum_{\substack{J_1,\ldots,J_k\\ |J_1|,\ldots,|J_k|\leq N}} D'^{bI_1\cdots I_kJ_1\cdots J_k}_{\kappa a_1\cdots a_k}(y) \big(e^*_{J_1}\partial^h_{I_1}\psi'^{a_1}_{(1)}\big)(y)\cdots\big(e^*_{J_k}\partial^h_{I_k}\psi'^{a_k}_{(k)}\big)(y)\ w_b.$$

Inserting the U-equivariant maps $\hat\psi_{(j)}:P\to V_j$ we get for all integers $1\leq\mu\leq m$ and $1\leq\nu\leq n$ using eqn (1.17)

$$(1.19)\qquad \partial^h_\mu(\hat\psi^{a_j}_{(j)})=\widehat{\partial_\mu(\psi^{a_j}_{(j)})}\ \text{ and }\ e^*_\nu(\hat\psi^{a_j}_{(j)})=-\sum_{a_j=1}^{p_j}\dot\rho_j(e_\nu)^{a_j}_{a'_j}\hat\psi^{a'_j}_{(j)}.$$

This shows already that if D' was U-equivariant modulo $\mathbf{K}:=\mathbf{K}_1+\cdots+\mathbf{K}_k$, i.e. for each $u\in U$ there is $K_u\in\mathbf{K}$ such that $uD'=D'+K_u$ then $\mathrm{res}(D')$ maps $(\psi_{(1)},\ldots,\psi_{(k)})$ to a U-equivariant smooth map $P\to W$ which can be identified with a smooth section in $\Gamma^\infty(M,P[W])$. The above local considerations show that $\mathrm{res}(D')$ is k-differential on M.
Let us show that the kernel of the restriction map is in $\mathbf{K}_1+\cdots+\mathbf{K}_k$ (the other inclusion has already been shown): Let $D'\in\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)$ such that the restriction of D' to any k U-equivariant maps vanishes. Looking at is local form (1.18) we can transform all the derivatives with respect to fundamental fields into matrix-multiplication using the equations (1.19) and conclude that the modified local operator

$$\check D'_\kappa\big(\psi'_{(1)},\ldots,\psi'_{(k)}\big)(y)= \sum_{a_1,a'_1=1}^{p_1}\cdots\sum_{a_k,a'_k=1}^{p_k}\sum_{b=1}^{q}\sum_{\substack{I_1,\ldots,I_k\\ |I_1|,\ldots,|I_k|\leq N}}\sum_{\substack{J_1,\ldots,J_k\\ |J_1|,\ldots,|J_k|\leq N}} D'^{bI_1\cdots I_kJ_1\cdots J_k}_{a_1\cdots a_k}(y) \dot\rho_1\big(S(e_{J_1})\big)^{a_1}_{a'_1}\big(\partial^h_{I_1}\psi'^{a'_1}_{(1)}\big)(y)\cdots\dot\rho_k\big(S(e_{J_1})\big)^{a_k}_{a'_k}\big(\partial^h_{I_k}\psi'^{a'_k}_{(k)}\big)(y)\ w_b$$

always vanishes on *all* smooth functions $\psi'_{(1)},\ldots,\psi'_{(k)}$ on $\tau^{-1}(\mathcal{U}_\kappa)$ having values in V_1,..., V_k, respectively: indeed, since $\tau^{-1}(\mathcal{U}_\kappa)$ is diffeomorphic to $\mathcal{U}_\kappa\times U$ it is clear that $\check D'_\kappa$ only contains derivatives in the direction of $\mathcal{U}_\kappa$, and can hence be considered as a family of k-differential operators on $\mathcal{U}_\kappa$ parametrised by U. By the local form of the U-equivariant sections $\hat\psi_{(j)}$, (1.17), we see that the functions $\psi^{a'_j}_{(j)}$ are completely arbitrary. Therefore the 'family' vanishes, hence $\check D'_\kappa$ vanishes. We can thus subtract $\check D'_\kappa$ from D' without changing D'. In this difference the following terms will occur

$$\begin{aligned}
&\delta^{a_1}_{a'_1}\big(e^*_{J_1}\partial^h_{I_1}\psi'^{a'_1}_{(1)}\big)(y)\cdots\delta^{a_k}_{a'_k}\big(e^*_{J_k}\partial^h_{I_k}\psi'^{a'_k}_{(k)}\big)(y)\\
&\qquad -\dot\rho_1\big(S(e_{J_1})\big)^{a_1}_{a'_1}\partial^h_{I_1}\psi'^{a'_1}_{(1)}\big)(y)\cdots\dot\rho_k\big(S(e_{J_k})\big)^{a_k}_{a'_k}\partial^h_{I_k}\psi'^{a'_k}_{(k)}\big)(y)\\
&= \sum_{r=1}^{k}\Big(\delta^{a_1}_{a'_1}\big(e^*_{J_1}\partial^h_{I_1}\psi'^{a'_1}_{(1)}\big)(y)\cdots\delta^{a_{r-1}}_{a'_{r-1}}\big(e^*_{J_{r-1}}\partial^h_{I_{r-1}}\psi'^{a'_{r-1}}_{(r-1)}\big)(y)\\
&\qquad\qquad \Big(\delta^{a_r}_{a'_r}\big(e^*_{J_r}\partial^h_{I_r}\psi'^{a'_r}_{(r)}\big)(y)-\dot\rho_r\big(S(e_{J_r})\big)^{a_r}_{a'_r}\partial^h_{I_r}\psi'^{a'_r}_{(r)}\big)(y)\Big)\\
&\qquad\qquad \dot\rho_{r+1}\big(S(e_{J_{r+1}})\big)^{a_{r+1}}_{a'_{r+1}}\partial^h_{I_{r+1}}\psi'^{a'_{r+1}}_{(r+1)}\big)(y)\cdots\dot\rho_k\big(S(e_{J_k})\big)^{a_k}_{a'_k}\partial^h_{I_k}\psi'^{a'_k}_{(k)}\big)(y)\Big),
\end{aligned}$$

and the difference in the rth summand always contains a factor of $\zeta^*\mathrm{id}_{V_r}-\dot\rho_r(\zeta)$ in its matrix form: indeed, according to the Poincaré-Birkhoff-Witt-Theorem, the vector space $\mathsf{U}(\mathfrak{u})$ is spanned by the monomials ζ^N, $\zeta\in\mathfrak{u}$, and N any nonnegative integer, and we get for all $N\geq 1$

$$(\zeta^N)^*\mathrm{id}_{V_r}-\dot\rho_r(\zeta^N)=\sum_{t=0}^{N-1}(\zeta^t)^*\dot\rho_r(\zeta^{N-1-t})\Big(\zeta^*\mathrm{id}_{V_r}-\dot\rho_r(\zeta)\Big).$$

It follows that for each $\kappa\in\mathfrak{S}$ there are locally defined differential operators $D'_{\kappa 1},\ldots,D'_{\kappa k}$ in $\tau^{-1}(\mathcal{U}_\kappa)$ such that $D'_{\kappa j}\in\mathbf{K}_j$ for all $1\leq j\leq k$ and $D'=D'_{\kappa 1}+\cdots+D'_{\kappa k}$ locally on $\tau^{-1}(\mathcal{U}_\kappa)$. Let $(\chi_\kappa)_{\kappa\in\mathfrak{S}}$ be a partition of unity subordinate to the open cover $(\mathcal{U}_\kappa)_{\kappa\in\mathfrak{S}}$. Defining the global differential operator $D'_j=\sum_{\kappa\in\mathfrak{S}}(\chi_\kappa\circ\tau)D'_{\kappa j}$ for each $1\leq j\leq k$ we see that $D'_j\in\mathbf{K}_j$ and $D'=D'_1+\cdots+D'_k$, showing

$$\mathrm{Ker}(\mathrm{res})=\mathbf{K}_1+\cdots+\mathbf{K}_k.$$

Next, let D be any k-differential operator in $\mathbf{Diff}_M(E_1,\ldots,E_k;F)$ given locally as in equation (1.8) where we use the particular base sections $f^{(j)}_{a_j}(x)=[\sigma_\kappa(x),v^{(j)}_{a_j}]$, $1\leq j\leq k$, $1\leq a_j\leq p_j$, for the bundles E_j, and $g_b(x)=[\sigma_\kappa(x),w_b]$, $1\leq b\leq q$ for the bundle F. Define for all $x\in\mathcal{U}_\kappa$ and $u\in U$ and each combination of indices the smooth map

$$D'^{bI_1\cdots I_k}_{\kappa a_1\cdots a_k}\big(\sigma_\kappa(x)u\big):=\sum_{a'_1=1}^{p_1}\cdots\sum_{a'_k=1}^{p_k}\sum_{b'=1}^{q}\rho(u^{-1})^b_{b'}D^{b'I_1\cdots I_k}_{\kappa a'_1\cdots a'_k}(x)\rho_1(u)^{a'_1}_{a_1}\cdots\rho_k(u)^{a'_k}_{a_k}.$$

Now form the local k-differential operator D'_κ as in eqn (1.18) where the multi-indices $J_1,\ldots,J_k$ are void, and globalize the expression to a k-differential operator $D'=\sum_{\kappa\in\mathfrak{S}}(\chi_\kappa\circ\tau)D'_\kappa$ on P by using the above partition of unity $(\chi_\kappa)_{\kappa\in\mathfrak{S}}$. It can easily be checked that D' is U-equivariant and that the restriction to U-equivariant smooth maps gives back D which proves surjectivity of the restriction map to the k-differential operators. We have the isomorphism

$$\frac{\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)^U}{\big(\mathbf{K}_1+\cdots+\mathbf{K}_k\big)^U}\cong\mathbf{Diff}_M(E_1,\ldots,E_k;F).$$

and since the k-differential operators on P which are only U-equivariant modulo $\mathbf{K}$ restrict to k-differential operators on M, the other stated isomorphism is also clear. $\square$

2 G-ϑ-equivariant Principal Bundles with Connections over Homogeneous Spaces

In this Section we shall mainly be interested in G-equivariant structures over *homogeneous spaces*: let G be a Lie group having Lie algebra $(\mathfrak{g}, [\ ,\ ])$, let $H \subset G$ be a closed subgroup where $\mathfrak{h} \subset \mathfrak{g}$ denotes its Lie algebra. Let $M = G/H$ denote the homogeneous space where $\pi : G \to G/H$ is the canonical projection, and $o = \pi(e) \in M$ the distinguished point. Recall that $(G, \pi, G/H, H)$ is a principal H-bundle over G/H.

2.1 Some G-equivariant Versions of Categories of Fibered Manifolds

Let G be a Lie group with Lie algebra $(\mathfrak{g}, [\ ,\ ])$. All the categories mentioned in the previous Section can now be considered in a 'G-equivariant way': Let $G \cdot \mathcal{FM}$ be the category of all *G-equivariant fibered manifolds*: the class of objects consists of all those fibered manifolds (E, τ, M) where the total space E and the base M are both left G-spaces and the projection $\tau : E \to M$ is G-equivariant, i.e. $\tau(gy) = g\tau(y)$ for all $g \in G$ and $y \in E$, and each set of morphisms consists of all those morphisms $\Phi : (E, \tau, M) \to (E', \tau', M')$ of fibered manifolds which in addition intertwine the left G-actions, i.e. $\Phi(gy) = g\Phi(y)$ for all $g \in G$ and all $y \in E$. In the same way we define the category of all *G-equivariant fibre bundles*, $G \cdot \mathcal{FB}$, and the category of all *G-equivariant vector bundles*, $G \cdot \mathcal{VB}$. Note also also the G-equivariant versions 'over M', i.e. the categories $G \cdot \mathcal{FM}_M$, $G \cdot \mathcal{FB}_M$, and $G \cdot \mathcal{VB}_M$.

For a G-equivariant version of the category $\mathcal{PB}(U)$ of all principal fibre bundles with fixed structure group U we first have to say how the left G-action and the right U-action on the total space P of a principal bundle (P, τ, M, U) are related: the simplest choice would be to declare that they commute, i.e. $g(pu) = (gp)u$ for all $g \in G$, $u \in U$, $p \in P$. In this case we shall speak of the category of all *G-equivariant principal bundles with fixed structure group U* (where morphisms intertwine all the actions) denoted by $G \cdot \mathcal{PB}(U)$, and its 'over M' version $G \cdot \mathcal{PB}(U)_M$. In the next Section we shall, however, treat the slightly more general version: let $\vartheta : G \times U \to U$ be a smooth left *automorphic* G-action on the Lie group U: that is, ϑ is a smooth left G-action on the manifold U such that for each $g \in G$ the map $\vartheta_g : U \to U : u \mapsto \vartheta(g, u)$ is an *Lie group automorphism of* U, i.e.

for all $u_1, u_2 \in U$

$$\vartheta_g(u_1 u_2) = \vartheta_g(u_1)\vartheta_g(u_2) \quad \text{and} \quad \vartheta_g(e_U) = e_U. \tag{2.1}$$

Note that if U modulo its identity component is a finitely generated group then the group $\mathrm{Aut}(U)$ of all Lie group automorphisms $U \to U$ carrying the compact-open topology is itself a Lie group, see e.g. [18] in which case the map $g \mapsto \vartheta_g$ is a smooth Lie group homomorphism $G \to \mathrm{Aut}(U)$. But we shall not need this restriction.

A principal U-bundle (P, τ, M, U) over a left G-space M having structure group U is called *G-ϑ-equivariant* iff there is a left G-action ℓ' on the total space P – which we shall mostly write $\ell'_g(p) = gp$ for all $g \in G$ and $p \in P$- projecting on the left G-action ℓ on M such that

$$\forall\ g \in G,\ p \in P,\ u \in U: \qquad g(pu) = (gp)\vartheta_g(u). \tag{2.2}$$

We shall denote the corresponding category by $G \cdot \mathcal{PB}(U;\vartheta)$ (where morphisms simply intertwine all the group actions) and its 'over M' version by $G \cdot \mathcal{PB}(U;\vartheta)_M$. For the trivial case $\vartheta_g = \mathrm{id}_U$ for all $g \in G$ we would return to the aforementioned *G-equivariant principal U-bundles (over M)*, $G \cdot \mathcal{PB}(U)$ and $G \cdot \mathcal{PB}(U)_M$.

Finally, a G-equivariant version of principal U-bundles with connection can be obtained as follows: objects are quintuples (P, τ, M, U, α) where the principal U-bundle is G-ϑ-equivariant, and $\alpha \in \Gamma^\infty(P, T^*P \otimes \mathfrak{u})$ is a connection 1-form satisfying

$$\forall\ g \in G: \quad \ell'^*_g \alpha = T_{e_U}\vartheta_g \circ \alpha. \tag{2.3}$$

Morphisms in this category between (P, τ, M, U, α) and $(P', \tau', M', U, \alpha')$ are just smooth maps $\Phi : P \to P'$ of total spaces intertwining the left G- and the right U-action (thereby inducing a unique G-equivariant smooth map $\phi : M \to M'$ on the bases) such that $\Phi^*\alpha' = \alpha$. We shall denote this category by $G \cdot \mathcal{PBC}(U, \vartheta)$ (and its 'over M'-version by $G \cdot \mathcal{PBC}(U, \vartheta)_M$), and in the particular case $\vartheta_g = \mathrm{id}_U$ for all $g \in G$ it will be denoted by $G \cdot \mathcal{PBC}(U)$ (with 'over M-version' $G \cdot \mathcal{PBC}(U)_M$).

An example with nontrivial ϑ will be given in Subsubsection 2.5.2.

2.2 G-equivariant Fibered manifolds over Homogeneous Spaces

Let $M = G/H$ be a homogeneous space. Note first that for any left G-space E any smooth G-equivariant map $f : E \to M$ is automatically a surjective submersion: let $x_0 \in M$ a value of f, i.e. $x_0 = f(y_0)$ for some $y_0 \in E$. Then for all $g \in G$ one has $gx_0 = gf(y_0) = f(gy_0)$ showing surjectivity because of the transitivity of the G-action on M. Furthermore, any tangent vector v at $x \in M$ is the value of a fundamental field $\xi_M(x)$ of the left G-action on M. Let $y \in E$ such that $f(y) = x$ then $T_y f\big(\xi_E(y)\big) = \xi_M\big(f(y)\big) = v$ showing that f is a submersion.

Let $(E, \tau, G/H)$ be a G-equivariant fibered manifold over M. There is an obvious functor $G \cdot \mathcal{FM}_M \to H \cdot \mathcal{M}f$ assigning to $(E, \tau, G/H)$ the fibre E_o over the distinguished point $o = \pi(e)$ (which is a left H-space since H fixes o). Any G-equivariant smooth map of total spaces inducing the identity map on M induces an H-equivariant map on the fibres over o. On the other hand, using the fact that $(G, \pi, G/H, H)$ is a G-equivariant principal H-bundle over G/H, we see that for any left H-space S the associated bundle functor $S \to G_H[S]$ which maps the morphism f of left H-spaces $S \to S'$ to the map $[g, z] \mapsto [g, f(z)]$ of associated fibre bundles over M. The following Proposition –which seems to be well-known– shows that the two functors constitute an equivalence of categories:

Proposition 2.1. *Let $G \cdot \mathcal{FM}_M$ the category of G-equivariant fibered manifolds over $M = G/H$ and $H \cdot \mathcal{M}f$ the category of smooth left H-spaces. Then the two functors $G_H[\ ] : H \cdot \mathcal{M}f \to G \cdot \mathcal{FM}_M$ and $(\ )_o : G \cdot \mathcal{FM}_M \to H \cdot \mathcal{M}f$ constitute an equivalence of categories*

$$H \cdot \mathcal{M}f \quad \simeq \quad G \cdot \mathcal{FM}_{G/H}$$

Proof: Consider first $(\ )_o \circ G_H[\ ]$. For any left H-space S let $\psi_S : G_H[S]_o \to S$ be the inverse of the map $\Phi_o : S \to G_H[S]_o$ given by $\Phi_o(z) = [e, z]$ which has been mentioned earlier. It is easy to check that $S \to \psi_S$ constitute a natural isomorphism $(\ )_o \circ G_H[\ ]$ to $\mathrm{id}_{H \cdot \mathcal{M}f}$. On the other hand, let (E, τ, M) be in $G \cdot \mathcal{FM}_M$. There is a canonical smooth map $\Phi_{(E,\tau,M)} = \Phi : G_H[E_o] \to E$ sending $[g, z]$ to gz which clearly induces the identity on M and is natural in (E, τ, M). It is easy to check that the map Φ is a bijection. We shall show that the map $\hat{\Phi} : G \times E_o \to E$ defined by $\hat{\Phi}(g, z) = gz$ is a submersion which shows that Φ is also a submersion, and since $\dim(G_H[E_o]) = \dim(E)$ Φ is a local diffeomorphism and therefore a diffeomorphism being bijective: indeed, let let $y \in E$ and $v \in T_y E$. Writing $x = \tau(y) \in M$ we have $w := T_y\tau(v) \in T_x M$, and since M is homogeneous there is $g \in G$ and $\xi \in \mathfrak{g}$ such that $x = go$ and $w = \frac{d}{dt}(g\exp(t\xi)o)|_{t=0}$. Let $z := g^{-1}y \in E_o$, and consider

$$v' := \frac{d}{dt}(g\exp(t\xi)z)|_{t=0} = T_{(g,z)}\hat{\Phi}\big(T_e L_g(\xi), 0\big) \in T_y E.$$

Clearly $T_y\tau(v') = w = T_y\tau(v)$, so $v - v' \in \mathrm{Ker}\ T_y\tau = T_y(E_x)$ since τ is a submersion. Writing $\ell_g : M \to M$ for the left G-action on M and $\ell'_g : E \to E$ for the left G-action on E, set $v'' := (T_z\ell'_g)^{-1}(v - v') \in T_z E$. Clearly, by equivariance of τ we get that $v'' \in \mathrm{Ker}\ T_z\tau = T_z E_o$. Therefore $v = v' + T_z\ell'_g(v'')$, and we get

$$v = T_{(g,z)}\hat{\Phi}\big(T_e L_g(\xi), v''\big),$$

whence $\hat{\Phi}$ is a submersion, which ends the proof. □

Note that this implies that every G-equivariant (weakly) fibered manifold over M is isomorphic to a G-equivariant fibre bundle whence the two categories $G\cdot\mathcal{FM}_M$ and $G\cdot\mathcal{FB}_M$ are also equivalent.

Recall that the space of all smooth sections $\Gamma^\infty(G/H, G_H[S])$ is in bijection with the space of all H-equivariant functions $\mathcal{C}^\infty(G,S)^H$. It is easy to see that the space of all G-invariant smooth sections is isomorphic to the subset of fixed points of the H-action, i.e.

$$S^H \cong \Gamma^\infty(G/H, G_H[S])^G \tag{2.4}$$

where the isomorphism is given by $z \mapsto \big(\pi(g)\mapsto [g,z]\big)$.

In a similar manner it is shown that the category of *left H-modules* (which are finite-dimensional $\mathbb{K}$ vector spaces) is equivalent to the category of all *G-equivariant vector bundles over M*, $G\cdot\mathcal{VB}_M$.

Recall that the *tangent bundle* of G/H is isomorphic to

$$G_H[\mathfrak{g}/\mathfrak{h}] \cong TM \tag{2.5}$$

where the Lie group H acts on the quotient $\mathfrak{g}/\mathfrak{h}$ as follows: let $\varpi:\mathfrak{g}\to\mathfrak{g}/\mathfrak{h}$ be the canonical projection, then the following representation $h\mapsto Ad'_h$ on $\mathfrak{g}/\mathfrak{h}$ is well-defined

$$Ad'_h\big(\varpi(\xi)\big) \quad := \quad \varpi\big(Ad_h(\xi)\big) \tag{2.6}$$

since $\mathfrak{h}$ is stable by all the Ad_h, $h\in H$. Note that the kernel of the linear map $T_e\pi : T_eG = \mathfrak{g}\to T_o(G/H)$ is equal to $\mathfrak{h}$, hence there is the linear isomorphism $\pi' : \mathfrak{g}/\mathfrak{h}\to T_o(G/H)$ which is clearly H-equivariant with respect to the actions (2.6) and $h\mapsto T_o\ell_h$. The above mentioned isomorphism of vector bundles is given by

$$[g,z]\mapsto T_e\ell_g\big(\pi'(z)\big) \tag{2.7}$$

for all $g\in G$ and $z\in\mathfrak{g}/\mathfrak{h}$. Let us compute the *Lie bracket of two vector fields X and Y on G/H*: identifying $T(G/H)$ with the associated bundle $G_H[\mathfrak{g}/\mathfrak{h}]$ there are the two frame forms $\hat{X},\hat{Y}:G\to\mathfrak{g}/\mathfrak{h}$, i.e. smooth H-equivariant functions corresponding to the two sections X,Y. Now fix a connection 1-form α in the principal H-bundle $G\to G/H$. In general α is NOT G-invariant, see the next Chapter. Let $\tilde{X},\tilde{Y}\in\Gamma^\infty(G,TG)$ be the horizontal lifts with respect to α, see the end of Subsection 1.1. Then the pairs $(\tilde{X},X)$ and $(\tilde{Y},Y)$ are π-related, i.e. $T\pi\circ\tilde{X} = X\circ\pi$ and $T\pi\circ\tilde{Y} = Y\circ\pi$, and $\tilde{X}$ and$\tilde{Y}$ are invariant under the right multiplication with H. Define the smooth functions $\hat{X}',\hat{Y}':G\to\mathfrak{g}$ by $\hat{X}'(g) = (T_eL_g)^{-1}\big(\tilde{X}(g)\big)$ and $\hat{Y}'(g) = (T_eL_g)^{-1}\big(\tilde{Y}(g)\big)$. Then both $\hat{X}'$ and $\hat{Y}'$ are H-equivariant, i.e. $\hat{X}'(gh) = Ad_{h^{-1}}\big(\hat{X}'(g)\big)$ and likewise for $\hat{Y}'$ for all $g\in G$ and $h\in H$, and project to the frame forms, i.e. $\varpi\circ\hat{X}' = \hat{X}$ and $\varpi\circ\hat{Y}' = \hat{Y}$. Since the pairs $(\tilde{X}, X)$ and $(\tilde{Y}, Y)$ are π-related, the same holds for the pair $\big([\tilde{X}, \tilde{Y}], [X, Y]\big)$ which allows to compute the Lie bracket of X and Y upon using the frame forms:

$$(2.8)\quad [X,Y]_{\mathrm{Lie}}(\pi(g)) = \Big[g, \hat{X}'(g)^{+}\big(\hat{Y}\big)(g) - \hat{Y}'(g)^{+}\big(\hat{X}\big)(g) + \varpi\big([\hat{X}'(g), \hat{Y}'(g)]\big)\Big]$$

It is straight forward to check that the above formula is well-defined and does not depend on the connection chosen.

Exercise: Show that the frame form of the fundamental field $\xi_{G/H}$ of the left G-action ℓ on G/H is given by (for all $g \in G$ and $\xi \in \mathfrak{g}$)

$$\widehat{\xi_{G/H}}(g) = \varpi\big(Ad_{g^{-1}}(\xi)\big).$$

Exercise: Let U be a Lie group, and $\theta : U \to G$, $j : H \to U$ be smooth Lie group homomorphisms such that the following diagram commutes

$$\begin{array}{ccccc} G & & \stackrel{\theta}{\longleftarrow} & & U \\ & i \nwarrow & & \nearrow j & \\ & & H & & \end{array}$$

where $i : H \to G$ is the natural inclusion of subgroups. Show that the associated bundle $G_H[U/j(H)]$ over G/H carries the structure of a G-equivariant *Lie groupoid* over the unit space G/H, see [26] or [28] for definitions, where (for all $g, g_1, g_2 \in G$, $u, u_1, u_2 \in U$) the target projection t equals the bundle projection, the source projection s is given by $s([g, u \bmod j(H)]) = \pi(g\theta(u))$, the unit map is given by $1(\pi(g)) = [g, e_U \bmod j(H)]$, the multiplication by

$$\mu\Big([g_1, u_1 \bmod j(H)], [g_2, u_2 \bmod j(H)]\Big) = \big[g_1, u_1 j\big(\theta(u_1)^{-1} g_1^{-1} g_2\big) u_2 \bmod j(H)\big],$$

and the inverse by $[g, u \bmod j(H)]^{-1} = [g\theta(u), u^{-1} \bmod j(H)]$. Moreover show that every G-equivariant Lie groupoid is isomorphic (in that category) to a Lie groupoid of the above form (equivalence of appropriate categories). Hint: define the Lie group U as the pull-back of the principal bundle $(G, \pi, G/H, H)$ over G/H to the t-fibre E_o over the distinguished point o by means of the restriction of the source projection s to E_o.

2.3 G-ϑ-equivariant Principal Bundles over Homogeneous Spaces

Let G be a Lie group with Lie algebra $(\mathfrak{g}, [\ ,\ ])$, let $H \subset G$ be a closed subgroup with Lie algebra $(\mathfrak{g}, [\ ,\ ])$, and let M be the homogeneous space $M = G/H$. Let U be a Lie group with Lie algebra $(\mathfrak{u}, [\ ,\ ])$, and let $\vartheta : G \times U \to U$ be an automorphic left G-action on U, written $\vartheta(g, u) = \vartheta_g(u)$.

For the following it is rather convenient to form the *semidirect product* $G_\vartheta \times U$ of the two Lie groups U and G with respect to the automorphic action ϑ: recall that the underlying manifold is $G \times U$, and for all $u, u_1, u_2 \in U$ and $g, g_1, g_2 \in G$ the multiplication is defined by

$$(g_1, u_1)(g_2, u_2) = \big(g_1 g_2, \vartheta_{g_2^{-1}}(u_1) u_2\big) \tag{2.9}$$

whence the unit element is (e, e_U), and the inverse of (g, u) is given by $\big(g^{-1}, \vartheta_g(u^{-1})\big)$. We shall denote left and right multiplication in $G_\vartheta \times U$ by L^ϑ and R^ϑ, respectively, i.e. $L^\vartheta_{(g_1,u_1)}(g_2, u_2) = (g_1, u_1)(g_2, u_2) = R^\vartheta_{(g_2,u_2)}(g_1, u_1)$ for all $g_1, g_2 \in G$ and $u_1, u_2 \in U$. Note that we have chosen a less common convention for the semidirect product in order to maintain the order $G \times U$ as opposed to the usual $U \times G$. Moreover recall that the subset $\{e\} \times U$ is a closed normal subgroup of $G_\vartheta \times U$ with factor group isomorphic to G. A concrete isomorphism is realized by the projection $\mathrm{pr}_1 : G_\vartheta \times U \to G$.

Now let $(P, \tau, G/H, U)$ a G-ϑ-equivariant principal U-bundle over G/H. In order to get an idea about the relevant structures involved, consider its fibre $P_o = \tau^{-1}(\{o\})$ over the distinguished point $o = \pi(e) \in G/H$, and *choose an element* $y_P \in P_o$. Since the map $U \to P_o$ given by $u \mapsto y_P u$ is a diffeomorphism, and since P_o is a *left* H-space there is a unique smooth map $\check{\chi}_P : H \to U$ such that for all $h \in H$:

$$y_P \check{\chi}_P(h) := h y_P. \tag{2.10}$$

We clearly have $\check{\chi}_P(e) = e_U$, and we get for all $h_1, h_2 \in H$

$$\begin{aligned} y_P \check{\chi}_P(h_1 h_2) &= (h_1 h_2) y_P = h_1 (h_2 y_P) = h_1 \big(y_P \check{\chi}_P(h_2)\big) \overset{(2.2)}{=} (h_1 y_P) \vartheta_{h_1}\big(\check{\chi}_P(h_2)\big) \\ &= \big(y_P \check{\chi}_P(h_1)\big) \vartheta_{h_1}\big(\check{\chi}_P(h_2)\big) = y_P \big(\check{\chi}_P(h_1) \vartheta_{h_1}\big(\check{\chi}_P(h_2)\big)\big) \end{aligned}$$

Hence $\check{\chi}_P(h_1 h_2) = \check{\chi}_P(h_1) \vartheta_{h_1}\big(\check{\chi}_P(h_2)\big)$, and the map

$$\chi : H \to U : h \mapsto \vartheta_{h^{-1}}\big(\check{\chi}_P(h)\big) \tag{2.11}$$

is easily checked to satisfy the identity

$$\forall\ h_1, h_2 \in H : \qquad \chi(h_1 h_2) = \vartheta_{h_2^{-1}}\big(\chi(h_1)\big) \chi(h_2). \tag{2.12}$$

Recall that the preceding equation (2.12) is the defining condition for the smooth map χ to be a *crossed homomorphism (with respect to ϑ)* $H \to U$. In the trivial case $\vartheta_g = \mathrm{id}_U$ for all $g \in G$ the map χ is a smooth homomorphism of Lie groups. Moreover note that the constant map $\chi(h) = e_U$ for all $h \in H$ is always a crossed homomorphism w.r.t. ϑ. Furthermore, recall that a smooth map $\chi : H \to U$ is a crossed homomorphism w.r.t. ϑ if and only if the combined map

$$\tilde{\chi} : H \to G_{\ \vartheta} \times U : h \mapsto \tilde{\chi}(h) := \big(h, \chi(h)\big)$$

is a homomorphism of Lie groups, i.e. $\tilde{\chi}(e) = (e, e_U)$ and $\tilde{\chi}(h_1 h_2) = \tilde{\chi}(h_1) \tilde{\chi}(h_2)$ for all $h_1, h_2 \in H$, and this is in turn equivalent to the fact that $\chi = \mathrm{pr}_2 \circ \tilde{\chi}$ for some Lie group homomorphism $\tilde{\chi} : H \to G\ _\vartheta \times U$ satisfying $\mathrm{pr}_1 \circ \tilde{\chi} = i_H$ (where $i_H : H \to G$ denotes the natural inclusion).

Define the following set:

$$\mathcal{P} = \mathcal{P}_\vartheta(H, U) := \{\chi : H \to U \mid \chi \text{ crossed homomorphism w.r.t. } \vartheta\}. \tag{2.13}$$

In order to make this set into the set of all objects of a small category we note that for any $u \in U$ and $h \in H$ the map $h \mapsto (e, u)\tilde{\chi}(h)(e, u)^{-1} =: \tilde{\chi}'(h)$ is again a morphism of Lie groups $H \to G\ _\vartheta \times U$ such that $\mathrm{pr}_1 \circ \tilde{\chi}' = i_K$, whence for any $u \in U$ and $h \in H$ the map $u.\chi : h \mapsto \vartheta_{h^{-1}}(u)\chi(h)u^{-1}$ is again a crossed homomorphism $H \to U$. It is not hard to see that the map $U \times \mathcal{P}_\vartheta(H, U) \to \mathcal{P}_\vartheta(H, U)$ is a left H-action (in the sense of sets). We define the morphism sets as follows: for each $\chi, \chi' \in \mathcal{P}$

$$\mathbf{Hom}_{\mathcal{P}}(\chi, \chi') := \{u \in U \mid (u.\chi)(h) = \vartheta_{h^{-1}}(u)\chi(h)u^{-1} = \chi'(h)\ \forall\ h \in H\}, \tag{2.14}$$

where composition of morphisms is defined by group multiplication in U. It immediately follows that each morphism is an isomorphism whence the small category $\mathcal{P}_\vartheta(H, U)$ is a groupoid, in fact the action groupoid of the above left U-action on $\mathcal{P}_\vartheta(H, U)$.

In order to define an associated bundle $G_H[U]$ with typical fibre U we unfortunately need to modify the semidirect product $G_\vartheta \times U$ by a diffeomorphism to relate the multiplication in this product to the convention of the right H-action for associated bundles: let $\Xi : G_\vartheta \times U \to G \times U$ the diffeomorphism

$$\Xi(g, u) := (g, u^{-1}) \tag{2.15}$$

which is an involution on the underlying manifold $G \times U$, i.e. $\Xi \circ \Xi = \mathrm{id}_{G \times U}$. Upon using left and right multiplications in the semidirect product define the following group actions on $G \times U$: $\hat{L} : G \times (G \times U) \to G \times U$, $\hat{R} : (G \times U) \times U \to G \times U$, $\hat{P} : U \times (G \times U) \to G \times U$, and $R^\chi : (G \times U) \times H \to G \times U$ where $\hat{L}$ will be a left G-action, $\hat{R}$ will be a right U-action, $\hat{P}$ will be a left U-action, and R^χ will be a right H-action: for all $g, g_0 \in G$, $h \in H$, and $u, u_0, \tilde{u} \in U$:

$$\begin{aligned}
g_0(g, u) := \hat{L}_{g_0}(g, u) &:= (g_0 g, u) = \big(\Xi^{-1} \circ L^\vartheta_{(g_0, e_U)} \circ \Xi\big)(g, u), && (2.16)\\
(g, u)u_0 := \hat{R}_{u_0}(g, u) &:= \big(g, u\vartheta_{g^{-1}}(u_0)\big) = \big(\Xi^{-1} \circ L^\vartheta_{(e, u_0^{-1})} \circ \Xi\big)(g, u) && (2.17)\\
\tilde{u}(g, u) := \hat{P}_{\tilde{u}}(g, u) &:= (g, \tilde{u}u) = \big(\Xi^{-1} \circ R^\vartheta_{(e, \tilde{u}^{-1})} \circ \Xi\big)(g, u) && (2.18)\\
R^\chi_{h_0}(g, u) &:= \big(gh_0, \chi(h_0)^{-1}\vartheta_{h_0^{-1}}(u)\big) = \big(\Xi^{-1} \circ R^\vartheta_{\tilde{\chi}(h)} \circ \Xi\big)(g, u) && (2.19)
\end{aligned}$$

All these actions are well-defined, and the definition of R^χ (2.19) shows that the map $\lambda^\chi : H \times U \to U$ defined by

$$\lambda^\chi_{h_0}(u) = \lambda^\chi(h_0, u) := \chi(h_0^{-1})^{-1}\vartheta_{h_0}(u) = \vartheta_{h_0}\big(\big(\chi(h_0)u\big)\big) \tag{2.20}$$

is a smooth left H-action on U such that $R^\chi_{h_0}(g,u) = \big(gh_0, \lambda^\chi_{h_0^{-1}}(u)\big)$. Since $\hat{L}$ and $\hat{R}$ come from left multiplications in the semidirect product, whereas $\hat{P}$ and R^χ come from a right multiplications it follows that all the maps $\hat{L}_{g_0}$ and $\hat{R}_{u_0}$ commute with all the maps $\hat{P}_{\tilde{u}}$ and $R^\chi_{h_0}$. Moreover we get

$$g_0\big((g,u)u_0\big) = \big(g_0 g, \vartheta_{g^{-1}}(u_0)\big) = (g_0 g, u)\vartheta_{g_0}(u_0) \tag{2.21}$$

$$\hat{P}_{\tilde{u}}\big(R^\chi_{h_0}(g,u)\big) = \big(gh_0, \tilde{u}\vartheta_{h_0^{-1}}\big(\chi(h_0^{-1})u\big)\big) = R^{\tilde{u}.\chi}_{h_0}\big(\hat{P}_{\tilde{u}}(g,u)\big) \tag{2.22}$$

Note that the subgroup H of G becomes a closed subgroup $\tilde{\chi}(H)$ of the semidirect product $G_\vartheta \times U$ via $\tilde{\chi}$: let $(h_n)_{n\in\mathbb{N}}$ be a sequence in H such that the sequence $\big(\tilde{\chi}(h_n)\big)|_{n\in\mathbb{N}}$ converges. In particular its first component $(h_n)_{n\in\mathbb{N}}$ converges to $h \in H$, hence $\big(\tilde{\chi}(h_n)\big)|_{n\in\mathbb{N}}$ converges to $\tilde{\chi}(h) \in \tilde{\chi}(H)$.

We can now define a functor $\mathbf{P} : \mathcal{P}_\vartheta(H,U) \to G \cdot \mathcal{PB}(U;\vartheta)_{G/H}$ as follows: for each $\chi \in \mathcal{P}_\vartheta(H,U)$ let P_χ be the associated bundle $G_H[U]$ where the subgroup H acts on U on the left via λ^χ, see (2.20). By definition of the right H-action (2.19) this is equal to $(G \times U)/H$. Let

$$\kappa_\chi = \kappa : G \times U \to P_\chi = G_H[U] \tag{2.23}$$

denote the projection, and for any $(g,u) \in G\times U$ we shall write $[g,u]_\chi$ for $\kappa(g,u) \in P_\chi$. Since H acts freely and properly on the right on $G \times U$ via R^χ (because Ξ is a diffeomorphism and $\tilde{\chi}(H)$ is a closed subgroup of $G_\vartheta \times U$) the above left G-action $\hat{L}$ (2.16) and the above right U-action $\hat{R}$ (2.17) pass to the quotient P_χ to define a left G-action ℓ' and a right U-action r such that κ intertwines the actions, i.e. $\ell'_{g_0} \circ \kappa = \kappa \circ \hat{L}_{g_0}$ and $r_{u_0} \circ \kappa = \kappa \circ \hat{R}_{u_0}$ for all $g_0 \in G$ and $u_0 \in U$. We get for all $g \in G$ and $u \in U$:

$$\ell'_{g_0}\big([g,u]\chi\big) = g_0[g,u]_\chi := [g_0 g, u]_\chi, \tag{2.24}$$

$$r_{u_0}\big([g,u]_\chi\big) = [g,u]_\chi u_0 := [g, u\vartheta_{g^{-1}}(u_0)]_\chi. \tag{2.25}$$

Equation (2.21) passes to the quotient as follows:

$$g_0\big(\big([g,u]_\chi\big)u_0\big) = \big([g_0 g, u]_\chi\big)\vartheta_{g_0}(u_0) \tag{2.26}$$

Note that the right U-action on P_χ is free: if for some $g \in G$, $u, u_0 \in U$ we have $[g,u]_\chi = [g,u]_\chi u_0 = [g, u\vartheta_{g^{-1}}(u_0)]_\chi$ it follows that $u = u\vartheta_{g^{-1}}(u_0)$ hence $u_0 = e_U$ whence the action is free. Moreover the right U-orbits coincide with the fibres of the associated bundle: by definition, the right U-orbits are contained in the fibres, on the other hand each $[g,u]_\chi \in \tau^{-1}\big(\pi(g)\big)$ is equal to $[g,e_U]_\chi \vartheta_g(u)$, hence in the right U-orbit passing through $[g,e_U]_\chi$. Using [23, p.87, Lemma 10.3] it follows that $(P_\chi, \tau, G/H, U)$ is a principal U-bundle over G/H which is G-ϑ-equivariant by eq (2.26). Next, let $\chi' \in \mathcal{P}_\vartheta(H,U)$, and let $\tilde{u} \in \mathbf{Hom}_{\mathcal{P}}(\chi,\chi')$ whence $\chi' = \tilde{u}.\chi$.

Using eqn (2.22) we see that the left U-action $\hat{P}_{\tilde{u}}$ on $G\times U$ induces a unique map $P_{\tilde{u}}:P_\chi\to P_{\tilde{u}.\chi}$ such that $P_{\tilde{u}}\circ\kappa_\chi=\kappa_{\tilde{u}.\chi}\circ\hat{P}_{\tilde{u}}$ for all $\tilde{u}\in U$. We get

$$P_{\tilde{u}}\big([g,u]_\chi\big):=[g,\tilde{u}u]_{\tilde{u}.\chi} \tag{2.27}$$

for all $g\in G$ and $u,\tilde{u}\in U$. Since $\hat{P}_{\tilde{u}}$ commutes with all $\hat{L}_{g_0}$ and $\hat{R}_u$ it follows that $P_{\tilde{u}}$ is a morphism of G-equivariant principal U-bundles over G/H. Clearly $P_{e_U}=\mathrm{id}_{P_\chi}$ and $P_{\tilde{u}_1}\circ P_{\tilde{u}_2}=P_{\tilde{u}_1\tilde{u}_2}$ for all $\tilde{u}_1,\tilde{u}_2\in U$, whence $\chi\to P_\chi$, $\tilde{u}\mapsto P_{\tilde{u}}$ is a covariant functor $\mathcal{P}_\vartheta(H,U)\to G\cdot\mathcal{PB}(U;\vartheta)_{G/H}$.

Conversely, in order to construct a functor $\mathbf{X}:G\cdot\mathcal{PB}(U;\vartheta)_{G/H}\to\mathcal{P}_\vartheta(H,U)$ let $(P,\tau,G/H,U)$ a G-ϑ-equivariant principal U-bundle over G/H, consider its fibre $P_o=\tau^{-1}(\{o\})$ over the distinguished point $o=\pi(e)\in G/H$, and *choose an element* $y_P\in P_o$. By the preceding considerations there is a unique crossed homomorphism $\chi_P:H\to U$ such that $y_P\vartheta_h\big(\chi_P(h)\big):=hy_P$ for all $h\in H$. Hence we get an assignment $\mathbf{X}:G\cdot\mathcal{PB}(U;\vartheta)_{G/H}\to\mathcal{P}_\vartheta(H,U)$ where $(P,\tau,G/H,U)$ is assigned the crossed homomorphism χ_P. Again note that the assignment depends on the choice $y_P\in P_o$. Furthermore, let $\Phi:(P,\tau,G/H,U)\to(P',\tau',G/H,U)$ a morphism of G-ϑ-equivariant principal U-bundles over G/H. Then $\Phi(y_P)\in P'_o$ whence there is a unique $\tilde{u}_\Phi\in U$ such that $\Phi(y_P)=y_{P'}\tilde{u}_\Phi$. We compute for all $h\in H$

$$\begin{aligned}\Phi(hy_P)&=\Phi\big(y_P\check{\chi}_P(h)\big)=\Phi(y_P)\check{\chi}_P(h)=(y_{P'}\tilde{u}_\Phi)\check{\chi}_P(h)=y_{P'}\big(\tilde{u}_\Phi\check{\chi}_P(h)\big)\\ \Phi(hy_P)&=h\Phi(y_P)=h\big(y_{P'}\tilde{u}_\Phi\big)=(hy_{P'})\vartheta_h(\tilde{u}_\Phi)=y_{P'}\big(\check{\chi}_{P'}(h)\vartheta_h(\tilde{u}_\Phi)\big)\end{aligned}$$

Applying $\vartheta_{h^{-1}}$ to the resulting equation for $\check{\chi}_P$ and $\check{\chi}_{P'}$ we get

$$\chi_{P'}(h)=\vartheta_{h^{-1}}(\tilde{u}_\Phi)\chi_P(h)\tilde{u}_\Phi^{-1}$$

whence $\tilde{u}_\Phi\in\mathbf{Hom}_{\mathcal{P}}(\chi_P,\chi_{P'})$. It is easily checked that the rule assigning to the bundle $(P,\tau,G/H,U)$ the crossed homomorphism χ_P and to the morphism Φ the group element $\tilde{u}_\Phi$ is a covariant functor $\mathbf{X}:G\cdot\mathcal{PB}(U;\vartheta)_{G/H}\to\mathcal{P}_\vartheta(H,U)$.

Proposition 2.2. *The two functors* $\mathbf{P}:\chi\to P_\chi$ *and* $\mathbf{X}:P\to\chi_P$ *constitute an equivalence of the small category* $\mathcal{P}_\vartheta(H,U)$ *and the large category* $G\cdot\mathcal{PB}(U;\vartheta)_{G/H}$*,*

$$\mathcal{P}_\vartheta(H,U)\quad\simeq\quad G\cdot\mathcal{PB}(U;\vartheta)_{G/H}.$$

Proof: The above considerations show that the two functors are well-defined. If we choose for each $\chi\in\mathcal{P}_\vartheta(H,U)$ the element $y_{P_\chi}=[e,e_U]_\chi\in(P_\chi)_o$ it is easy to check that the composite functor $\mathbf{X}\circ\mathbf{P}$ equals the identity functor $\mathcal{P}_\vartheta(H,U)\to\mathcal{P}_\vartheta(H,U)$. For the composition $\mathbf{P}\circ\mathbf{X}$ define the map $\Phi_P:(\mathbf{P}\circ\mathbf{X})(P)\to P$ by

$$\Phi_P([g,u]_{\chi_P}):=g(y_Pu). \tag{2.28}$$

It is easy to check using the right H-action (2.19), the definition of χ_P (2.10), and the proof of Proposition 2.1 that it is a well-defined isomorphism of G-ϑ-equivariant principal U-bundles over G/H. For a morphism $\Psi:P\to P'$ in $G\cdot\mathcal{PB}(U;\vartheta)_{G/H}$ we use the formula $(\mathbf{P}\circ\mathbf{X})(\Psi)=P_{\tilde{u}_\Psi}$ to show that $P\to\Phi_P$ is a natural isomorphism. □

2.4 G-ϑ-equivariant connections and Atiyah classes

In this Section we should like to define small category which will be equivalent to $G \cdot \mathcal{PBC}(U)_{G/H}$.

2.4.1 A slight generalization of Wang's Theorem

In order to get an idea, we fix a crossed homomorphism $\chi : H \to U$ and consider the associated principal U-bundle $(P_\chi, \tau, G/H, U)$ of the preceding Subsection 2.3. It will be much more convenient to work on the manifold $G \times U$ and use the projection $\kappa_\chi : G \times U \to P_\chi$. However, as it turns out, the semidirect product $G_\vartheta \times U$ will be even better thanks to its structure of a Lie group. Let $\tilde{\kappa} = \tilde{\kappa}_\chi : G_\vartheta \times U \to P_\chi$ be the canonical projection (recall that $P_\chi = (G_\vartheta \times U)/\tilde{\chi}(H)$), and we get the relation between the two projections via the involution Ξ see (2.15)

$$\kappa_\chi \circ \Xi = \tilde{\kappa}_\chi \tag{2.29}$$

In any of the two cases $G \times U$ or $G_\vartheta \times U$, we have a principal H-bundle over P_χ. Recall the notion of a *tensorial* 1*-form* $\hat{\alpha}$ (resp. $\tilde{\alpha}$) with values in $\mathfrak{u}$ on $G \times U$ (resp. $G_\vartheta \times U$), see e.g. [21, p.75]: for any $\eta \in \mathfrak{h}$ let $\hat{\eta}^*$ (resp. $\tilde{\eta}^*$) the fundamental vector field $\hat{\eta}^*(g,u) = \frac{d}{dt}\big(R^\chi_{\exp(t\eta)}(g,u)\big)|_{t=0}$ (resp. $\tilde{\eta}^*(g,u) = \frac{d}{dt}\big(R^\vartheta_{\tilde{\chi}(\exp(t\eta))}(g,u)\big)|_{t=0}$) for all $g \in G$ and $u \in U$. Then $\hat{\alpha}$ (resp. $\tilde{\alpha}$) is called tensorial iff for all $h \in H$ and $\eta \in \mathfrak{h}$

$$R^{\chi *}_h \hat{\alpha} = \hat{\alpha} \quad \big(\text{resp.} \quad {R^\vartheta_{\tilde{\chi}(h)}}^* \tilde{\alpha} = \tilde{\alpha}\big), \tag{2.30}$$

$$\hat{\alpha}(\hat{\eta}^*) = 0 \quad \big(\text{resp.} \quad \tilde{\alpha}(\tilde{\eta}^*) = 0\big). \tag{2.31}$$

In particular, each pull-back $\kappa_\chi^* \alpha$ (resp. $\tilde{\kappa}_\chi^* \alpha$) of a $\mathfrak{u}$-valued 1-form α on P_χ is tensorial. It is well-known that the pull-back is a linear bijection of the vector space of all $\mathfrak{u}$-valued 1-forms on P_χ and the vector space of the tensorial forms on the total space of the H-bundle, see e.g. [21, p.76]. Concentrating on the case $G_\vartheta \times U$, we see that the affine space of all G-ϑ-equivariant connection 1-forms on the principal U-bundle P_χ is in bijection (vial pull-back with $\tilde{\kappa}_\chi$) with the affine space of all $\mathfrak{u}$-valued 1-forms $\tilde{\alpha}$ on the Lie group $G_\vartheta \times U$ satisfying the tensoriality conditions (2.30) and (2.31), and in addition the following conditions for which we use equations (1.4), (1.5), and (2.3) for a G-ϑ-equivariant connection), and the fact that $\tilde{\kappa}_\chi$ intertwines the left G-action and the right U-action on $G_\vartheta \times U$ and on P_χ, i.e. $\tilde{\kappa}_\chi \circ L^\vartheta_{(g_0, e_U)} = \ell'_{g_0} \circ \tilde{\kappa}_\chi$ and $\tilde{\kappa}_\chi \circ L^\vartheta_{(e, u_0^{-1})} = r_{u_0} \circ \tilde{\kappa}_\chi$ according to eqs (2.16) and (2.17) for all $g_0 \in G$, $u_0 \in U$, and $\zeta \in \mathfrak{u}$:

$$\begin{aligned} {L^\vartheta_{(e,u_0^{-1})}}^* \tilde{\alpha} &= Ad_{u_0^{-1}} \circ \tilde{\alpha}, & (2.32)\\ \tilde{\alpha}(\tilde{\zeta}^*) &= \zeta, & (2.33)\\ {L^\vartheta_{(g_0,e_U)}}^* \tilde{\alpha} &= T_{e_U}\vartheta_{g_0} \circ \tilde{\alpha}. & (2.34)\end{aligned}$$

where the fundamental field $\tilde{\zeta}^*$ is defined by $\tilde{\zeta}^*(g,u) = \frac{d}{dt}\big(L^{\vartheta}_{(e,\exp(-t\zeta)}(g,u)\big)|_{t=0}$. In the ensuing computations the following smooth map $\dot{\vartheta} : U \to \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g},\mathfrak{u})$ appears very often: for all $u \in U$, $\xi \in \mathfrak{g}$) set

$$\dot{\vartheta}(u)(\xi) = \dot{\vartheta}_u(\xi) := \frac{d}{dt}\big(u^{-1}\vartheta_{exp(t\xi)}(u)\big)|_{t=0}. \tag{2.35}$$

which is well-defined because $t \mapsto u^{-1}\vartheta_{exp(t\xi)}(u)$ is a smooth curve emanating at the unit element $e_U \in U$. Note that $\dot{\vartheta}$ vanishes for the trivial case $\vartheta_g = \mathrm{id}_U$ for all $g \in G$. It can be seen as the evaluation of the Maurer-Cartan form on U on the fundamental field of the left G-action on U.

There is the following

Proposition 2.3. *With the above definitions: Let $\chi \in \mathcal{P}_\vartheta(H,U)$, and P_χ the corresponding G-ϑ-equivariant U-bundle over G/H. Let $\tilde{\kappa}_\chi : G_\vartheta \times U \to P_\chi$ be the canonical projection. Then the following affine spaces are in bijection:*

1. *The affine space of all G-ϑ-equivariant connection 1-forms on P_χ.*
2. *The affine space of all linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ satisfying the following equations for all $h \in H$, $\xi \in \mathfrak{g}$, and $\eta \in \mathfrak{h}$:*

$$\begin{aligned} \varphi[\chi,\mathsf{p}(h)](\xi) &:= T_{e_U}\vartheta_h\Big(Ad_{\chi(h)}\big(\mathsf{p}(Ad_{h^{-1}}\xi)\big)\Big) - \mathsf{p}(\xi) + \dot{\vartheta}_{\chi(h^{-1})}(\xi) \\ &= 0, \end{aligned} \tag{2.36}$$

$$\mathsf{p}(\eta) - T_e\chi(\eta) = 0. \tag{2.37}$$

The bijection is given by

$$\alpha \mapsto \Big(\xi \mapsto \big(\tilde{\kappa}^*_\chi\alpha\big)_{(e,e_U)}(\xi,0)\Big).$$

and in the other direction by $\mathsf{p} \mapsto \alpha[\chi,\mathsf{p}]$ where its pull-back to the semidirect product $G_\vartheta \times U$ reads

$$\big(\tilde{\kappa}^*_\chi\alpha[\chi,\mathsf{p}]\big)_{(g,u)}\big(T_{(e,e_U)}L^{\vartheta}_{(g,u)}(\xi,\zeta)\big) = T_{e_U}\vartheta_g\Big(Ad_u\big(\mathsf{p}(\xi) - \zeta\big)\Big) \tag{2.38}$$

for all $g \in G$, $u \in U$, $\xi \in \mathfrak{g}$, and $\zeta \in \mathfrak{u}$; and its pull-back to $G \times U$ reads

$$\big(\kappa^*_\chi\alpha[\chi,\mathsf{p}]\big)_{(g,u)}\big(T_eL_g(\xi), T_{e_U}L_u(\zeta)\big) = T_{e_U}\vartheta_g\Big(Ad_{u^{-1}}\big(\mathsf{p}(\xi)\big) + \dot{\vartheta}_u(\xi) + \zeta\Big) \tag{2.39}$$

for all $g \in G$, $u \in U$, $\xi \in \mathfrak{g}$, and $\zeta \in \mathfrak{u}$.

Proof: It is easy to compute the following identities in the semidirect product for all $g \in G$, $u \in U$, $\xi \in \mathfrak{g}$, and $\zeta \in \mathfrak{u}$

$$T_{(e,e_U)}L^{\vartheta}_{(g,u)}(\xi,\zeta)\big) = \Big(T_eL_g(\xi),\ \ T_{e_U}L_u(\zeta) - T_{e_U}L_u\big(\dot{\vartheta}_u(\xi)\big)\Big), \tag{2.40}$$

$$T_{(e,e_U)}R^{\vartheta}_{(g,u)}(\xi,\zeta)\big) \;=\; \Big(T_eR_g(\xi),\;\; T_{e_U}R_u\big(T_{e_U}\vartheta_{g^{-1}}(\zeta)\big)\Big), \tag{2.41}$$

$$\tilde{\zeta}^*(g,u) \;=\; T_{(e,e_U)}L^{\vartheta}_{(g,u)}\Big(0,\;\; -Ad_{u^{-1}}\big(\vartheta_{g^{-1}}(\zeta)\big)\Big) \tag{2.42}$$

We deduce the following formula for the adjoint representation in the semidirect product which we shall use very often:

$$Ad^{\vartheta}_{(g,u)}(\xi,\zeta) = \Big(Ad_g(\xi),\;\; T_{e_U}\vartheta_g\big(Ad_u(\zeta)\big) - T_{e_U}\vartheta_g\big(Ad_u(\dot{\vartheta}_u(\xi))\big)\Big). \tag{2.43}$$

Moreover, we get for all $\eta \in \mathfrak{h}$

$$\tilde{\eta}^*(g,u) = T_{(e,e_U)}L^{\vartheta}_{(g,u)}\Big(\eta,\;\; T_e\chi(\eta)\Big). \tag{2.44}$$

Let first α be a G-ϑ-equivariant connection 1-form on P_χ, and set $\tilde{\alpha} = \tilde{\kappa}^*_\chi\alpha$. Since $(g,e_U)(e,u) = (g,u)$ for all $g \in G$ and $u \in U$ conditions (2.32) and (2.34) show that

$${L^{\vartheta}_{(g,u)}}^*\tilde{\alpha} = {L^{\vartheta}_{(e,u)}}^*\big({L^{\vartheta}_{(g,e_U)}}^*\tilde{\alpha}\big) = {L^{\vartheta}_{(e,u)}}^*\big(T_{e_U}\vartheta_g \circ \tilde{\alpha}\big) = T_{e_U}\vartheta_g \circ Ad_u \circ \tilde{\alpha},$$

hence writing the linear map $\tilde{\alpha}_{(e,e_U)} : \mathfrak{g}\times\mathfrak{u} \to \mathfrak{u}$ as (p,q) with linear maps $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ and $\mathsf{q} : \mathfrak{u} \to \mathfrak{u}$, we get

(2.45)

$$\tilde{\alpha}_{(g,u)}\big(T_{(e,e_U)}L^{\vartheta}_{(g,u)}(\xi,\zeta)\big) = T_{e_U}\vartheta_g\Big(Ad_u\Big(\mathsf{p}(\xi)+\mathsf{q}(\zeta)\Big)\Big) = T_{e_U}\vartheta_g\Big(Ad_u\Big(\mathsf{p}(\xi)-\zeta\Big)\Big)$$

because condition (2.33) reads for all $g \in G$, $u \in U$, $\zeta \in \mathfrak{u}$

$$\zeta = \tilde{\alpha}_{(g,u)}\big(\tilde{\zeta}^*(g,u)\big) \overset{(2.42)}{=} -\mathsf{q}(\zeta).$$

For future use, let us define for any linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ the linear map $\tilde{\mathsf{p}}$: $\mathfrak{g}\times\mathfrak{u} \to \mathfrak{u}$ by (for all $\xi \in \mathfrak{g}$ and $\zeta \in \mathfrak{u}$):

$$\tilde{\mathsf{p}}(\xi,\zeta) := \mathsf{p}(\xi) - \zeta. \tag{2.46}$$

which at present is of course equal to $\tilde{\alpha}_{(e,e_U)}$.
Conversely, it is easy to see that for any choice of linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ the right hand side of equation (2.45) can be used as a definition of a $\mathfrak{u}$-valued 1-form $\tilde{\alpha}$ on the Lie group $G_\vartheta \times U \to P_\chi$ which automatically satisfies conditions (2.32), (2.34), and (2.42) since any Lie group is parallelizable.
Since left and right multiplications in any Lie group commute, the condition (2.30) $({R^{\vartheta}_{\tilde{\chi}(h)}}^*\tilde{\alpha})_{(g,u)} = \tilde{\alpha}_{(g,u)}$ for any $g \in G$, $u \in U$ is in fact equivalent (thanks to the identities (2.32) and (2.34)) to $({R^{\vartheta}_{\tilde{\chi}(h)}}^*\tilde{\alpha})_{(e,e_U)} = \tilde{\alpha}_{(e,e_U)}$. Now

$$\big({R^{\vartheta}_{\tilde{\chi}(h)}}^*\tilde{\alpha}\big)_{(e,e_U)}(\xi,\zeta) - \tilde{\alpha}_{(e,e_U)}(\xi,\zeta)$$

$$
\begin{aligned}
&= \quad \tilde{\alpha}_{\tilde{\chi}(h)}\Big(T_{(e,e_U)}L^{\vartheta}_{\tilde{\chi}(h)}\Big(Ad^{\vartheta}_{\tilde{\chi}(h^{-1})}(\xi,\zeta)\Big)\Big) - \tilde{\alpha}_{(e,e_U)}(\xi,\zeta) \\
&= \quad T_{e_U}\vartheta_h\Big(Ad_{\chi(h)}\Big(\tilde{\mathsf{p}}\Big(Ad^{\vartheta}_{\tilde{\chi}(h^{-1})}(\xi,\zeta)\Big)\Big)\Big) - \tilde{\mathsf{p}}(\xi,\zeta) \\
&\overset{(2.38),(2.43)}{=} \quad T_{e_U}\vartheta_h\Big(Ad_{\chi(h)}\Big(\mathsf{p}\big(Ad_{h^{-1}}(\xi)\big) - Ad_{\chi(h)^{-1}}\big(T_{e_U}\vartheta_{h^{-1}}\big(\zeta - \dot{\vartheta}_{\chi(h^{-1})}(\xi)\big)\big)\Big)\Big) \\
&\qquad\qquad - \mathsf{p}(\xi) + \zeta \quad = \quad \varphi[\chi,\mathsf{p}](h)\big(\xi\big)
\end{aligned}
$$

using the identities $\vartheta_{h^{-1}}(\chi(h^{-1})) = \chi(h)^{-1}$ for any $h \in H$. This shows that invariance of $\tilde{\alpha}$ by H is equivalent to eqn (2.36) on p. Finally, for all $\eta \in \mathfrak{h}$ we get $\tilde{\alpha}_{(g,u)}(\tilde{\eta}^*(g,u)) = T_{e_U}\vartheta_g\big(Ad_u\big(\mathsf{p}(\eta) - T_e\chi(\eta)\big)\big)$ whence eqn (2.37) is equivalent to (2.31).
The computation of formula (2.39) is straight-forward using eqn (2.38) and the relation between κ_χ and $\tilde{\kappa}_\chi$, eqn (2.29). □

In the particular case of G-invariant connections (i.e. $\vartheta_g = \mathrm{id}_U$ for all $g \in G$) the above Proposition had already been formulated by H.-C. Wang in 1958, [36].

2.4.2 Smooth Lie group cohomology and Chevalley-Eilenberg cohomology

In order to understand the two conditions (2.36) and (2.37) on the linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ we recall the definition of *smooth Hochschild cohomology of the Lie group* H: let $\mathbf{V}$ be a smooth finite-dimensional H-module, i.e. a finite-dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and a smooth Lie group homomorphism $H \to GL(\mathbf{V})$. Define the smooth Lie group (co)complex by setting $CG^k(H,\mathbf{V}) = \{0\}$ for each strictly negative integer k, and

$$
CG^0(H,\mathbf{V}) := \mathbf{V} \ \text{ and } \ CG^k(H,\mathbf{V}) := \mathcal{C}^\infty(\underbrace{H \times \cdots \times H}_{k \text{ factors}}, \mathbf{V}) \ \ \forall\, k \in \mathbb{N}\setminus\{0\},
$$

and let $CG(H,\mathbf{V}) := \oplus_{k\in\mathbb{N}} CG^k(H,\mathbf{V})$. Recall the *coboundary operator* $\delta : CG(H,\mathbf{V}) \to CG(H,\mathbf{V})$ of degree 1 which is defined on each $f \in CG^k(H,\mathbf{V})$ by

$$
\begin{aligned}
(\delta f)(h_1,\ldots,h_{k+1}) \;=\;& h_1\big(f(h_2,\ldots,h_{k+1})\big) \\
& + \sum_{r=1}^{k} (-1)^r f(h_1,\ldots,h_r h_{r+1},\ldots,h_{k+1}) \\
& + (-1)^{k+1} f(h_1,\ldots,h_k).
\end{aligned}
\tag{2.47}
$$

It is easy to check that $\delta^2 = 0$, and define the *kth cohomology group* $HG^k(H,\mathbf{V})$ *of* H *with values in* $\mathbf{V}$ by

$$
HG^k(H,\mathbf{V}) := \frac{\mathrm{Ker}\big(\delta : CG^k(H,\mathbf{V}) \to CG^{k+1}(H,\mathbf{V})\big)}{\mathrm{Im}\big(\delta : CG^{k-1}(H,\mathbf{V}) \to CG^k(H,\mathbf{V})\big)} =: \frac{ZG^k(H,\mathbf{V})}{BG^k(H,\mathbf{V})}
\tag{2.48}
$$

Recall that the elements of the 'numerator' of the above factor space are called *k-cocycles* and the elements of the 'denominator' are called *k-coboundaries.* In particular for $k = 0$ we have $HG^0(H, \mathbf{V}) \cong ZG^0(H, \mathbf{V})$ which is equal to the subspace of all *invariants* of $\mathbf{V}$, $\mathbf{V}^H$.
The above cohomology framework being the most naive one, there are other choices for the cochain space: a map $f : H^{\times k} \to V$ is called *locally smooth* iff there is an open neighbourhood of $(e, \dots, e) \in H^{\times k}$ (depending on f) such that the restriction of f to that neighbourhood is smooth. There are still others, see e.g. the article [34] for a good review of this.

Next recall for any Lie algebra $(\mathfrak{h}, [\ ,\ ])$ over any field $\mathbb{K}$ (which is not necessarily finite-dimensional), and any $\mathfrak{h}$-module V (where we write $v \mapsto \dot{\rho}_\eta(v)$ for the module map) its *Chevalley-Eilenberg complex* by setting $C^k_{CE}(\mathfrak{h}, V) = \{0\}$ for each strictly negative integer k, and

$$C^0_{CE}(\mathfrak{h}, \mathbf{V}) := \mathbf{V} \quad \text{and} \quad C^k_{CE}(\mathfrak{h}, \mathbf{V}) := \mathbf{Hom}_{\mathbb{K}}(\Lambda^k \mathfrak{h}, \mathbf{V}) \ \ \forall\, k \in \mathbb{N} \setminus \{0\},$$

and let $C_{CE}(\mathfrak{h}, \mathbf{V}) := \oplus_{k \in \mathbb{N}} C^k_{CE}(\mathfrak{h}, \mathbf{V})$. Recall the *coboundary operator* $\delta_{CE} : C_{CE}(\mathfrak{h}, \mathbf{V}) \to C_{CE}(\mathfrak{h}, \mathbf{V})$ of degree 1 which is defined on each $\varphi \in CG^k(H, \mathbf{V})$ by

$$\begin{aligned}(2.49)\quad (\delta_{CE}\varphi)(\eta_1 \wedge \cdots \wedge \eta_{k+1}) = \\ &\sum_{i=1}^{k+1} (-1)^{i-1} \dot{\rho}_{\eta_i}\big(\varphi(\eta_1 \wedge \cdots \wedge \widehat{\eta_i} \wedge \cdots \wedge \eta_{k+1}\big) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \varphi\big([\eta_i, \eta_j] \wedge \cdots \wedge \widehat{\eta_i} \wedge \cdots \wedge \widehat{\eta_j} \wedge \cdots \wedge \eta_{k+1}\big)\end{aligned}$$

It is easy to check that $\delta^2_{CE} = 0$, and define the *kth cohomology group* $H^k_{CE}(\mathfrak{h}, \mathbf{V})$ *of* $\mathfrak{h}$ *with values in* $\mathbf{V}$ by

$$(2.50) \qquad H^k_{CE}(\mathfrak{h}, \mathbf{V}) := \frac{\operatorname{Ker}\big(\delta_{CE} : C^k_{CE}(\mathfrak{h}, \mathbf{V}) \to C^{k+1}_{CE}(\mathfrak{h}, \mathbf{V})\big)}{\operatorname{Im}\big(\delta_{CE} : C^{k-1}_{CE}(\mathfrak{h}, \mathbf{V}) \to C^k_{CE}(\mathfrak{h}, \mathbf{V})\big)} =: \frac{Z^k_{CE}(\mathfrak{h}, \mathbf{V})}{B^k_{CE}(\mathfrak{h}, \mathbf{V})}$$

Recall that the elements of the 'numerator' of the above factor space are called *k-cocycles* and the elements of the 'denominator' are called *k-coboundaries.* In particular for $k = 0$ we have $H^0_{CE}(\mathfrak{h}, \mathbf{V}) \cong Z^0_{CE}(\mathfrak{h}, \mathbf{V})$ which is equal to the subspace of all *invariants* of $\mathbf{V}$, $\mathbf{V}^{\mathfrak{h}}$.

Note also that there is the following chain map $\mathbf{D}$ to the Chevalley-Eilenberg cohomology complex of the Lie algebra $\mathfrak{h}$ with values in the $\mathfrak{h}$-module $\mathbf{V}$: let $f \in CG^k(H, \mathbf{V})$, $\eta \in \mathfrak{h}$, and $i \in \mathbb{N}$ such that $1 \le i \le k$, then let $\eta^{(i)}$ denote the left invariant vector field on the Lie group $H^{\times k}$ whose value at $(e, \dots, e)$ is given by $(0, \dots, 0, \eta, 0, \dots, 0)$ where η appears in the ith factor. Define
(2.51)

$$\big(\mathbf{D}_k(f)\big)(\eta_1 \wedge \cdots \wedge \eta_k) := \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \big(\eta^{(1)}_{\sigma(1)} \big(\eta^{(2)}_{\sigma(2)} \big(\cdots \big(\eta^{(k)}_{\sigma(k)}(f)\big) \cdots \big)(e, \dots, e).$$

It is not hard to check that this is a chain map from the (locally) smooth cochains to the Chevalley-Eilenberg cochains.

2.4.3 Atiyah classes as obstructions to the existence of G-ϑ equivariant connections

Now note that the map $\Psi^\chi : H \to \mathrm{GL}(\mathfrak{u})$ defined by

$$\Psi^\chi_h = T_{e_U}\vartheta_h \circ Ad_{\chi(h)} \tag{2.52}$$

is a smooth representation because $\big(0, \Psi^\chi_h(\zeta)\big) = Ad^\vartheta_{\tilde{\chi}(h)}(0,\zeta)$, see eqn (2.43). Moreover, there is the exact sequence of H-modules (w.r.t. the adjoint representation of H on $\mathfrak{g}$, $\mathfrak{h}$, and hence on $\mathfrak{g}/\mathfrak{h}$)

$$\{0\} \to \mathfrak{h} \stackrel{\iota}{\to} \mathfrak{g} \stackrel{\varpi}{\to} \mathfrak{g}/\mathfrak{h} \to \{0\},$$

and since they are vector space there results the exact sequence of H-modules

$$\{0\} \to \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h},\mathfrak{u}) \stackrel{\varpi^*}{\to} \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g},\mathfrak{u}) \stackrel{\iota^*}{\to} \mathbf{Hom}_{\mathbb{R}}(\mathfrak{h},\mathfrak{u}) \to \{0\},$$

where H acts on the target modules of the above **Hom**-spaces by means of Ψ^χ. From this sequence we get a short exact sequence of Hochschild complexes

$$\begin{aligned}\{0\} \to CG\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h},\mathfrak{u})\big) &\stackrel{\widehat{\varpi^*}}{\to} CG\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g},\mathfrak{u})\big) \\ &\stackrel{\widehat{\iota^*}}{\to} CG\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{h},\mathfrak{u})\big) \to \{0\}\end{aligned} \tag{2.53}$$

where the Hochschild coboundary δ depends on χ, and we shall sometimes write δ^χ. Recall the following important Hochschild cochains:

$$T_e\chi \in CG^0\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{h},\mathfrak{u})\big) \tag{2.54}$$

$$\dot{\vartheta}_{\chi((\cdot)^{-1})} : h \mapsto \big(\xi \mapsto \dot{\vartheta}_{\chi(h^{-1})}(\xi)\big) \in CG^1\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g},\mathfrak{u})\big) \tag{2.55}$$

Lemma 2.4. *With the above notations:*

$$\delta\big(\dot{\vartheta}_{\chi((\cdot)^{-1})}\big) = 0 \tag{2.56}$$

$$\delta\big(T_e\chi\big) + \widehat{\iota^*}\big(\dot{\vartheta}_{\chi((\cdot)^{-1})}\big) = 0 \tag{2.57}$$

Proof: The first equation follows from the identity

$$Ad^\vartheta_{\tilde{\chi}(h_2^{-1}h_1^{-1})}(\xi,0) = \Big(Ad^\vartheta_{\tilde{\chi}(h_2^{-1})} \circ Ad^\vartheta_{\tilde{\chi}(h_1^{-1})}\Big)(\xi,0)$$

for all $h_1, h_2 \in H$, $\xi \in \mathfrak{g}$, see eqn (2.43). The second equation is deduced from the fact that $\tilde{\chi}$ is a homomorphism of Lie groups whence its derivative $T_e\tilde{\chi}$ intertwines the adjoint actions, viz.

$$T_e\tilde{\chi} \circ Ad_h = Ad^\vartheta_{\tilde{\chi}(h)} \circ T_e\tilde{\chi}$$

for all $h \in H$: it suffices to look at the $\mathfrak{u}$-component. $\square$

There is now a characteristic class $c_{G,H,U,\vartheta,\chi}$ in $HG^1\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big)$ defined as follows: since ι^* is surjective, we can choose a linear map $\mathsf{p} \in \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{u})$ with $\iota^*\mathsf{p} = T_e\chi$. We recall for all $h \in H$

$$\varphi[\chi, \mathsf{p}](h) = \Psi_h^\chi \circ \tilde{\mathsf{p}} \circ Ad^\vartheta_{\tilde{\chi}(h^{-1})} - \tilde{\mathsf{p}} = \delta(\mathsf{p})(h) + \dot{\vartheta}_{\chi(h^{-1})} \tag{2.58}$$

and get

$$\widehat{\iota^*}\big(\varphi[\chi, \mathsf{p}]\big) = \delta\big(\widehat{\iota^*}(\mathsf{p})\big) + \widehat{\iota^*}\big(\dot{\vartheta}_{\chi((\cdot)^{-1})}\big) = \delta\big(T_e\chi\big) + \widehat{\iota^*}\big(\dot{\vartheta}_{\chi((\cdot)^{-1})}\big) \overset{(2.57)}{=} 0$$

hence –using the fact that the sequence (2.53) is exact– there is a unique $f \in CG^1\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big)$ such that

$$\widehat{\varpi^*}(f) = \varphi[\chi, \mathsf{p}] = \delta(\mathsf{p}) + \dot{\vartheta}_{\chi((\cdot)^{-1})}. \tag{2.59}$$

Thanks to (2.56) we get $0 = \delta\big(\widehat{\varpi^*}(f)\big) = \widehat{\varpi^*}\big(\delta(f)\big)$ whence $\delta(f) = 0$ since $\widehat{\varpi^*}$ is injective. For another choice p' in $\mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{u})$, note that $\iota^*(\mathsf{p}' - \mathsf{p}) = 0$ hence –again by exactness of the Hochschild complex, eqn (2.53)– there is a unique $g \in \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})$ with $\varpi^*(g) = \mathsf{p}' - \mathsf{p}$. It follows that

$$\widehat{\varpi^*}(f' - f)) = \delta\big(\mathsf{p}' - \mathsf{p}\big) = \widehat{\varpi^*}\big(\delta(g)\big) \tag{2.60}$$

whence $f' - f = \delta(g)$ since $\widehat{\varpi^*}$ is injective. It follows that the cohomology class $[f]$ of the 1-cocycle f does not depend on the choice of p, and hence the definition

$$c_\chi := c_{G,H,U,\vartheta,\chi} := [f] \in HG^1\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big) \tag{2.61}$$

makes sense and is called the *Atiyah class* (of $(G, H, U, \vartheta, \chi)$).
Note that in the important particular case $\vartheta_g = \mathrm{id}_U$ for all $g \in G$ of an *invariant connection*, the map χ is already a homomorphism of Lie groups, whence $T_e\chi$ intertwines the adjoint actions, so it is a 0-cocycle and defines a cohomology class $[T_e\chi]$ in $HG^0\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathfrak{u})\big)$. The image of this class under the connecting homomorphism

$$HG^0\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathfrak{u})\big) \to HG^1\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big)$$

in the long exact cohomology sequence coincides with the Atiyah class.

The relation of the Atiyah class with the characterization of G-ϑ-equivariant connections in Proposition 2.3 is contained in the following

Proposition 2.5. *With the hypotheses of Proposition 2.3:*

1. *There is a G-ϑ-equivariant connection in the bundle P_χ if and only if the Atiyah class $c_{G,H,U,\vartheta,\chi}$ vanishes.*

2. *In case the Atiyah class vanishes: the tangent space of the affine space of all G-ϑ-equivariant connections in the bundle P_χ is isomorphic to*

$$HG^0\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big) \cong ZG^0\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big)$$

Proof: **1.** According to Proposition 2.3 a G-ϑ-equivariant connection 1-form on P_χ exists if and only if the linear map $\mathsf{p} \in \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{u})$ satisfies the two conditions (2.36) and (2.37) which in fact can be expressed in cohomological terms as

$$\delta(\mathsf{p}) + \dot{\vartheta}_{\chi((\cdot)^{-1})} = 0, \tag{2.62}$$

$$\iota^*\mathsf{p} = T_e\chi. \tag{2.63}$$

Now if such a map p satisfying the preceding conditions exists, it is clear that the map f in eqn (2.59) vanishes, hence its class, the Atiyah class, vanishes. Conversely, suppose that the Atiyah class $[f]$ vanishes. According to the definition of this class it follows that there is a map $\mathsf{p}' \in \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{u})$ such that $\iota^*\mathsf{p}' = T_e\chi$ and a map $g \in \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})$ such that

$$\widehat{\varpi^*}(f) = \widehat{\varpi^*}\big(\delta(g)\big) = \delta\big(\widehat{\varpi^*}(g)\big) = \delta(\mathsf{p}') + \dot{\vartheta}_{\chi((\cdot)^{-1})}.$$

It is immediate that the linear map $\mathsf{p} := \mathsf{p}' - \widehat{\varpi^*}(g)$ satisfies the two above conditions (2.62) and (2.63) whence a G-ϑ-equivariant connection 1-form exists on P_χ.
2. Suppose that p and p' satisfy (2.62) and (2.63). Then their difference $\mathsf{p}' - \mathsf{p}$ is a cocoycle satisfying $\iota^*(\mathsf{p}' - \mathsf{p}) = 0$ whence there is a cocycle $g \in ZG^0\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big)$ such that $\varpi^* g = \mathsf{p}' - \mathsf{p}$. Conversely it is clear that for any such cocycle g the sum $\mathsf{p} + \varpi^* g$ satisfies (2.62) and (2.63) if p does. □

2.4.4 Lie algebra versions

One gets a '*Lie algebra version*' or an *infinitesimal version* of the preceding Atiyah classes by replacing group elements by the exponential functions of Lie algebra elements times a parameter t and differentiating w.r.t. t at $t = 0$; more precisely, fix the following data: let $(\mathfrak{g}, [\ ,\ ])$ and $(\mathfrak{u}, [\ ,\ ])$ be a Lie algebras (not necessarily finite-dimensional), let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. In order to get an analogue of the automorphic G-action ϑ (2.1) define the linear map $\ddot{\vartheta} : \mathfrak{g} \to \mathbf{Hom}_{\mathbb{K}}(\mathfrak{u}, \mathfrak{u})$ as the following 'second derivative of ϑ' for all $\xi \in \mathfrak{g}$ and $\zeta \in \mathfrak{u}$:

$$\begin{aligned} \ddot{\vartheta}_\xi(\zeta) &:= \frac{\partial}{\partial s}\Big(T_{e_U}\vartheta_{\exp(s\xi)}(\zeta)\Big)\Big|_{s=0} = \frac{\partial^2}{\partial s \partial t}\Big(\vartheta_{\exp(s\xi)}\big(\exp(t\zeta)\big)\Big)\Big|_{s=0=t} \\ &= \frac{\partial}{\partial t}\Big(\dot{\vartheta}_{\exp(t\zeta)}(\xi)\Big|_{t=0}. \end{aligned}$$

It follows immediately that $\ddot{\vartheta}$ is a *derivational Lie algebra representation* in the following sense: for all $\xi \in \mathfrak{g}$ and $\zeta \in \mathfrak{u}$, satisfying (for all $\xi, \xi' \in \mathfrak{g}$ and $\zeta, \zeta' \in \mathfrak{u}$)
(2.64)

$$\ddot{\vartheta}_{[\xi,\xi']}(\zeta) = \ddot{\vartheta}_\xi\big(\ddot{\vartheta}_{\xi'}(\zeta)\big) - \ddot{\vartheta}_{\xi'}\big(\ddot{\vartheta}_\xi(\zeta)\big) \quad \text{and} \quad \ddot{\vartheta}_\xi\big([\zeta, \zeta']\big) = [\ddot{\vartheta}_\xi(\zeta), \zeta'] + [\zeta, \ddot{\vartheta}_\xi(\zeta')].$$

In our more general situation of Lie algebras we fix such a derivational Lie algebra representation $\ddot{\vartheta}$. Note the important particular case $\ddot{\vartheta} = 0$ corresponding to trivial ϑ. With these data one can form the *semidirect sum* $\mathfrak{g} \oplus \mathfrak{u}$ of the two Lie algebras: on the direct sum of the vector spaces the bracket is defined as follows for all $\xi, \xi' \in \mathfrak{g}$ and $\zeta, \zeta' \in \mathfrak{u}$:

$$[(\xi,\zeta),(\xi',\zeta')] := \big([\xi,\xi'], \ddot{\vartheta}_\xi(\zeta') - \ddot{\vartheta}_{\xi'}(\zeta) + [\zeta,\zeta']\big), \tag{2.65}$$

and it is easy to check that this is a Lie bracket. Next, in order to get an analogue of the crossed homomorphism $\chi : H \to U$ we consider first its derivative $\dot{\chi} = T_e\chi$ at $e \in H$. Since $\tilde{\chi} : H \to G_\vartheta \times U$ defined by $\tilde{\chi}(h) = \big(h, \chi(h)\big)$ is a homomorphism of Lie groups, it follows that its derivative is a morphism of Lie algebras, hence upon using the above semidirect sum structure (2.65) we get the following identity of a *crossed Lie algebra homomorphism w.r.t.* $\ddot{\vartheta}$, i.e. for all $\eta, \eta' \in \mathfrak{h}$:

$$[\dot{\chi}(\eta), \dot{\chi}(\eta')] - \dot{\chi}([\eta,\eta']) + \ddot{\vartheta}_\eta\big(\dot{\chi}(\eta')\big) - \ddot{\vartheta}_{\eta'}\big(\dot{\chi}(\eta)\big) = 0. \tag{2.66}$$

Again in our more general situation we fix a crossed Lie algebra homomorphism $\dot{\chi}$. Having fixed the data $(\mathfrak{g}, \mathfrak{h}, \mathfrak{u}, \ddot{\vartheta}, \dot{\chi})$ we get the following: the linear map $\psi^{\dot{\chi}} : \mathfrak{h} \to \mathbf{Hom}_{\mathbb{K}}(\mathfrak{u},\mathfrak{u})$, written $\psi^{\dot{\chi}}_\eta(\zeta)$ for all $\eta \in \mathfrak{h}$ and $\zeta \in \mathfrak{u}$ and defined by

$$\psi^{\dot{\chi}}_\eta(\zeta) = \ddot{\vartheta}_\eta(\zeta) + ad_{\dot{\chi}(\eta)}(\zeta), \tag{2.67}$$

is readily checked to be a representation of $\mathfrak{h}$ in $\mathfrak{u}$. The formula can be obtained by differentiating (2.52). Hence we have the Chevalley-Eilenberg complexes of $\mathfrak{h}$ with values in $\mathbf{Hom}_{\mathbb{K}}(\mathfrak{g}/\mathfrak{h},\mathfrak{u})$, in $\mathbf{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{u})$, and in $\mathbf{Hom}_{\mathbb{K}}(\mathfrak{h},\mathfrak{u})$. The linear map $\ddot{\vartheta}\circ(\mathrm{id}_\mathfrak{g}\otimes\dot{\chi})$ is readily checked to be Chevalley-Eilenberg 1-cocycle with values in $\mathbf{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{u})$, i.e.

$$\delta_{CE}\big(\ddot{\vartheta} \circ (\mathrm{id}_\mathfrak{g} \otimes \dot{\chi})\big) = 0, \tag{2.68}$$

(analogue of the group cocycle $h \mapsto \big(\xi \mapsto \dot{\vartheta}_{\chi(h^{-1})}(\xi)\big)$, see equation (2.56). Moreover, as a direct consequence of eqn (2.66) we get the analogue of eqn (2.57),

$$\delta_{CE}(\dot{\chi}) - \widehat{\iota^*}\big(\ddot{\vartheta} \circ (\mathrm{id}_\mathfrak{g} \otimes \dot{\chi})\big). = 0. \tag{2.69}$$

Finally, given a linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ such that the restriction of p to $\mathfrak{h}$ is equal to $\dot{\chi}$ the Lie analogue of the map $\varphi[\chi,\mathsf{p}] : \mathfrak{g} \to \mathfrak{u}$ (see eqn (2.36) is given by

$$\varphi[\dot{\chi},\mathsf{p}] = \delta_{CE}(\mathsf{p}) - \ddot{\vartheta} \circ (\mathrm{id}_\mathfrak{g} \otimes \dot{\chi}). \tag{2.70}$$

With these identities, a linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{h}$ may be called an *infinitesimal* $\mathfrak{g}$-$\ddot{\vartheta}$*-equivariant connection* iff the restriction of p to $\mathfrak{h}$ equals $\dot{\chi}$ and iff $\varphi[\dot{\chi},\mathsf{p}] = 0$. Moreover, the whole homological reasoning of the preceding Subsubsection can be copied to define an Atiyah class $c_{\mathfrak{g},\mathfrak{h},\mathfrak{u},\ddot{\vartheta},\dot{\chi}}$ as an element of $H^1_{CE}\big(\mathfrak{h}, \mathbf{Hom}_{\mathbb{K}}(\mathfrak{g}/\mathfrak{h},\mathfrak{u})\big)$ which is the obstruction to the existence of an *infinitesimal* $\mathfrak{g}$-$\ddot{\vartheta}$*-equivariant connection.*

2.4.5 Equivalence of the categories $\mathcal{PC}_\vartheta(H,U)$ and $G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}$

We shall return to the Lie group case: before we can define the relevant categories, we have to see how the group U acts on the structures defined before: clearly, the vector space $\mathbf{Hom}_\mathbb{R}(\mathfrak{g}\times\mathfrak{u},\mathfrak{u})$ is a left U-module by the obvious conjugation $\mathsf{P}\mapsto\hat{u}.\mathsf{P}=Ad_{\hat{u}}\circ\mathsf{P}\circ Ad^\vartheta_{(e,\hat{u}^{-1})}$ for all $\hat{u}\in U$. The image of the map $\mathsf{p}\mapsto\tilde{\mathsf{p}}$, see eqn (2.46) is an affine subspace of the vector space $\mathbf{Hom}_\mathbb{R}(\mathfrak{g}\times\mathfrak{u},\mathfrak{u})$, and formula (2.43) shows that it is invariant under the above U-action. Hence there is an affine U-action on the vector space $\mathbf{Hom}_\mathbb{R}(\mathfrak{g},\mathfrak{u})$, given by

$$\hat{u}.\mathsf{p}=Ad_{\hat{u}}\circ\mathsf{p}+\dot{\vartheta}_{\hat{u}^{-1}}.\overset{(2.43)}{=}Ad_{\hat{u}}\circ\big(\mathsf{p}-\dot{\vartheta}_{\hat{u}}\big). \tag{2.71}$$

Next, for $\hat{u}\in U$ consider the crossed homomorphism $u.\chi$. Then using $\widetilde{\hat{u}.\chi}(h)=(e,\hat{u})\tilde{\chi}(h)(e,\hat{u}^{-1})$ we first get for all $\hat{u}\in U$ that

$$\Psi^{\hat{u}.\chi}_h(\zeta)=\mathrm{pr}_2\circ Ad^\vartheta_{(e,\hat{u})\tilde{\chi}(h)(e,\hat{u}^{-1})}(0,\zeta)=\Big(Ad_{\hat{u}}\circ\Psi^\chi_h\circ Ad_{\hat{u}^{-1}}\Big)(\zeta),$$

and for all $\eta\in\mathfrak{h}$:

$$T_e(\hat{u}.\chi)(\eta)=\mathrm{pr}_2\big(Ad^\vartheta_{(e,\hat{u})}\big(T_e\tilde{\chi}(\eta)\big)\big)=\mathrm{pr}_2\big(Ad^\vartheta_{(e,\hat{u})}\big(\eta,T_e\chi(\eta)\big)\big)\overset{(2.43)}{=}Ad_{\hat{u}}\big(T_e\chi(\eta)-\dot{\vartheta}_{\hat{u}}(\eta)\big).$$

whence a comparison with equation (2.71) shows that for all $\hat{u}\in U$

$$T_e(\hat{u}.\chi)=\hat{u}.T_e\chi \tag{2.72}$$

where all the preceding considerations and definitions also work for Hom-spaces with $\mathfrak{g}$ replaced by $\mathfrak{h}$. An immediate consequence is the equation

$$\iota^*\mathsf{p}=T_e\chi\iff\iota^*\big(\hat{u}.\mathsf{p}\big)=T_e(\hat{u}.\chi).$$

Next, recall that the defining 1-cochain for the Atiyah class, $\varphi[\chi,\mathsf{p}]$, see eqn (2.58), can be written as

$$\varphi[\chi,\mathsf{p}](h)\big(\xi\big)=\Big(\Psi^\chi_h\circ\tilde{\mathsf{p}}\circ Ad^\vartheta_{\tilde{\chi}(h^{-1})}\Big)\big(\xi,\zeta\big)-\tilde{\mathsf{p}}(\xi,\zeta)$$

and is independent of $\zeta\in\mathfrak{u}$, see the proof of Proposition 2.3. We compute that for all $\hat{u}\in U$, $h\in H$, and $\xi\in\mathfrak{g}$:

$$\varphi[\hat{u}.\chi,\hat{u}.\mathsf{p}](h)\big(\xi\big)=Ad_{\hat{u}}\Big(\varphi[\chi,\mathsf{p}](h)\big(\xi\big)\Big). \tag{2.73}$$

Now observe that for each $\hat{u}\in U$ the map $f\mapsto Ad_{\hat{u}}\circ f$ from $\mathbf{Hom}_\mathbb{K}(W,\mathfrak{u})$ (for $W=\mathfrak{g}$, $\mathfrak{h}$, or $\mathfrak{g}/\mathfrak{h}$) intertwines the H-actions Ψ^χ and $\Psi^{\hat{u}.\chi}$. Therefore it induces a canoncial map $Ad'_{\hat{u}}$ on Hochschild cohomology. The above equation (2.73) implies that

$$c_{G,H,U,\vartheta,\hat{u}.\chi}=Ad'_{\hat{u}}\big(c_{G,H,U,\vartheta,\chi}\big). \tag{2.74}$$

We shall define two small categories: let $\mathcal{P}^0_\vartheta(H,U)$ be the following set:

$$\mathcal{P}^0 = \mathcal{P}^0_\vartheta(H,U) := \{\chi \in \mathcal{P}_\vartheta(H,U) \mid c_{G,H,U,\vartheta,\chi} = 0\}. \tag{2.75}$$

and for any two $\chi, \chi' \in \mathcal{P}^0_\vartheta(H,U)$

$$\mathbf{Hom}_{\mathcal{P}^0}(\chi,\chi') := \mathbf{Hom}_{\mathcal{P}}(\chi,\chi'). \tag{2.76}$$

which is well-defined thanks to eqn (2.74). Note also that $\mathcal{P}^0_\vartheta(H,U)$ is not empty because for the constant map (i.e. $\chi(h) = e_U$ for all $h \in H$)–which gives a trivial bundle– it follows that $T_e\chi = 0$ and $\dot{\vartheta}_{\chi(\cdot)^{-1}} = \dot{\vartheta}_{e_U} = 0$ whence $c_{G,H,U,\vartheta,\chi} = 0$. Hence $\mathcal{P}^0_\vartheta(H,U)$ is a full subcategory of $\mathcal{P}_\vartheta(H,U)$.
We now define a second small category by declaring its set of objects by

$$\begin{aligned}\mathcal{PC}_\vartheta(H,U) = \mathcal{PC} \;:=\; &\{(\chi,\mathsf{p}) \in \mathcal{P}^0_\vartheta(H,U) \times \mathbf{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{u}) \mid \text{such that} \\ &\forall\, \eta \in \mathfrak{h} : \mathsf{p}(\eta) = T_e\chi(\eta) \text{ and } \varphi[\chi,\mathsf{p}] = 0\},\end{aligned} \tag{2.77}$$

and its set of morphisms by

$$\begin{aligned}&\mathbf{Hom}_{\mathcal{PC}}\big((\chi,\mathsf{p}),(\chi',\mathsf{p}')\big) := \\ &\qquad \{\hat{u} \in \mathbf{Hom}_{\mathcal{P}^0}(\chi,\chi') \mid \mathsf{p}' = \hat{u}.\mathsf{p} = Ad_{\hat{u}} \circ \mathsf{p} + \dot{\vartheta}_{\hat{u}^{-1}}\},\end{aligned} \tag{2.78}$$

where composition of morphism is given by group multiplication in U. It is clear from the above considerations that $\mathcal{PC}_\vartheta(H,U)$ is a category, and that the projection $(\chi,\mathsf{p}) \to \chi$ is a full covariant functor to $\mathcal{P}^0_\vartheta(H,U)$.

We wish to define covariant functors between the categories $\mathcal{PC}_\vartheta(H,U)$ and $G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}$: let $\mathbf{PC} : \mathcal{PC}_\vartheta(H,U) \to G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}$ be the assignment

$$\mathbf{PC}(\chi,\mathsf{p}) = \big(P_\chi, \alpha[\chi,\mathsf{p}]\big) \quad \text{and} \quad \mathbf{PC}(\hat{u}) = P_{\hat{u}} \tag{2.79}$$

where $(\chi,\mathsf{p}),(\chi',\mathsf{p}') \in \mathcal{PC}_\vartheta(H,U)$, and $\hat{u} \in \mathbf{Hom}_{\mathcal{PC}}\big((\chi,\mathsf{p}),(\chi',\mathsf{p}')\big)$, see eqs (2.27) and (2.38). According to Proposition 2.3 the assignment $\mathbf{PC}$ is well-defined on the object level, and $\mathbf{PC}_{\hat{u}} = P_{\hat{u}} : P_\chi \to P_{\chi'} = P_{\hat{u}.\chi}$ is a morphism of G-ϑ-equivariant principal U-bundles over G/H according to Proposition 2.2. In order to show that it is a morphism in $G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}$, note first that eqs (2.15), (2.18), (2.27), and (2.29) imply that

$$P_{\hat{u}} \circ \tilde{\kappa}_\chi = \tilde{\kappa}_{\hat{u}.\chi} \circ R^\vartheta_{(e,\hat{u}^{-1})}$$

and for all $g \in G$, $u \in U$, $\xi \in \mathfrak{g}$, $\zeta \in \mathfrak{u}$:

$$\begin{aligned}&\Big(\tilde{\kappa}^*_\chi\big(P^*_{\hat{u}}(\alpha[\hat{u}.\chi,\hat{u}.\mathsf{p}])\big)\Big)_{(g,u)}\Big(T_{(e,e_U)}L^\vartheta_{(g,u)}(\xi,\zeta)\Big) \\ &= \quad \Big({R^\vartheta_{(e,\hat{u}^{-1})}}^*\big(\tilde{\kappa}^*_{\hat{u}.\chi}(\alpha[\hat{u}.\chi,\hat{u}.\mathsf{p}])\big)\Big)_{(g,u)}\Big(T_{(e,e_U)}L^\vartheta_{(g,u)}(\xi,\zeta)\Big) \\ &= \quad \big(\tilde{\kappa}^*_{\hat{u}.\chi}(\alpha[\hat{u}.\chi,\hat{u}.\mathsf{p}])\big)_{(g,u\hat{u}^{-1})}\Big(T_{(e,e_U)}L^\vartheta_{(g,u\hat{u}^{-1})}\big(Ad^\vartheta_{(e,\hat{u})}(\xi,\zeta)\big)\Big)\end{aligned}$$

$$
\begin{aligned}
&\stackrel{(2.38),(2.43)}{=} && T_{e_U}\vartheta_g\Big(Ad_{u\hat{u}^{-1}}\Big(\big(\hat{u}.\mathsf{p}\big)(\xi) - Ad_{\hat{u}}(\zeta) + Ad_{\hat{u}}\big(\dot{\vartheta}_{\hat{u}}(\xi)\big)\Big)\Big) \\
&\stackrel{(2.71)}{=} && T_{e_U}\vartheta_g\Big(Ad_u\Big(\mathsf{p}(\xi) - \zeta\Big)\Big) \\
&= && \tilde{\kappa}_\chi^*\big(\alpha[\chi,\mathsf{p}]\big)_{(g,u)}\Big(T_{(e,e_U)}L^{\vartheta}_{(g,u)}(\xi,\zeta)\Big),
\end{aligned}
$$

whence

$$P_{\hat{u}}^*(\alpha[\hat{u}.\chi, \hat{u}.\mathsf{p}]) = \alpha[\chi,\mathsf{p}], \tag{2.80}$$

showing that $P_{\hat{u}}$ preserves the connections and is thus a morphism in the category $G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}$. It is easy to check that **PC** is a covariant functor.
In order to define a covariant functor in the other direction $G\cdot\mathcal{PBC}(U;\vartheta)_{G/H} \to \mathcal{PC}_\vartheta(H,U)$ recall the functor **X** from $G\cdot\mathcal{PB}(U;\vartheta)_{G/H} \to \mathcal{P}_\vartheta(H,U)$, see Proposition 2.2, sending a G-ϑ-equivariant principal U-bundle over G/H, $(P,\tau,G/H,U,\vartheta)$, to the crossed homomorphism $\chi_P : H \to U$ upon choosing an element $y_P \in P_o$. Recall furthermore the natural isomorphism $\Phi_P : P_{\chi_P} \to P$ between the functors $\mathbf{P}\circ\mathbf{X}$ and $\mathrm{id}_{G\cdot\mathcal{PB}(U;\vartheta)_{G/H}}$, see eqn (2.28) in the proof of Proposition 2.2. Define the functor **XC** from $G\cdot\mathcal{PBC}(U;\vartheta)_{G/H} \to \mathcal{PC}_\vartheta(H,U)$ by

$$\mathbf{XC}(P,\tau,G/H,U,\vartheta,\alpha) = \Big(\chi_P,\quad \xi\mapsto \tilde{\kappa}_{\chi_P}^*\big(\Phi_P^*\alpha\big)_{(e,e_U)}\big(\xi,0\big)\Big) \tag{2.81}$$

Note that for any morphism Φ of G-ϑ-equivariant principal U-bundles over G/H the pull-back of a G-ϑ-equivariant connection 1-form with respect to Φ is again a G-ϑ-equivariant connection 1-form whence the image object of **XC** is an object in $\mathcal{PC}_\vartheta(H,U)$ thanks to Proposition 2.3. Define **XC** on morphisms Φ as the functor **X**, i.e. $\mathbf{XC}(\Phi) = \tilde{u}_\Phi \in U$, see the considerations preceding Proposition 2.2. A computation similar to the one leading to eqn (2.80) shows that $\tilde{u}_\Phi$ is a morphism in $\mathcal{PC}_\vartheta(H,U)$, i.e. maps p to $\tilde{u}_\Phi.\mathsf{p}$. It follows that **XC** is a covariant functor. We get the following analogue of Proposition 2.2:

Proposition 2.6. *The two functors* $\mathbf{PC} : (\chi,\mathsf{p}) \to \big(P_\chi, \alpha[\chi,\mathsf{p}]\big)$ *and* **XC** *as defined above, see eqn (2.81), constitute an equivalence of the small category* $\mathcal{PC}_\vartheta(H,U)$ *and the large category* $G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}$*,*

$$\mathcal{PC}_\vartheta(H,U) \;\;\simeq\;\; G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}.$$

Proof: Analogous to the proof of Proposition 2.2. □

2.4.6 Covariant derivatives in associated vector bundles for G-invariant connections

In this Subsubsection we shall –for simplicity– consider the situation where ϑ is trivial: let G, H, U as before, and $\chi : H \to U$ is a homomorphism of Lie groups.

Furthermore, suppose that the associated Atiyah class $c_{G,H,U,\chi}$ vanishes, and let $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ an H-invariant map, i.e. for all $h \in H$: $Ad_{\chi(h)} \circ \mathsf{p} = \mathsf{p} \circ Ad_h$. Suppose that there is a representation $\rho : U \to GL(V)$ of the Lie group U in a finite-dimensional vector space V. As before we shall write $\dot{\rho} : \mathfrak{u} \to \mathfrak{gl}(V)$ for the induced representation of the Lie algebra $\mathfrak{u}$ of U. Consider the G-equivariant principal U bundle $G_H[U]$ over G/H, and let E' denote the associated vector bundle $(G_H[U])_U[V]$ over G/H where U acts on V via ρ. Recall the morphism of principal bundles over G/H defined by $\Phi : G \to G_H[U] : g \mapsto [g, e_U]$, see eqn (1.2). By eqn (1.3) E' is naturally isomorphic as a G-equivariant vector bundle over G/H to the associated vector bundle $E = G_H[V]$ where H acts on V via the representation $\rho \circ \chi$ by means of the maps $\Phi_V : E \to E'$. Let ∇' denote the covariant derivative in the associated vector bundle E', see eqn (1.7). Let $\ell' : G \times E' \to E'$ be the left G-action on E' given by $\ell_g[g', v] = [g'g, v]$ which projects onto the canonical G-action ℓ on G/H. Since the connection form $\alpha[\chi, \mathsf{p}]$ on $G_H[U]$ is G-invariant it follows that the horizontal lift $X \mapsto X^h$ (see Section 1.1, the paragraph before eqn (1.7)) is G-equivariant in the sense that $\ell_g'^*\big(X^h\big) = (\ell_g^* X)^h$ for all $g \in G$. Moreover, G acts on the total space of the vector bundle E' as vector bundle morphisms via ℓ', whence there is a pull-back of smooth sections ψ' of E', viz $\big(\ell_g'^*\psi'\big)(x) := \ell'_{g^{-1}}\big(\psi'(gx)\big)$ for all $x \in G/H$ and some $g \in G$. Now it is straight-forward to check that ∇' is a *G-invariant covariant derivative* in the sense that for all $g \in G$ we have $\big(\ell_g'^*\nabla'_X\psi'\big) = \nabla'_{\ell_g^x(X)}\big(\ell_g'^*(\psi')\big)$ for all $g \in G$, vector fields X on G/H, and all smooth section ψ of E'.

Proposition 2.7. *Let G, H, U, χ, V, ρ as above. Suppose that the Atiyah class $c_{G,H,U,\chi}$ vanishes, and let $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ be a H-equivariant linear map extending $T_e\chi : \mathfrak{h} \to \mathfrak{u}$. Then there exists a G-invariant covariant derivative ∇ in the vector bundle E such that for all vector fields X on G/H and smooth sections $\psi \in \Gamma^\infty(G/H, E)$*

$$\Phi_V \circ \big(\nabla_X \psi\big) = \nabla'_X\Big(\Phi_V \circ \psi\Big). \tag{2.82}$$

Moreover, let $\tilde{X}$ any H-invariant lift of the vector field X to G, i.e. we have $T_g\pi\big(\tilde{X}(g)\big) = X\big(\pi(g)\big)$ for all $g \in G$ and $R_h^*\tilde{X} = \tilde{X}$ for all $H \in H$. Then there is the formula

$$\nabla_X\psi\big(\pi(g)\big) = \Big[g, \big(\tilde{X}(\hat{\psi})\big)(g) + \dot{\rho}\big(\mathsf{p}\big(T_gL_{g^{-1}}(\tilde{X}(g))\big)\big)\big(\hat{\psi}(g)\big)\Big] \tag{2.83}$$

for all $g \in G$ which does not depend on the lift $X \mapsto \tilde{X}$.

In general the lift $\tilde{X}$ will NOT be G-equivariant in the sense that the vector fields $L_{g_0}^*\tilde{X}$ and $\widetilde{\ell_{g_0}^* X}$ are equal for all $g_0 \in G$. However, the formula (2.83) does not depend on the lift $X \mapsto \tilde{X}$. Hence there can be G-invariant covariant derivatives in associated vector bundles of $\pi : G \to G/H$ which are NOT coming from a G-invariant connection in $(G, \pi, G/H, H)$ (which is equivalent to G/H being reductive, see Section 2.5.1).

Proof: Let X be a vector field on G/H and fix an H-invariant lift $\tilde{X}$ on G. According to formula (1.7) we need to compute the horizontal lift X^h of X from G/H to the total space $G_H[U]$. We use the description of $G_H[U]$ as $G \times U$ modulo the right H-action R^χ given by $(g,u)h = (gh, \chi(h)^{-1}u)$ for all $g \in G$, $h \in H$, and $u \in U$, see eqn (2.19), i.e. by means of the surjective submersion $\kappa_\chi : G \times U \to G_H[U]$, see eqn (2.23). Let $\tilde{X}^h$ be the vector field on $G \times U$ defined by (for all $g \in G$, $u \in U$):

$$\tilde{X}^h(g,u) = \Big(\tilde{X}(g), \;\; -T_{e_U}R_u\big(\mathsf{p}\big(T_gL_{g^{-1}}(\tilde{X}(g))\big)\big)\Big). \tag{2.84}$$

It can be easily checked using the H-equivariance of p that the following smooth map $G \times U \to T\big(G_H[U]\big)$ is invariant by the right H-action R^χ:

$$(g,u) \mapsto T_{(g,u)}\kappa_\chi\big(\tilde{X}^h(g,u)\big) \in T_{\kappa_\chi(g,u)}\big(G_H[U]\big)$$

and defines thus a unique vector field X^h on $G_H[U]$ such that $T\kappa_\chi \circ \tilde{X}^h = X^h \circ \kappa_\chi$. Using the form (2.39) of the connection 1-form $\alpha[\chi, \mathsf{p}]$ it is quickly computed that X^h is horizontal, U-invariant, and projects onto X via the bundle projection $\tau : G_H[U] \to G/H$, i.e. $T\tau \circ X^h = X \circ \tau$, whence X^h is the horizontal lift of X with respect to $\alpha[\chi, \mathsf{p}]$. Moreover, although $\tilde{X}^h$ depends on the lift $X \mapsto \tilde{X}$ the projected vector field X^h does not depend on the lift because the restriction of p to $\mathfrak{h}$ equals $T_e\chi$. Now let $\psi \in \Gamma^\infty(G/H, E)$, and let $\hat{\psi} : G \to V$ the associated H-equivariant smooth function. It is easy to compute that

$$\widehat{\Phi_V \circ \psi}\big(\kappa_\chi(g,u)\big) = \rho(u^{-1})\big(\hat{\psi}(g)\big) \tag{2.85}$$

for al $g \in G$ and $u \in U$, and therefore

$$X^h\Big(\widehat{\Phi_V \circ \psi}\Big)\big(\kappa_\chi(g,u)\big) = \rho(u^{-1})\Big(\tilde{X}\big(\hat{\psi}\big)(g) + \dot{\rho}\big(\mathsf{p}\big(T_gL_{g^{-1}}(\tilde{X}(g))\big)\big)\big(\hat{\psi}(g)\big)\Big)$$

which gives the associated U-equivariant function of the r.h.s. of the stated equation (2.82) using eqn (1.7). Again by (2.85) it is clear that this Φ_V composed with something described by the stated equation (2.83). This proves both stated equations. It is easy to check that the r.h.s. of eqn (2.83) describes a covariant derivative as it is obviously $\mathbb{R}$-bilinear, $\mathcal{C}^\infty(G/H, \mathbb{K})$ linear in the argument X (since $\widetilde{fX} = (\pi^*f)\tilde{X}$ for all $f \in \mathcal{C}^\infty(G/H, \mathbb{K})$) and first order in ψ (since $\widehat{f\psi} = (\pi^*f)\hat{\psi}$ for all $f \in \mathcal{C}^\infty(G/H, \mathbb{K})$). The G-invariance of the covariant derivative ∇ follows from the G-invariance of the covariant derivative ∇' and the fact that Φ_V is G-equivariant and invertible. $\square$

2.5 Examples of G-ϑ-equivariant principal U-bundles and connections

2.5.1 Reductive homogeneous spaces

Let $U = H$, $\vartheta_g = \mathrm{id}_H$ for all $g \in G$, and $\chi = \mathrm{id}_H$.

It is easy to see that the Atiyah class vanishes (hence there is a G-invariant connection in the bundle $(G, \pi, G/H)$) iff there is a H-invariant projection $\mathsf{p} : \mathfrak{g} \to \mathfrak{h}$ (which is idempotent since $\mathsf{p}(\eta) = \eta$ for all $\eta \in \mathfrak{h}$), hence iff there is an H-invariant subspace $\mathfrak{m} \subset \mathfrak{g}$ which is a complement of $\mathfrak{h}$, i.e. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Recall that homogeneous spaces G/H for which such an H-invariant complement $\mathfrak{m}$ to the subalgebra $\mathfrak{h}$ exists are called *reductive homogeneous spaces*, see e.g. [22, p.190]. It is well-known that for compact G all the homogeneous spaces are reductive.

Note that for a reductive homogeneous space the Atiyah class vanishes for any Lie group U and any smooth homomorphism $\chi : H \to U$ (again $\vartheta_g = \mathrm{id}_U$ for all $g \in G$): It suffices to extend the linear map $T_e\chi : \mathfrak{h} \to \mathfrak{u}$ on all of $\mathfrak{g}$ by setting it equal to zero on $\mathfrak{m}$. Hence invariant connections always exist in this case, see [36, p.10, Cor. 3].

2.5.2 Coadjoint orbits

Let $U = U(1)$ the circle group.

We shall first consider the case where ϑ is trivial, i.e.$\vartheta_g = \mathrm{id}_H$ for all $g \in G$. Let $\chi : H \to U(1)$ be a smooth homomorphism of Lie groups. Then the Atiyah class vanishes iff the linear map $T_e\chi : \mathfrak{h} \to \mathfrak{u}(1) \cong \mathbb{R}$ extends to a H-invariant linear map $\mathsf{p} : \mathfrak{g} \to \mathbb{R}$, i.e. iff $\mathsf{p} \in \mathfrak{g}^*$ and $\mathsf{p}(\eta) = T_e\chi(\eta)$ for all $\eta \in \mathfrak{h}$ and

$$\forall\, h \in H : \quad \mathsf{p} \circ Ad_h = \mathsf{p} \;\text{ iff }\; H \subset G_{\mathsf{p}} := \{g \in G \mid Ad^*_g(\mathsf{p}) = \mathsf{p}\}.$$

Clearly G_p is the isotropy group at p of the *coadjoint action* of G on the dual of its Lie algebra, $\mathfrak{g}^*$. Recall the *coadjoint orbit* $\mathcal{O}^G_{\mathsf{p}}$ passing through p defined by

$$\mathcal{O}^G_{\mathsf{p}} := \{Ad^*_g(\mathsf{p}) \in \mathfrak{g}^* \mid g \in G\} \cong G/G_{\mathsf{p}}.$$

So if the Atiyah class vanishes, the homogeneous space G/H is a G-equivariant fibre bundle over some coadjoint orbit G/G_{p} (where the obvious projection is given by $\tau(gH) = gG_{\mathsf{p}}$). In case $H = G_{\mathsf{p}}$ the coadjoint orbit is called *integral.* Note also that in case the restriction of p to $\mathfrak{h}$ the homomorphism χ is always surjective. For further information on this matter, see e.g. [24], [16] or [37].

Returning to the general case it is not hard to see that the group of all smooth automorphisms of $U(1)$ has only two elements: the identity map, and the map sending each element of the circle to its inverse. Hence for any nontrivial ϑ it follows that the Lie group G is disconnected. Moreover $\dot{\vartheta}$ necessarily vanishes in this case.

Example 2.8. Let $G = O(1,3) \times \mathbb{R}^4$ the physicist's Poincaré group where $O(1,3)$ denotes the orthogonal group of $\mathbb{R}^4$ equipped with the nondegenerate symmetric bilinear form $g(x,y) = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3$ (the so-called Minkowski metric, where $x, y \in \mathbb{R}^4$, written $x = (x^0, x^1, x^2, x^3)^T$). G is the semidirect product of $O(1,3)$ and $\mathbb{R}^4$. G is known to have 4 connected components, represented by the 4 elements $e = (\mathrm{diag}(1,1,1,1), 0)$, $P = (\mathrm{diag}(1,-1,-1,-1), 0)$ ('parity'), $T = (\mathrm{diag}(-1,1,1,1), 0)$ ('time reversal'), and $PT = (\mathrm{diag}(-1,-1,-1,-1), 0)$. The factor group of G modulo its identity component is thus isomorphic to the finite group $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by the classes of the above transformations. Let $\epsilon : G \to \mathbb{Z}_2 = \{1, -1\}$ be the homomorphism of Lie groups sending each $g \in G$ first to its class modulo the identity component, and then using the map assigning to e and to P the value 1, and to T and PT the value -1. Now set U equal to the circle group $U(1)$, and define $\vartheta : G \times U \to U$ by

$$\vartheta_g(u) = u^{\epsilon(g)}. \tag{2.86}$$

Among the coadjoint orbits of the Poincaré group started by J.-M. Souriau and others there is for instance one corresponding to a massive particle with spin which has two connected components for which the symplectic 2-form used in mathematical physics is NOT equal to the usual G-invariant Kirillov-Kostant-Souriau 2-form (see e.g. eqn (3.8)) but differs by a sign on one of the components, see e.g. [37, p.121]: this is due to the fact that physicists want time reversal to be an *anti-unitary* map in Hilbert space when quantized, which corresponds to an *anti-symplectic transformation* on the orbit.

2.5.3 G-invariant linear connections in the tangent bundle of G/H

Let $U = GL(\mathfrak{g}/\mathfrak{h})$, $\vartheta_g = \mathrm{id}_U$ for all $g \in G$, and $\chi : H \to U$ given by $\chi(h) = Ad'_h$, see (2.6).

Consider the principal $GL(\mathfrak{g}/\mathfrak{h})$-bundle $G_H[U]$ over G/H. Since the group of all linear maps $GL(\mathfrak{g}/\mathfrak{h})$ acts linearly on $\mathfrak{g}/\mathfrak{h}$ there is an associated vector bundle $G_H[U][\mathfrak{g}/\mathfrak{h}]$. By means of the isomorphism (1.3) –which is clearly G-equivariant– the associated vector bundle $(G_H[U])_U[\mathfrak{g}/\mathfrak{h}]$ is isomorphic to the associated vector bundle $G_H[\mathfrak{g}/\mathfrak{h}]$ which in turn is isomorphic to the tangent bundle of G/H. Vanishing Atiyah class of the above situation is equivalent to having a G-invariant connection in the G-equivariant principal U-bundle $G_H[U]$ which will yield a *G-invariant covariant derivative ∇ in the tangent bundle of G/H* according to formula (1.7). Note also that the *torsion* $\mathrm{Tor}_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ for all vector fields X, Y on G/H of ∇ is a G-invariant tensor field. Hence the modified covariant derivative

$$\nabla'_X Y = \nabla_X Y - \frac{1}{2}\mathrm{Tor}_\nabla(X,Y)$$

is still G-invariant and torsion-free. Note that there is the following isomorphism $G_H[U]$ to the *bundle of all linear frames*, $P^1(G/H)$, of G/H: choose a base $e_1,\ldots,e_m$ of $\mathfrak{g}/\mathfrak{h}$, and define $\Phi : G_H[U] \to P^1(G/H)$ by (for all $g \in G$, $u \in GL(\mathfrak{g}/\mathfrak{h})$)

$$\Phi : [g,u] \mapsto \Big(T_o\ell_g\big(\pi'\big(u(e_1)\big)\big),\ldots,T_o\ell_g\big(\pi'\big(u(e_m)\big)\big)\Big),$$

and we have for all $\tilde{u} \in U$ $\Phi([g,u]\tilde{u}) = \Phi([g,u])\theta(\tilde{u})$ where $\theta(\tilde{u})$ is the matrix of $\tilde{u}$ with respect to the base $e_1,\ldots,e_m$. Clearly $\theta : GL(\mathfrak{g}/\mathfrak{h}) \to GL(m,\mathbb{R})$ is a smooth isomorphism of Lie groups, and hence Φ is an isomorphism of principal fibre bundles over G/H.

When smooth Lie group cohomology is replaced by the Chevalley-Eilenberg cohomology of the Lie subalgebra $\mathfrak{h}$, see Subsubsection 2.4.4, then the above case appeared in the literature before, see the work by Nguyen-van Hai, [31, p.46, eqs (10)-(13), Prop.3] for general Lie algebra inclusions, [29], [30] for foliations, [5], [6] for foliations of coisotropic submanifolds, and [7] for general Lie algebra inclusions $\mathfrak{h} \subset \mathfrak{g}$.

2.6 Generalisation of Infinitesimal Coadjoint Orbits

We have seen in the previous Sections essentially by Wang's Theorem how to associate G-ϑ equivariant connections in principal U-bundles over a *given* homogeneous space G/H by certain linear maps $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$. One may consider the following 'inverse problem': *given* an automorphic left G-action ϑ on U and *given* an arbitrary linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$, is there a closed subgroup H of G and a crossed homomorphism $\chi : H \to U$ such that the conditions (2.36) and (2.37) of Proposition 2.3 are satisfied? In case G arbitrary, $U = U(1)$, and ϑ trivial the question is answered by any integral coadjoint orbit of G: since $\mathfrak{u}(1) \cong \mathbb{R}$ the linear maps p in question are thus elements of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$. The subgroup H is defined as the *isotropy subgroup of* p, see Subsection 2.5.2. For the general situation I do not know how to do this for Lie groups. However, for the infinitesimal situation (dealt with in Subsection 2.4.4) where $\mathfrak{g}$, $\mathfrak{u}$ are two Lie algebras, $\ddot{\vartheta} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{u})$ is a derivational Lie algebra representation (for example $\ddot{\vartheta} = 0$), see eqs (2.64), and $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ is an arbitrary linear map, the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ can –roughly speaking– be defined as the 'subalgebra of all elements of $\mathfrak{g}$ fixing p which defines the potential connection'. More precisely, define

(2.87)

$$\mathfrak{g}_{\mathsf{p}} \;:=\; \{\eta \in \mathfrak{g} \mid \forall\ \xi \in \mathfrak{g} :\ [\mathsf{p}(\eta),\mathsf{p}(\xi)] - \mathsf{p}([\eta,\xi]) + \ddot{\vartheta}_\eta\big(\mathsf{p}(\xi)\big) - \ddot{\vartheta}_\xi\big(\mathsf{p}(\eta)\big) = 0\}.$$

By an elementary computation (which is lengthy in case $\ddot{\vartheta} \neq 0$) using only the Jacobi identity of the occurring Lie brackets and the defining equations (2.64) of a derivational Lie algebra representation we get the following result generalizing the isotropy algebra $\mathfrak{g}_{\mathsf{p}}$ of $\mathsf{p} \in \mathfrak{g}^*$ for an infinitesimal coadjoint orbit:

Proposition 2.9. *With the above hypotheses:*

1. $\mathfrak{g}_{\mathsf{p}}$ *is a Lie subalgebra of* $\mathfrak{g}$.
2. *The restriction* $\dot{\chi}$ *of* p *to* $\mathfrak{g}_{\mathsf{p}}$ *defines an infinitesimal crossed morphism of Lie algebras* $\mathfrak{g}_{\mathsf{p}} \to \mathfrak{u}$*, see eqn (2.66), whence* p *satisfies eqn (2.37).*
3. p *is an infinitesimal* $\mathfrak{g}$-$\ddot{\vartheta}$*-equivariant connection in the sense of Subsection 2.4.4, i.e. for which* $\varphi[\dot{\chi}, \mathsf{p}] = 0$*, see eqn (2.70).*

Warning: note that the definition of the subalgebra $\mathfrak{g}_{\mathsf{p}}$ of $\mathfrak{g}$ given in eqn (2.87) does not work to ensure the statement of the preceding Proposition in the important particular case where $\mathfrak{u} = \mathbf{Hom}_{\mathbb{K}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$: and $\ddot{\vartheta} = 0$: here the Lie algebra $\mathfrak{u}$ depends on the subalgebra $\mathfrak{h}$! I do not know how to define some $\mathfrak{g}_{\mathsf{p}}$ in this case.

3 Multidifferential Operators over Homogeneous Spaces

3.1 Characterization of multi-differential operators

In this subsection we should like to state a Theorem about multi-differential operators on homogeneous spaces which is at least folklore because it is used for instance in [1] for a particular case.

Let G be a Lie group with Lie algebra $(\mathfrak{g}, [\ ,\ ])$, let H be a closed subgroup of G with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, and let V_1,...,V_k, W be finite-dimensional H-modules (which are vector spaces over the field $\mathbb{K}$ which is either equal to $\mathbb{R}$ or equal to $\mathbb{C}$). Let $V_1^*, \ldots, V_k^*$ denote the dual H-modules. Form the associated vector bundles $E_1 = G_H[V_1]$,..., $E_k = G_H[V_k]$, $F = G_H[W]$ (with respect to the principal H-bundle $(G, \pi, G/H, H)$. We are interested in suitable description of the space of k-multi-differential operators $\mathbf{Diff}_{G/H}(E_1, \ldots, E_k; F)$.

In order to state the theorem note that any finite-dimensional H-module is also a $\mathfrak{h}$-module, and hence a left module for the universal envelopping algebra $\mathsf{U}(\mathfrak{h})$ via $(\eta_1\eta_k)v = \eta_1(\eta_2(\cdots(\eta_k v)\cdots)$ for all $\eta_1, \ldots, \eta_k \in \mathfrak{h}$. Moreover the universal envelopping algebra of $\mathfrak{g}$, $\mathsf{U}(\mathfrak{g})$, is a right module of the $\mathsf{U}(\mathfrak{h})$ since the latter is a subalgebra of the former. In the following we shall denote by $\otimes$ the tensor products of vector spaces over $\mathbb{K}$ whereas $\otimes_{\mathsf{U}(\mathfrak{h})}$ denotes the tensor product with respect to the ring $\mathsf{U}(\mathfrak{h})$. Recall that universal envelopping algebras over finite-dimensional Lie algebras are filtered algebras $\mathsf{U}(\mathfrak{g}) = \bigcup_{i \in \mathbb{N}} \mathsf{U}_i(\mathfrak{g})$ where all the subspaces $\mathsf{U}_i(\mathfrak{g})$ are finite-dimensional.

Theorem 3.1. *With the above hypotheses and notations:*

1. *The $\mathcal{C}^\infty(G/H,\mathbb{K})$-module of all k-multi-differential operators over G/H, $\mathbf{Diff}_{G/H}(E_1,\ldots,E_k;F)$, is isomorphic to the $\mathcal{C}^\infty(G/H,\mathbb{K})$-module of all smooth 'filtered' sections of the associated vector bundle of the principal bundle $(G,\pi,G/H,H)$ with typical fibre*

$$W\otimes(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_1^*)\otimes\cdots\otimes(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_k^*). \tag{3.1}$$

where the Lie group H acts on this vector space diagonally on the tensor factors via the given action on V_1^*,..., V_k^*, W, via the adjoint action Ad_h on $\mathfrak{g}$ which give a unique action (also denoted by Ad_h) on $\mathsf{U}(\mathfrak{g})$ (and on $\mathsf{U}(\mathfrak{h})$) preserving the bialgebra structure.

2. *The $\mathbb{K}$-vector space of all G-invariant k-multi-differential operators over G/H is isomorphic to the subspace of H-invariants*

$$\Big(W\otimes(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_1^*)\otimes\cdots\otimes(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_k^*)\Big)^H. \tag{3.2}$$

3. *In the particular case where all the H-modules $V_1,\ldots,V_k,W$ are equal to the trivial module $\mathbb{K}$ the above typical fibre (3.1) reduces to*

$$\left(\frac{\mathsf{U}(\mathfrak{g})}{\mathsf{U}(\mathfrak{g})\mathfrak{h}}\right)^{\otimes k}. \tag{3.3}$$

Proof: **1**. Write A for the associative commutative unital $\mathbb{K}$-algebra $\mathcal{C}^\infty(G/H,\mathbb{K})$, and B for $\mathcal{C}^\infty(G/H,\mathbb{K})$. Clearly the pull-back $\pi^*: A\to B$ is an injection of $\mathbb{K}$-algebras on the subalgebra of all right H-invariants in B. According to Proposition 1.2 we have in the particular case of the principal H-bundle $(G,\pi,G/H,H)$ over G/H the isomorphism of A-modules

$$\left(\frac{\mathbf{Diff}_G(G\times V_1,\ldots,G\times V_k;G\times W)}{\mathbf{K}_1+\cdots+\mathbf{K}_k}\right)^H\cong\mathbf{Diff}_{G/H}(E_1,\ldots,E_k;F).$$

It is not hard to see, using Proposition 1.1, that there is an isomorphism of B-modules

$$\mathcal{C}^\infty(G,\mathbb{K})\otimes W\otimes\big(\mathsf{U}(\mathfrak{g})\otimes V_1^*\big)\otimes\cdots\otimes\big(\mathsf{U}(\mathfrak{g})\otimes V_k^*\big)\cong\mathbf{Diff}_G(G\times V_1,\ldots,G\times V_k;G\times W).$$

which is the following map

$$\begin{aligned}&f\otimes w\otimes\mathsf{u}_1\otimes v_1^*\otimes\cdots\otimes\mathsf{u}_k\otimes v_k^*\\&\quad\mapsto\ \Big((\psi_1',\ldots,\psi_k')\mapsto fw\big(\mathsf{u}_1^+(\langle v_1^*,\psi_1'\rangle)\big)\cdots\big(\mathsf{u}_k^+(\langle v_k^*,\psi_k'\rangle)\big)\Big).\end{aligned}$$

where $f \in \mathcal{C}^\infty(G,\mathbb{K})$, $w \in W$, $\mathsf{u}_1,\ldots,\mathsf{u}_k \in \mathsf{U}(\mathfrak{g})$, $v_1^* \in V_1^*$,..., $v_k^* \in V_k^*$, $\psi_1' \in \mathcal{C}^\infty(G,V_1)$,...,$\psi_k' \in \mathcal{C}^\infty(G,V_k)$. Note that the H-action on multi-differential operators, see (1.14) is transferred by the above isomorphism to

$$\begin{aligned} & h.\Big(f\otimes w\otimes \mathsf{u}_1\otimes v_1^*\otimes\cdots\otimes \mathsf{u}_k\otimes v_k^*\Big) \\ & \quad = \quad (f\circ R_h)\otimes\rho(h)(w)\otimes Ad_h(\mathsf{u}_1)\otimes\rho_1^*(h)\big(v_1^*\big)\otimes\cdots\otimes Ad_h(\mathsf{u}_k)\otimes\rho_k^*(h)\big(v_k^*\big). \end{aligned}$$

We shall compute the H-(and B-)submodules $\mathbf{K}_1$,..., $\mathbf{K}_k$, see eqn (1.15). Using the above map, the submodule $\mathbf{K}_j$ is spanned by elements of the following form

$$f\otimes w\otimes \mathsf{u}_1\otimes v_1^*\otimes\cdots\otimes\big(\mathsf{u}_j\eta\otimes v_j^* - \mathsf{u}_j\otimes\dot\rho_j^*(\eta)(v_j^*)\big)\otimes\cdots\otimes\mathsf{u}_k\otimes v_k^*$$

and since $\mathsf{U}(\mathfrak{h})$ is generated by all monomials $\eta_1\cdots\eta_n$ with $\eta_1,\ldots,\eta_n \in \mathfrak{h}$ we can conclude that

$$\begin{aligned} & \left(\frac{\mathbf{Diff}_G(G\times V_1,\ldots,G\times V_k;G\times W)}{\mathbf{K}_1+\cdots+\mathbf{K}_k}\right)\cong \\ & \qquad\qquad \mathcal{C}^\infty(G,\mathbb{K})\otimes W\otimes\big(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_1^*\big)\otimes\cdots\otimes\big(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_k^*\big). \end{aligned}$$

Finally, passing to the H-invariants on both sides of the above isomorphism we see that the space of multi-differential operators on G/H is given by smooth sections of a vector bundle with typical fibre (3.1).

2. Clear by eqn (2.4).

3. This is evident. □

Note that each factor in the fibre (3.1), $\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_i^*$ is a left $\mathsf{U}(\mathfrak{g})$-module, the so-called *induced module* or *Verma module* with respect to the module V_i^* of the subalgebra $\mathsf{U}(\mathfrak{h})$.

As a Corollary to the preceding Theorem 3.1 we can conclude that for any associated vector bundle $E = G_H[V]$ (where V is a finite-dimensional H-module) its rth jet prolongation ($r \in \mathbb{N}$) is isomorphic to the following associated bundle

$$J^rE_j \cong G_H\Big[\big(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_j^*\big)_r^*\Big]. \tag{3.4}$$

Exercise: The Lie bracket of vector fields on G/H obviously is a G-invariant bidifferential operator on the tangent bundle. Let $\otimes$ denote the tensor product over the ground field $\mathbb{K}$. Show that the Lie bracket is induced by an H-invariant element of $(\mathfrak{g}/\mathfrak{h})\otimes\big(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}(\mathfrak{g}/\mathfrak{h})^*\big)\otimes\big(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}(\mathfrak{g}/\mathfrak{h})^*\big)$ which is obtained as follows: first project the following element of $(\mathfrak{g}/\mathfrak{h})\otimes\mathsf{U}(\mathfrak{g})\otimes\mathfrak{g}^*\otimes\mathsf{U}(\mathfrak{g})\otimes\mathfrak{g}^*$,

$$\varpi\circ[\ ,\ ]_{135}+\varpi_{15}\otimes(\mathrm{id}_\mathfrak{g})_{34}-\varpi_{13}\otimes(\mathrm{id}_\mathfrak{g})_{25},$$

(where the indices between 1 and 5 refer to the position of a given map in the fivefold tensor product (standard notation in Hopf algebra theory)) to $(\mathfrak{g}/\mathfrak{h})\otimes$

$\big(\mathsf{U}(\mathfrak{g}) \otimes_{\mathsf{U}(\mathfrak{h})} \mathfrak{g}^*\big) \otimes \big(\mathsf{U}(\mathfrak{g}) \otimes_{\mathsf{U}(\mathfrak{h})} \mathfrak{g}^*\big)$, and then check that it is in the appropriate subspace (note that by PBW $\mathsf{U}(\mathfrak{g})$ is a free, hence flat $\mathsf{U}(\mathfrak{h})$-module).

Exercise: Generalize the preceding exercise to the situation where $(\mathfrak{u}, [\ ,\ ]_{\mathfrak{u}})$ is another Lie algebra, $\varphi : \mathfrak{u} \to \mathfrak{g}$ and $j : \mathfrak{h} \to \mathfrak{u}$ are Lie algebra morphisms such that $\phi \circ j = i$ where $i : \mathfrak{h} \to \mathfrak{g}$ is the natural inclusion, H acts on $\mathfrak{u}$ by Lie algebra morphisms leaving invariant $j(\mathfrak{h}$ such that φ intertwines the H-action with the adjoint action, and $\varpi_{\mathfrak{u}} : \mathfrak{u} \to \mathfrak{u}/(j(\mathfrak{h}))$ is the natural projection. Show that the associated vector bundle $E := G_H[\mathfrak{u}/(j(\mathfrak{h}))]$ carries the structure a G-equivariant *Lie algebroid* (see e.g. [26] or [28] for definitions) where the anchor map $E \to T(G/H) \cong G_H[\mathfrak{g}/\mathfrak{h}]$ is induced by φ and the bracket is obtained as in the preceding exercise by considering in $\big(\mathfrak{u}/(j(\mathfrak{h}))\big) \otimes \mathsf{U}(\mathfrak{g}) \otimes \mathfrak{u}^* \otimes \mathsf{U}(\mathfrak{g}) \otimes \mathfrak{u}^*$ the element

$$\varpi_{\mathfrak{u}} \circ \big([\ ,\ ]_{\mathfrak{u}}\big)_{135} + \big(\varpi_{\mathfrak{u}}\big)_{15} \otimes \varphi_{34} - \big(\varpi_{\mathfrak{u}}\big)_{13} \otimes \varphi_{25}.$$

Show that all G-equivariant Lie algebroids over G/H are obtained that way (equivalence of categories).

3.2 A Theorem by Calaque, Căldăraru and Tu

In the preceding Subsection 3.1 the H-module (resp. $\mathfrak{h}$-module) $\mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}$ turned out to be rather important. Note that this module is filtered by $\mathsf{U}_n(\mathfrak{g})/\big(\mathsf{U}(\mathfrak{g})\mathfrak{h}\big)_n$. Another filtered H-module (resp. $\mathfrak{h}$-module) is $\mathsf{S}(\mathfrak{g}/\mathfrak{h})$ where H (resp. $\mathfrak{h}$) acts via Ad' (resp. via ad') on $\mathfrak{g}/\mathfrak{h}$. One may ask the question whether these two filtered H-modules (resp. $\mathfrak{h}$-modules) are isomorphic as filtered H-modules. (resp. $\mathfrak{h}$-modules).

The question is trivial in the case when the two Hopf algebras $\mathsf{S}(\mathfrak{g})$ and $\mathsf{U}(\mathfrak{g})$ are concerned: by the Poincaré-Birkhoff-Witt (PBW) Theorem the two coalgebras $\mathsf{S}(\mathfrak{g})$ and $\mathsf{U}(\mathfrak{g})$ are isomorphic. As an isomorphism the symmetrization map $\omega : \mathsf{S}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$ can be used and is known to be an isomorphism of filtered H-modules (resp. filtered $\mathfrak{h}$-modules, see e.g. [13, p.80-81]. Let $\bullet$ denote the commutative multiplication in $\mathsf{S}(\mathfrak{g})$, Δ_S be the comultiplication, and ϵ_S be the counit, whereas the multiplication in $\mathsf{U}(\mathfrak{g})$ will be denoted by $\mathsf{u}_1\mathsf{u}_2$, the comultiplication by Δ, and the counit by ϵ. In order to make computations we shall use the convolution $*$ in $\mathbf{Hom}_{\mathbb{K}}\big(\mathsf{S}(\mathfrak{g}), \mathsf{U}(\mathfrak{g})\big)$ using the comultiplication Δ_S and the multiplication in $\mathsf{U}(\mathfrak{g})$. Let $\mathsf{b} : \mathsf{S}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$ be the linear map $i \circ \mathrm{pr}$ where pr denotes the projection $\mathsf{S}(\mathfrak{g}) \to \mathfrak{g}$, and $i_{\mathfrak{g}} : \mathfrak{g} \to \mathsf{U}(\mathfrak{g})$ the injection obtained by PBW. Then the solution of the differential equation

$$\frac{d\omega_t}{dt} = \mathsf{b} * \omega_t \qquad \text{with } \ \omega_0 = 1\epsilon_S$$

gives a well-defined family of coalgebra maps $t \mapsto \omega_t : \mathsf{S}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$ such that $\omega_1 = \omega$, the above-mentioned symmetrization map. For $t \neq 0$ ω_t is invertible. Again for computational purposes, fix a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (where the subspace $\mathfrak{m}$ is in general not H-invariant (resp. $\mathfrak{h}$-invariant)), and for any $\xi \in \mathfrak{g}$ let $\xi_\mathfrak{m}$ its component in $\mathfrak{m}$ and $\xi_\mathfrak{h}$ its component in $\mathfrak{h}$. Let $\mathsf{b}_\mathfrak{m}$ (resp. $\mathsf{b}_\mathfrak{h}$) be the map b followed by the projection onto $\mathfrak{m}$ (resp. $\mathfrak{h}$). Consider the modified differential equation

$$\frac{d\Psi_t}{dt} = \mathsf{b}_\mathfrak{m} * \Psi_t + \Psi_t * \mathsf{b}_\mathfrak{h} \qquad \text{with} \quad \Psi_0 = 1\epsilon_S$$

It can be rapidly checked using the fact that the right multiplication with $\eta \in \mathfrak{h}$ is a coderivation of $\mathsf{S}(\mathfrak{g})$ that $t \mapsto \Psi_t$ is a family of coalgebra morphisms (which are isomorphisms for each $t \neq 0$) mapping the ideal and coideal $\mathsf{S}(\mathfrak{g}) \bullet \mathfrak{h}$ of $\mathsf{S}(\mathfrak{g})$ onto the left ideal and coideal $\mathsf{U}(\mathfrak{g})\mathfrak{h}$ of $\mathsf{U}(\mathfrak{g})$, and passes hence to the quotient to define a coalgebra isomorphism between $\mathsf{S}(\mathfrak{g}/\mathfrak{h}) \cong \mathsf{S}(\mathfrak{g})/\big(\mathsf{S}(\mathfrak{g})\bullet\mathfrak{h}\big)$ and $\mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}$. Moreover in the reductive case, i.e. in the case where $\mathfrak{m}$ is H-invariant (resp. $\mathfrak{h}$-invariant), it follows easily that Ψ_1 and thus the induced isomorphism intertwines the H-actions (resp. the $\mathfrak{h}$-actions) proving the statement in that case which was well-known.

Recently, Calaque, Căldăraru and Tu [7] have shown the following result:

Theorem 3.2. *Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic 0 and $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. Then the following two statements are equivalent:*

1. *The Atiyah class of the Lie algebra inclusion (w.r.t. $\mathfrak{u} = \mathfrak{gl}(\mathfrak{g}/\mathfrak{u})$ and $\dot{\chi} = ad'$) vanishes.*
2. *The two filtered $\mathfrak{h}$-modules are isomorphic by a filtered isomorphism:*

$$\mathsf{S}(\mathfrak{g}/\mathfrak{h}) \quad \cong \quad \frac{\mathsf{U}(\mathfrak{g})}{\mathsf{U}(\mathfrak{g})\mathfrak{h}}.$$

We should like to prove the implication "**1.** $\Rightarrow$ **2.**" first of the analogous statement in smooth Lie group cohomology using differential geometry of affine connections:
Suppose that the Atiyah class $c_{G,H,GL(\mathfrak{g}/\mathfrak{h}),Ad'}$ vanishes. By Proposition 2.5 we get a G-invariant connection in the frame bundle $P^1(G/H)$ and in turn a G-invariant covariant derivative ∇ in the tangent bundle $T(G/H)$ which we can choose to be torsion-free. For any smooth manifold, let $\mathsf{S}TM$ (resp. $\mathsf{S}\big(T^*M\big)$) denote the bundle of symmetric tensors of the tangent (resp. of the cotangent) bundle. There is a differential operator of order 1, D –which depends on the covariant derivative ∇– mapping each $\Gamma^\infty\big(M, \mathsf{S}^k(T^*M)\big)$ to $\Gamma^\infty\big(M, \mathsf{S}^{k+1}(T^*M)\big)$ defined by

$$(3.5) \qquad (\mathsf{D}\gamma)\big(X_1, \ldots, X_{k+1}\big) := \frac{1}{k!} \sum_{\sigma \in S_{k+1}} (\nabla_{X_{\sigma(1)}}\gamma)\big(X_{\sigma(2)}, \ldots, X_{\sigma(k+1)}\big).$$

for each $\gamma \in \Gamma^\infty\big(M, \mathsf{S}^k(T^*M)\big)$ and vector fields $X_1, \ldots, X_{k+1}$ on M. It is easy to check that D is a derivation of the commutative associative pointwise multiplication in $\Gamma^\infty\big(M, \mathsf{S}(T^*M)\big)$. Moreover, for any vector field X on M let $i_X : \Gamma^\infty\big(M, \mathsf{S}^k(T^*M)\big)$ to $\Gamma^\infty\big(M, \mathsf{S}^{k-1}(T^*M)\big)$ the usual interior product defined by $(i_X\gamma)\big(X_2, \ldots, X_k\big) = \gamma(X_1, X_2, \ldots, X_k)$ which can be canonically extended to a representation of the filtered vector bundle of commutative associative unital algebras $\Gamma^\infty\big(M, \mathsf{S}(TM)\big))$ on $\Gamma^\infty\big(M, \mathsf{S}(T^*M)\big)$ in the usual way, i.e. $i_{X_1\cdots X_r}\gamma = i_{X_1}\big(\cdots i_{X_k}(\gamma)\cdots\big)$. Recall that the *standard symbol calculus with respect to* ∇ is the linear map $\rho_S : \Gamma^\infty\big(M, \mathsf{S}(TM)\big) \to \mathbf{Diff}_M(M\times\mathbb{R}, M\times\mathbb{R})$ given by the following expression for all nonnegative integers k, $A \in \Gamma^\infty\big(M, \mathsf{S}^k(TM)\big)$, and $f \in \mathcal{C}^\infty(M, \mathbb{R})$

$$\rho_S(A)(f) = i_A\big(\mathsf{D}^k(f)\big). \tag{3.6}$$

It is well-known that the map ρ_S is a filtered isomorphism of filtered $\mathcal{C}^\infty(M, \mathbb{R})$-modules, see e.g. [4]. Moreover note that the $\mathcal{C}^\infty(M, \mathbb{R})$-module $\mathbf{Diff}_M(M \times \mathbb{R}, M \times \mathbb{R})$ is the space of all smooth sections of the filtered dual space of the jet bundle $J^\infty(M, \mathbb{R})_0$, see [23] for more informations. This means that the two vector bundles $\mathsf{S}(TM)$ and ${J^\infty(M, \mathbb{R})_0}^*$ are isomorphic by a filtered isomorphism given by ρ_S. Returning to the homogeneous space G/H: since the covariant derivative ∇ is G-invariant it follows that the isomorphism ρ_S is G-equivariant. Moreover the bundle $\mathsf{S}T(G/H)$ is isomorphic to the associated bundle $G_H[\mathsf{S}(\mathfrak{g}/\mathfrak{h})]$ whereas the graded dual of the jet bundle is given by the associated bundle $G_H[\mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}]$. Using the fibre functor $E \to E_o$ from $G \cdot \mathcal{VB}_{G/H}$ we see that there is a filtered isomorphism of the filtered H-modules $\mathsf{S}(\mathfrak{g}/\mathfrak{h})$ to $\mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}$.

Secondly, for the pure Lie algebra case we shall make some precisions on the isomorphism: in order to get an idea we consider the formula for the covariant derivative of $\gamma \in \Gamma^\infty(M, \mathsf{S}^kT^*M)$ in the direction of a vector field X on G/H at the distinguished point $o = \pi(e)$ according to formula (2.83): let $\xi \in \mathfrak{g}$ such that $T_e\pi(\xi) = X(o)$ (we can suppose that $\xi = \tilde{X}(e)$ for some H-invariant lift $X \mapsto \tilde{X}$), and $\xi_1, \ldots, \xi_k \in \mathfrak{g}$. Note that the associated H-equivariant function $\hat{\gamma}$ is a smooth map $G \to \mathbf{Hom}_{\mathbb{K}}\big(\mathsf{S}^k(\mathfrak{g}/\mathfrak{h}), \mathbb{K}\big)$, hence

$$\begin{aligned}(\nabla_X\gamma)(o)\Big(\varpi(\xi_1)\bullet\cdots\bullet\varpi(\xi_k)\Big) &= \big(\xi^+(\hat{\gamma})\big)(e)\Big(\varpi(\xi_1)\bullet\cdots\bullet\varpi(\xi_k)\Big)\\ &-\sum_{r=1}^{k}\hat{\gamma}(e)\Big(\mathsf{p}(\xi,\varpi(\xi_r))\bullet\varpi(\xi_1)\bullet\cdots\bullet\varpi(\xi_{r-1})\bullet\varpi(\xi_{r+1})\bullet\cdots\bullet\varpi(\xi_k)\Big).\end{aligned} \tag{3.7}$$

Returning to the case of a general Lie algebra $\mathfrak{g}$ containing a subalgebra $\mathfrak{h}$: Let $j : \mathfrak{g}/\mathfrak{h} \to \mathfrak{m} \subset \mathfrak{g}$ be the inverse of the restriction of the canonical projection $\varpi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ to $\mathfrak{m}$. Define a bilinear map $\tilde{\mathsf{p}} : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ by $\tilde{\mathsf{p}}(\xi, \xi') := j\big(\mathsf{p}(\xi)\big(\varpi(\xi')\big)\big)$. Note that $\tilde{\mathsf{p}}$ is NOT $\mathfrak{h}$-invariant, but rather

$$\big[\eta, \tilde{\mathsf{p}}(\xi, \xi')\big] - \tilde{\mathsf{p}}([\eta, \xi], \xi') - \tilde{\mathsf{p}}(\xi, [\eta, \xi']) = \big[\eta, \tilde{\mathsf{p}}(\xi, \xi')\big]_{\mathfrak{h}}$$

for all $\xi, \xi' \in \mathfrak{g}$ and $\eta \in \mathfrak{h}$. Let $a_1, \ldots$ be a sequence of nonzero elements of $\mathbb{K}$. We translate eqn (3.7) and its symmetrization (3.5) into the definition of a ‘curve of linear maps' $t \mapsto \Phi_t : \mathsf{S}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$ which we write as a formal power series $\sum_{k=0}^{\infty} t^k \Phi_k$ where for each nonnegative integer k the linear map Φ_k is defined on $\mathsf{S}^k(\mathfrak{g})$ and takes its values in $\mathsf{U}(\mathfrak{g})_k$. Note that in the space of linear maps the formal power series converges in the sense that t can be replaced by any element in $\mathbb{K}$. The recursive definition goes as follows: $\Phi_0 = 1\epsilon_S$, and

$$\Phi_{k+1}(\xi_1 \bullet \cdots \bullet \xi_{k+1}) = \frac{1}{k+1} \sum_{r=1}^{k+1} \xi_r \big(\Phi_k(\xi_1 \bullet \cdots \bullet \xi_{r-1} \bullet \xi_{r+1} \bullet \cdots \bullet \xi_{k+1})\big)$$
$$- \frac{1}{k+1} \sum_{r=1}^{k+1} \sum_{\substack{s=1 \\ s \neq r}}^{k+1} \Phi_k\big(\tilde{\mathsf{p}}(\xi_r, \xi_s) \bullet \xi_1 \bullet \cdots \bullet \xi_{r-1} \bullet \xi_{r+1} \bullet \cdots \bullet \xi_{s-1} \bullet \xi_{s+1} \bullet \cdots \bullet \xi_{k+1}\big).$$

First, since obviously $\Phi_k(\xi_1 \bullet \cdots \bullet \xi_k)$ is a nonzero multiple of $\xi_1 \cdots \xi_k$ modulo $\mathsf{U}(\mathfrak{g})_{k-1}$ it follows by PBW that Φ is a linear bijection. Let $\Pi : \mathsf{U}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}$ denote the canonical projection. It can be shown by a lengthy, but elementary induction over k that **i)** the map $\Pi \circ \Phi_k$ vanishes if one of the arguments is in $\mathfrak{h}$ and that **ii)** the map $\Pi \circ \Phi_k$ intertwines the $\mathfrak{h}$-actions (where the induction step has to be done for both statements at the same time). Hence Φ sends the (co)ideal $\mathsf{S}(\mathfrak{g})\bullet\mathfrak{h}$ of $\mathsf{S}(\mathfrak{g})$ into the left ideal and coideal $\mathsf{U}(\mathfrak{g})\mathfrak{h}$ of $\mathsf{U}(\mathfrak{g})$, and the above filtration argument plus an induction shows that this is onto. Hence there is an induced isomorphism of $\mathfrak{h}$-modules $\mathsf{S}(\mathfrak{g}/\mathfrak{h}) \to \mathsf{U}(\mathfrak{g})/(\mathsf{U}(\mathfrak{g})\mathfrak{h})$ which is an isomorphism of coalgebras for $t \neq 0$. Alternatively, in order to avoid fiddling around tedious combinatorics, one may consider the following differential equation: let r denote the linear map $\mathsf{S}^2(\mathfrak{g}) \to \mathfrak{g}$ defined by $\mathsf{r}(\xi_1 \bullet \xi_2) = \tilde{\mathsf{p}}(\xi_1, \xi_2) + \tilde{\mathsf{p}}(\xi_2, \xi_1)$. Moreover let $\tilde{*}$ denote the convolution of linear maps from $\mathsf{S}(\mathfrak{g})$ to $\mathsf{S}(\mathfrak{g})$ with respect to $\bullet$ and Δ_S. Then

$$\frac{d\Phi_t}{dt} = \mathsf{b} * \Phi_t - \Phi_t \circ (\mathsf{r}\tilde{*}\mathrm{id}_{\mathsf{S}(\mathfrak{g})}) \qquad \text{with} \quad \Psi_0 = 1\epsilon_S$$

defines the above 'curve'. Note that $\mathsf{r}\tilde{*}\mathrm{id}_{\mathsf{S}(\mathfrak{g})}$ is a coderivation of the coalgebra $\mathsf{S}(\mathfrak{g})$. The identities can be proved by showing that they satisfy certain differential equations with initial condition 0, hence they hold for all t by uniqueness of the solution. □

3.3 Invariant star-products on certain coadjoint orbits

Let G be a Lie group, $(\mathfrak{g}, [\ ,\ ])$ be its Lie algebra, and $\mathsf{p} \in \mathfrak{g}^*$. Let $\mathcal{O}_{\mathsf{p}}^G$ the coadjoint orbit $\{Ad_g^*(\mathsf{p}) \mid g \in G\}$, and G/H its associated homogeneous space, i.e. H is the isotropy subgroup $H = G_{\mathsf{p}} = \{g \in g \mid Ad_g^*(\mathsf{p}) = \nu\}$. The orbit and G/H are well-known to be *Hamiltonian G-spaces*, i.e. there is a canonical G-invariant symplectic 2-form, the Kirillov-Kostant-Souriau form given by

(3.8) $$\omega_{\pi(g)}\big([g, \pi'(\xi)], [g, \pi'(\xi')]\big) = \langle \mathsf{p}, [\xi, \xi'] \rangle$$

for all $g \in G$, $\xi, \xi' \in \mathfrak{g}$, see e.g. [16] for details, and a momentum mapping $J : G/H \to \mathfrak{g}^*$ given by $J\big(\pi(g)\big) = Ad^*_g(\mathsf{p})$. Recall that a *star-product* on a symplectic or more general Poisson manifold is a formal associative deformation of the associative commutative unital algebra of all smooth $\mathbb{K}$-valued functions on the manifold such that all the bilinear maps are bidifferential operators and that the first order commutator is proportional to the Poisson bracket, see e.g. [3], [15] or [35].

Proposition 3.3. *With the above notations: suppose that the Atiyah class with respect to the induced adjoint representation $Ad' : H \to Gl(\mathfrak{g}/\mathfrak{h})$, $c_{G,H,Gl(\mathfrak{g}/\mathfrak{h}),Ad'}$, vanishes. Then there is a G-invariant star-product $*$ on G/H.*

Proof: It follows that there is a G-invariant torsion-free covariant derivative in the tangent bundle of G/H. Using the Heß-Lichnerowicz-Tondeur trick, see e.g. [35, p.454], there is a G-invariant torsion-free symplectic connection in the tangent bundle. Using a Theorem by B.Fedosov, see [15, 180-183] there is G-invariant star-product on G/H. □.

Note that the series of bidifferential operators of the star-product is an element of $\Big(\mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h} \otimes \mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}\Big)^H[[\lambda]]$ (which has been noted in [1]). For compact coadjoint orbits there is Karabegov's construction in [19] whereas for certain $\mathbb{Z}$-graded coadjoint orbits Alekseev and Lakhowska constructed a star-product via the Shapovalov trace, see [1]. Pikulin and Tevelev classified those nilpotent orbits of reductive Lie groups for which the Atiyah class vanishes and only found a small class, see [33] for details.

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Twisting Poisson algebras, coPoisson algebras and Quantization

by Martin Bordemann, Olivier Elchinger and Abdenacer Makhlouf

Abstract

The purpose of this paper is to study twistings of Poisson algebras or bialgebras, coPoisson algebras or bialgebras and star-products. We consider Hom-algebraic structures generalizing classical algebraic structures by twisting the identities by a linear self map. We summarize the results on Hom-Poisson algebras and introduce Hom-coPoisson algebras and bialgebras. We show that there exists a duality between Hom-Poisson bialgebras and Hom-coPoisson bialgebras. A relationship between enveloping Hom-algebras endowed with Hom-coPoisson structures and corresponding Hom-Lie bialgebra structures is studied. Moreover we set quantization problems and generalize the notion of star-product. In particular, we characterize the twists for the Moyal-Weyl product for polynomials of several variables.

Introduction

The study of nonassociative algebras was originally motivated by certain problems in physics and other branches of mathematics. The first motivation to study nonassociative Hom-algebras comes from quasi-deformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras [1, 9, 11, 22, 21].

Hom-Lie algebras were first introduced by Hartwig, Larsson and Silvestrov in order to describe q-deformations of Witt and Virasoro algebras using σ-derivations (see [20]). The corresponding associative type objects, called Hom-associative algebras were introduced by Makhlouf and Silvestrov in [31]. The enveloping algebras of Hom-Lie algebras were studied by Yau in [39]. The dual notions, Hom-coalgebras, Hom-bialgebras, Hom-Hopf algebras and Hom-Lie coalgebras, were studied first in [30, 32]. The study was enhanced in [41, 42] and done with a categorical point of view in [8]. Further developments and results could be found in [2, 3, 17, 18, 40, 43]. The notion of Hom-Poisson algebras appeared first in [31] and then studied in [45]. The main feature of Hom-algebra and Hom-coalgebra structures is that the classical identities are twisted by a linear self map.

In this paper, we review the results on Hom-Poisson algebras and introduce the notions of Hom-coPoisson algebra and Hom-coPoisson bialgebra. We show that there is a duality between the Hom-coPoisson bialgebras and Hom-Poisson bialgebras, generalizing to Hom-setting the result in [35]. Moreover we set the quantization problems and generalize the notion of star-product. In particular, we characterize the twists for the Moyal-Weyl product for polynomials of several variables. In Section 1, we summarize the definitions of the Hom-algebra and Hom-coalgebra structures twisting classical structures of associative algebras, Lie algebras and their duals. We extend to Hom-structures the classical relationship between the algebra and its finite dual and provide an easy tool to obtain a Hom-structure from a classical structure and provide some examples. In Section 2, we review the main results on Hom-Poisson algebras from [31, 45] and construct some new examples. In Section 3, we introduce and study Hom-coPoisson algebras, Hom-coPoisson bialgebras and Hom-coPoisson Hopf algebras. We provide some key constructions and show a link between enveloping algebras (viewed as a Hom-bialgebra) endowed with a coPoisson structure and Hom-Lie bialgebras. Moreover, we show that there is a duality between Hom-Poisson bialgebra (resp. Hom-Poisson Hopf algebra) and Hom-coPoisson bialgebra (resp. Hom-coPoisson Hopf algebra). In Section 4, we review the 1-parameter formal deformation of Hom-algebras (resp. Hom-coalgebras) and their relationship to Hom-Poisson algebra (resp. Hom-coPoisson algebras). Then we state the quantization problem and define star-product in Hom-setting. Finally we study twistings of Moyal-Weyl star-product.

1 Twisting usual structures: Hom-algebras and Hom-coalgebras

Throughout this paper $\mathbb{K}$ denotes a field of characteristic 0 and A is a $\mathbb{K}$-module. In the sequel we denote by σ the linear map $\sigma : A^{\otimes 3} \to A^{\otimes 3}$, defined as $\sigma(x_1 \otimes x_2 \otimes x_3) = x_3 \otimes x_1 \otimes x_2$ and by τ_{ij} linear maps $\tau : A^{\otimes n} \to A^{\otimes n}$ where $\tau_{ij}(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_j \otimes \cdots \otimes x_n) = (x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes x_n)$.

We mean by a Hom-algebra a triple (A, μ, α) in which $\mu : A^{\otimes 2} \to A$ is a linear map and $\alpha : A \to A$ is a linear self map. The linear map $\mu^{op} : A^{\otimes 2} \to A$ denotes the opposite map, i.e. $\mu^{op} = \mu \circ \tau_{12}$. A Hom-coalgebra is a triple (A, Δ, α) in which $\Delta : A \to A^{\otimes 2}$ is a linear map and $\alpha : A \to A$ is a linear self map. The linear map $\Delta^{op} : A \to A^{\otimes 2}$ denotes the opposite map, i.e. $\Delta^{op} = \tau_{12} \circ \Delta$. For a linear self-map $\alpha : A \to A$, we denote by α^n the n-fold composition of n copies of α, with $\alpha^0 \cong Id$. A Hom-algebra (A, μ, α) (resp. a Hom-coalgebra (A, Δ, α))

is said to be *multiplicative* if $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$ (resp. $\alpha^{\otimes 2} \circ \Delta = \Delta \circ \alpha$). The Hom-algebra is called *commutative* if $\mu = \mu^{op}$ and the Hom-coalgebra is called *cocommutative* if $\Delta = \Delta^{op}$.

Unless otherwise specified, we assume in this paper that the Hom-algebras and Hom-coalgebras are multiplicative. Classical algebras or coalgebras are also regarded as a Hom-algebras or Hom-coalgebras with identity twisting map. Given a Hom-algebra (A, μ, α), we often use the abbreviation $\mu(x, y) = xy$ for $x, y \in A$. Likewise, for a Hom-coalgebra (A, Δ, α), we will use Sweedler's notation $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$ but often omit the symbol of summation. When the Hom-algebra (resp. Hom-coalgebra) is multiplicative, we also say that α is multiplicative for μ (resp. Δ).

In the following, we summarize the definitions and properties of usual Hom-structures, provide some examples and recall the twisting principle giving a way to turn a classical structure to Hom-structure using algebra morphisms.

1.1 Hom-associative algebras and Hom-Lie algebras

We recall the definitions of Hom-Lie algebras and Hom-associative algebras.

Definition 1.1. Let (A, μ, α) be a Hom-algebra.

1. The *Hom-associator* $\mathfrak{as}_{\mu,\alpha} : A^{\otimes 3} \to A$ is defined as

(1.1) $$\mathfrak{as}_{\mu,\alpha} = \mu \circ (\mu \otimes \alpha - \alpha \otimes \mu).$$

2. The Hom-algebra A is called a *Hom-associative algebra* if it satisfies the Hom-associative identity

(1.2) $$\mathfrak{as}_{\mu,\alpha} = 0.$$

3. A Hom–associative algebra A is called *unital* if there exists a linear map $\eta : \mathbb{K} \to A$ such that

(1.3) $$\mu \circ (Id_A \otimes \eta) = \mu \circ (\eta \otimes Id_A) = \alpha.$$

   The unit element is $1_A = \eta(1)$.

4. The *Hom-Jacobian* $J_{\mu,\alpha} : A^{\otimes 3} \to A$ is defined as

(1.4) $$J_{\mu,\alpha} = \mu \circ (\alpha \otimes \mu) \circ (Id + \sigma + \sigma^2).$$

5. The Hom-algebra A is called a *Hom-Lie algebra* if it satisfies $\mu + \mu^{op} = 0$ and the Hom-Jacobi identity

(1.5) $$J_{\mu,\alpha} = 0.$$

We recover the usual definitions of the associator, an associative algebra, the Jacobian, and a Lie algebra when the twisting map α is the identity map. In terms of elements $x, y, z \in A$, the Hom-associator and the Hom-Jacobian are

$$\mathfrak{as}_{\mu,\alpha}(x,y,z) = (xy)\alpha(z) - \alpha(x)(yz),$$
$$J_{\mu,\alpha}(x,y,z) = \circlearrowleft_{x,y,z} [\alpha(x),[y,z]],$$

where the bracket denotes the product and $\circlearrowleft_{x,y,z}$ denotes the cyclic sum on x, y, z.

Let (A,μ,α) and $A' = (A',\mu',\alpha')$ be two Hom-algebras (either Hom-associative or Hom-Lie). A linear map $f : A \to A'$ is a *morphism of Hom-algebras* if

$$\mu' \circ (f \otimes f) = f \circ \mu \quad \text{and} \qquad f \circ \alpha = \alpha' \circ f.$$

It is said to be a *weak morphism* if holds only the first condition.

Remark 1.2. It was shown in [29] that the commutator of a Hom-associative algebra leads to a Hom-Lie algebra. The construction of the enveloping algebras of Hom-Lie algebras is described in [39].

Example 1.3. Let $\{x_1, x_2, x_3\}$ be a basis of a 3-dimensional vector space A over $\mathbb{K}$. The following multiplication μ and linear map α on $A = \mathbb{K}^3$ define a Hom-associative algebra over $\mathbb{K}$:

$$\begin{array}{lcllcl}
\mu(x_1,x_1) &=& ax_1, & \mu(x_2,x_2) &=& ax_2,\\
\mu(x_1,x_2) &=& \mu(x_2,x_1) = ax_2, & \mu(x_2,x_3) &=& bx_3,\\
\mu(x_1,x_3) &=& \mu(x_3,x_1) = bx_3, & \mu(x_3,x_2) &=& \mu(x_3,x_3) = 0,
\end{array}$$

$$\alpha(x_1) = ax_1, \quad \alpha(x_2) = ax_2, \quad \alpha(x_3) = bx_3$$

where a, b are parameters in $\mathbb{K}$. This algebra is not associative when $a \neq b$ and $b \neq 0$, since

$$\mu(\mu(x_1,x_1),x_3)) - \mu(x_1,\mu(x_1,x_3)) = (a-b)bx_3.$$

Example 1.4 (Jackson $\mathfrak{sl}_2$)**.** The Jackson $\mathfrak{sl}_2$ is a q-deformation of the classical Lie algebra $\mathfrak{sl}_2$. It carries a Hom-Lie algebra structure but not a Lie algebra structure by using Jackson derivations. It is defined with respect to a basis $\{x_1, x_2, x_3\}$ by the brackets and a linear map α such that

$$\begin{array}{lclclcl}
[x_1,x_2] &=& -2qx_2, & \quad & \alpha(x_1) &=& qx_1,\\
[x_1,x_3] &=& 2x_3, & & \alpha(x_2) &=& q^2x_2,\\
[x_2,x_3] &=& -\frac{1}{2}(1+q)x_1, & & \alpha(x_3) &=& qx_3,
\end{array}$$

where q is a parameter in $\mathbb{K}$. If $q = 1$ we recover the classical $\mathfrak{sl}_2$.

The following proposition gives an easy way to twist classical structures in Hom-structures.

Theorem 1.5 ([40]).

(1) Let $A = (A, \mu)$ be an associative algebra and $\alpha : A \to A$ be a linear map which is multiplicative with respect to μ, i.e. $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$. Then $A_\alpha = (A, \mu_\alpha = \alpha \circ \mu, \alpha)$ is a Hom-associative algebra.

(2) Let $A = (A, [\ ,\ ])$ be a Lie algebra and $\alpha : A \to A$ be a linear map which is multiplicative with respect to $[\ ,\ ]$, i.e. $\alpha \circ [\ ,\ ] = [\ ,\ ] \circ \alpha^{\otimes 2}$. Then $A_\alpha = (A, [\ ,\ ]_\alpha = \alpha \circ [\ ,\ ], \alpha)$ is a Hom-Lie algebra.

Proof. (1) We have

$$\begin{aligned} \mathfrak{as}_{\mu_\alpha,\alpha} &= \mu_\alpha \circ (\mu_\alpha \otimes \alpha - \alpha \otimes \mu_\alpha) \\ &= \alpha \circ \mu \circ \alpha^{\otimes 2} \circ (\mu \otimes Id - Id \otimes \mu) \\ &= \alpha^2 \circ \mathfrak{as}_{\mu,Id} = 0 \end{aligned}$$

since (A, μ) is associative, so A_α is a Hom-associative algebra.

(2) We have $\forall\ x, y \in A,\ [y, x] = -[x, y]$ and

$$\begin{aligned} J_{[\ ,\ ]_\alpha,\alpha} &= [\ ,\ ]_\alpha \circ (\alpha \otimes [\ ,\ ]_\alpha) \circ (Id + \sigma + \sigma^2) \\ &= \alpha \circ [\ ,\ ] \circ \alpha^{\otimes 2} \circ (Id \otimes [\ ,\ ]) \circ (Id + \sigma + \sigma^2) \\ &= \alpha^2 \circ J_{[\ ,\ ],Id} = 0 \end{aligned}$$

since $[\ ,\ ]$ is a Lie bracket, so A_α is a Hom-Lie algebra.

□

Remark 1.6. More generally, the categories of Hom-associative algebras and Hom-Lie algebras are closed under twisting self-weak morphisms. If $A = (A, \mu, \alpha)$ is a Hom-associative algebra (resp. Hom-Lie algebra) and β a weak morphism, then $A_\beta = (A, \mu_\beta = \beta \circ \mu, \beta \circ \alpha)$ is a Hom-associative algebra (resp. Hom-Lie algebra) as well (see [45]).

Example 1.7. To twist the usual Lie algebra $\mathfrak{sl}_2$, generated by the elements e, f, h and relations $[h, e] = 2e,\ [h, f] = -2f,\ [e, f] = h$, we figure out morphisms $\alpha : \mathfrak{sl}_2 \to \mathfrak{sl}_2$ written as $\alpha = (\alpha_{ij})$ with respect to the basis (e, f, h), by solving the equations $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ with respect to the basis in the coefficients α_{ij}. We obtain $(\mathfrak{sl}_2)_\alpha$, Hom-Lie versions of $\mathfrak{sl}_2$, with α given by

1. $\alpha(e) = \lambda e,\ \alpha(f) = \lambda^{-1} f,\ \alpha(h) = h,$
2. $\alpha(e) = \lambda f,\ \alpha(f) = \lambda^{-1} e,\ \alpha(h) = -h,$
3. $\alpha(e) = -\lambda^2 e + f + \lambda h,\ \alpha(f) = \frac{1}{2} e - \frac{1}{2\lambda^2} f + \frac{1}{2\lambda} h,\ \alpha(h) = \lambda e + \lambda^{-1} f,$

where λ is a nonzero element in $\mathbb{K}$.

1.2 Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras

In this section we summarize and describe some of basic properties of Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras which generalize the classical coalgebra, bialgebra and Hopf algebra structures.

Definition 1.8. A *Hom-coalgebra* is a triple (A, Δ, α) where A is a $\mathbb{K}$-module and $\Delta : A \to A \otimes A,\ \alpha : A \to A$ are linear maps.

A *Hom-coassociative coalgebra* is a Hom-coalgebra satisfying

$$(\alpha \otimes \Delta) \circ \Delta = (\Delta \otimes \alpha) \circ \Delta. \tag{1.6}$$

A Hom-coassociative coalgebra is said to be *counital* if there exists a map $\varepsilon : A \to \mathbb{K}$ satisfying

$$(id \otimes \varepsilon) \circ \Delta = \alpha \qquad \text{and} \qquad (\varepsilon \otimes id) \circ \Delta = \alpha. \tag{1.7}$$

Let (A, Δ, α) and (A', Δ', α') be two Hom-coalgebras (resp. Hom-coassociative coalgebras). A linear map $f\ : A \to A'$ is a *morphism of Hom-coalgebras* (resp. *Hom-coassociative coalgebras*) if

$$(f \otimes f) \circ \Delta = \Delta' \circ f \quad \text{and} \quad f \circ \alpha = \alpha' \circ f.$$

If furthermore the Hom-coassociative coalgebras admit counits ε and ε', we have moreover $\varepsilon = \varepsilon' \circ f$.

The following theorem shows how to construct a new Hom-coassociative Hom-coalgebra starting with a Hom-coassociative Hom-coalgebra and a Hom-coalgebra morphism. We only need the coassociative comultiplication of the coalgebra.

Theorem 1.9. *Let $(A, \Delta, \alpha, \varepsilon)$ be a counital Hom-coalgebra and $\beta : A \to A$ be a weak Hom-coalgebra morphism. Then $(A, \Delta_\beta, \alpha \circ \beta, \varepsilon)$, where $\Delta_\beta = \Delta \circ \beta$, is a counital Hom-coassociative coalgebra.*

Proof. We show that $(A, \Delta_\beta, \alpha \circ \beta)$ satisfies the axiom (1.6).

Indeed, using the fact that $(\beta \otimes \beta) \circ \Delta = \Delta \circ \beta$, we have

$$\begin{aligned}(\alpha \circ \beta \otimes \Delta_\beta) \circ \Delta_\beta &= (\alpha \circ \beta \otimes \Delta \circ \beta) \circ \Delta \circ \beta \\ &= ((\alpha \otimes \Delta) \circ \Delta) \circ \beta^2 \\ &= ((\Delta \otimes \alpha) \circ \Delta) \circ \beta^2 \\ &= (\Delta_\beta \otimes \alpha \circ \beta) \circ \Delta_\beta.\end{aligned}$$

Moreover, the axiom (1.7) is also satisfied, since we have

$$(id \otimes \varepsilon) \circ \Delta_\beta = (id \otimes \varepsilon) \circ \Delta \circ \beta = \alpha \circ \beta = (\varepsilon \otimes id) \circ \Delta \circ \beta = (\varepsilon \otimes id) \circ \Delta_\beta.$$

□

Remark 1.10. The previous theorem shows that the category of coassociative Hom-coalgebras is closed under weak Hom-coalgebra morphisms. It leads to the following examples:

1. Let (A, Δ) be a coassociative coalgebra and $\beta : A \to A$ be a coalgebra morphism. Then (A, Δ_β, β) is a Hom-coassociative coalgebra.
2. Let (A, Δ, α) be a (multiplicative) coassociative Hom-coalgebra. For all non negative integer n, $(A, \Delta_{\alpha^n}, \alpha^{n+1})$ is a Hom-coassociative coalgebra.

In the following we show that there is a duality between Hom-associative and the Hom-coassociative structures (see [30, 32]).

Theorem 1.11. *Let (A, Δ, α) be a Hom-coassociative coalgebra. Then the dual $(A^*, \Delta^*, \alpha^*)$ is a Hom-associative algebra. Moreover, A^* is unital whenever A is counital.*

Proof. The product $\mu = \Delta^*$ is defined from $A^* \otimes A^*$ to A^* by

$$(fg)(x) = \Delta^*(f, g)(x) = \langle \Delta(x), f \otimes g \rangle = (f \otimes g)(\Delta(x)) = \sum_{(x)} f(x_1)g(x_2), \quad \forall x \in A$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between the vector space $A \otimes A$ and its dual vector space. For $f, g, h \in A^*$ and $x \in A$, we have

$$(fg)\alpha^*(h)(x) = \langle (\Delta \otimes \alpha) \circ \Delta(x), f \otimes g \otimes h \rangle$$

and

$$\alpha^*(f)(gh)(x) = \langle (\alpha \otimes \Delta) \circ \Delta(x), f \otimes g \otimes h \rangle.$$

So the Hom-associativity $\mu \circ (\mu \otimes \alpha^* - \alpha^* \otimes \mu) = 0$ follows from the Hom-coassociativity $(\Delta \otimes \alpha - \alpha \otimes \Delta) \circ \Delta = 0$.
Moreover, if A has a counit ε satisfying $(id \otimes \varepsilon) \circ \Delta = \alpha = (\varepsilon \otimes id) \circ \Delta$ then for $f \in A^*$ and $x \in A$ we have

$$(\varepsilon f)(x) = \sum_{(x)} \varepsilon(x_1) f(x_2) = \sum_{(x)} f(\varepsilon(x_1)x_2) = f(\alpha(x)) = \alpha^*(f)(x)$$

and

$$(f\varepsilon)(x) = \sum_{(x)} f(x_1)\varepsilon(x_2) = \sum_{(x)} f(x_1\varepsilon(x_2)) = f(\alpha(x)) = \alpha^*(f)(x)$$

which shows that ε is the unit in A^*. □

The dual of a Hom-algebra (A, μ, α) is not always a Hom-coalgebra, because the coproduct does not land in the good space: $\mu^* : A^* \to (A \otimes A)^* \supsetneq A^* \otimes A^*$. It is the case if the Hom-algebra is finite dimensional, since $(A \otimes A)^* = A^* \otimes A^*$.

In the general case, for any algebra A, define

$$A^\circ = \{f \in A^*, f(I) = 0 \text{ for some cofinite ideal } I \text{ of } A\},$$

where a *cofinite ideal* I is an ideal $I \subset A$ such that A/I is finite dimensional.

A° is a subspace of A^* since it is closed under multiplication by scalars and the sum of two elements of A° is again in A° since the intersection of two cofinite ideals is again a cofinite ideal. If A is finite dimensional, of course $A^\circ = A^*$.

For an (Hom-)algebra map $f : A \to B$ we note $f^\circ := f^*|_{B^\circ} : B^\circ \to A^\circ$. Since $A^\circ \otimes A^\circ = (A \otimes A)^\circ$ (see [38, Lemma 6.0.1]) the dual $\mu^* : A^* \to (A \otimes A)^*$ of the multiplication $\mu : A \otimes A$ satisfies $\mu^*(A^\circ) \subset A^\circ \otimes A^\circ$. Indeed, for $f \in A^*,\ x, y \in A$, we have $\langle \mu^*(f), x \otimes y\rangle = \langle f, xy\rangle$. So if I is a cofinite ideal such that $f(I) = 0$, then $I \otimes A + A \otimes I$ is a cofinite ideal of $A \otimes A$ which vanish on $\mu^*(f)$. Similarly we have $\alpha^\circ(A^\circ) \subset A^\circ$.

Define $\Delta = \mu^\circ = \mu^*|_{A^\circ}$, $\alpha^\circ = \alpha^*|_{A^\circ}$ and $\varepsilon : A^\circ \to \mathbb{K}$ by $\varepsilon(f) = f(1)$.

Theorem 1.12. *Let (A, μ, α) be a multiplicative Hom-associative algebra. Then $(A^\circ, \Delta, \alpha^\circ)$ is a Hom-coassociative coalgebra. Moreover, it is counital if A is unital.*

Proof. The coproduct Δ is defined from A° to $A^\circ \otimes A^\circ$ by

$$\Delta(f)(x \otimes y) = \mu^*|_{A^\circ}(f)(x \otimes y) = \langle \mu(x \otimes y), f\rangle = f(xy), \quad x, y \in A.$$

For $f, g, h \in A^\circ$ and $x, y \in A$, we have

$$(\Delta \circ \alpha^\circ) \circ \Delta(f)(x \otimes y \otimes z) = \langle \mu \circ (\mu \otimes \alpha)(x \otimes y \otimes z), f\rangle$$

and

$$(\alpha^\circ \circ \Delta) \circ \Delta(f)(x \otimes y \otimes z) = \langle \mu \circ (\otimes \mu\alpha)(x \otimes y \otimes z), f\rangle.$$

So the Hom-coassociativity $(\Delta \otimes \alpha^\circ - \alpha^\circ \otimes \Delta) \circ \Delta = 0$ follows from the Hom-associativity $\mu \circ (\mu \otimes \alpha - \alpha \otimes \mu) = 0$.
Moreover, if A has a unit η satisfying $\mu \circ (id \otimes \eta) = \alpha = \mu \circ (\eta \otimes id)$ then for $f \in A^\circ$ and $x \in A$ we have

$$(\varepsilon \otimes id) \circ \Delta(f)(x) = f(1 \cdot x) = f(\alpha(x)) = \alpha^\circ(f)(x)$$

and

$$(id \otimes \varepsilon) \circ \Delta(f)(x) = f(x \cdot 1) = f(\alpha(x)) = \alpha^\circ(f)(x)$$

which shows that $\varepsilon : A^\circ \to \mathbb{K}$, $f \mapsto f(1)$ is the counit in A°. $\square$

If the Hom-associative algebra is finite dimensional then for any Hom-associative algebra the dual is provided with a structure of Hom-coassociative coalgebra.

Definition 1.13. A *Hom-bialgebra* is a tuple $(A, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ where

1. (A, μ, α, η) is a Hom-associative algebra with a unit η.
2. $(A, \Delta, \beta, \varepsilon)$ is a Hom-coassociative coalgebra with a counit ε.
3. The linear maps Δ and ε are compatible with the multiplication μ and the unit η, that is

$$\Delta(e) = e \otimes e \qquad \text{where } e = \eta(1), \tag{1.8}$$

$$\Delta(\mu(x \otimes y)) = \Delta(x) \bullet \Delta(y) = \sum_{(x)(y)} \mu(x_1 \otimes y_1) \otimes \mu(x_2 \otimes y_2), \tag{1.9}$$

$$\varepsilon(e) = 1, \tag{1.10}$$

$$\varepsilon(\mu(x \otimes y)) = \varepsilon(x)\,\varepsilon(y), \tag{1.11}$$

$$\varepsilon \circ \alpha(x) = \varepsilon(x), \tag{1.12}$$

where the bullet $\bullet$ denotes the multiplication on tensor product.
If $\alpha = \beta$ the Hom-bialgebra is denoted $(A, \mu, \eta, \Delta, \varepsilon, \alpha)$.

Combining previous observations, with only one twisting map, we obtain:

Proposition 1.14. *Let (A, μ, Δ, α) be a multiplicative Hom-bialgebra. Then the finite dual $(A^\circ, \mu^\circ, \Delta^\circ, \alpha^\circ)$ is a Hom-bialgebra as well.*

Remark 1.15.

1. Given a Hom-bialgebra $(A, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$, it is shown in [30, 32] that the vector space $Hom(A, A)$ with the multiplication given by the convolution product carries a structure of Hom-associative algebra.
2. An endomorphism S of A is said to be an *antipode* if it is the inverse of the identity over A for the Hom-associative algebra $Hom(A, A)$ with the multiplication given by the convolution product defined by

$$f \star g = \mu \circ (f \otimes g)\,\Delta$$

   and the unit being $\eta \circ \varepsilon$.
3. A *Hom-Hopf algebra* is a Hom-bialgebra with an antipode.

By combining Theorems 1.5 and 1.9, we obtain:

Proposition 1.16. *Let (A, μ, Δ, α) be a Hom-bialgebra and $\beta : A \to A$ be a Hom-bialgebra morphism commuting with α. Then $(A, \mu_\beta, \Delta_\beta, \beta \circ \alpha)$ is a Hom-bialgebra.*

Notice that the category of Hom-bialgebra is not closed under weak Hom-bialgebra morphisms.

Example 1.17 (Universal enveloping Hom-algebra)**.** Given a multiplicative Hom-associative algebra $A = (A, \mu, \alpha)$, one can associate to it a multiplicative Hom-Lie algebra $HLie(A) = (A, [\ ,\ ], \alpha)$ with the same underlying module (A, α) and the bracket $[\ ,\ ] = \mu - \mu^{op}$. This construction gives a functor $HLie$ from multiplicative Hom-associative algebras to multiplicative Hom-Lie algebras. In [39], Yau constructed the left adjoint U_{HLie} of $HLie$. He also made some minor modifications in [42] to take into account the unital case.

The functor U_{HLie} is defined as

$$U_{HLie}(A) = F_{HNAs}(A)/I^{\infty} \qquad \text{with} \quad F_{HNAs}(A) = \bigoplus_{n\geqslant 1} \bigoplus_{\tau \in T_n^{wt}} A_\tau^{\otimes n} \tag{1.13}$$

for a multiplicative Hom-Lie algebra $(A, [\ ,\ ], \alpha)$. Here T_n^{wt} is the set of weighted n-trees encoding the multiplication of elements (by trees) and twisting by α (by weights), $A_\tau^{\otimes n}$ is a copy of $A^{\otimes n}$ and I^{∞} is a certain submodule of relations build in such a way that the quotient is Hom-associative.

Moreover, the comultiplication $\Delta : U_{HLie}(A) \to U_{HLie}(A) \otimes U_{HLie}(A)$ defined by $\Delta(x) = \alpha(x) \otimes 1 + 1 \otimes \alpha(x)$ equips the multiplicative Hom-associative algebra $U_{HLie}(A)$ with a structure of Hom-bialgebra.

1.3 Hom-Lie coalgebras and Hom-Lie bialgebras

As it is the case for the Hom-associative algebras, the Hom-Lie algebras also have a dualized version, Hom-Lie coalgebras. They share the same kind of properties. We review here the principal results, similar results can be found in [44].

Definition 1.18. A *Hom-Lie coalgebra* (A, Δ, α) is a (multiplicative) Hom-coalgebra satisfying $\Delta + \Delta^{op} = 0$ and the Hom-coJacobi identity

$$(Id + \sigma + \sigma^2) \circ (\alpha \otimes \Delta) \circ \Delta = 0. \tag{1.14}$$

We call Δ the cobracket.

We recover Lie coalgebra when $\alpha = Id$. Just like (co)associative (co)algebras we have the following dualization properties.

Theorem 1.19.

1. *Let (A, Δ, α) be a Hom-Lie coalgebra. Then $(A^*, \Delta^*, \alpha^*)$ is a Hom-Lie algebra.*
2. *Let $(A, [\ ,\ ], \alpha)$ be a multiplicative Hom-Lie algebra. Then $(A^\circ, [\ ,\ ]^\circ, \alpha^\circ)$ is a Hom-Lie coalgebra.*

Notice that if A is finite dimensional then we may remove the assumption of multiplicativity.

The twisting principle also works, showing that the category of Hom-Lie coalgebras is closed under weak Hom-coalgebras morphisms.

Theorem 1.20. *Let (A, Δ, α) be a Hom-Lie coalgebra and $\beta : A \to A$ a weak Hom-coalgebra morphism. Then $(A, \Delta_\beta = \Delta \circ \beta, \alpha \circ \beta)$ is a Hom-Lie coalgebra.*

Proof. We have $\Delta_\beta + \Delta_\beta^{op} = (\Delta + \Delta^{op}) \circ \beta = 0$, and

$$\begin{aligned}(Id + \sigma + \sigma^2) \circ (\alpha \circ \beta \otimes \Delta_\beta) \circ \Delta_\beta &= (Id + \sigma + \sigma^2) \circ (\alpha \circ \beta \otimes \Delta \circ \beta) \circ \Delta \circ \beta \\ &= (Id + \sigma + \sigma^2) \circ (\alpha \otimes \Delta) \circ \Delta \circ \beta^2 \\ &= 0.\end{aligned}$$

□

The previous theorem can be used to construct Hom-Lie coalgebras.

Corollary 1.21.

1. *Let (A, Δ) be a Lie coalgebra and $\beta : A \to A$ be a Lie coalgebra morphism. Then (A, Δ_β, β) is a Hom-Lie coalgebra.*
2. *Let (A, Δ, α) be a (multiplicative) Hom-Lie coalgebra. For all non negative integer n, $(A, \Delta_{\alpha^n}, \alpha^{n+1})$ is a Hom-Lie coalgebra.*

The Hom-Lie bialgebra structure was introduced first in [44]. The definition presented below is slightly more general. They border the class defined by Yau and permit to consider the compatibility condition for different A-valued cohomology of Hom-Lie algebras.

Definition 1.22. A *Hom-Lie bialgebra* is a tuple $(A, [\ ,\ ], \alpha, \Delta, \beta)$ where

1. $(A, [\ ,\ ], \alpha)$ is a Hom-Lie algebra.
2. (A, Δ, β) is a Hom-Lie coalgebra.
3. The following compatibility condition holds for $x, y \in A$:

   (1.15) $$\Delta([x, y]) = ad_{\alpha(x)}(\Delta(y)) - ad_{\alpha(y)}(\Delta(x)),$$

   where the adjoint map $ad_x : A^{\otimes n} \to A^{\otimes n}$ ($n \geqslant 2$) is given by
   (1.16)
   $$ad_x(y_1 \otimes \cdots \otimes y_n) = \sum_{i=1}^{n} \beta(y_1) \otimes \cdots \otimes \beta(y_{i-1}) \otimes [x, y_i] \otimes \beta(y_{i+1}) \otimes \cdots \otimes \beta(y_n).$$

A *morphism* $f : A \to A'$ of Hom-Lie bialgebras is a linear map commuting with α and β such that $f \circ [\ ,\ ] = [\ ,\ ] \circ f^{\otimes 2}$ and $\Delta \circ f = f^{\otimes 2} \circ \Delta$.

If $\alpha = \beta = Id$ we recover Lie bialgebras and if $\alpha = \beta$, we recover the class defined in [44], these Hom-Lie bialgebras are denoted $(A, [\ ,\ ], \Delta, \alpha)$.

In terms of elements, the compatibility condition (1.15) writes

$$\begin{aligned}\Delta([x,y]) =& ad_{\alpha(x)}(\Delta(y)) - ad_{\alpha(y)}(\Delta(x))\\ =& [\alpha(x), y_1]\otimes\beta(y_2) + \beta(y_1)\otimes[\alpha(x), y_2]\\ &- [\alpha(y), x_1]\otimes\beta(x_2) - \beta(x_1)\otimes[\alpha(y), x_2].\end{aligned} \tag{1.17}$$

Remark 1.23. If $\alpha = \beta$, the compatibility condition (1.17) is equivalent to say that Δ is a 1-cocycle with respect to α^0-adjoint cohomology of Hom-Lie algebras and for $\beta = Id$, it corresponds to α^{-1}-adjoint cohomology, (see [37] and [2]).

The following Proposition generalizes slightly [44, Theorem 3.5].

Proposition 1.24. *Let $(A, [\ ,\ ], \Delta, \alpha)$ be a Hom-Lie bialgebra and $\beta : A \to A$ a Hom-Lie bialgebra morphism commuting with α. Then $(A, [\ ,\ ]_\beta = \beta\circ[\ ,\ ], \Delta_\beta = \Delta\circ\beta, \alpha\circ\beta)$ is a Hom-Lie bialgebra.*

Proof. We already know that $(A, [\ ,\ ]_\beta, \beta\circ\alpha)$ is a Hom-Lie algebra and that $(A, \Delta_\beta, \alpha\circ\beta)$ is a Hom-Lie coalgebra. It remains to prove the compatibility condition (1.17) for Δ_β and $[\ ,\ ]_\beta$, with the twisting map $\alpha\circ\beta = \beta\circ\alpha$. On one side, we have

$$\Delta_\beta([x,y]) = \Delta\circ\beta^2\circ[x,y] = \left(\beta^{\otimes 2}\right)^2\circ\Delta([x,y]),$$

since $\Delta\circ\beta = \beta^{\otimes 2}\circ\Delta$. Using in addition $\beta\circ[\ ,\ ] = [\ ,\ ]\circ\beta^{\otimes 2}$ and the fact that α and β commute, we have

$$\begin{aligned}ad_{\alpha\circ\beta(x)}(\Delta_\beta(y)) &= ad_{\alpha\circ\beta(x)}(\beta(y_1)\otimes\beta(y_2))\\ &= [\alpha\circ\beta(x), \beta(y_1)]_\beta\otimes\alpha\circ\beta^2(y_2) + \alpha\circ\beta^2(y_1)\otimes[\alpha\circ\beta(x), \beta(y_2)]_\beta\\ &= \left(\beta^{\otimes 2}\right)^2([\alpha(x), y_1]\otimes\alpha(y_2) + \alpha(y_1)\otimes[\alpha(x), y_2]).\end{aligned}$$

It follows that $\Delta_\beta([x,y]) = ad_{\alpha\circ\beta(x)}(\Delta_\beta(y)) - ad_{\alpha\circ\beta(y)}(\Delta_\beta(x))$ as wished. □

As for the Hom-bialgebra, the category of Hom-Lie bialgebra is not closed under weak Hom-Lie bialgebra morphisms.

Hom-Lie bialgebra can be dualized. We obtain the following proposition generalized the result stated in [44] for finite dimensional case and using natural pairing.

Proposition 1.25. *Let $(A, [\ ,\ ], \alpha, \Delta, \beta)$ be a multiplicative Hom-Lie bialgebra. Then the finite dual $(A^\circ, [\ ,\ ]^\circ, \alpha^\circ, \Delta^\circ, \beta^\circ)$ is a Hom-Lie bialgebra as well.*

2 Hom-Poisson algebras

The Hom-Poisson algebras were introduced in [31], where they emerged naturally in the study of 1-parameter formal deformations of commutative Hom-associative algebras. Then this structure was studied in [45]. It is shown that they are closed under twisting by suitable self maps and under tensor products. Moreover it is shown that (de)polarization Hom-Poisson algebras are equivalent to admissible Hom-Poisson algebras, each of which has only one binary operation. We survey in this section the fundamental results and provide examples.

Definition 2.1. A *Hom-Poisson* algebra is a tuple $(A, \mu, \{\ ,\ \}, \alpha)$ consisting of

(1) a commutative Hom-associative algebra (A, μ, α) and

(2) a Hom-Lie algebra $(A, \{\ ,\ \}, \alpha)$

such that the Hom-Leibniz identity

$$\{\ ,\ \} \circ (\mu \otimes \alpha) = \mu \circ (\alpha \otimes \{\ ,\ \} + (\{\ ,\ \} \otimes \alpha) \circ \tau_{23}) \tag{2.1}$$

is satisfied.

In a Hom-Poisson algebra $(A, \{\ ,\ \}, \mu, \alpha)$, the operation $\{\ ,\ \}$ is called *Hom-Poisson bracket*. In terms of elements $x, y, z \in A$, the Hom-Leibniz identity says

$$\{xy, \alpha(z)\} = \alpha(x)\{y, z\} + \{x, z\}\alpha(y)$$

where as usual $\mu(x, y)$ is abbreviated to xy. By the antisymmetry of the Hom-Poisson bracket $\{\ ,\ \}$, the Hom-Leibniz identity is equivalent to

$$\{\alpha(x), yz\} = \{x, y\}\alpha(z) + \alpha(y)\{x, z\}.$$

We recover Poisson algebras when the twisting map is the identity.

Definition 2.2. A *Hom-Poisson bialgebra* $(A, \mu, \eta, \Delta, \varepsilon, \alpha, \{\ ,\ \})$ is a Hom-Poisson algebra $(A, \mu, \{\ ,\ \}, \alpha)$ which is also a Hom-bialgebra $(A, \mu, \eta, \Delta, \varepsilon, S, \alpha)$, the two structures being compatible in the sense that $\{\ ,\ \}$ is a μ-coderivation,

$$\Delta \circ \{\ ,\ \} = (\{\ ,\ \} \otimes \mu + \mu \otimes \{\ ,\ \}) \circ \Delta^{[2]}.$$

In term of elements, this compatibility condition writes

$$\Delta(\{a, b\}) = \{\Delta(a), \Delta(b)\}$$

with the Hom-Poisson bracket on $A \otimes A$ defined by

$$\{a_1 \otimes a_2, b_1 \otimes b_2\} := \{a_1, b_1\} \otimes a_2 b_2 + a_1 b_1 \otimes \{a_2, b_2\}.$$

We have the same definition for Hom-Poisson Hopf algebras.

Example 2.3. Let $\{x_1, x_2, x_3\}$ be a basis of a 3-dimensional vector space A over $\mathbb{K}$. The following multiplication μ, skew-symmetric bracket and linear map α on A define a Hom-Poisson algebra over $\mathbb{K}^3$:

$$\begin{array}{rclrcl} \mu(x_1, x_1) &=& x_1, & \{x_1, x_2\} &=& ax_2 + bx_3, \\ \mu(x_1, x_2) &=& \mu(x_2, x_1) = x_3, & \{x_1, x_3\} &=& cx_2 + dx_3, \end{array}$$

$$\alpha(x_1) = \lambda_1 x_2 + \lambda_2 x_3, \quad \alpha(x_2) = \lambda_3 x_2 + \lambda_4 x_3, \quad \alpha(x_3) = \lambda_5 x_2 + \lambda_6 x_3$$

where $a, b, c, d, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ are parameters in $\mathbb{K}$.

Theorem 2.4 ([45])**.** *Let $A = (A, \mu, \{\ ,\ \})$ be a Poisson algebra, and $\alpha : A \to A$ be a linear map which is multiplicative for μ and $\{\ ,\ \}$. Then*

$$A_\alpha = (A, \quad \mu_\alpha = \alpha \circ \mu, \quad \{\ ,\ \}_\alpha = \alpha \circ \{\ ,\ \}, \quad \alpha)$$

is a Hom-Poisson algebra.

Proof. Using Theorem (1.5), we already have that (A, μ_α, α) is a commutative Hom-associative algebra and that $(A, \{\ ,\ \}_\alpha, \alpha)$ is a Hom-Lie algebra. It remains to check the Hom-Leibniz identity

$$\begin{aligned} \{\ ,\ \}_\alpha \circ (\mu_\alpha \otimes \alpha) &= \alpha \circ \{\ ,\ \} \circ \alpha^{\otimes 2} \circ (\mu \otimes Id) \\ &= \alpha^2 \circ \{\ ,\ \} \circ (\mu \otimes Id), \end{aligned}$$

since A is a Poisson algebra

$$\begin{aligned} &= \alpha^2 \circ \mu \circ (Id \otimes \{\ ,\ \} + (\{\ ,\ \} \otimes Id) \circ \tau_{23}) \\ &= \alpha \circ \mu \circ \alpha^{\otimes 2} \circ (Id \otimes \{\ ,\ \} + (\{\ ,\ \} \otimes Id) \circ \tau_{23}) \\ &= \mu_\alpha \circ (\alpha \otimes \{\ ,\ \}_\alpha + (\{\ ,\ \}_\alpha \otimes \alpha) \circ \tau_{23}), \end{aligned}$$

so A_α is a Hom-Poisson algebra. □

Example 2.5. The Sklyanin Poisson algebra $q_4(\mathcal{E})$ (see [36] for a more detailed definition and properties) is defined on $\mathbb{C}[x_0, x_1, x_2, x_3]$ by a parameter $k \in \mathbb{C}$ with the usual polynomial multiplication, and bracket given by $\{\ ,\ \}$ where brackets between the coordinate functions are defined as

$$\begin{array}{l} \{x_i, x_{i+1}\} = k^2 x_i x_{i+1} - x_{i+2} x_{i+3}, \\ \{x_i, x_{i+2}\} = k(x_{i+3}^2 - x_{i+1}^2), \end{array} \qquad i = 1, 2, 3, 4 \pmod 4.$$

We again search a morphism $\alpha : q_4(\mathcal{E}) \to q_4(\mathcal{E})$ written as $\alpha = (\alpha_{ij})$ with respect to the basis (x_0, x_1, x_2, x_3), by solving the coefficients α_{ij} in the equations $\alpha([x_i, x_j]) = [\alpha(x_i), \alpha(x_j)]$ with respect to the basis. For simplicity, we take α diagonal, $\alpha_{ij} = 0$ if $i \neq j$.
We obtain $q_4(\mathcal{E})_\alpha$, Hom-Poisson versions of $q_4(\mathcal{E})$, with α given by

1. $\alpha(x_0) = -\lambda x_0,\ \alpha(x_1) = i\lambda x_1,\ \alpha(x_2) = \lambda x_2,\ \alpha(x_3) = -i\lambda x_3,$
2. $\alpha(x_0) = -\lambda x_0,\ \alpha(x_1) = -i\lambda x_1,\ \alpha(x_2) = \lambda x_2,\ \alpha(x_3) = i\lambda x_3,$
3. $\alpha(x_0) = \lambda x_0,\ \alpha(x_1) = -\lambda x_1,\ \alpha(x_2) = \lambda x_2,\ \alpha(x_3) = -\lambda x_3,$
4. $\alpha = \lambda id,$

with $\lambda \in \mathbb{K}$.

For example, $q_4(\mathcal{E})$ carries a structure of Hom-Poisson algebra, for any $\lambda \in \mathbb{C}$, with the following bracket

$$\begin{aligned} \{x_i, x_{i+1}\} &= -\lambda^2(k^2 x_i x_{i+1} - x_{i+2}x_{i+3}), \\ \{x_i, x_{i+2}\} &= \lambda^2 k(x_{i+3}^2 - x_{i+1}^2), \end{aligned} \qquad i = 1,2,3,4 \pmod 4,$$

and linear map

$$\alpha(x_0) = \lambda x_0,\ \alpha(x_1) = -\lambda x_1,\ \alpha(x_2) = \lambda x_2,\ \alpha(x_3) = -\lambda x_3.$$

2.1 Constructing Hom-Poisson algebras from Hom-Lie algebras

Suppose that $(A, [\ ,\ ], \alpha)$ is a finite dimensional Hom-Lie algebra and $\{e_i\}_{1\leqslant i\leqslant n}$ be a basis of A. Set C_{ij}^k be the structure constants of the bracket with respect to the basis, that is $[e_i, e_j] = \sum_{k=1}^n C_{ij}^k e_k$ and α_i^s be the coefficients of the morphism α, that is $\alpha(e_i) = \sum_{s=1}^n \alpha_i^s e_s$.
The skew-symmetry of the bracket and the Hom-Jacobi condition can be written with the structure constants as

$$C_{ji}^k = -C_{ij}^k \qquad \text{antisymmetry},$$
$$\sum_{1\leqslant p,q\leqslant n} (\alpha_i^p C_{jk}^q + \alpha_j^p C_{ki}^q + \alpha_k^p C_{ij}^q) C_{pq}^s = 0 \qquad \text{Hom-Jacobi identity}.$$

To construct a Hom-Poisson algebra from a Hom-Lie algebra, we should define a commutative multiplication $\cdot$ which is Hom-associative and a bracket $\{\ ,\ \}$ satisfying the Hom-Leibniz identity. Define the bracket $\{\ ,\ \}$ as being equal to the bracket $[\ ,\ ]$ on the basis, and extended by the Hom-Leibniz identity.
Set M_{ij}^k be the structure constants for the multiplication, that is $e_i \cdot e_j = \sum_{k=1}^n M_{ij}^k e_k$. By commutativity, $M_{ji}^k = M_{ij}^k$. The Hom-Leibniz identity writes

$$\begin{aligned} 0 &= \{e_i \cdot e_j, \alpha(e_k)\} - \alpha(e_i)\cdot\{e_j, e_k\} - \{e_i, e_k\}\cdot\alpha(e_j) \\ \Leftrightarrow 0 &= \sum_{s=1}^n \underbrace{\left(M_{ij}^p \alpha_k^q C_{pq}^s - (\alpha_i^p C_{jk}^q + C_{ik}^p \alpha_j^q) M_{pq}^s\right)}_{S_{ijks}} e_s \end{aligned}$$

$$\Leftrightarrow 0 = S_{ijks},$$

giving a linear system in the M_{ij}^l of n^4 equations in n^3 unknowns[1].

The Hom-associativity writes

$$\begin{aligned} 0 &= (e_i \cdot e_j) \cdot \alpha(e_k) - \alpha(e_i) \cdot (e_j \cdot e_k) \\ \Leftrightarrow 0 &= \sum_{s=1}^{n} \underbrace{\left((M_{ij}^p \alpha_k^q - \alpha_i^p M_{jk}^q) M_{pq}^s\right)}_{R_{ijks}} e_s \\ \Leftrightarrow 0 &= R_{ijks}, \end{aligned}$$

giving a non linear system in the M_{ij}^l of n^4 equations in n^3 unknowns.

Solving first the equations of Hom-Leibniz and then checking if the solutions satisfy the Hom-associativity equations, we obtain example of Hom-Poisson algebras.

Example 2.6. We consider the 3-dimensional Hom-Lie algebra with basis $\{e_1, e_2, e_3\}$, brackets given by

$$\begin{aligned} [e_1, e_2] &= C_{12}^2 e_2 + C_{12}^3 e_3 \\ [e_1, e_3] &= C_{13}^2 e_2 + C_{13}^3 e_3 \\ [e_2, e_3] &= 0, \end{aligned}$$

and morphism $\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}$ in the basis $\{e_1, e_2, e_3\}$. The only multiplications giving a Hom-Poisson algebra are of the form

$$\begin{aligned} e_1 \cdot e_2 &= \lambda e_2 \\ e_1 \cdot e_3 &= \lambda e_3 \\ e_2 \cdot e_3 &= 0 \\ e_i \cdot e_i &= 0 \quad \text{for } i = 1, 2, 3. \end{aligned}$$

Example 2.7. Other examples of Hom-Poisson algebras of dimension 3 with basis $\{e_1, e_2, e_3\}$ are given by twisting the following Poisson algebra:

$$e_1 \cdot e_1 = e_2 \qquad \{e_1, e_3\} = a e_2 + b e_3,$$

with all other multiplication and brackets equal to zero.

The morphism α is computed to be multiplicative for the multiplication and the bracket.

With $a \neq 0,\ b \neq 0$	With $a \neq 0, b = 0$
$\alpha(e_1) = e_1 + \alpha_{12} e_2 + \alpha_{13} e_3$	$\alpha(e_1) = c e_1 + \alpha_{12} e_2 + \alpha_{13} e_3$
$\alpha(e_2) = e_2$	$\alpha(e_2) = c^2 e_2$
$\alpha(e_3) = \alpha_{32} e_2 + \frac{b}{a} \alpha_{32} e_3$	$\alpha(e_3) = \alpha_{31} e_1 + \alpha_{32} e_2 + \alpha_{33} e_3$
	where c is a solution (if it exist) of $X^2 - \alpha_{33} X + \alpha_{13}\alpha_{31} = 0$.

[1] actually $\frac{n^2(n+1)}{2}$ unknowns using the commutativity

2.2 Flexibles structures

We recall here some results on flexible structures described in [29] and provide a connection to Hom-Poisson algebras.

Definition 2.8. A Hom-algebra $A = (A, \mu, \alpha)$ is called flexible if for any $x, y \in A$

$$\mu(\mu(x,y),\alpha(x)) = \mu(\alpha(x),\mu(y,x))). \tag{2.2}$$

Remark 2.9. Using the Hom-associator $\mathfrak{as}_{\mu,\alpha}$, the condition (2.2) may be written as

$$\mathfrak{as}_{\mu,\alpha}(x,y,x) = 0.$$

Lemma 2.10. *Let $A = (A, \mu, \alpha)$ be a Hom-algebra. The following assertions are equivalent*

(1) A is flexible.

(2) For any $x, y \in A$, $\quad \mathfrak{as}_{\mu,\alpha}(x,y,x) = 0.$

(3) For any $x, y, z \in A$, $\quad \mathfrak{as}_{\mu,\alpha}(x,y,z) = -\mathfrak{as}_{\mu,\alpha}(z,y,x).$

Proof. The equivalence of the first two assertions follows from the definition. The assertion $\mathfrak{as}_{\mu,\alpha}(x-z,y,x-z) = 0$ holds by definition and it is equivalent to $\mathfrak{as}_{\mu,\alpha}(x,y,z) + \mathfrak{as}_{\mu,\alpha}(z,y,x) = 0$ by linearity. □

Corollary 2.11. *Any Hom-associative algebra is flexible.*

Let $A = (A, \mu, \alpha)$ be a Hom-algebra, where μ is the multiplication and α a homomorphism. We define two new multiplications using μ :

$$\forall\, x, y \in A \quad x \bullet y = \mu(x,y) + \mu(y,x), \quad \{x,y\} = \mu(x,y) - \mu(y,x).$$

We set $A^+ = (A, \bullet, \alpha)$ and $A^- = (A, \{\ ,\ \}, \alpha)$.

Proposition 2.12. *A Hom-algebra $A = (A, \mu, \alpha)$ is flexible if and only if*

$$\{\alpha(x), y \bullet z\} = \{x,y\} \bullet \alpha(z) + \alpha(y) \bullet \{x,z\}. \tag{2.3}$$

Proof. Let A be a flexible Hom-algebra. By Lemma (2.10), this is equivalent to $\mathfrak{as}_{\mu,\alpha}(x,y,z) + \mathfrak{as}_{\mu,\alpha}(z,y,x) = 0$ for any $x, y, z \in A$. This implies

$$\begin{aligned} \mathfrak{as}_{\mu,\alpha}(x,y,z) + \mathfrak{as}_{\mu,\alpha}(z,y,x) + \mathfrak{as}_{\mu,\alpha}(x,z,y) + \\ \mathfrak{as}_{\mu,\alpha}(y,z,x) - \mathfrak{as}_{\mu,\alpha}(y,x,z) - \mathfrak{as}_{\mu,\alpha}(z,x,y) = 0 \end{aligned} \tag{2.4}$$

By expansion, the previous relation is equivalent to $\{\alpha(x), y \bullet z\} = \{x,y\} \bullet \alpha(z) + \alpha(y) \bullet \{x,z\}$. Conversely, assume we have the condition (2.3) in Proposition. By setting $x = z$ in (2.4), one gets $\mathfrak{as}_{\mu,\alpha}(x,y,x) = 0$. Therefore A is flexible. □

Hence, we obtain the following connection to Hom-Poisson algebras.

Proposition 2.13. *Let $A = (A, \mu, \alpha)$ be a flexible Hom-algebra which is Hom-associative. Then $(A, \bullet, \{\ ,\ \}, \alpha)$, where $\bullet$ and $\{\ ,\ \}$ are the operations defining A^+ and A^- respectively, is a Hom-Poisson algebra.*

2.3 1-operation Poisson algebras

Algebras with one operation were introduced by Loday and studied by Markl and Remm in [33]. The twisted version was studied in [45] where they are called admissible Hom-Poisson algebras.

Definition 2.14. A 1-*operation Hom-Poisson algebra* is a Hom-algebra $(A, \cdot, \alpha)$ satisfying, for any $x, y, z \in A$,

$$3\mathfrak{as}_{\cdot,\alpha}(x,y,z) = (x \cdot z) \cdot \alpha(y) + (y \cdot z) \cdot \alpha(x) - (y \cdot x) \cdot \alpha(z) - (z \cdot x) \cdot \alpha(y). \tag{2.5}$$

If α is the identity map, A is called a 1-operation Poisson algebra.

We consider a Hom-algebra $(A, \cdot, \alpha)$. We define two operations $\bullet : A \otimes A \to A$ and $\{\ ,\ \} : A \otimes A \to A$ by

$$\forall\, x, y \in A, \qquad x \bullet y = x \cdot y + y \cdot x, \qquad \{x, y\} = x \cdot y - y \cdot x. \tag{2.6}$$

Theorem 2.15. $(A, \bullet, \{\ ,\ \}, \alpha)$ *is a Hom-Poisson algebra if and only if* $(A, \cdot, \alpha)$ *is a* 1-*operation Hom-Poisson algebra.*

Proof. Suppose that $(A, \bullet, \{\ ,\ \}, \alpha)$ is a Hom-Poisson algebra. Since

$$\forall\, x, y \in A,\ x \cdot y = \frac{1}{2}(\{x, y\} + x \bullet y), \tag{2.7}$$

we have, by expansion,

$$\mathfrak{as}_{\cdot,\alpha}(x,y,z) = (x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) = \frac{1}{4}\{\alpha(y), \{z, x\}\},$$

and, on the other hand, using that the multiplication $\bullet$ is Hom-associative and commutative, and that $\{\ ,\ \}$ is a Hom-Lie bracket,

$$(x \cdot z) \cdot \alpha(y) + (y \cdot z) \cdot \alpha(x) - (y \cdot x) \cdot \alpha(z) - (z \cdot x) \cdot \alpha(y) = \frac{3}{4}\{\alpha(y), \{z, x\}\}.$$

We thus have the equation (2.5).

Suppose now that the equation (2.5) is verified. We have to show that $\bullet$ is Hom-associative and that $\{\ ,\ \}$ is a Hom-Lie bracket.

Using the relation (2.5), we obtain the identities

$$\forall\, x, y, z \in A \quad \mathfrak{as}_{\cdot,\alpha}(x,y,z) + \mathfrak{as}_{\cdot,\alpha}(y,z,x) + \mathfrak{as}_{\cdot,\alpha}(z,x,y) = 0 \tag{2.8}$$

$$\forall\, x, y, z \in A \quad \mathfrak{as}_{\cdot,\alpha}(x,y,z) + \mathfrak{as}_{\cdot,\alpha}(z,y,x) = 0. \tag{2.9}$$

This last identity (2.9) shows that $(A, \cdot, \alpha)$ is a Hom-flexible algebra using Lemma 2.10.

We now obtain

$$\begin{aligned}&\mathfrak{as}_{\bullet,\alpha}(x,y,z) = (x\bullet y)\bullet\alpha(z) - \alpha(x)\bullet(y\bullet z)\\ =&\mathfrak{as}_{\cdot,\alpha}(y,z,x) + \mathfrak{as}_{\cdot,\alpha}(x,z,y) - (\mathfrak{as}_{\cdot,\alpha}(z,y,x) + \mathfrak{as}_{\cdot,\alpha}(x,y,z))\\ \overset{(2.9)}{=}&0.\end{aligned}$$

So the product $\bullet$ is Hom-associative and commutative by definition. Moreover,

$$\begin{aligned}&J_{\{\ ,\ \},\alpha}(x,y,z) = \{\alpha(x),\{y,z\}\} + \{\alpha(y),\{z,x\}\} + \{\alpha(z),\{x,y\}\}\\ =&-(\mathfrak{as}_{\cdot,\alpha}(x,y,z) + \mathfrak{as}_{\cdot,\alpha}(y,z,x) + \mathfrak{as}_{\cdot,\alpha}(z,x,y))+\\ &\mathfrak{as}_{\cdot,\alpha}(x,z,y) + \mathfrak{as}_{\cdot,\alpha}(y,x,z) + \mathfrak{as}_{\cdot,\alpha}(z,y,x)\\ \overset{(2.8)}{=}&0,\end{aligned}$$

so $\{\ ,\ \}$ is a Hom-Lie bracket. Since A is flexible, Proposition 2.12 leads to the compatibility between $\bullet$ and $\{\ ,\ \}$,

$$\{\alpha(x), y\bullet z\} = \{x,y\}\bullet\alpha(z) + \alpha(y)\bullet\{x,z\}.$$

So $(A,\bullet,\{\ ,\ \},\alpha)$ is a Hom-Poisson algebra. □

Proposition 2.16. *Let $(A,\cdot)$ be a 1-operation Poisson algebra, and $\alpha : A\to A$ be a linear map multiplicative for the multiplication $\cdot$, i.e. $\alpha\circ\cdot = \cdot\circ\alpha^{\otimes 2}$, then $A_\alpha = (A,\cdot_\alpha = \alpha\circ\cdot,\alpha)$ is a 1-operation Hom-Poisson algebra.*

Proof. We have

$$\begin{aligned}3\mathfrak{as}_{\cdot_\alpha,\alpha}(x,y,z) &= (x\cdot_\alpha y)\cdot_\alpha\alpha(z) - \alpha(x)\cdot_\alpha(y\cdot_\alpha z)\\ &= \alpha(\alpha(x\cdot y)\cdot\alpha(z)) - \alpha(\alpha(x)\cdot\alpha(y\cdot z)) = \alpha^2((x\cdot y)\cdot z - x\cdot(y\cdot z)),\end{aligned}$$

and since $\cdot$ verifies the 1-operation equation,

$$\begin{aligned}3\mathfrak{as}_{\cdot_\alpha,\alpha}(x,y,z) &= \alpha^2((x\cdot z)\cdot y + (y\cdot z)\cdot x - (y\cdot x)\cdot z - (z\cdot x)\cdot y)\\ &= \alpha(\alpha(x\cdot z)\cdot\alpha(y)) + \alpha(\alpha(y\cdot z)\cdot\alpha(x))\\ &\quad - \alpha(\alpha(y\cdot x)\cdot\alpha(z)) - \alpha(\alpha(z\cdot x)\cdot\alpha(y))\\ &= (x\cdot_\alpha z)\cdot_\alpha(y) + (y\cdot_\alpha z)\cdot_\alpha\alpha(x) - (y\cdot_\alpha x)\cdot_\alpha\alpha(z) - (z\cdot_\alpha x)\cdot_\alpha\alpha(y).\end{aligned}$$

□

3 Hom-coPoisson structures

Definition 3.1. A *Hom-coPoisson algebra* consists of a cocommutative coassociative Hom-coalgebra $(A,\Delta,\varepsilon,\alpha)$ equipped with a skew-symmetric linear map $\delta : A\to A\otimes A$, the Hom-coPoisson cobracket, satisfying the following conditions

(i) (Hom-coJacobi identity)

$$(Id + \sigma + \sigma^2) \circ (\alpha \otimes \delta) \circ \delta = 0, \tag{3.1}$$

(ii) (Hom-coLeibniz rule)

$$(\Delta \otimes \alpha) \circ \delta = (\alpha \otimes \delta) \circ \Delta + \tau_{23} \circ (\delta \otimes \alpha) \circ \Delta. \tag{3.2}$$

It is denoted by a tuple $(A, \Delta, \varepsilon, \alpha, \delta)$.

Proposition 3.2. *Let $(A, \Delta, \varepsilon, \alpha, \delta)$ be a Hom-coPoisson algebra and $\beta : A \to A$ be a Hom-coPoisson algebra morphism. Then $(A, \Delta_\beta = \Delta \circ \beta, \varepsilon, \alpha \circ \beta, \delta_\beta = \delta \circ \beta)$ is a Hom-coPoisson algebra.*

Proof. Theorem 1.9 insures that $(A, \Delta_\beta, \varepsilon, \alpha \circ \beta)$ is a coassociative Hom-coalgebra and Theorem 1.20 that $(A, \delta_\beta, \alpha \circ \beta)$ is a Hom-Lie coalgebra. It remains to show the compatibility condition 3.2. On the left hand side, we have

$$(\Delta_\beta \otimes \alpha \circ \beta) \circ \delta_\beta = (\Delta \circ \beta \otimes \alpha \circ \beta) \circ \delta \circ \beta = (\Delta \otimes \alpha) \circ \beta^2,$$

and the right hand side gives

$$\begin{aligned}
&(\alpha \circ \beta \otimes \delta_\beta) \circ \Delta_\beta + \tau_{23} \circ (\delta_\beta \otimes \alpha \circ \beta) \circ \Delta_\beta \\
&\quad = (\alpha \circ \beta \otimes \delta \circ \beta) \circ \Delta \circ \beta + \tau_{23} \circ (\delta \circ \beta \otimes \alpha \circ \beta) \circ \Delta \circ \beta \\
&\quad = [(\alpha \otimes \delta) \circ \Delta + \tau_{23} \circ (\delta \otimes \alpha) \circ \Delta] \circ \beta^2,
\end{aligned}$$

which ends the proof. □

We may state the following Corollaries. Starting from a classical coPoisson algebra, we may construct Hom-coPoisson algebras using coPoisson algebra morphisms. On the other hand a Hom-coPoisson algebra gives rise to infinitely many Hom-coPoisson algebras.

Corollary 3.3.

1. *Let $(A, \Delta, \varepsilon, \delta)$ be a coPoisson algebra and $\beta : A \to A$ be a coPoisson algebra morphism. Then $(A, \Delta_\beta = \Delta \circ \beta, \varepsilon, \beta, \delta_\beta = \delta \circ \beta)$ is a Hom-coPoisson algebra.*
2. *Let $(A, \Delta, \varepsilon, \alpha, \delta)$ be a Hom-coPoisson algebra. Then for any non negative integer n, we have $(A, \Delta \circ \alpha^n, \varepsilon, \alpha^{n+1}, \delta \circ \alpha^n)$ is a Hom-coPoisson algebra.*

Definition 3.4. A *Hom-coPoisson bialgebra* $(A, \mu, \eta, \Delta, \varepsilon, \alpha, \delta)$ is a Hom-coPoisson algebra $(A, \Delta, \varepsilon, \alpha, \delta)$ which is also a Hom-bialgebra $(A, \mu, \eta, \Delta, \varepsilon, \alpha)$, the two structures being compatible in the sense that δ is a Δ-derivation,

$$\delta \circ \mu = (\mu \otimes \mu) \circ \tau_{23} \circ (\delta \otimes \Delta + \Delta \otimes \delta).$$

A *Hom-coPoisson Hopf algebra* $(A, \mu, \eta, \Delta, \varepsilon, S, \alpha, \delta)$ is a Hom-coPoisson bialgebra $(A, \mu, \eta, \Delta, \varepsilon, \alpha, \delta)$ with an antipode S, such that the tuple $(A, \mu, \eta, \Delta, \varepsilon, S, \alpha)$ is a Hom-Hopf algebra.

We extend the connection between Lie bialgebras and coPoisson-Hopf algebras presented in [10] to the Hom setting.

Proposition 3.5. *Let* $(\mathfrak{g},[\ ,\ ],\alpha)$ *be a Hom-Lie algebra. If its universal enveloping algebra* $U_{HLie}(\mathfrak{g})$ *has a Hom-coPoisson structure* δ*, making it a Hom-coPoisson bialgebra, then* $\delta(\mathfrak{g})\subset\mathfrak{g}\otimes\mathfrak{g}$ *and* $\delta|_{\mathfrak{g}}$ *equips* $(\mathfrak{g},[\ ,\ ],\alpha,\delta,Id)$ *with a structure of Hom-Lie bialgebra. Conversely, for a Hom-Lie bialgebra* $(\mathfrak{g},[\ ,\ ],\delta,\alpha)$*, the cobracket* $\delta:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ *extends uniquely to a Hom-coPoisson cobracket on* $U_{HLie}(\mathfrak{g})$*, which makes* $U_{HLie}(\mathfrak{g})$ *a Hom-coPoisson bialgebra.*

Proof. Let $\delta:U_{HLie}(\mathfrak{g})\to U_{HLie}(\mathfrak{g})\otimes U_{HLie}(\mathfrak{g})$ be a Hom-coPoisson cobracket on $U_{HLie}(\mathfrak{g})$. To show that $\delta(\mathfrak{g})\subset\mathfrak{g}\otimes\mathfrak{g}$, let $x\in\mathfrak{g}$, and write $\delta(x)=\sum_{(x)}x^{(1)}\otimes x^{(2)}$ where $x^{(1)},x^{(2)}\in U_{HLie}(\mathfrak{g})$. We may assume that the $x^{(2)}$ are linearly independent. By the Hom-coLeibniz condition (3.2), we have

$$\begin{aligned}\sum_{(x)}\Delta(x^{(1)})\otimes\alpha(x^{(2)})=&\alpha(1)\otimes\delta(\alpha(x))+\alpha(\alpha(x))\otimes\delta(1)\\&+\tau_{23}\circ(\delta(1)\otimes\alpha(\alpha(x))+\delta(\alpha(x))\otimes\alpha(1))\end{aligned}$$

since $x\in\mathfrak{g}$ and $\Delta(x)=1\otimes\alpha(x)+\alpha(x)\otimes 1$. Moreover, α is a morphism so $\alpha(1)=1$ and δ is a Δ-derivation so $\delta(1)=0$, hence

$$\begin{aligned}\sum_{(x)}\Delta(x^{(1)})\otimes\alpha(x^{(2)})&=1\otimes\delta(\alpha(x))+\tau_{23}\circ(\delta(\alpha(x))\otimes 1)\\&=\sum_{(x)}\big(1\otimes\alpha(x^{(1)})+\alpha(x^{(1)})\otimes 1\big)\otimes\alpha(x^{(2)})\end{aligned}$$

using the multiplicativity $\delta\circ\alpha=\alpha^{\otimes 2}\circ\delta$ of the Hom-coPoisson morphism α. It follows that the $x^{(1)}$ are Hom-primitive elements ($\Delta(x)=1\otimes\alpha(x)+\alpha(x)\otimes 1$) of $U_{HLie}(\mathfrak{g})$, hence $\delta(\mathfrak{g})\subset\mathfrak{g}\otimes U_{HLie}(\mathfrak{g})$. Since δ is skew-symmetric,

$$\delta(\mathfrak{g})\subset(\mathfrak{g}\otimes U_{HLie}(\mathfrak{g}))\cap(U_{HLie}(\mathfrak{g})\otimes\mathfrak{g})=\mathfrak{g}\otimes\mathfrak{g}.$$

To prove the compatibility condition (1.17) for $\delta|_{\mathfrak{g}}$ and the twisting maps α and Id, let $x,y\in\mathfrak{g}$ and compute

$$\begin{aligned}\delta([x,y])=&\delta(xy-yx)\\=&\delta(x)\Delta(y)+\Delta(x)\delta(y)-\delta(y)\Delta(x)-\Delta(y)\delta(x)\\=&[\Delta(x),\delta(y)]-[\Delta(y),\delta(x)]\\=&[\alpha(x),y^{(1)}]\otimes y^{(2)}+y^{(1)}\otimes[\alpha(x),y^{(2)}]\\&-[\alpha(y),x^{(1)}]\otimes x^{(2)}-x^{(1)}\otimes[\alpha(y),x^{(2)}]\\=&ad_{\alpha(x)}(\Delta(y))-ad_{\alpha(y)}(\Delta(y)).\end{aligned}$$

Conversely, $\delta:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ uniquely extends in $\overline{\delta}:U_{HLie}(\mathfrak{g})\to U_{HLie}(\mathfrak{g})\otimes U_{HLie}(\mathfrak{g})$ such that $\overline{\delta}|_{\mathfrak{g}}=\delta$, with the formula

$$\overline{\delta}(xy)=\overline{\delta}(x)\Delta(y)+\Delta(x)\overline{\delta}(y).$$

This gives $U_{HLie}(\mathfrak{g})$ a structure of Hom-coPoisson bialgebra. □

3.1 Duality

In this section, we extend to Hom-algebras the result stated in [35], that the Hopf dual of a coPoisson Hopf algebra is a Poisson-Hopf algebra.

Definition 3.6. An algebra A over $\mathbb{K}$ is said to be an almost normalizing extension over $\mathbb{K}$ if A is a finitely generated $\mathbb{K}$-algebra with generators $x_1, \ldots, x_n$ satisfying the condition

$$x_i x_j - x_j x_i \in \sum_{l=1}^{n} \mathbb{K} x_l + \mathbb{K} \tag{3.3}$$

for all i, j.

Lemma 3.7. *Let A be an almost normalizing extension of $\mathbb{K}$ with generators $x_1, \ldots, x_n$. Then A is spanned by all standard monomials*

$$x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}, \qquad r_i \in \mathbb{N}$$

together with the unity 1.

Proof. This follows immediately from induction on the degree of monomials. □

Recall that the bialgebra (resp. Hopf) dual A° of a Hom-bialgebra (resp. Hom-Hopf algebra) A consists of

$$A^\circ = \{f \in A^*, f(I) = 0 \text{ for some cofinite ideal } I \text{ of } A\},$$

where A^* is the dual vector space of A.

Theorem 3.8. *Let A be a Hom-coPoisson bialgebra (resp. Hopf algebra) with Poisson co-bracket δ and twisting multiplicative map α. If A is an almost normalizing extension over $\mathbb{K}$, then the bialgebra (resp. Hopf) dual A° is a Hom-Poisson bialgebra (resp. Hopf algebra) with twisting map α° and bracket*

$$\{f, g\}(x) = \langle \delta(x), f \otimes g \rangle, \qquad x \in A \tag{3.4}$$

for any $f, g \in A^\circ$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between the vector space $A \otimes A$ and its dual vector space.

Proof. The proof is almost the same as in [35]. We do not reproduce here the first step showing that the bracket (3.4) is well-defined, it uses the fact that A is an almost normalizing extension over $\mathbb{K}$.

The skew-symmetry follows from $\tau_{12} \circ \delta = -\delta$, we have

$$\begin{aligned} \{g, f\}(x) &= \langle \delta(x), g \otimes f \rangle = \langle \tau_{12} \circ \delta, f \otimes g \rangle \\ &= -\langle \delta(x), f \otimes g \rangle = -\{f, g\}(x), \end{aligned}$$

for all $x \in A$.

The equation (3.4) satisfies the Hom-Leibniz rule: since

$$\{fg, \alpha^\circ(h)\}(x) = \langle (\Delta \otimes \alpha) \circ \delta(x), f \otimes g \otimes h \rangle \tag{3.5}$$

and

$$\begin{aligned}&(\alpha^\circ(f)\{g,h\} + \{f,h\}\alpha^\circ(g))(x)\\ &= \langle (\alpha \otimes \delta) \circ \Delta(x), f \otimes g \otimes h \rangle + \langle \tau_{23} \circ (\delta \otimes \alpha) \circ \Delta(x), f \otimes g \otimes h \rangle\end{aligned}$$

for $x \in A$ and $f, g, h \in A^\circ$, it is enough to show that

$$(\Delta \otimes \alpha) \circ \delta = (\alpha \otimes \delta) \circ \Delta + \tau_{23} \circ (\delta \otimes \alpha) \circ \Delta, \tag{3.6}$$

but this is just the Hom-coLeibniz rule for δ.

The equation (3.4) satisfies the Hom-Jacobi identity: we have

$$\begin{aligned}\{\alpha^\circ(f), \{g,h\}\}(x) &= \langle (\alpha \otimes \delta) \circ \delta(x), f \otimes g \otimes h \rangle,\\ \{\alpha^\circ(g), \{h,f\}\}(x) &= \langle \sigma \circ (\alpha \otimes \delta) \circ \delta(x), f \otimes g \otimes h \rangle,\\ \{\alpha^\circ(h), \{f,g\}\}(x) &= \langle \sigma^2 \circ (\alpha \otimes \delta) \circ \delta(x), f \otimes g \otimes h \rangle,\end{aligned}$$

for $x \in A$ and $f, g, h \in A^\circ$. Hence (3.4) satisfies the Hom-Jacobi identity if and only if δ satisfies

$$(Id + \sigma + \sigma^2) \circ (\alpha \otimes \delta) \circ \delta = 0,$$

which is just the Hom-coJacobi identity of δ.

The bracket defined by (3.4) satisfies the compatibility condition with the comultiplication of the Hom-bialgebra (resp. Hopf algebra), it is a μ-coderivation: since δ is a Δ-derivation, we have for $f, g \in A^\circ$

$$\begin{aligned}\Delta(\{f,g\})(x \otimes y) &= \{f,g\}(xy) = \langle \delta(xy), f \otimes g \rangle\\ &= \langle \delta(x)\Delta(y), f \otimes g \rangle + \langle \Delta(x)\delta(y), f \otimes g \rangle\\ &= \langle \delta(x), f_1 \otimes g_1 \rangle \langle \Delta(y), f_2 \otimes g_2 \rangle\\ &\quad + \langle \Delta(x), f_1 \otimes g_2 \rangle \langle \delta(y), f_2 \otimes g_2 \rangle\\ &= \{f_1, g_1\}(x)(f_2 g_2)(y) + (f_1 g_1)(x)\{f_2, g_2\}(y)\\ &= \{\Delta(f), \Delta(g)\}(x \otimes y).\end{aligned}$$

Finally, the bracket defined by (3.4) equips A° with the structure of a Hom-Poisson bialgebra (resp. Hopf algebra), the twisting map being α°. □

Let $(\mathfrak{g}, [\ ,\ ], \delta, \alpha)$ be a Hom-Lie bialgebra, $U_{HLie}(\mathfrak{g})$ the universal enveloping Hom-bialgebra of $\mathfrak{g}$ with comultiplication Δ. The cobracket $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is extended uniquely to a Δ-derivation $\overline{\delta} : U_{HLie}(\mathfrak{g}) \to U_{HLie}(\mathfrak{g}) \otimes U_{HLie}(\mathfrak{g})$ such that $\overline{\delta}|_{\mathfrak{g}} = \delta$ and $\overline{\delta}(xy) = \overline{\delta}(x)\Delta(y) + \Delta(x)\overline{\delta}(y)$. Then $U_{HLie}(\mathfrak{g})$ is a Hom-coPoisson bialgebra with cobracket $\overline{\delta}$.

Corollary 3.9. *Let* $(\mathfrak{g}, [\ ,\ ], \delta, \alpha)$ *be a finite dimensional Hom-Lie bialgebra. Then the dual* $U_{HLie}(\mathfrak{g})^\circ$ *of the universal enveloping Hom-bialgebra* $U_{HLie}(\mathfrak{g})$ *is a Hom-Poisson bialgebra with Poisson bracket*

$$\{f, g\}(x) = \langle \overline{\delta}(x), f \otimes g \rangle, \qquad x \in U_{HLie}(\mathfrak{g})$$

for $f, g \in U_{HLie}(\mathfrak{g})^\circ$.

Proof. Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g}$. Then $U_{HLie}(\mathfrak{g})$ is an almost normalizing extension over $\mathbb{K}$ with generators $x_1, \ldots, x_n$. Thus the result follows from Theorem 3.8. □

4 Deformation theory and Quantization

Deformation is one of the oldest techniques used by mathematicians and physicists. The first instances of the so-called deformation theory were given by Kodaira and Spencer for complex structures and by Gerstenhaber for associative algebras [14]. The Lie algebras case was studied by Nijenhuis and Richardson [34] and the deformation theory for bialgebras and Hopf algebras were developed later by Gerstenhaber and Schack [15]. The main and popular tool is the power series ring or more generally any commutative algebras. By standard facts of deformation theory, the infinitesimal deformations of an algebra of a given type are parametrized by a second cohomology of the algebra. More generally, it is stated that deformations are controlled by a suitable cohomology. A modern approach, essentially due to Quillen, Deligne, Drinfeld, and Kontsevich, is that, in characteristic zero, every deformation problem is controlled by a differential graded Lie algebra, via solutions of Maurer-Cartan equation modulo gauge equivalence.

Some mathematical formulations of quantization are based on the algebra of observables and consist in replacing the classical algebra of observables (typically complex-valued smooth functions on a Poisson manifold) by a noncommutative one constructed by means of an algebraic formal deformation of the classical algebra. The so-called deformation quantization problem was introduced in the seminal paper [4] by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer (1978).

In 1997, Kontsevich solved a longstanding problem in mathematical physics, that is every Poisson manifold admits formal quantization which is canonical up to a certain equivalence.

4.1 Formal deformation of Hom-associative algebras

In [31] the formal deformation theory is extended to Hom-associative and Hom-Lie algebras and a suitable cohomology is provided, see also [2]. The usual results involving deformation first order element and second order cohomology groups extends in the Hom case. We describe briefly the results in this section. We consider deformations of Hom-associative algebras where the twist map remains unchanged.

Let $A = (A, \mu_0, \alpha)$ be a Hom-associative algebra. Let $\mathbb{K}[[t]]$ be the power series ring in one variable t and coefficients in $\mathbb{K}$ and $A[[t]]$ be the set of formal power series whose coefficients are elements of A, ($A[[t]]$ is obtained by extending the coefficients domain of A from $\mathbb{K}$ to $\mathbb{K}[[t]]$). Then $A[[t]]$ is a $\mathbb{K}[[t]]$-module. When A is finite-dimensional, we have $A[[t]] = A \otimes \mathbb{K}[[t]]$. Note that A is a submodule of $A[[t]]$.

Definition 4.1. Let $A = (A, \mu_0, \alpha)$ be a Hom-associative algebra. A formal Hom-associative deformation of A is given by a $\mathbb{K}[[t]]$-bilinear map $\mu_t : A[[t]] \otimes A[[t]] \to A[[t]]$ of the form

$$\mu_t = \sum_{i \geqslant 0} \mu_i t^i \tag{4.1}$$

where each μ_i is a $\mathbb{K}$-bilinear map $\mu_i : A \otimes A \to A$ (extended to be $\mathbb{K}[[t]]$-bilinear) such that holds the following formal Hom-associativity condition:

$$\mathfrak{as}_{\mu_t,\alpha} = \mu_t \circ (\mu_t \otimes \alpha - \alpha \otimes \mu_t) = 0. \tag{4.2}$$

If $\alpha = Id$ the definition reduces to formal deformations of an associative algebra.

The equation (4.2) can be written

$$\sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \big(\mu_i(\alpha(x), \mu_j(y, z)) - \mu_i(\mu_j(x, y, \alpha(z))\big) t^{i+j} = 0. \tag{4.3}$$

Introducing the following notation

$$(x, y, z) \mapsto \mu_i \circ_\alpha \mu_j(x, y, z) := \mu_i(\alpha(x), \mu_j(y, z)) - \mu_i(\mu_j(x, y, \alpha(z)),$$

the deformation equation may be written as follows

$$\sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} (\mu_i \circ_\alpha \mu_j) t^{i+j} = 0 \qquad \text{or} \qquad \sum_{s\in\mathbb{N}} t^s \sum_{i=0}^{s} \mu_i \circ_\alpha \mu_{s-i} = 0. \tag{4.4}$$

It is equivalent to the infinite system: $\sum_{i=0}^{s} \mu_i \circ_\alpha \mu_{s-i} = 0,\ s = 0, 1, \dots$.

The A-valued Hochschild Type cohomology of Hom-associative algebras initiated in [31] and extended in [2] suits and leads to the following cohomological interpretation:

1. There is a natural bijection between $H^2(A, A)$ and the set of equivalence classes of deformation (mod t^2) of A.
2. If $H^2(A, A) = 0$ then every deformation of A is trivial.

The fact that the antisymmetrization of the first order element of a deformation of an associative algebra defines a Poisson bracket remains true in the Hom setting. More precisely, we have the following theorem.

Theorem 4.2 ([31]). *Let $A = (A, \mu_0, \alpha)$ be a commutative Hom-associative algebra and $A_t = (A, \mu_t, \alpha)$ be a deformation of A. Consider the bracket defined for $x, y \in A$ by $\{x, y\} = \mu_1(x, y) - \mu_1(y, x)$ where μ_1 is the first order element of the deformation μ_t. Then $(A, \mu_0, \{\ ,\ \}, \alpha)$ is a Hom-Poisson algebra.*

The proof is mainly computational, it leans on the properties of the α-associators, and on rewriting the deformation equations in terms of coboundary operators.

4.2 Deformations of Hom-coalgebras and Hom-Bialgebras

The formal deformation theory for bialgebras and Hopf algebras was introduced in [15]. It is extended to Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras in [12], where a suitable cohomology is obtained and the classical results are extended to Hom-setting.

Definition 4.3. Let (A, Δ, α) be a Hom-coalgebra. A formal Hom-coalgebra deformation of A is given by a linear map $\Delta_t : A[[t]] \to A[[t]] \otimes A[[t]]$ such that $\Delta_t = \sum_{i \geqslant 0} \Delta_i t^i$ where each Δ_i is a linear map $\Delta_i : A \to A \otimes A$ (extended to be $\mathbb{K}[[t]]$-linear) such that holds the following formal Hom-coassociativity condition:

$$(\Delta_t \otimes \alpha - \alpha \otimes \Delta_t) \circ \Delta_t = 0. \tag{4.5}$$

Definition 4.4. Let (A, μ, Δ, α) be a Hom-bialgebra. A formal Hom-bialgebra deformation of A is given by linear maps $\mu_t : A[[t]] \otimes A[[t]] \to A[[t]]$ and $\Delta_t : A[[t]] \to A[[t]] \otimes A[[t]]$ such that

1. $(A[[t]], \mu_t, \alpha)$ is a Hom-associative algebra,
2. $(A[[t]], \Delta_t, \alpha)$ is a Hom-coassociative coalgebra,
3. The multiplication and the comultiplication are compatible, that is

$$\Delta_t \circ \mu_t = \mu_t \otimes \mu_t \circ \tau_{23} \circ (\Delta_t \otimes \Delta_t).$$

It is shown in [12] that deformations are controlled by Hochschild type cohomology and any deformation of unital Hom-associative algebra (resp. counital Hom-coassociative colgebra) is equivalent to unital Hom-associative algebra (resp. counital Hom-coassociative colgebra). Furthermore, any deformation of a Hom-Hopf algebra as a Hom-bialgebra is automatically a Hom-Hopf algebra.

In a similar way as for Hom-associative algebra, we have:

Theorem 4.5. *Let $A = (A, \Delta_0, \alpha)$ be a cocommutative Hom-coassociative coalgebra and $A_t = (A, \Delta_t, \alpha)$ be a deformation of A. Consider the cobracket defined for $x \in A$ by $\delta(x) = \Delta_1(x) - \Delta_1^{op}(x)$ where Δ_1 is the first order element of the deformation Δ_t. Then $(A, \Delta_0, \delta, \alpha)$ is a Hom-coPoisson algebra.*

4.3 Quantization and Twisting star-products

The deformation quantization problem in the Hom-setting is stated as follows: given a Hom-Poisson algebra (resp. Hom-coPoisson algebra), find a formal deformation of a commutative Hom-associative algebra (resp. cocommutative Hom-coassociative coalgebra) such that the first order element of the deformation defines the Hom-Poisson algebra (resp. Hom-coPoisson algebra). In the classical case the Hom-Poisson structure is called the quasi-classical limit and the deformation is the star-product. This point of view initiated in [4] attempts to view the quantum mechanics as a deformation of the classical mechanics, the Lorentz group is a deformation of the Galilee group.

Let $(A, \cdot, \{\ ,\ \}, \alpha)$ be a commutative Hom-associative algebra endowed with a Hom-Poisson bracket $\{\ ,\ \}$.

Definition 4.6. A $\star$-product on A is a one parameter formal deformation defined on A by

$$f \star_t g = \sum_{r=0}^{\infty} t^r \mu_r(f, g)$$

such that

1. The $\star$-product in $A[[t]]$ is Hom-associative, that is

$$\forall r \in \mathbb{N}, \quad \sum_{i=0}^{r} (\mu_i(\mu_{r-i}(f, g), \alpha(h)) - (\mu_i(\alpha(f), \mu_{r-i}(g, h))) = 0,$$

2. $\mu_0(f, g) = f \cdot g$
3. $\mu_1(f, g) - \mu_1(g, f) = \{f, g\}$
4. $\mu_r(f, 1) = \mu_r(1, f) = 0 \quad \forall\ r > 0$

Remark 4.7.

- The condition (2) shows that $[f, g] := \frac{1}{2t}(f \star_t g\ -\ g \star_t f)$ is a deformation of the Hom-Lie structure $\{\ ,\ \}$.
- The condition $\mu_1(f, g) - \mu_1(g, f) = \{f, g\}$ expresses the correspondence between the deformation and the Hom-Poisson structure

$$\frac{f \star_t g - g \star_t f}{t}|_{t=0} = \{f, g\}.$$

Similarly we set the dual version of quantization problem as follows.

Let A be cocommutative Hom-coPoisson bialgebra (resp. Hopf algebra) and let δ be its Poisson cobracket. A quantization of A is a Hom-bialgebra (resp. Hom-Hopf algebra) deformation A_t of A such that

$$\delta(x) = \frac{\Delta_t(a) - \Delta_t^{op}(a)}{t} \pmod{t},$$

where $x \in A$ and a is any element of $A[[t]]$ such that $x = a \pmod{t}$.

Theorem 4.8. *Let $(A, \star)$ be an associative deformation of an associative algebra (A, μ_0), with $\star = \mu_0 + t\mu_1 + t^2\mu_2 + \cdots$*
Let $\alpha : A \to A$ be a morphism such that for all $i \in \mathbb{N}$, $\alpha \circ \mu_i = \mu_i \circ \alpha^{\otimes 2}$. Then $(A, \star_\alpha = \alpha \circ \star, \alpha)$ is a Hom-associative deformation of A.

Proof. Since for all $i \in \mathbb{N}$, $\alpha \circ \mu_i = \mu_i \circ \alpha^{\otimes 2}$, we also have $\alpha \circ \star = \star \circ \alpha^{\otimes 2}$. For $f, g, h \in (A, \star_\alpha, \alpha)$, we get

$$\begin{aligned}(f \star_\alpha g) \star_\alpha \alpha(h) &= \alpha(\alpha(f) \star \alpha(g)) \star \alpha(\alpha(h)) = \alpha((\alpha(f) \star \alpha(g)) \star \alpha(h)) \\ &= \alpha(\alpha(f) \star (\alpha(g) \star \alpha(h))) = \alpha(\alpha(f)) \star \alpha(\alpha(g) \star \alpha(h)) = \alpha(f) \star_\alpha (g \star_\alpha h),\end{aligned} \tag{4.6}$$

the passage from the first line to the second is due to the associativity of $\star$. This shows that the product $\star_\alpha$ is Hom-associative. ☐

4.4 Moyal-Weyl Hom-associative algebra

In the following, we twist the Moyal-Weyl product. It is the associative $\star$-product corresponding to the deformation of the Poisson phase-space bracket, one of the first examples of Kontsevich formal deformation [24].

We consider the Poisson algebra of polynomials of two variables $(\mathbb{R}[x, y], \cdot, \{\ ,\ \})$ where the Poisson bracket of two polynomials is given by $\{f, g\} = \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x}$.

The associated associative algebra is $(\mathbb{R}[x, y], \star)$ where the star-product is given by the Moyal-Weyl formula

$$f \star g = \sum_{n \in \mathbb{N}} \frac{\partial^n f}{\partial x^n}\frac{\partial^n g}{\partial y^n}\frac{t^n}{n!} = \sum_{n \in \mathbb{N}} \mu_n(f, g) t^n, \tag{4.7}$$

where $\mu_n(f, g) = \dfrac{1}{n!}\dfrac{\partial^n f}{\partial x^n}\dfrac{\partial^n g}{\partial y^n}$.

Proposition 4.9. *A morphism $\alpha : \mathbb{R}[x, y] \to \mathbb{R}[x, y]$ satisfying $\alpha \circ \mu_i = \mu_i \circ \alpha^{\otimes 2}$ for all $i \in \mathbb{N}$ which gives $(\mathbb{R}[x, y],\ \star_\alpha = \alpha\star, \alpha)$ a structure of Hom-associative algebra is of the form*

$$\alpha(x) = ax + b \ \text{ and } \ \alpha(y) = \frac{1}{a}y + c \qquad \text{where} \quad a, b, c \in \mathbb{R},\ a \neq 0. \tag{4.8}$$

Proof. Let $\alpha : \mathbb{R}[x,y] \to \mathbb{R}[x,y]$ be a morphism such that for all $i \in \mathbb{N}$, $\alpha\mu_i = \mu_i\alpha^{\otimes 2}$. In particular, for $i = 0$,

$$\alpha(fg) = \alpha(f)\alpha(g), \tag{4.9}$$

which shows that α is multiplicative, so it is sufficient to define it on x and y. For $i = 1$, we get

$$\alpha\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\right) = \frac{\partial \alpha(f)}{\partial x}\frac{\partial \alpha(g)}{\partial y}, \tag{4.10}$$

which implies that $\alpha(\{f,g\}) = \{\alpha(f), \alpha(g)\}$.

We set $P_1(x,y) := \alpha(x)$ and $P_2 := \alpha(y)$. For $f(x,y) = x$ and $g(x,y) = y$, the equation (4.10) gives

$$1 = \alpha(1) = \frac{\partial P_1}{\partial x}\frac{\partial P_2}{\partial y},$$

so we must have $P_1(x,y) = ax + Q_1(y)$ and $P_2(x,y) = \frac{1}{a}y + Q_2(x)$ with $a \in \mathbb{R}\setminus\{0\}$ and $Q_1, Q_2 \in \mathbb{R}[x,y]$. For $f(x,y) = x$ and $g(x,y) = y$, the equation (4.10) gives

$$0 = \alpha(0) = \frac{\partial P_2}{\partial x}\frac{\partial P_1}{\partial y} = Q'_{2,x}Q'_{1,y}.$$

So we can suppose that $Q_1(y) = b$ is constant. Reporting in the equation (4.10), with $f(x,y) = g(x,y) = y$, we find

$$0 = \alpha(0) = \frac{\partial P_2}{\partial x}\frac{\partial P_2}{\partial y} = Q'_{2,x}\frac{1}{a},$$

so $Q_2(x) = c$ is constant. It remains to verify that for $i > 1$, $\alpha\mu_i = \mu_i\alpha^{\otimes 2}$ *i.e.* for $f, g \in \mathbb{R}[x,y]$, $\alpha\left(\frac{\partial^i f}{\partial x^i}\frac{\partial^i g}{\partial y^i}\frac{1}{i!}\right) = \frac{\partial^i \alpha(f)}{\partial x^i}\frac{\partial^i \alpha(g)}{\partial y^i}\frac{1}{i!}$. By multiplicavity of α, the only non trivial case is $f(x,y) = x^n$ and $g(x,y) = y^m$. We have

$$\begin{aligned} \alpha\left(\frac{\partial^i f}{\partial x^i}\frac{\partial^i g}{\partial y^i}\frac{1}{i!}\right) &= \alpha\left(i!\binom{n}{i}x^{n-i}i!\binom{m}{i}y^{m-i}\frac{1}{i!}\right) \\ &= i!\binom{n}{i}i!\binom{m}{i}(ax+b)^{n-i}\left(\frac{1}{a}y+c\right)^{m-i}\frac{1}{i!} \end{aligned} \tag{4.11}$$

$$= i!\binom{n}{i}i!\binom{m}{i}(ax+b)^{n-i}a^i\left(\frac{1}{a}y+c\right)^{m-i}\left(\frac{1}{a}\right)^i\frac{1}{i!} = \frac{\partial^i(ax+b)^n}{\partial x^i}\frac{\partial^i\left(\frac{1}{a}y+c\right)^m}{\partial y^i}\frac{1}{i!}$$

$$= \frac{\partial^i \alpha(f)}{\partial x^i}\frac{\partial^i \alpha(g)}{\partial y^i}\frac{1}{i!}.$$

□

The Hom-algebra $(\mathbb{R}[x,y],\star_\alpha,\alpha)$ is Hom-associative and not associative if $\alpha \neq Id$. Indeed, for $f(x,y)=1,\ g(x,y)=y$ and $h(x,y)=x$, we have

$$(f\star_\alpha g)\star_\alpha h = \alpha(\alpha(f)\star\alpha(g))\star\alpha(h)$$
$$= \alpha\left(1\star\frac{1}{a}y+c\right)\star(ax+b) = \alpha\left(\frac{1}{a}y+c\right)\star(ax+b),$$

and

$$f\star_\alpha(g\star_\alpha h) = \alpha(\alpha(f))\star\alpha(\alpha(g)\star\alpha(h))$$
$$= 1\star\alpha\left(\left(\frac{1}{a}y+c\right)\star(ax+b)\right) = \alpha\left(\frac{1}{a}y+c\right)\star\alpha(ax+b)$$

which are different in general.

More generally, we can consider the Poisson algebra of polynomials of $n \geqslant 3$ variables $(\mathbb{R}[x_1,\dots,x_n],\cdot,\{\ ,\ \})$ where the Poisson bracket of two polynomials is given by $\{f,g\} = \sum_{1\leqslant i,j\leqslant n}\tau_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$, with $\tau=(\tau_{ij})$ an antisymmetric $n\times n$ real matrix.

The associated associative algebra is $(\mathbb{R}[x_1\dots,x_n],\star)$ where the star-product is given by the Moyal-Weyl formula

$$f\star g = \sum_{n\in\mathbb{N}}\sum_{1\leqslant i_1,j_1,\dots,i_n,j_n\leqslant n}\sigma_{i_1j_1}\cdots\sigma_{i_nj_n}\frac{\partial^n f}{\partial x_{i_1}\cdots\partial x_{i_n}}\frac{\partial^n g}{\partial x_{j_1}\cdots\partial x_{j_n}}\frac{t^n}{n!}, \tag{4.12}$$

where $\sigma=(\sigma_{ij})$ is the matrix whose antisymmetrization is τ.

Set $n>2$ and $\mu_n = \frac{\partial^n f}{\partial x_{i_1}\cdots\partial x_{i_n}}\frac{\partial^n g}{\partial x_{j_1}\cdots\partial x_{j_n}}$.

Proposition 4.10. *The only morphisms $\alpha:\mathbb{R}[x_1,\dots,x_n]\to\mathbb{R}[x_1,\dots,x_n]$ satisfying $\alpha\circ\mu_i=\mu_i\circ\alpha^{\otimes 2}$ for all $i\in\mathbb{N}$ which give $(\mathbb{R}[x_1,\dots,x_n],\ \star_\alpha=\alpha\star,\alpha)$ a structure of Hom-associative algebra are of the form*

$$\forall\, 1\leqslant i\leqslant n,\ \alpha(x_i)=x_i+b_i \quad \text{or} \quad \forall\, 1\leqslant i\leqslant n,\ \alpha(x_i)=-x_i+b_i \qquad \text{where} \quad b_i\in\mathbb{R}. \tag{4.13}$$

Proof. The proof is similar to the case with two variables, we get $\alpha(x_i)=a_ix_i+b_i$, except that this time, $a_ia_j=1$ for all $i\neq j$, which gives the two cases of the proposition. □

4.5 Moyal-Weyl Hom-Poisson algebra

We consider the Poisson algebra of polynomials of two variables $(\mathbb{R}[x,y],\cdot,\{\ ,\ \})$ where the Poisson bracket of two polynomials is given by $\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$.

Proposition 4.11. *A morphism $\alpha : \mathbb{R}[x,y] \to \mathbb{R}[x,y]$ which gives $(\mathbb{R}[x,y], \quad \cdot_\alpha = \alpha \circ \cdot, \quad \{\ ,\ \}_\alpha = \alpha \circ \{\ ,\ \}, \quad \alpha)$ a structure of Hom-Poisson algebra satisfies the equation*

$$1 = \frac{\partial \alpha(x)}{\partial x}\frac{\partial \alpha(y)}{\partial y} - \frac{\partial \alpha(x)}{\partial y}\frac{\partial \alpha(y)}{\partial x}. \tag{4.14}$$

Proof. Since α is a morphism of Poisson algebra, it satisfies, for all $f, g \in \mathbb{R}[x,y]$

$$\alpha(f \cdot g) = \alpha(f) \cdot \alpha(g)$$
$$\alpha(\{f,g\} = \{\alpha(f), \alpha(g)\}.$$

The first equation shows that it is sufficient to define α on x and y. For the second equation, we can suppose by linearity that $f(x,y) = x^k y^l$ and $g(x,y) = x^p y^q$. It then rewrites

$$kq\alpha(x)^{k-1}\alpha(y)^l\alpha(x)^p\alpha(y)^{q-1} - pl\alpha(x)^k\alpha(y)^{l-1}\alpha(x)^{p-1}\alpha(y)^q$$
$$= \frac{\partial(\alpha(x)^k\alpha(y)^l)}{\partial x}\frac{\partial(\alpha(x)^p\alpha(y)^q)}{\partial y} - \frac{\partial(\alpha(x)^k\alpha(y)^l)}{\partial y}\frac{\partial(\alpha(x)^p\alpha(y)^q)}{\partial x}$$

which simplifies in

$$1 = \frac{\partial \alpha(x)}{\partial x}\frac{\partial \alpha(y)}{\partial y} - \frac{\partial \alpha(x)}{\partial y}\frac{\partial \alpha(y)}{\partial x}.$$

□

Example 4.12. We give some examples for the morphism α. We set

$$\alpha(x) = \Gamma_1(x,y) = \sum_{0 \leqslant i,j \leqslant d} a_{ij} x^i y^j$$
$$\alpha(y) = \Gamma_2(x,y) = \sum_{0 \leqslant i,j \leqslant d} b_{ij} x^i y^j$$

with $\Gamma_1, \Gamma_2 \in \mathbb{R}[x,y]$, and d the bigger degree for each variable. Since the equation (4.14) only contains derivatives of Γ_1 and Γ_2, without loss of generality we assume $a_{00} = b_{00} = 0$.

Degree 1

$$\Gamma_1(x,y) = a_{10}x + a_{01}y$$
$$\Gamma_2(x,y) = b_{10}x + b_{01}y$$

The equation (4.14) becomes

$$1 = a_{10}b_{01} - a_{01}b_{10}. \tag{4.15}$$

The polynomials Γ_1, Γ_2 are of one of the following form

(i) $\Gamma_1(x,y) = a_{10}x + a_{01}y, \qquad \Gamma_2(x,y) = -\frac{1}{a_{10}}x \qquad$ with $a_{10} \neq 0$,

(ii) $\Gamma_1(x,y) = \frac{1+a_{01}b_{10}}{b_{01}}x + a_{01}y, \qquad \Gamma_2(x,y) = b_{10}x + b_{01}y \qquad$ with $b_{01} \neq 0$.

Degree 2 For simplicity, we only take one of the polynomials of degree two.

$$\Gamma_1(x,y) = a_{10}x + a_{01}y$$
$$\Gamma_2(x,y) = b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{20}y^2$$

Arranging the equation (4.14) by degree, we obtain the following system of equations.

$$\begin{cases} 1 = a_{10}b_{01} - a_{01}b_{10} \\ 0 = 2a_{10}b_{02} - a_{01}b_{11} \\ 0 = a_{10}b_{11} - 2a_{01}b_{20} \end{cases}$$

The polynomials Γ_1, Γ_2 are of one of the following form

(i) $\Gamma_1(x,y) = \frac{1+a_{01}b_{10}}{b_{01}}x + a_{01}y, \qquad \Gamma_2(x,y) = b_{10}x + b_{01}y$

(ii) $\Gamma_1(x,y) = \frac{1}{b_{01}}x, \qquad \Gamma_2(x,y) = b_{20}x^2 + b_{10}x + b_{01}y$

(iii) $\Gamma_1(x,y) = a_{10}x + \frac{2a_{10}b_{02}}{b_{11}}y, \qquad \Gamma_2(x,y) = \frac{1}{a_{01}}y + \frac{b_{11}^2}{4b_{02}}x^2 + b_{11}xy + b_{02}y^2$

(iv) $\Gamma_1(x,y) = a_{10}x + \frac{2a_{10}b_{02}}{b_{11}}y, \qquad \Gamma_2(x,y) = b_{10}x + \frac{2a_{10}b_{02}b_{10}+b_{11}}{a_{10}b_{11}}y + \frac{b_{11}^2}{4b_{02}}x^2 + b_{11}xy + b_{02}y^2$

We now want to deform the classical Moyal-Weyl star-product on $\mathbb{R}[x,y]$ using morphisms α previously found. For $P, Q \in \mathbb{R}[x,y]$ we define

$$P \star_\alpha Q = \sum_{n\geqslant 0} \mu_{n\alpha}(P,Q)t^n \quad \text{with} \quad \mu_{n\alpha}(P,Q) = \frac{\partial\alpha(P)}{\partial x_n}\frac{\partial\alpha(Q)}{\partial y_n}\frac{1}{n!}. \tag{4.16}$$

The Hom-associator writes

(4.17)
$$\mathfrak{as}_{\star_\alpha,\alpha}(P,Q,R) = \sum_{p\geqslant 1}\sum_{n=0}^{p} \big(\mu_{p-n_\alpha}(\mu_{n\alpha}(P,Q),\alpha(R) - \mu_{p-n_\alpha}(\alpha(P),\mu_{n\alpha}(Q,R))\big)t^p$$
$$= \sum_{p\geqslant 1}\sum_{n=0}^{p} \frac{1}{n!(p-n)!}\left(\frac{\partial\alpha\left(\frac{\partial\alpha(P)}{\partial x^n}\frac{\partial\alpha(Q)}{\partial y^n}\right)}{\partial x^{p-n}}\frac{\partial\alpha^2(R)}{\partial y^{p-n}} - \frac{\partial\alpha^2(P)}{\partial x^{p-n}}\frac{\partial\alpha\left(\frac{\partial\alpha(Q)}{\partial x^n}\frac{\partial\alpha(R)}{\partial y^n}\right)}{\partial y^{p-n}}\right),$$

since the multiplication $\cdot_\alpha$ is Hom-associative, the constant term vanishes.

Trying to make the coefficient in t^n vanish for particular polynomials P, Q, R, we obtain conditions on the morphism α.

4.5.1 Degree 1 case

In degree 1, the solutions of (4.15) are of the form (i) or (ii).

In the first case (i), the morphism α satisfies

$$\alpha(x) = \Gamma_1(x,y) = a_{10}x + a_{01}y,$$
$$\alpha(y) = \Gamma_2(x,y) = -\frac{1}{a_{01}}x,$$

with $a_{10} \neq 0$.
For the particular polynomials $P(x,y) = x, Q(x,y) = y = R(x,y)$, we have

$$\mathfrak{as}_{\star_\alpha,\alpha}(P,Q,R) = y \neq 0$$

so $\mathfrak{as}_{\star_\alpha,\alpha} \neq 0$ and this morphism α does not give a deformation of the Moyal-Weyl star-product.

In the second case (ii) the morphism α satisfies

$$\alpha(x) = \Gamma_1(x,y) = \frac{1 + a_{01}b_{10}}{b_{01}}x + a_{01}y,$$
$$\alpha(y) = \Gamma_2(x,y) = b_{10}x + b_{01}y,$$

with $b_{01} \neq 0$.
Renaming $c := a_{01}b_{10}$ if $b_{10} \neq 0$, we can write

$$\alpha(x) = \Gamma_1(x,y) = \frac{1+c}{b_{01}}x + \frac{c}{b_{10}}y,$$
$$\alpha(y) = \Gamma_2(x,y) = b_{10}x + b_{01}y.$$

For the particular polynomials $P(x,y) = x, Q(x,y) = y = R(x,y)$, the coefficient in t of $\mathfrak{as}_{\star_\alpha,\alpha}(P,Q,R)$ vanishes only if $c = 0$ and $b_{01} = 0$ or if $c = 0$ and $b_{10} = 0$, which is not possible.
If $b_{10} = 0$ then the morphism α satisfies

$$\alpha(x) = \Gamma_1(x,y) = \frac{1}{b_{01}}x + a_{01}y,$$
$$\alpha(y) = \Gamma_2(x,y) = b_{01}y,$$

and the coefficient in t of $\mathfrak{as}_{\star_\alpha,\alpha}(P,Q,R)$ vanishes only if $a_{01} = 0$. In that case, the morphism α is as in the Proposition 4.9, and thus gives a deformation of the Moyal-Weyl star-product.

Finally, the only morphisms α of degree 1 which give raise to a deformation of the Moyal-Weyl star-product are as in the Proposition 4.9.

4.5.2 Degree 2 case

In degree 2, computations done with the computer algebra system *Mathematica* also led to the case given by the Proposition 4.9.

We conjecture that the only morphisms α of the Proposition 4.11 are the one found in the Proposition 4.9.

An interesting question would be to know if twisting by a morphism α is functorial. If this is the case, the previous conjecture would be true and we would have the following commutative diagram

$$\begin{array}{ccc} (A,\mu,\{\ ,\ \}) & \xrightarrow{\text{twist}_\alpha} & (A,\mu_\alpha,\{\ ,\ \}_\alpha,\alpha) \\ \text{Kont}\updownarrow & & \updownarrow\text{Kont} \\ (A,\star) & \xrightarrow[\text{twist}_\alpha]{} & (A,\star_\alpha,\alpha) \end{array} \tag{4.18}$$

with Kont the Kontsevich bijection between the set of equivalence classes of Poisson brackets on the commutative $\mathbb{K}[[t]]$-algebra $A[[t]]$ and the set of equivalence classes of star products, and $\text{twist}_\alpha : \mathsf{Poiss} \to \mathsf{Hom\text{-}Poiss}$ the functor from the category Poiss of Poisson algebras to the category Hom-Poiss of Hom-Poisson algebras. More generally, twist_α is a functor from Struct to Hom-Struct, where Struct is a category of structures such as associative algebras Ass, Lie algebras Lie, Poisson algebra Poiss, and so on, and Hom-Struct the corresponding category of Hom-structures.

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Strong and Covariant Morita equivalences in Deformation Quantization

by Henrique Bursztyn and Stefan Waldmann

Abstract

This note presents an overview of various aspects of the representation theory of star products, including different notions of module and Morita equivalence, as well as classification results. Along the way, we highlight many connections with the work of Nikolai Neumaier.

1 Introduction

A central theme in Nikolai Neumaier's work was formal deformation quantization [2] (see e.g. [41] for an introduction), a subject to which he gave many important contributions; within deformation quantization, the study of representations of star-product algebras was among his main topics of interest. This note presents an overview of various aspects of the representation theory of star products, including different notions of Morita equivalence as well as classification results, some of which had the direct influence of Nikolai's work.

Morita equivalence, in its original and most basic form, is an equivalence relation among unital rings which identifies those with equivalent "representation theories" (i.e., categories of left modules). The notion of Morita equivalence can be transferred to many other situations: basically, it can be formulated whenever one specifies a reasonable notion of representation of (or module over) a mathematical object. This note presents some instances of this idea when the mathematical object in question is a star-product algebra; as we will see, depending on the properties of star products that one wants to take into account, different notions of representation and Morita equivalence arise.

In order to find appropriate frameworks for star-product representations, it is convenient to recall the physical motivation of star products as models for observable algebras of quantum systems. Star products are formal associative deformations, in the sense of Gerstenhaber [18], of the commutative algebra of smooth, complex-valued functions $C^\infty(M)$ on a Poisson manifold (M, π), thought of as the classical phase space. A star product $\star$ makes the space of formal power series $C^\infty(M)[[\hbar]]$ (here $\hbar$ is viewed as a formal parameter) into a unital, associative algebra over the ring $\mathbb{C}[[\hbar]]$; a key requirement is that star products deform the pointwise product of functions "in the direction" of the given Poisson structure π, meaning that the $\star$-commutator on $C^\infty(M)[[\hbar]]$ agrees (up to a constant), in first order, with the Poisson bracket on M.

The role of star-product algebras as observable algebras indicates that one should consider not only their ring structures, but also additional properties. In fact, a desirable scenario would be to use formal deformation quantization to eventually obtain C^*-algebras represented on Hilbert spaces. But this aim is hard to achieve: there are many technical difficulties in handling convergence issues for formal power series and finding C^*-norms for suitable classes of functions, although this can be done in specific examples. An alternative approach is to proceed within the framework of formal power series, observing that some properties of C^*-algebras and their representations carry over to the purely algebraic formal setting. Indeed, there are two important "C^*-like" features that one may consider for star products: first, by considering Hermitian star products, i.e. star products compatible with complex conjugation of functions (we assume the parameter $\hbar$ to be real),

$$\overline{f \star g} = \overline{g} \star \overline{f}, \qquad f, g \in C^\infty(M)[[\hbar]],$$

one endows star-product algebras with *-*involutions*; second, one may take into account notions of *positivity* (e.g. for algebra elements and linear functionals) resulting from the natural order structure on the ring $\mathbb{R}[[\hbar]]$ (a formal series $\sum_{r=0}^\infty \hbar^r a_r$ is declared to be *positive* if its first nonzero term is positive). These additional features of star products lead to notions of representations parallel to those for C^*-algebras [38, 39], and to an algebraic version of the concept of *strong Morita equivalence* [13]. On top of that, one may consider star products carrying symmetries, given by actions of a Hopf algebra H, and representations which are compatible with these symmetries. This leads to the notion of *H-covariant Morita equivalence*, studied by Nikolai in one of his last publications [21].

This note is organized as follows. Section 2 is divided in two parts: first, we review the usual classification of star products and their characteristic classes (highlighting Nikolai's contributions in this context) and, afterwards, we discuss the classification of star products with respect to ring-theoretic Morita equivalence. In Section 3 we consider algebras with additional properties and present various ways in which one can enhance the notions of (bi-)module and representation, by taking into account positivity and the presence of symmetries; these new (bi-)modules lead to refined notions of Morita equivalence, such as strong and covariant Morita equivalences, treated in Section 4. Here we emphasize the bicategorical approach to Morita equivalence: we describe different versions of Morita equivalence as isomorphisms in appropriate bicategories of bimodules with extra structure, which are composed via suitable tensor products. In the last Section 5, we revisit the Morita classification of star products for strong and covariant Morita equivalences, recalling Nikolai's work on the latter.

2 Ring-theoretic classifications of star products

2.1 Equivalences of star products and characteristic classes

We start by recalling the classical notion of equivalence for star products. We say that two star products $\star$ and $\star'$ on a Poisson manifold M are *equivalent* if there is a formal series $T = \mathsf{id} + \sum_{r=1}^{\infty} \hbar^r T_r$ of differential operators $T_r : C^\infty(M) \longrightarrow C^\infty(M)$ such that

$$f \star' g = T^{-1}(Tf \star Tg) \quad \text{and} \quad T1 = 1, \tag{2.1}$$

for all $f, g \in C^\infty(M)[[\hbar]]$. We refer to T as an *equivalence transformation.* In particular, $\star$ and $\star'$ define isomorphic $\mathbb{C}[[\hbar]]$-algebra structures on $C^\infty(M)[[\hbar]]$.

Analogously, we call $\star$ and $\star'$ *diffeomorphic* if there is a Poisson diffeomorphism $\Phi\colon M \longrightarrow M$ with

$$f \star' g = \Phi_*(\Phi^* f \star \Phi^* g), \tag{2.2}$$

for all $f, g \in C^\infty(M)[[\hbar]]$. Note that the fact that Φ preserves the Poisson structure is necessary if $\star$ and $\star'$ quantize the same Poisson bracket in first order. One may now verify that two star-product algebras $(C^\infty(M)[[\hbar]], \star)$ and $(C^\infty(M)[[\hbar]], \star')$ are isomorphic as algebras over $\mathbb{C}[[\hbar]]$ if and only if there is a Poisson diffeomorphism Φ and an equivalence transformation T such that, for all functions $f, g \in C^\infty(M)[[\hbar]]$, one has

$$f \star' g = T^{-1}\Phi_*(\Phi^* Tf \star \Phi^* Tg). \tag{2.3}$$

The set of all star products on M is denoted by $\underline{\mathrm{Def}}(M)$, while $\underline{\mathrm{Def}}(M, \pi_1)$ denotes the set of star products for a fixed first-order Poisson bracket $\pi_1 \in \Gamma^\infty(\Lambda^2 TM)$. The equivalence transformations form a group under composition which acts on $\underline{\mathrm{Def}}(M)$ and leaves $\underline{\mathrm{Def}}(M, \pi_1)$ invariant. Hence we can form the orbit spaces for this group action, which we denote by $\mathrm{Def}(M)$ and $\mathrm{Def}(M, \pi_1)$, respectively. In other words, $\mathrm{Def}(M, \pi_1)$ is the set of classes of star products (up to equivalence) quantizing π_1.

For the classification of star products up to equivalence we rely on Kontsevich's formality theorem [25] and on the globalization of the formality map in [16]. In order to formulate the classification, recall that a *formal Poisson tensor* is a formal series $\pi = \hbar\pi_1 + \hbar^2\pi_2 + \cdots \in \hbar\Gamma^\infty(\Lambda^2 TM)[[\hbar]]$ with $[\![\pi, \pi]\!] = 0$, where we extend the Schouten bracket $[\![\cdot, \cdot]\!]$ $\hbar$-linearly. We denote the set of formal Poisson tensors on M by $\underline{\mathrm{FPoisson}}(M)$, and the subset of formal Poisson tensors with fixed first-order term π_1 by $\underline{\mathrm{FPoisson}}(M, \pi_1)$.

A formal vector field is a formal series $X = \hbar X_1 + \hbar^2 X_2 + \cdots \hbar\Gamma^\infty(TM)[[\hbar]]$. Since by definition a formal vector field starts in order $\hbar$, we can exponentiate its Lie derivative to get a well-defined operator

$$\exp(\mathscr{L}_X) : \Gamma^\infty(\Lambda^\bullet TM)[[\hbar]] \longrightarrow \Gamma^\infty(\Lambda^\bullet TM)[[\hbar]], \tag{2.4}$$

preserving tensor degrees. Analogously, we can act on formal series of other kinds of tensor fields on M. By the Baker-Campbell-Hausdorff series one sees that the composition of $\exp(\mathscr{L}_X)$ and $\exp(\mathscr{L}_Y)$, for two formal vector fields X and Y, is again of the form $\exp(\mathscr{L}_Z)$ for a formal vector field $Z = \mathrm{BCH}(X, Y)$. Noticing that $\exp(\mathscr{L}_{-X})$ is the inverse of $\exp(\mathscr{L}_X)$, we see that the operators (2.4) form a group, called the *formal diffeomorphism group* of M and denoted by $\mathrm{FDiffeo}(M)$. If π is a formal Poisson tensor, then $\pi' = \exp(\mathscr{L}_X)(\pi)$ is still a formal Poisson tensor with the same first order term: $\pi'_1 = \pi_1$. Thus we get an action of $\mathrm{FDiffeo}(M)$ on the set of formal Poisson tensors which leaves $\underline{\mathrm{FPoisson}}(M, \pi_1)$ invariant. The orbit spaces of this group action are the equivalence classes of formal Poisson tensors up to formal diffeomorphisms, denoted by $\mathrm{FPoisson}(M)$ and $\mathrm{FPoisson}(M, \pi_1)$.

Kontsevich's formality theorem gives (among many other things) a construction of a star product $\star_\pi$ out of a given formal Poisson tensor π, once a global formality on M is chosen. The map $\pi \mapsto \star_\pi$ is such that, first, $\star_\pi$ quantizes π_1 as desired and, second, it induces a *bijection*

$$\mathrm{FPoisson}(M, \pi_1) \ni [\pi] \;\mapsto\; [\star_\pi] \in \mathrm{Def}(M, \pi_1) \tag{2.5}$$

between the formal Poisson tensors deforming π_1, up to formal diffeomorphisms, and the formal star products quantizing π_1, up to equivalence. In other words, classes of star products in $\mathrm{Def}(M, \pi_1)$ are classified by elements in $\mathrm{FPoisson}(M, \pi_1)$. Also, using e.g. the globalized formality from [16], one can show that, for a Poisson diffeomorphism Φ, the star product $\Phi^*(\star_\pi)$ obtained from $\star_\pi$ as in (2.2) is equivalent to $\star_{\Phi^*\pi}$, though generally not equal; so (2.5) has a natural equivariance property relative to Poisson diffeomorphisms.

In the symplectic setting the above classification (2.5) can be made more concrete. In fact, the classification of star products on symplectic manifolds (M, ω) is prior to Kontsevich's work and can be phrased as follows: via the Fedosov construction [17] of symplectic star products one can associate to every formal series of closed two-forms $\Omega = \hbar\Omega_1 + \hbar^2\Omega_2 + \cdots \in \hbar\Gamma^\infty(\Lambda^2 T^*M)$ a star product $\star_\Omega$ such that any two $\star_\Omega$ and $\star_{\Omega'}$ are equivalent if and only if Ω and Ω' are cohomologous. Moreover, an inductive construction shows that for every star product $\star$ on (M, ω) there is an Ω such that $\star$ is equivalent to the Fedosov star product $\star_\Omega$. This leads to the classification of symplectic star products by their *Fedosov classes*,

$$\mathrm{Def}(M, \omega) \ni [\star] \;\mapsto\; F(\star) = [\Omega] \in \hbar\mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})[[\hbar]], \tag{2.6}$$

where Ω is a formal series of closed two-forms such that $[\star] = [\star_\Omega]$. This point of view was developed by various authors, see [6, 30, 42].

Alternatively, one has an intrinsic classification not relying on the Fedosov construction but rather on a Cech cohomological argument: there is an intrinsic *characteristic class*

$$c(\star) \in \frac{[\omega]}{\mathrm{i}\hbar} + \check{\mathrm{H}}^2(M, \mathbb{C})[[\hbar]] \tag{2.7}$$

such that $\star$ and $\star'$ are equivalent if and only if $c(\star) = c(\star')$, and any formal series in the affine space $\frac{[\omega]}{\mathrm{i}\hbar} + \check{\mathrm{H}}^2(M, \mathbb{C})[[\hbar]]$ arises as a characteristic class. Here the choice of $\frac{[\omega]}{\mathrm{i}\hbar}$ as the origin for the affine space is conventional. Remarkably, the construction of $c(\star)$ does not rely on any particular construction of star products but only on elementary facts about the Weyl star product on $\mathbb{R}^{2n}$ and a Cech cohomological patching on Darboux charts of (M, ω), see [15, 20] for this approach.

It is now a theorem of Nikolai that the two classes coincide after a trivial rescaling [33]: with the identification $\check{\mathrm{H}}^2(M, \mathbb{C}) = \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})$, one gets

$$c(\star) = \frac{[\omega] + F(\star)}{\mathrm{i}\hbar}. \tag{2.8}$$

Since symplectic manifolds are particular cases of Poisson manifolds, the classification of star products via Kontsevich's formality (2.5) should also match the classification via (2.7). This was verified in [14], where it was shown that Kontsevich's class $[\pi]$ of $\star$ is just the "inverse" of $c(\star)$. This makes sense as any representative of the formal series $c(\star)$ agrees, in lowest order, with the symplectic two-form ω; the fact that ω can be inverted to a Poisson tensor $\pi_1 = \omega^{-1}$ guarantees that the formal series can be inverted to a formal Poisson tensor.

2.2 Ring-theoretic Morita classification

We now consider a different classification problem in formal deformation quantization: viewing star products as unital $\mathbb{C}[[\hbar]]$-algebras, we discuss their classification up to (ring-theoretic) Morita equivalence. In subsequent sections we will present different ways in which Morita equivalence can be enhanced, and then revisit the classification of star products accordingly.

Let us briefly recall the notion of Morita equivalence [29] in its original form (see e.g. [26] for a textbook). Two unital algebras (over a fixed commutative, unital ground ring) $\mathcal{A}$ and $\mathcal{B}$ are *Morita equivalent* if there exists a $(\mathcal{B}, \mathcal{A})$-bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ which is "invertible" in the following sense: there is an $(\mathcal{A}, \mathcal{B})$-bimodule ${}_{\mathcal{A}}\mathcal{F}_{\mathcal{B}}$ for which there are bimodule isomorphisms

$${}_{\mathcal{A}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \cong {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}, \qquad {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{F}_{\mathcal{B}} \cong {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}\,.$$

Such a bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is referred to as an *equivalence bimodule*. As we will revisit Morita equivalence later in the paper, in more detail and from a broader perspective, we now only mention a few of its basic properties. First, as an equivalence relation among unital algebras, Morita equivalence is a nontrivial extension of the usual notion of algebra isomorphism: indeed, an isomorphism $\Phi\colon \mathcal{B} \longrightarrow \mathcal{A}$ gives rise to an equivalence bimodule which is simply $\mathcal{A}$ as a right $\mathcal{A}$-module, and where $\mathcal{B}$ acts on the left via Φ. Also, denoting by $\mathsf{Mod}(\mathcal{A})$ the category of left $\mathcal{A}$-modules, any equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ induces an equivalence of categories $\mathsf{Mod}(\mathcal{A}) \longrightarrow \mathsf{Mod}(\mathcal{B})$ via the tensor product $\otimes_{\mathcal{A}}$, and this is the sense in which $\mathcal{A}$ and $\mathcal{B}$ have equivalent "representation theories". We finally remark that Morita's theorem provides a complete characterization of equivalence bimodules; in particular, it shows that they are finitely generated and projective over each algebra.

Example 2.1. We briefly discuss the equivalence bimodules ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ for which $\mathcal{A} = C^\infty(M)$ is the commutative algebra of complex-valued, smooth functions on a manifold M. It follows from the smooth version of Serre-Swan's theorem that, since any such equivalence bimodule is finitely generated and projective as a right $\mathcal{A}$-module, it must be given by the sections of a vector bundle $E \longrightarrow M$, on which $C^\infty(M)$ acts by pointwise multiplication; the algebra acting on the left is then necessarily isomorphic to $\Gamma^\infty(\mathsf{End}(E))$. In fact, any nonzero vector bundle defines an equivalence bimodule in this way. An auto-equivalence bimodule of $C^\infty(M)$ must be given by a line bundle $L \longrightarrow M$, since this is the only case where $C^\infty(M) \cong \Gamma^\infty(\mathsf{End}(L))$.

Going back to star products, the classification problem amounts to determining the conditions on the characteristic classes, as in (2.5) and (2.7), such that the corresponding star-product algebras are Morita equivalent. The easier part of the classification accounts for isomorphic star products: according to (2.3), if we mod out the equivalence transformations, we are still left with an action of Poisson diffeomorphisms on characteristic classes of star products whose orbits identify isomorphic ones. The more interesting part of the Morita classification comes from nontrivial equivalence bimodules. One may check that an equivalence bimodule for $\star$ and $\star'$ has a classical limit which remains an equivalence bimodule for the undeformed products. As seen in Example 2.1, such bimodules must be given by sections of line bundles. Hence the problem of Morita classification reduces to the question of which line bundles $L \longrightarrow M$ can be deformed into equivalence bimodules for star products. It turns out that one can always deform the sections $\Gamma^\infty(L)[[\hbar]]$ into a right $\star$-module in a unique way, up to equivalence [11]. This relies on the fact that the classical module is projective. Moreover, the endomorphisms $C^\infty(M) \cong \Gamma^\infty(\mathsf{End}(L))$ inherit a deformation $\star'$ from this procedure, in such a way that we get a deformed bimodule. The new star product $\star'$ quantizes the same Poisson bracket π_1 on M, see [11, 10]. The question is then how to compute the class of $\star'$ in terms of the class of $\star$ and the line bundle L.

We first mention the Morita classification for symplectic star products [12]:

Theorem 2.2 (Morita classification, symplectic case)**.** *Two star products $\star$ and $\star'$ on a symplectic manifold (M, ω) are Morita equivalent if and only if there is a symplectomorphism Φ such that*

$$\Phi^* c(\star') - c(\star) \in 2\pi \mathrm{i} \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{Z}). \tag{2.9}$$

In this case, the line bundle L with Chern class $c_1(L) = \frac{1}{2\pi\mathrm{i}}(\Phi^* c(\star') - c(\star))$ can be deformed into an equivalence bimodule for $\star$ and $\star'$.

More specifically, one obtains an equivalence bimodule for star products through a deformed bimodule structure on $\Gamma^\infty(L)[[\hbar]]$, where $\star$ acts on the right and $\star'$ acts (via Φ) on the left. Note that here only the image of the Chern class of L in *de Rham cohomology* matters; in particular, since torsion elements in $\check{\mathrm{H}}^2(M,\mathbb{Z})$ vanish in $\mathrm{H}^2_{\mathrm{dR}}(M,\mathbb{Z})$, they only account for isomorphic star products.

The previous theorem already hints at how the classification for star products on general Poisson manifolds should be: one should "invert" the relation $\Phi^* c(\star') = c(\star) + 2\pi \mathrm{i} c_1(L)$ via a geometric series to get the corresponding relation for the Kontsevich classes. This heuristic reasoning first appeared in [23], where Morita equivalence was studied in the context of non-commutative gauge field theories.

To make this heuristics precise we have to elaborate on how two-forms act on Poisson structures. Given a formal Poisson structure

$$\pi = \hbar\pi_1 + \cdots \in \hbar\Gamma^\infty(\Lambda^2 TM)[[\hbar]],$$

we can equivalently view it, as usual, as a $\mathbb{C}[[\hbar]]$-linear bundle map

$$\pi^\sharp \colon \Gamma^\infty(T^*M)[[\hbar]] \longrightarrow \hbar\Gamma^\infty(TM)[[\hbar]] \tag{2.10}$$

via $\pi^\sharp(\alpha) = \pi(\alpha, \cdot)$, where $\alpha \in \Gamma^\infty(T^*M)[[\hbar]]$. Analogously, given a two-form $B \in \Gamma^\infty(\Lambda^2 T^*M)[[\hbar]]$ we have a bundle map in the opposite direction

$$B^\sharp \colon \Gamma^\infty(TM)[[\hbar]] \longrightarrow \Gamma^\infty(T^*M)[[\hbar]], \tag{2.11}$$

via $B^\sharp(X) = B(X, \cdot)$, for $X \in \Gamma^\infty(TM)[[\hbar]]$. Since we require π to start in first order of $\hbar$, the composition $B^\sharp\pi^\sharp$ is a $\mathbb{C}[[\hbar]]$-linear endomorphism of $\Gamma^\infty(T^*M)[[\hbar]]$ raising the $\hbar$-degree at least by one. Hence $\mathsf{id} + B^\sharp\pi^\sharp$ is necessarily invertible via a geometric series, so we may consider the inverse

$$\left(\mathsf{id} + B^\sharp\pi^\sharp\right)^{-1} : \Gamma^\infty(T^*M)[[\hbar]] \longrightarrow \Gamma^\infty(T^*M)[[\hbar]]. \tag{2.12}$$

We have the following results:

Proposition 2.3. *Let $B \in \Gamma^\infty(\Lambda^2 T^*M)[[\hbar]]$ and $\pi \in \hbar\Gamma^\infty(\Lambda^2 TM)[[\hbar]]$.*

1. *There exists a unique $\mathfrak{a}(B,\pi) \in \hbar\Gamma^\infty(\Lambda^2 TM)[[\hbar]]$ with $\mathfrak{a}(B,\pi)^\sharp = \pi^\sharp \circ (\mathsf{id} + B^\sharp\pi^\sharp)^{-1}$.*

2. *If π is a formal Poisson structure and B is closed then $\mathfrak{a}(B,\pi)$ is also a formal Poisson structure.*

3. *$\mathfrak{a}$ defines an action of the abelian group of formal series of closed two-forms on the set of formal Poisson structures.*

In analogy to the case without $\hbar$-powers, we call the map $\pi \mapsto \mathfrak{a}(B, \pi)$ a *gauge transformation* of π by the two-form B, see [40]; note that, in the purely geometric situation (with no powers in $\hbar$), the invertibility of $\mathsf{id} + B^\sharp \pi^\sharp$ is not automatic, depending on the choices of B and π.

A key feature of the action $\mathfrak{a}$ is that exact two-forms $B = \mathrm{d} A$, with $A \in \Gamma^\infty(T^*M)[[\hbar]]$, yield equivalent formal Poisson structures. Thus we obtain a well-defined action of the second de Rham cohomology $\mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})[[\hbar]]$ on the equivalence classes of formal Poisson structures which preserves the lowest order term π_1. We denote this action by

$$\mathfrak{a} \colon \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})[[\hbar]] \times \mathrm{FPoisson}(M, \pi_1) \longrightarrow \mathrm{FPoisson}(M, \pi_1). \tag{2.13}$$

This is the action which determines the Morita classification of star product [14]:

Theorem 2.4 (Morita classification, Poisson case)**.** *Let $\star$ and $\star'$ be two star products on a Poisson manifold (M, π_1) with classes $[\pi]$ and $[\pi']$, respectively. Then $\star$ and $\star'$ are Morita equivalent if and only if there is a Poisson diffeomorphism Φ and an integral two-form B, $[B] \in 2\pi \mathrm{i} \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{Z})$, such that*

$$\Phi^*[\pi'] = [\mathfrak{a}(B, \pi)]. \tag{2.14}$$

As in Theorem 5.3, the corresponding line bundle with Chern class $c_1(L) = \frac{1}{2\pi \mathrm{i}}[B]$ can be deformed into an equivalence bimodule for $\star'$ and $\star$.

The construction of equivalence bimodules for star products can be refined in more specific geometric situations. We will mention two examples related to Nikolai's work, namely the cases of Kähler manifolds and cotangent bundles:

- For a Kähler manifold M, Fedosov's construction gives (at least) three canonical star products on M: the Weyl-ordered star product $\star_{\mathrm{Weyl}}$, the Wick star product $\star_{\mathrm{Wick}}$, and the anti-Wick star product $\star_{\overline{\mathrm{Wick}}}$. It was known that these three star product are not equivalent in general, and their characteristic classes are given by
(2.15)
$$c(\star_{\mathrm{Weyl}}) = \frac{[\omega]}{\mathrm{i}\hbar}, \quad c(\star_{\mathrm{Wick}}) = \frac{[\omega]}{\mathrm{i}\hbar} - \mathrm{i}\pi c_1(L_{\mathrm{can}}), \text{ and } c(\star_{\overline{\mathrm{Wick}}}) = \frac{[\omega]}{\mathrm{i}\hbar} + \mathrm{i}\pi c_1(L_{\mathrm{can}}),$$
where L_{can} denotes the canonical line bundle of M, i.e. the line bundle of holomorphic volume forms, see [24] as well as Nikolai's PhD thesis [32]. Thus we see from Theorem 2.2 that $\star_{\mathrm{Wick}}$ and $\star_{\overline{\mathrm{Wick}}}$ are always Morita equivalent, and they are Morita equivalent to $\star_{\mathrm{Weyl}}$ if and only if the canonical line bundle has a square root [34]. The construction of the deformed bimodule structure of L_{can} can be obtained from a rather explicit Fedosov construction. Also in [34] it was shown that for a holomorphic line bundle one can achieve deformed bimodule structures with the *separation of variables property*.

- For a cotangent bundle $M = T^*Q$, a line bundle $L \longrightarrow T^*Q$ is isomorphic to the pull-back of a line bundle on Q. Hence the curvature two-form of L corresponds to a closed two-form B on Q which has the physical interpretation of a *magnetic field*. If B is not exact, and thus L is not the trivial line bundle, then B corresponds to a magnetic monopole. The integrality condition in Theorem 2.2 can then be understood as Dirac's quantization condition for a magnetic monopole, giving a new interpretation of this condition in terms of Morita theory [12]. This result relates to previous work of Nikolai on the representation theory of star products, see [8, 9, 7], as well as his Diploma thesis [31]. We will come back to these results in Section 5.1.

3 Modules with additional structures

3.1 Inner products

We now consider additional properties of star-product algebras, beyond their ring structure, and discuss how they lead to enhanced notions of modules and representations. As mentioned in the introduction, we may restrict ourselves to Hermitian star products, which renders star product algebras with the structure of *-algebras, with involution given by complex conjugation of functions. We will also consider the order structure on the ring $\mathbb{R}[[\hbar]]$, which leads to various notions of positivity for star-product algebras. It will be convenient to work, more generally, in the following algebraic set-up: we will consider *-algebras over a ring $\mathsf{C} = \mathsf{R}(\mathrm{i})$, with $\mathrm{i}^2 = -1$ and R being an ordered ring. This framework encompasses Hermitian star product algebras (with $\mathsf{C} = \mathbb{C}[[\hbar]]$) and also C^*-algebras (with $\mathsf{C} = \mathbb{C}$).

Guided by the notions of Hilbert modules and strong Morita equivalence for C^*-algebras, see e.g. [38, 39, 27, 37], one considers the following. Let $\mathcal{A}$ be a *-algebra over C, and let $\mathcal{E}_{\mathcal{A}}$ be a right $\mathcal{A}$-module. We henceforth assume that all modules carry a compatible C-module structure such that all other structure maps are (multi-)linear over C. Even though this is not strictly necessary, we assume for simplicity that all algebras are unital and all modules are unital as well, i.e. the algebra unit acts as the identity on the module.

An *$\mathcal{A}$-valued inner product* is a map

$$\langle \cdot , \cdot \rangle_{\mathcal{A}} : \mathcal{E}_{\mathcal{A}} \times \mathcal{E}_{\mathcal{A}} \longrightarrow \mathcal{A}, \tag{3.1}$$

which is C-linear in the *second* argument, and such that $\langle x, y \cdot a\rangle_{\mathcal{A}} = \langle x, y\rangle_{\mathcal{A}}\, a$, for all $x, y \in \mathcal{E}_{\mathcal{A}}$ and $a \in \mathcal{A}$, and $\langle x, y\rangle_{\mathcal{A}} = (\langle y, x\rangle_{\mathcal{A}})^*$. We call $\langle \cdot , \cdot \rangle_{\mathcal{A}}$ *non-degenerate* if $\langle x, y\rangle_{\mathcal{A}} = 0$ for all $y \in \mathcal{E}_{\mathcal{A}}$ implies $x = 0$. Note that these inner products already make use of the *-involution.

In order to take into account the ordering of R, we proceed as follows. First, we call a linear functional $\omega\colon \mathcal{A} \longrightarrow \mathsf{C}$ *positive* if $\omega(a^*a) \geq 0$ for all $a \in \mathcal{A}$. In this case, ω satisfies a Cauchy-Schwarz inequality and behaves much like the positive functionals in operator algebra theory. We use these positive functionals to define positivity of algebra elements: $a \in \mathcal{A}$ is *positive* if $\omega(a) \geq 0$ for all positive ω. In quantum physical terms this means that all the expectation values of the observable a are positive. Since this is all the information we can possibly get about the observable a in an operational way, this notion of "positivity by measurement" is well motivated by the desired applications in quantum physics. Standard arguments show that positive functionals form a convex cone in the dual of $\mathcal{A}$ which is stable under the operation $\omega \mapsto \omega_b$, with $\omega_b(a) = \omega(b^* a b)$ for every $b \in \mathcal{A}$. Moreover, the set of positive elements in $\mathcal{A}$, which we denote by $\mathcal{A}^+$, form a convex cone as well, stable under the maps $a \mapsto b^* a b$. Clearly, it contains the cone of "sums of squares" $\mathcal{A}^{++}$, i.e. those a which can be written as $a = \sum_{i=1}^n \alpha_i b_i^* b_i$ with $0 < \alpha_i \in \mathsf{R}$ and $b_i \in \mathcal{A}$. In general it is a nontrivial question to decide whether $\mathcal{A}^+ = \mathcal{A}^{++}$; for polynomials this is the famous Hilbert's 17th problem. For C^*-algebras one always has equality, a fact heavily relying on continuous spectral calculus.

We can now define the positivity requirements for an algebra-valued inner product. We call an $\mathcal{A}$-valued inner product $\langle \cdot\,, \cdot \rangle_{\mathcal{A}}$ *positive* if $\langle x, x \rangle_{\mathcal{A}} \in \mathcal{A}^+$, for all $x \in \mathcal{E}_{\mathcal{A}}$. To get better properties with respect to tensor products, it will be convenient to refine this notion and call $\langle \cdot\,, \cdot \rangle_{\mathcal{A}}$ *completely positive* if, for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in \mathcal{E}_{\mathcal{A}}$, the matrix $(\langle x_i, x_j \rangle_{\mathcal{A}}) \in M_n(\mathcal{A})$ is positive. Here we use that $M_n(\mathcal{A})$ is naturally a *-algebra, so the notion of positivity makes sense. Such a right $\mathcal{A}$-module $\mathcal{E}_{\mathcal{A}}$ with completely positive and non-degenerate inner product $\langle \cdot\,, \cdot \rangle_{\mathcal{A}}$ will be called a (right) *pre-Hilbert $\mathcal{A}$-module*. If we only have a non-degenerate inner product, we call $\mathcal{E}_{\mathcal{A}}$ a (right) *inner-product $\mathcal{A}$-module*. It is clear that we can define an inner product on a left $\mathcal{A}$-module in an analogous way, replacing the $\mathcal{A}$-linearity in the second argument to the right by $\mathcal{A}$-linearity in the first argument to the left.

Let $\mathcal{B}$ be another *-algebra acting on $\mathcal{E}_{\mathcal{A}}$ from the left, such that we have a $(\mathcal{B}, \mathcal{A})$-bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. We always assume that the left $\mathcal{B}$-module structure is compatible with $\mathcal{E}_{\mathcal{A}}$, i.e., $\langle b \cdot x, y \rangle_{\mathcal{A}} = \langle x, b^* \cdot y \rangle_{\mathcal{A}}$ for all $b \in \mathcal{B}$ and $x, y \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. If the inner product is non-degenerate then we call this an *inner-product* $(\mathcal{B}, \mathcal{A})$-*bimodule*. If in addition $\langle \cdot\,, \cdot \rangle_{\mathcal{A}}$ is completely positive, then we call ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ a *pre-Hilbert* $(\mathcal{B}, \mathcal{A})$-*bimodule*. Note that the two algebras $\mathcal{B}$ and $\mathcal{A}$ enter the picture in a non-symmetrical way.

Given two inner-product, or pre-Hilbert, bimodules ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and ${}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}}$, a *morphism* $T \colon {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \longrightarrow {}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}}$ is a bimodule morphism such that there exists a (necessarily unique) bimodule morphism $T^* \colon {}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}} \longrightarrow {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ with

$$\langle x, Ty \rangle_{\mathcal{A}}' = \langle T^* x, y \rangle_{\mathcal{A}} \tag{3.2}$$

for all $x \in {}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}}$ and $y \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. We call T^* the adjoint of T. With these morphisms, one may consider the category of inner product $(\mathcal{B}, \mathcal{A})$-bimodules as well as the category of pre-Hilbert $(\mathcal{B}, \mathcal{A})$-bimodules. These categories define two possible notions of "*-representation theory" for a *-algebra $\mathcal{B}$: we denote the categories of *-representations of $\mathcal{B}$ on inner-product $\mathcal{A}$-modules by $^*\text{-}\mathsf{Mod}_{\mathcal{A}}(\mathcal{B})$, and on pre-Hilbert $\mathcal{A}$-modules by $^*\text{-}\mathsf{Rep}_{\mathcal{A}}(\mathcal{B})$.

We conclude this section with some examples.

Example 3.1. For a unital *-algebra $\mathcal{A}$, consider the free right $\mathcal{A}$-module $\mathcal{A}^n$; we define the $\mathcal{A}$-valued inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i^* y_i, \tag{3.3}$$

which is easily shown to be completely positive and non-degenerate. On $\mathcal{A}^n$ we have a natural left action of the matrix algebra $M_n(\mathcal{A})$, by matrix multiplication, which turns $\mathcal{A}^n$ into a pre-Hilbert $(M_n(\mathcal{A}), \mathcal{A})$-bimodule.

More generally, let $P = P^2 = P^* \in M_n(\mathcal{A})$ be a Hermitian idempotent matrix, i.e. a *projection*. Let us consider the projective right $\mathcal{A}$-module $P\mathcal{A}^n$, with the inner product given by the restriction of (3.3). Since P is a projection, we have $\langle Px, Py \rangle = \langle x, Py \rangle = \sum_{i=1}^n x_i^* P_{ij} y_j$, where $P_{ij} \in \mathcal{A}$ are the coefficients of P. One may check that this is a completely positive, non-degenerate inner product. If we consider $PM_n(\mathcal{A})P$ with its canonical *-algebra structure, then $P\mathcal{A}^n$ is a pre-Hilbert $(PM_n(\mathcal{A})P, \mathcal{A})$-bimodule. It is easy to see that $PM_n(\mathcal{A})P$ consists of *all* right $\mathcal{A}$-linear endomorphisms of $P\mathcal{A}^n$ in this case.

Example 3.2. Let us consider the geometric example $\mathcal{A} = C^\infty(M)$, as in Example 2.1. As mentioned there, a finitely-generated, projective module $P\mathcal{A}^n$ is, up to isomorphism, just the module of smooth sections $\Gamma^\infty(E)$ of a complex vector bundle $E \longrightarrow M$. A fiber metric h on E gives a non-degenerate inner product via

$$\langle \psi, \phi \rangle (p) = h_p(\psi(p), \phi(p)), \tag{3.4}$$

for $p \in M$ and $\psi, \phi \in \Gamma^\infty(E)$. In this case we have not only non-degeneracy, but the map

$$\Gamma^\infty(E) \ni \psi \;\mapsto\; \langle \psi, \cdot \rangle \in (\Gamma^\infty(E))^* = \Gamma^\infty(E^*) \tag{3.5}$$

from the right $\mathcal{A}$-module $\Gamma^\infty(E)$ into the dual left $\mathcal{A}$-module is bijective. In general, we call an inner product with this property *strongly non-degenerate*. Finally, note that writing $\Gamma^\infty(E) = PC^\infty(M)^n$ as a projective module amounts to establishing an isomorphism $E = \operatorname{im} P \subseteq M \times \mathbb{C}^n$ of E with a subbundle of the trivial bundle. Then $PM_n(C^\infty(M))P$ corresponds to the sections $\Gamma^\infty(\mathsf{End}(E))$ of the endomorphism bundle of E.

3.2 Hopf-algebra symmetries

We now discuss notions of (bi)modules when the algebras carry symmetries. In the C^*-algebraic framework this has been done for actions of locally compact groups under the name of C^*-dynamical systems. We choose here a slightly more general notion of Hopf-algebra action so as to include infinitesimal actions of Lie algebras by derivations. Details can be found in [22].

Let H be a Hopf *-algebra over C, i.e. a Hopf algebra with a *-involution such that the coproduct Δ and the counit ϵ are *-homomorphisms, and such that $S(S(g)^*)^* = g$ for every $g \in H$, where S is the antipode of H. An H-*symmetry* of a *-algebra $\mathcal{A}$ is an action of H on $\mathcal{A}$, that we denote by

$$\triangleright \colon H \times \mathcal{A} \longrightarrow \mathcal{A};$$

i.e. it is an H-module algebra structure, such that in addition we have $(g \triangleright a)^* = S(g)^* \triangleright a^*$. Suppose that all algebras in question have such a symmetry of a fixed Hopf *-algebra H. If we are given a (right) inner-product $\mathcal{A}$-module $\mathcal{E}_{\mathcal{A}}$, then we call it H-*covariant* (or H-equivariant) if we have an H-action on $\mathcal{E}_{\mathcal{A}}$ such that

$$g \triangleright (x \cdot a) = (g_{(1)} \triangleright x) \cdot (g_{(2)} \triangleright a) \tag{3.6}$$

and

$$g \triangleright \langle x, y \rangle_{\mathcal{A}} = \langle S(g_{(1)})^* \triangleright x, g_{(2)} \triangleright y \rangle_{\mathcal{A}}, \tag{3.7}$$

where we use the Sweedler notation $\Delta(g) = g_{(1)} \otimes g_{(2)}$ for the coproduct. If we have an inner product $(\mathcal{B}, \mathcal{A})$-bimodule then we require an analogous compatibility for the left $\mathcal{B}$-module structure. Finally, morphisms between H-covariant bimodules are adjointable morphisms as above which, in addition, commute with the H-action. In this way we obtain the categories of H-covariant *-representations of a *-algebra $\mathcal{B}$ on H-covariant inner-product, or pre-Hilbert, $(\mathcal{B}, \mathcal{A})$-bimodules. We denote these categories by $^*\text{-}\mathsf{Mod}_{\mathcal{A},H}(\mathcal{B})$ and $^*\text{-}\mathsf{Rep}_{\mathcal{A},H}(\mathcal{B})$, respectively.

3.3 Tensor products

As we now see, all the notions of bimodule previously introduced can be seen as "generalized morphisms" between *-algebras; their composition is given by suitable tensor products, which we now discuss.

Let ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be inner-product, or pre-Hilbert, bimodules over the *-algebras $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, with or without H-symmetry. One defines an $\mathcal{A}$-valued inner product on the algebraic tensor product ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ as follows: first, we set

$$\langle \phi \otimes x, \psi \otimes y \rangle_{\mathcal{A}}^{\mathcal{F}\otimes\mathcal{E}} = \left\langle x, \langle \phi, \psi \rangle_{\mathcal{B}}^{\mathcal{F}} \cdot y \right\rangle_{\mathcal{A}}^{\mathcal{E}}, \tag{3.8}$$

and we define an inner product by C-sesquilinear extension of this formula to all elements of the tensor product. Note that this is indeed well-defined on the tensor product over $\mathcal{B}$. It is not hard to check that $\langle \cdot\,, \cdot \rangle_{\mathcal{A}}^{\mathcal{F}\otimes\mathcal{E}}$ is an $\mathcal{A}$-valued inner product and the left $\mathcal{C}$-module structure is compatible with it. Slightly less trivial is the fact that this inner product is again completely positive, provided that both inner products are completely positive, see [13, Thm. 4.7]. It may however be degenerate. To circumvent this problem, we mod out the tensor product by the subspace $({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})^{\perp}$ to get

$$ {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \mathbin{\widehat{\otimes}_{\mathcal{B}}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} := ({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}) / ({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})^{\perp}. \tag{3.9} $$

It can be checked that this is an inner-product (resp. pre-Hilbert) $(\mathcal{C}, \mathcal{A})$-bimodule. Moreover, if all algebras and bimodules are H-covariant, then on the tensor product one defines an H-action in the usual way: $g \triangleright (\phi \otimes_{\mathcal{B}} x) = (g_{(1)} \triangleright \phi) \otimes_{\mathcal{B}} (g_{(2)} \triangleright x)$. This action passes to the quotient ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \mathbin{\widehat{\otimes}_{\mathcal{B}}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and turns it into an H-covariant bimodule. All the above constructions are compatible with the morphisms we have specified, so we conclude that the tensor product defines functors

$$ \widehat{\otimes}_{\mathcal{B}} \colon {}^{*}\text{-}\mathsf{Mod}_{\mathcal{B},H}(\mathcal{C}) \times {}^{*}\text{-}\mathsf{Mod}_{\mathcal{A},H}(\mathcal{B}) \longrightarrow {}^{*}\text{-}\mathsf{Mod}_{\mathcal{A},H}(\mathcal{C}) \tag{3.10} $$

as well as

$$ \widehat{\otimes}_{\mathcal{B}} \colon {}^{*}\text{-}\mathsf{Rep}_{\mathcal{B},H}(\mathcal{C}) \times {}^{*}\text{-}\mathsf{Rep}_{\mathcal{A},H}(\mathcal{B}) \longrightarrow {}^{*}\text{-}\mathsf{Rep}_{\mathcal{A},H}(\mathcal{C}), \tag{3.11} $$

where we can omit H for the versions without symmetry.

The tensor product $\widehat{\otimes}$ also enjoys the usual associativity properties, up to a canonical isomorphism. This means that we have an isomorphism

$$ \mathsf{asso} \colon \left({}_{\mathcal{D}}\mathcal{G}_{\mathcal{C}} \mathbin{\widehat{\otimes}_{\mathcal{C}}} {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \right) \mathbin{\widehat{\otimes}_{\mathcal{B}}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \longrightarrow {}_{\mathcal{D}}\mathcal{G}_{\mathcal{C}} \mathbin{\widehat{\otimes}_{\mathcal{C}}} \left({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \mathbin{\widehat{\otimes}_{\mathcal{B}}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \right), \tag{3.12} $$

which respects all the structures on the bimodules, i.e. the inner products and, in the covariant case, the H-symmetry. Indeed, the usual associativity of the algebraic tensor product $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$ holds also on the quotients needed for $\widehat{\otimes}$, and respects all extra structures.

Since we have unital *-algebras, there is a canonical pre-Hilbert $(\mathcal{A}, \mathcal{A})$-bimodule given by ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$, with the inner product $\langle a, a' \rangle = a^* a'$. Note that the unit is needed to show that $\langle \cdot\,, \cdot \rangle$ is non-degenerate. This inner product is also *full*, in the sense that the span of all $\langle a, a' \rangle$ is the whole algebra $\mathcal{A}$. (More generally, we could use *-algebras which are idempotent and non-degenerate in the sense that $ab = 0$ for all b implies $a = 0$; then ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ would have the same properties.) If $\mathcal{A}$ is equipped with an H-symmetry, then ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ inherits this symmetry. These particular bimodules serve as "units" for the tensor product; i.e., there is a canonical isomorphism

$$ \mathsf{left} \colon {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \mathbin{\widehat{\otimes}_{\mathcal{B}}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \longrightarrow {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \tag{3.13} $$

for every ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$, and similarly we have a canonical isomorphism

$$ \mathsf{right} \colon {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \mathbin{\widehat{\otimes}_{\mathcal{A}}} {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \longrightarrow {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}, \tag{3.14} $$

respecting all the additional structures we have. Indeed, on the level of algebraic tensor products these maps are the usual ones, i.e. $b \otimes x \mapsto b \cdot x$. (For the case of non-unital algebras, we have to add the conditions $\mathcal{B} \cdot {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} = {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \cdot \mathcal{A} = {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ for all the bimodules, so as to restore surjectivity of left and right.)

We observe that the isomorphisms asso, left, and right satisfy the usual coherence conditions, as the ones for the algebraic tensor product. This allows the construction of the following *bicategories* (weak 2-categories), see [4]. As objects we take unital *-algebras (more generally, we could work with non-degenerate and idempotent *-algebras). We can also add an H-symmetry for the *-algebras. For the 1-morphisms from $\mathcal{A}$ to $\mathcal{B}$, we take the inner-product (resp. pre-Hilbert) $(\mathcal{B}, \mathcal{A})$-bimodules, with H-symmetry if the *-algebras carry H-symmetry. For the 2-morphisms from ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ to ${}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}}$, we take the adjointable bimodule morphisms, which should be H-covariant in the presence of H-symmetries. The tensor product $\widehat{\otimes}$ together with the canonical maps asso, left, and right define a bicategory. We wind up with four possible flavors of bicategories of bimodules denoted by

1. $\underline{\mathsf{Bimod}}^*$ for inner-product bimodules,
2. $\underline{\mathsf{Bimod}}^{\mathrm{str}}$ for pre-Hilbert bimodules,
3. $\underline{\mathsf{Bimod}}^*_H$ for inner-product bimodules with H-symmetry,
4. $\underline{\mathsf{Bimod}}^{\mathrm{str}}_H$ for pre-Hilbert bimodules with H-symmetry.

For completeness, we mention that there are the ring-theoretic versions $\underline{\mathsf{Bimod}}$ and $\underline{\mathsf{Bimod}}_H$, where we only have algebras over C as objects but no *-involutions. In this case the tensor product is just the algebraic tensor product.

Important for us is the fact that in any bicategory we have a bigroupoid of *invertible* 1-*morphisms.* Here invertible means invertible with respect to the tensor product, up to 2-isomorphisms. This bigroupoid is called the *Picard bigroupoid* of the bicategory. In our situation, we have again four flavours of Picard groupoids:

1. The *-Picard bigroupoid $\underline{\mathsf{Pic}}^*$ is the bigroupoid of invertible 1-morphisms of $\underline{\mathsf{Bimod}}^*$.
2. The strong Picard bigroupoid $\underline{\mathsf{Pic}}^{\mathrm{str}}$ is the bigroupoid of invertible 1-morphisms in $\underline{\mathsf{Bimod}}^{\mathrm{str}}$.
3. The H-covariant *-Picard bigroupoid $\underline{\mathsf{Pic}}^*_H$ is the bigroupoid of invertible 1-morphisms of $\underline{\mathsf{Bimod}}^*_H$.
4. The H-covariant strong Picard bigroupoid $\underline{\mathsf{Pic}}^{\mathrm{str}}_H$ is the bigroupoid of invertible 1-morphisms in $\underline{\mathsf{Bimod}}^{\mathrm{str}}_H$.

Again, there are ring-theoretic versions of the Picard bigroupoid which we denote by $\underline{\mathsf{Pic}}$ and $\underline{\mathsf{Pic}}_H$, in the H-covariant case.

4 Strong and covariant Morita equivalences

Any bigroupoid corresponds to a groupoid, obtained through the identification of isomorphic 1-morphisms. For the Picard bigroupoids that we just introduced, we have to use *isometric* isomorphisms in order to respect all relevant structures. This leads to the following *Picard groupoids*: the *-Picard groupoid Pic^*, the strong Picard groupoid $\mathsf{Pic}^{\mathrm{str}}$, the H-covariant *-Picard groupoid Pic^*_H, and the H-covariant strong Picard groupoid $\mathsf{Pic}^{\mathrm{str}}_H$. These groupoids consist of the *-algebras as units and the equivalence classes of invertible bimodules (of the corresponding type) as arrows. In particular, for every *-algebra $\mathcal{A}$ we have the isotropy group of arrows starting and ending at $\mathcal{A}$. This is the *Picard group* of $\mathcal{A}$, which we denote by $\mathsf{Pic}^*(\mathcal{A})$, $\mathsf{Pic}^{\mathrm{str}}(\mathcal{A})$, $\mathsf{Pic}^*_H(\mathcal{A})$, and $\mathsf{Pic}^{\mathrm{str}}_H(\mathcal{A})$, depending on the case.

We now define the associated versions of Morita equivalence:

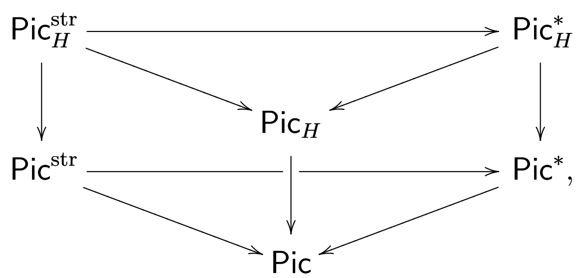
Definition 4.1 (Morita equivalence)**.** Two *-algebras over C are called

1. *-*Morita equivalent* if they are isomorphic in $\underline{\mathsf{Bimod}}^*$,
2. *strongly Morita equivalent* if they are isomorphic in $\underline{\mathsf{Bimod}}^{\mathrm{str}}$,
3. *H-covariantly *-Morita equivalent* if they are isomorphic in $\underline{\mathsf{Bimod}}^*_H$,
4. *H-covariantly strongly Morita* equivalent if they are isomorphic in $\underline{\mathsf{Bimod}}^{\mathrm{str}}_H$.

As usual, isomorphism of objects in a bicategory means that there is an invertible 1-morphism between them. Equivalently, two *-algebras are Morita equivalent in one of the above senses if and only if they are in the same orbit of the corresponding Picard groupoid. We also note that we have the ring-theoretic versions based on the Picard groupoids Pic and Pic_H, the former leading to the notion of Morita equivalence discussed in Section 2.2. A bimodule which is invertible, and hence defines a Morita equivalence, is also referred to as an *equivalence bimodule*, and a key problem is to characterize them in each case.

Note that forgetting the additional structures on bimodules (e.g. the complete positivity of inner products, the H-covariance, the inner products) preserves their invertibility. This gives the following diagram

(4.1)



of commuting groupoid morphisms. Hence a lot of questions in Morita theory can be answered by first understanding the Picard groupoids Pic and Pic_H in the ring-theoretic setting and, afterwards, investigating the kernels and images of these groupoid morphisms.

An immediate consequence of Morita equivalence is the equivalence of appropriate categories of modules:

Theorem 4.2 (Equivalence of representation theories)**.** *Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be a* **-Morita equivalence bimodule, and let $\mathcal{D}$ be a fixed* **-algebra. Then the functor*

$$\mathsf{R}_{\mathcal{E}} = {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \mathbin{\widehat{\otimes}_{\mathcal{A}}} \colon {}^*\text{-}\mathsf{Mod}_{\mathcal{D}}(\mathcal{A}) \longrightarrow {}^*\text{-}\mathsf{Mod}_{\mathcal{D}}(\mathcal{B}) \tag{4.2}$$

is an equivalence of categories. Analogous statements hold for a strong Morita equivalence bimodule, an H-covariant **-Morita equivalence bimodule, or an H-covariant strong Morita equivalence bimodule.*

The idea is to show that there are natural transformations from $\mathsf{R}_{\mathcal{A}}$ to the identity functor (via left) and from $\mathsf{R}_{\mathcal{F}} \circ \mathsf{R}_{\mathcal{E}}$ to $\mathsf{R}_{\mathcal{F}\widehat{\otimes}_{\mathcal{B}}\mathcal{E}}$ (via asso). Having the bicategory properties of $\underline{\mathsf{Bimod}}^*$, this is immediate.

Remark 4.3 (Picard groupoid actions)**.** We can view Theorem 4.2 as a consequence of an *action* of the Picard groupoid on the representation theories of the *-algebras under consideration. In a similar way, many other Morita invariants can be viewed as arising from suitable actions of the Picard groupoid. Basic examples include the Picard groups themselves, the centers, the (H-equivariant) K-theory, and the lattices of certain *-ideals carrying information about the H-symmetry. We refer to [22] for a further discussion.

We now discuss how an equivalence bimodule actually looks like. Note that if $\mathcal{E}_{\mathcal{A}}$ is an inner-product right $\mathcal{A}$-module then we have particular *rank one* operators $\Theta_{x,y}\colon \mathcal{E}_{\mathcal{A}} \longrightarrow \mathcal{E}_{\mathcal{A}}$ defined by

$$\Theta_{x,y}(z) = x \cdot \langle y, z \rangle_{\mathcal{A}}, \tag{4.3}$$

for $x, y, z \in \mathcal{E}_{\mathcal{A}}$. From the properties of $\langle \cdot\,, \cdot \rangle_{\mathcal{A}}$ we see that $\Theta_{x,y}$ is right $\mathcal{A}$-linear. Moreover, $\Theta_{x,y}$ has an adjoint operator explicitly given by $\Theta_{y,x}$. We denote by

$$\mathfrak{F}(\mathcal{E}_{\mathcal{A}}) = \mathsf{C}\text{-}\operatorname{span}\{\Theta_{x,y} \mid x, y \in \mathcal{E}_{\mathcal{A}}\} \tag{4.4}$$

the *finite rank operators* on $\mathcal{E}_{\mathcal{A}}$. They form a *-algebra such that $\mathcal{E}_{\mathcal{A}}$ becomes an inner product $(\mathfrak{F}(\mathcal{E}_{\mathcal{A}}), \mathcal{A})$-bimodule. Moreover, if $\mathcal{E}_{\mathcal{A}}$ is equipped with an H-symmetry, then we get an induced *-action of H on $\mathfrak{F}(\mathcal{E}_{\mathcal{A}})$.

Theorem 4.4 (Equivalence bimodules)**.** *Two unital* **-algebras $\mathcal{A}$ and $\mathcal{B}$ are* **-Morita equivalent if and only if there exists an inner product $(\mathcal{B}, \mathcal{A})$-bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ such that*

1. *The inner product $\langle \cdot\,, \cdot \rangle_{\mathcal{A}}$ is full (and necessarily strongly non-degenerate).*

2. $\mathcal{B}$ is isomorphic to $\mathfrak{F}(\mathcal{E}_{\mathcal{A}})$ via the action map.

In this case ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is equipped with a full $\mathcal{B}$-valued inner product $\Theta_{\cdot,\cdot}$ and $\mathcal{B} \cong \mathfrak{F}(\mathcal{E}_{\mathcal{A}})$ coincides with all adjointable operators on $\mathcal{E}_{\mathcal{A}}$. Moreover, ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is finitely generated and projective as a right $\mathcal{A}$-module and as a left $\mathcal{B}$-module. If $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent, then $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and $\Theta_{\cdot,\cdot}$ are, in addition, completely positive. In the H-covariant case, the bimodule carries an H-action compatible with both inner products.

For all cases, with additional effort, one also has non-unital formulations for idempotent and non-degenerate algebras. The *-Morita equivalence version is due to Ara [1], the strong Morita equivalence version comes from [13, Thm. 6.1], while the H-covariant versions were treated in [22].

5 Back to Morita classification of star products

We now revisit the Morita classification of star products, see Theorems 2.2 and 2.4, in light of the refined notions of Morita equivalence discussed in Section 4.

5.1 Strong Morita equivalence

It is known that, for unital C^*-algebras, ring-theoretic and strong Morita equivalences coincide, see [3]. It turns out that the same holds for Hermitian star-product algebras. The fact underlying this result is that, on any ring-theoretic equivalence bimodule between Hermitian star products, one can find suitable algebra-valued inner products. At the classical level of undeformed algebras, this follows from (3.4) since on every vector bundle we have a positive definite Hermitian fiber metric. Then one should verify that such fiber metrics can be deformed into algebra-valued inner product for $\star$. This fact was shown in [11] and treated more systematically in [13, Sect. 7 and Sect. 8], where the general relations between the ring-theoretic and the strong Picard groupoid, Pic and $\mathsf{Pic}^{\mathrm{str}}$, are studied in detail. The conclusion from [13, Thm. 8.9] can be formulated in terms of the groupoid morphisms (4.1):

Theorem 5.1 (Strong Morita equivalence of Hermitian star products)**.**

(a) Within the class of Hermitian star products, the canonical groupoid morphism $\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}$ is injective, and $\mathsf{Pic}^{\mathrm{str}}$ has the same orbits as Pic. In particular, two Hermitian star products are strongly Morita equivalent if and only if they are Morita equivalent.

(b) If $\star$ and $\star'$ are Morita equivalent Hermitian star products, then $\mathsf{Pic}^{\mathrm{str}}(\star, \star') \longrightarrow \mathsf{Pic}(\star, \star')$ is surjective if and only if all derivations of $\star$ are quasi-inner.

In part (b), we use the notation $\mathsf{Pic}^{\mathrm{str}}(\star, \star')$ for the space of arrows in $\mathsf{Pic}^{\mathrm{str}}$ from $\star$ to $\star'$ (similarly for Pic); we also call a derivation D of $\star$ *quasi-inner* if it is of the form $D = \frac{1}{\mathrm{i}\hbar}[H, \cdot\,]_\star$, for some $H \in C^\infty(M)[[\hbar]]$. Hence, the coincidence of the ring-theoretic and strong Picard groups boils down to whether there are derivations which are not quasi-inner. In the symplectic case, it is known that all derivations are quasi-inner if and only if $\mathrm{H}^1_{\mathrm{dR}}(M, \mathbb{C}) = \{0\}$. So, although strong and ring-theoretic Morita equivalences define the same equivalence relation for Hermitian star products, the corresponding Picard groups are generally distinct.

In light of part (a) of the theorem, one may directly use Theorems 2.2 and 2.4 for a description of strongly Morita equivalent Hermitian star products in terms of their characteristic classes. We mention, for completeness, that a result of Nikolai [33, Sec. 5] characterizes symplectic Hermitian star products in terms of the classes (2.7): they must satisfy $\overline{c(\star)} = -c(\star)$, a property that is stable under Morita equivalence (c.f. Theorems 2.2). A similar characterization, extending Nikolai's result, should also hold for the classes (2.5) in the Poisson case.

Remark 5.2. We note that *-Morita equivalence of Hermitian star products falls into the same classification since, on a connected component of M, the (strongly non-degenerate) inner products on the sections of a line bundle can either be completely positive or completely negative.

As discussed in [12] and mentioned at the end of Section 2.2, strong Morita equivalence turns out to be related to Nikolai's work on the representation theory of star products on cotangent bundles $M = T^*Q$ [8, 9, 7]. More specifically, the usual Schrödinger type representation on functions on the configuration space Q requires a star product $\star$ with trivial class (i.e. without magnetic monopoles, see Section 2.2). In the presence of a magnetic monopole described by an integral two-form B, one can deform the associated line bundle on the cotangent bundle T^*Q into a *strong* Morita equivalence bimodule between $\star$ and a new star product $\star_B$. We can then use this equivalence bimodule to relate (pre-Hilbert) modules over $\star$ and $\star_B$ (see Theorem 4.2). In particular, tensoring this equivalence bimodule with the Schrödinger representation of $\star$ on $C^\infty_0(Q)[[\hbar]]$ yields a representation of $\star_B$ on the space of sections $\Gamma^\infty_0(L)[[\hbar]]$ of the line bundle L over Q determined by B. On the other hand, the star product $\star_B$ had been previously considered in Nikolai's joint work [7], where a representation of $\star_B$ on the space $\Gamma^\infty_0(L)[[\hbar]]$ was constructed directly, locally out of $\star$ by applying a local version of "minimal coupling" using the local potentials $A \in \Gamma^\infty(T^*U)$ of $B\big|_U = \mathrm{d}\,A$. It was shown in [12] that, modulo canonical identifications, both constructions agree: the representation of $\star_B$ on $\Gamma^\infty_0(L)[[\hbar]]$ from [7] exactly corresponds to the Schrödinger representation of $\star$ under strong Morita equivalence.

Still in this direction, we mention the unfinished project by Nikolai to transfer the ideas of the representation theory of star products on cotangent bundles to star products on general Lie algebroids. Building on [35], the plan was to construct representations and equivalence bimodules as in the cotangent bundle case, thereby establishing the relation to the pseudo-differential operator algebraic quantizations in [36]. Nikolai was unfortunately not able to finish this project, but Nikolai's student Alexander Held took initial steps in his Diploma thesis.

5.2 Covariant Morita equivalence

We finally address covariant Morita equivalence for star products on symplectic manifolds; this was the subject of one of Nikolai's last joint projects. The general case of star products on Poisson manifolds is yet to be worked out, but should follow along the same lines, relying on Theorem 2.4 and equivariant formality maps [16].

Let (M, ω) be a symplectic manifold acted upon by a Lie algebra $\mathfrak{g}$; we assume the action to be symplectic, though not necessarily Hamiltonian. We will also assume that the action preserves a connection (and hence also a torsion-free symplectic connection). This is in fact a mild requirement: if the $\mathfrak{g}$-action comes from a symplectic action of a Lie group G and if this G-action is proper, then we always have such an invariant connection. But even in the non-proper case there are interesting examples where such a connection exists.

A star product $\star$ is called $\mathfrak{g}$-*invariant* if the fundamental vector fields $\xi_M \in \Gamma^\infty(TM)$ of the $\mathfrak{g}$-action act as derivations of $\star$ for all $\xi \in \mathfrak{g}$. One has a classification of $\mathfrak{g}$-invariant star products, up to $\mathfrak{g}$-invariant equivalence transformations [5]: every such star product is $\mathfrak{g}$-invariantly equivalent to a Fedosov star product $\star_\Omega$, where the closed two-form $\Omega \in \hbar\Gamma^\infty(\Lambda^2 T^*M)^{\mathfrak{g}}[[\hbar]]$ is $\mathfrak{g}$-invariant, and two such star products $\star_\Omega$, $\star'_\Omega$ are $\mathfrak{g}$-invariantly equivalent if and only if the corresponding two-forms Ω and Ω' are cohomologous in the *invariant* de Rham cohomology. Thus one can define a $\mathfrak{g}$-*invariant characteristic class* by

$$c^{\mathfrak{g}}(\star) = \frac{[\omega] + [\Omega]}{\mathrm{i}\hbar} \in \frac{[\omega]}{\mathrm{i}\hbar} + \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})^{\mathfrak{g}}[[\hbar]], \tag{5.1}$$

where $\mathrm{H}^\bullet_{\mathrm{dR}}(M, \mathbb{C})^{\mathfrak{g}}$ denotes the $\mathfrak{g}$-invariant de Rham cohomology of M.

Forgetting the invariance gives us a canonical map

$$\mathrm{H}^\bullet_{\mathrm{dR}}(M, \mathbb{C})^{\mathfrak{g}} \longrightarrow \mathrm{H}^\bullet_{\mathrm{dR}}(M, \mathbb{C}). \tag{5.2}$$

We also need to consider the $\mathfrak{g}$-equivariant de Rham cohomology. We use the Cartan model, see e.g. [19]. Here we only need its Lie algebra version: the complex is

$$\Omega^\bullet_{\mathfrak{g}}(M, \mathbb{C}) = \bigoplus_{k=0}^{\infty} \bigoplus_{2i+j=k} \left(\mathrm{Pol}^i(\mathfrak{g}) \otimes \Gamma^\infty(\Lambda^j T^*M)\right)^{\mathfrak{g}}, \tag{5.3}$$

with the differential $\mathrm{d}_{\mathfrak{g}}$ given by $(\mathrm{d}_{\mathfrak{g}}\, \alpha)(\xi) = \mathrm{d}\, \alpha(\xi) + \mathrm{i}_{\xi_M}\, \alpha(\xi)$ for $\xi \in \mathfrak{g}$. In particular, for the second equivariant de Rham cohomology we have a two-form part and a function part linear in $\mathfrak{g}$. Projecting on the two-form part, we get an induced map in cohomology

$$\mathrm{H}^2_{\mathfrak{g}}(M,\mathbb{C}) \longrightarrow \mathrm{H}^2_{\mathrm{dR}}(M,\mathbb{C})^{\mathfrak{g}}. \tag{5.4}$$

Using these canonical maps we can refine Theorem 2.2 as follows [21]:

Theorem 5.3. *Let (M,ω) be a symplectic manifold carrying a symplectic Lie algebra action of $\mathfrak{g}$ which preserves a connection. Let $\star$ and $\star'$ be two $\mathfrak{g}$-invariant star products (resp. Hermitian star products) on (M,ω). Then $\star$ and $\star'$ are $\mathfrak{g}$-covariantly (resp. strongly $\mathfrak{g}$-covariantly) Morita equivalent if and only if there exists a $\mathfrak{g}$-invariant symplectomorphism Φ such that $\Phi^* c^{\mathfrak{g}}(\star') - c^{\mathfrak{g}}(\star)$ is in the image of the first map in*

$$\mathrm{H}^2_{\mathfrak{g}}(M,\mathbb{C}) \longrightarrow \mathrm{H}^2_{\mathrm{dR}}(M,\mathbb{C})^{\mathfrak{g}} \longrightarrow \mathrm{H}^2_{\mathrm{dR}}(M,\mathbb{C}), \tag{5.5}$$

and maps to a $2\pi\mathrm{i}$-integral de Rham cohomology class under the second map.

As previously mentioned, a similar classification should hold in the Poisson case, based on Theorem 2.4 and on equivariant formality maps, as in [16]; we observe that, just as Theorem 5.3, the construction of equivariant formalities make use of $\mathfrak{g}$-invariant connections.

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An explicit formula for a star product with separation of variables

by Alexander Karabegov

Abstract

For a star product with separation of variables $*$ on a pseudo-Kähler manifold we give a simple closed formula of the total symbol of the left star multiplication operator L_f by a given function f. The formula for the star product $f * g$ can be immediately recovered from the total symbol of L_f.

(Dedicated to the memory of Nikolai Neumaier)

1 Introduction

Given a vector space W and a formal parameter ν, we denote by $W[[\nu]]$ the space of formal vectors $w = w_0 + \nu w_1 + \nu^2 w_2 + \ldots, w_r \in W$. One can also consider formal vectors that are formal Laurent series in ν with a finite polar part,

$$w = \sum_{r \geq k} \nu^r w_r$$

with $k \in \mathbb{Z}$.

Let M be a Poisson manifold endowed with a Poisson bracket $\{\cdot, \cdot\}$. A star product $*$ on M is an associative product on the space $C^\infty(M)[[\nu]]$ of formal functions on M given by a ν-adically convergent series

$$f * g = \sum_{r=0}^{\infty} \nu^r C_r(f, g),$$

where C_r are bidifferential operators, $C_0(f, g) = fg$, and $C_1(f, g) - C_1(g, f) = i\{f, g\}$ (see [1]). We also assume that the unit constant is the unity of the star-product $*$. A star product can be restricted to an open subset of M and recovered from its restrictions to subsets forming an open covering of M. Given functions $f, g \in C^\infty(M)[[\nu]]$, denote by L_f and R_g the left star multiplication operator by f and the right star multiplication by g, respectively. Then $L_f g = f * g = R_g f$ and the associativity of $*$ is equivalent to the property that $[L_f, R_g] = 0$ for any f, g.

The operators L_f and R_g are formal differential operators on M. It was proved by Kontsevich in [9] that deformation quantizations exist on arbitrary Poisson manifolds.

A star product is called natural if, for each r, the bidifferential operator C_r is of order not greater than r in each of its arguments (see [6]). We call a formal differential operator $A = A_0 + \nu A_1 + \nu^2 A_2 + \ldots$ natural if the order of A_r is not greater than r. If a star product is natural, the operators L_f and R_f for any $f \in C^\infty(M)[[\nu]]$ are natural. The star products of Fedosov [4] and Kontsevich [9] are natural.

Now let M be a pseudo-Kähler manifold of complex dimension m endowed with a pseudo-Kähler form ω_{-1} and the corresponding Poisson bracket $\{\cdot,\cdot\}$. A star product with separation of variables $*$ on M is a star product such that the bidifferential operators C_r differentiate the first argument in antiholomorphic directions and the second argument in holomorphic ones (see [7], [3]). Star products with separation of variables appear naturally in the context of Berezin quantization (see [2]). It was proved in [3] and [8] that the star products with separation of variables are natural in the sense of [6].

A star product on a pseudo-Kähler manifold M is a star product with separation of variables if and only if for any local holomorphic function a and a local antiholomorphic function b on M the operators L_a and R_b are pointwise multiplication operators by the functions a and b, respectively,

$$L_a = a, \;\; R_b = b.$$

Otherwise speaking, if f is a local holomorphic or g is a local antiholomorphic function, then $f * g = fg$.

A formal form $\omega = \frac{1}{\nu}\omega_{-1} + \omega_0 + \nu\omega_1 + \ldots$ such that the forms $\omega_r, r \geq 1$, are of type (1,1) with respect to the complex structure on M and may be degenerate is called a formal deformation of the pseudo-Kähler form ω_{-1}. It was proved in [7] that the star products with separation of variables on a pseudo-Kähler manifold (M, ω_{-1}) are bijectively parametrized by the formal deformations of the form ω_{-1} (see also [10]).

A star product with separation of variables $*$ on (M, ω_{-1}) corresponds to a formal deformation ω of the form ω_{-1} if for any contractible holomorphic chart $(U, \{z^k, \bar{z}^l\})$, where $1 \leq k, l \leq m$, and a formal potential $\Phi = \frac{1}{\nu}\Phi_{-1} + \Phi_0 + \nu\Phi_1 + \ldots$ of ω (i.e., $\omega = i\partial\bar{\partial}\Phi$) one has

$$R_{\nu\frac{\partial\Phi}{\partial\bar{z}^l}} = \nu\left(\frac{\partial\Phi}{\partial\bar{z}^l} + \frac{\partial}{\partial\bar{z}^l}\right).$$

The star product with separation of variables $*$ parametrized by a given deformation ω of ω_{-1} can be constructed as follows. As shown in [7], for any formal function f on U one can find a unique formal differential operator A on U commuting with the operators $R_{\bar{z}^l} = \bar{z}^l$ and $R_{\nu\frac{\partial\Phi}{\partial\bar{z}^l}}$ and such that $A1 = f$. This is the left multiplication operator by f with respect to $*$, $A = L_f$. In particular, one can immediately check that

$$L_{\nu \frac{\partial \Phi}{\partial z^k}} = \nu \left(\frac{\partial \Phi}{\partial z^k} + \frac{\partial}{\partial z^k} \right).$$

Now, for any formal function g on U we recover the product of f and g as $f * g = L_f g$. The local star products parametrized by ω agree on the intersections of coordinate charts and define a global star product on M.

We call the star product with separation of variables parametrized by the trivial deformation $\omega = \frac{1}{\nu}\omega_{-1}$ of ω_{-1} **standard**.

Explicit formulas for star products with separation of variables on pseudo-Kähler manifolds can be given in terms of graphs encoding the bidifferential operators C_r (see [11], [5], [12]).

In this paper we give a closed formula expressing the total symbol of the left star multiplication operator L_f of the standard star product with separation of variables $*$ on a coordinate chart U of a pseudo-Kähler manifold M in terms of a family of differential operators on the cotangent bundle T^*U acting on symbols of differential operators on U. One can immediately recover a formula for the star product $f * g$ on U from the total symbol of the operator L_f.

2 A recursive formula for the symbol of the left multiplication operator

A differential operator A on a real n-dimensional manifold M can be written in local coordinates $\{x^i\}$ on a chart $U \subset M$ in the normal form,

$$A = p_{i_1 i_2 \ldots i_n}(x) \left(\frac{\partial}{\partial x^1} \right)^{i_1} \ldots \left(\frac{\partial}{\partial x^n} \right)^{i_n},$$

where summation over repeated indices is assumed. Denote by $\{\xi_i\}$ the dual fibre coordinates on T^*U. Then the total symbol of A is given by the fibrewise polynomial function

$$\tau(A)(x, \xi) = p_{i_1 i_2 \ldots i_n}(x) \, (\xi_1)^{i_1} \ldots (\xi_n)^{i_n}$$

on T^*U. The mapping $A \mapsto \tau(A)$ is a bijection of the space of differential operators on U onto the space of fibrewise polynomial functions on the cotangent space T^*U. The composition of differential operators induces via this bijection an associative operation $\circ$ on the fibrewise polynomial functions on T^*U. The composition $\circ$ of fibrewise polynomial functions $p(x, \xi)$ and $q(x, \xi)$ is given by the formula

$$(p \circ q)(x,\xi) = \exp\left(\frac{\partial}{\partial \eta_i}\frac{\partial}{\partial y^i}\right) p(x,\eta)q(y,\xi)\bigg|_{y=x,\eta=\xi} =$$

$$\sum_{r=0}^{\infty} \frac{1}{r!} \frac{\partial^r p}{\partial \xi_{i_1} \dots \partial \xi_{i_r}} \frac{\partial^r q}{\partial x^{i_1} \dots \partial x^{i_r}}, \tag{2.1}$$

where the sum has a finite number of nonzero terms. If $p = p(x)$ or $q = q(\xi)$, then $p \circ q = pq$, which means that the operation $\circ$ has the separation of variables property with respect to the variables x and ξ. Formula (2.1) is valid for complex coordinates as well.

Now let $*$ be the standard star product with separation of variables on a pseudo-Kähler manifold (M, ω_{-1}) of complex dimension m. Choose a contractible coordinate chart $(U, \{z^k, \bar{z}^l\})$ on M and let Φ_{-1} be a potential of ω_{-1} on U. Given a formal function $f = f_0 + \nu f_1 + \dots$ on U, the left star multiplication operator L_f is the formal differential operator on U determined by the conditions that (i) $L_f 1 = f * 1 = f$, (ii) it commutes with the pointwise multiplication operators $R_{\bar{z}^l} = \bar{z}^l$, and (iii) it commutes with the operators

$$R_{\frac{\partial \Phi_{-1}}{\partial \bar{z}^l}} = \frac{\partial \Phi_{-1}}{\partial \bar{z}^l} + \nu \frac{\partial}{\partial \bar{z}^l}$$

for $1 \leq l \leq m$. Also, the operator L_f is natural, i.e., $L_f = A_0 + \nu A_1 + \dots$, where A_r is a differential operator on U of order not greater than r.

Denote by $\{\zeta_k, \bar{\zeta}_l\}$ the dual fibre coordinates on T^*U. We want to describe conditions (i) - (iii) on the operator L_f in terms of its total symbol $F = \tau(L_f) = F_0 + \nu F_1 + \dots$, where $F_r = \tau(A_r)$. Condition (ii) means that F does not depend on the antiholomorphic fibre variables $\bar{\zeta}_l$, $F = F(\nu, z, \bar{z}, \zeta)$. Condition (i) means that $F|_{\zeta=0} = f$ and $F_r|_{\zeta=0} = f_r$. Condition (iii) is expressed as follows:

$$F \circ \left(\frac{\partial \Phi_{-1}}{\partial \bar{z}^l} + \nu \bar{\zeta}_l\right) = \left(\frac{\partial \Phi_{-1}}{\partial \bar{z}^l} + \nu \bar{\zeta}_l\right) \circ F. \tag{2.2}$$

Using the definition (2.1) of the operation $\circ$ and its separation of variables property we simplify (2.2):

$$F \circ \frac{\partial \Phi_{-1}}{\partial \bar{z}^l} + \nu \zeta_l F = \frac{\partial \Phi_{-1}}{\partial \bar{z}^l} F + \nu \zeta_l F + \nu \frac{\partial F}{\partial \bar{z}^l}. \tag{2.3}$$

We will use the conventional notation,

$$g_{k_1 \dots k_r \bar{l}} = \frac{\partial^{r+1} \Phi_{-1}}{\partial z^{k_1} \dots \partial z^{k_r} \partial \bar{z}^l}.$$

Using (2.1) we simplify (2.3) further:

$$\sum_{r=1}^{\infty} \frac{1}{r!} g_{k_1 \dots k_r \bar{l}} \frac{\partial^r F}{\partial \zeta_{k_1} \dots \partial \zeta_{k_r}} = \nu \frac{\partial F}{\partial \bar{z}^l}. \tag{2.4}$$

In particular, $g_{k\bar{l}}$ is the metric tensor corresponding to ω_{-1}. We denote its inverse by $g^{\bar{l}k}$ and introduce the following operators:

$$\Gamma_r = g_{k_1 \ldots k_r \bar{l}}\, g^{\bar{l}k} \zeta_k \frac{\partial^r}{\partial \zeta_{k_1} \ldots \partial \zeta_{k_r}} \text{ and } D = \nu g^{\bar{l}k} \zeta_k \frac{\partial}{\partial \bar{z}^l}.$$

In particular,

$$\Gamma_1 = \zeta_k \frac{\partial}{\partial \zeta_k}$$

is the Euler operator for the holomorphic fibre variables. Multiplying both sides of (2.4) by $g^{\bar{l}k}\zeta_k$ and summing over the index l, we obtain the formula

$$\sum_{r=1}^{\infty} \frac{1}{r!} \Gamma_r F = DF. \tag{2.5}$$

We want to assign a grading to the variables ν and ζ_k such that $|\nu| = 1$ and $|\zeta_k| = -1$. Denote by $\mathcal{E}_p$ the space of formal series in the variables ν and ζ_k with coefficients in $C^\infty(U)$ such that the grading of each monomial $f(z, \bar{z})\nu^r \zeta_{k_1} \ldots \zeta_{k_s}$ in such a series satisfies $r - s \geq p$. The spaces $\mathcal{E}_p$ form a descending filtration on the space $\mathcal{E} := \mathcal{E}_0$:

$$\mathcal{E} = \mathcal{E}_0 \supset \mathcal{E}_1 \supset \ldots.$$

Since L_f is a natural operator, its total symbol $F = \tau(L_f)$ is an element of $\mathcal{E}$. The operator Γ_r acts on $\mathcal{E}$ and raises the filtration by $r-1$. The operator D acts on $\mathcal{E}$ and respects the filtration. Observe that the series on the left-hand side of (2.5) converges in the topology induced by the filtration on $\mathcal{E}$. The space $\mathcal{E}$ breaks into the direct sum of subspaces, $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$, where $\mathcal{E}'$ consists of the elements of $\mathcal{E}$ that do not depend on the fibre variables ζ_k, i.e., $\mathcal{E}' = C^\infty(U)[[\nu]]$, and $\mathcal{E}''$ is the kernel of the mapping $\mathcal{E} \ni H \mapsto H|_{\zeta=0}$. Observe that the Euler operator $\Gamma_1 : \mathcal{E} \to \mathcal{E}$ respects the decomposition $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$, $\mathcal{E}'$ is its kernel, and $\mathcal{E}''$ is its image. Moreover, the operator Γ_1 is invertible on $\mathcal{E}''$. Every operator $\Gamma_k : \mathcal{E} \to \mathcal{E}$ maps $\mathcal{E}$ to $\mathcal{E}''$ and has $\mathcal{E}'$ in its kernel.

The following lemma is straightforward.

Lemma 2.1. *The operator* $\exp D = \sum_{r=0}^{\infty} \frac{1}{r!} D^r$ *acts on* $\mathcal{E}$ *and* $\exp(-D)$ *is its inverse operator on* $\mathcal{E}$*. The operator* $\exp D$ *leaves invariant the subspace* $\mathcal{E}''$ *and the operator* $\exp D - 1$ *maps* $\mathcal{E}$ *to* $\mathcal{E}''$*.*

Lemma 2.2. *We have the following identity,*

$$\Gamma_1 - D = \mathrm{e}^D \,\Gamma_1 \mathrm{e}^{-D}.$$

Proof. The lemma follows from the fact that $[\Gamma_1, D] = D$ and the calculation

$$\mathrm{e}^D \,\Gamma_1 \mathrm{e}^{-D} = \sum_{r=0}^{\infty} \frac{1}{r!} (\mathrm{ad}\, D)^r \,\Gamma_1 = \Gamma_1 - D.$$

□

Using Lemma 2.2, we rewrite formula (2.5) as follows:

$$\left(\mathrm{e}^D\,\Gamma_1 \mathrm{e}^{-D} + \sum_{r=2}^{\infty} \frac{1}{r!}\Gamma_r\right) F = 0. \tag{2.6}$$

Introduce the operator

$$Q = -\mathrm{e}^{-D}\left(\sum_{r=2}^{\infty} \frac{1}{r!}\Gamma_r\right)\mathrm{e}^D \tag{2.7}$$

on $\mathcal{E}$. It raises the filtration on $\mathcal{E}$ by one and maps $\mathcal{E}$ to $\mathcal{E}''$. Applying the operator $\exp(-D)$ on both sides of (2.6) we obtain that

$$(\Gamma_1 - Q)\,\mathrm{e}^{-D} F = 0. \tag{2.8}$$

Using the decomposition $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$ and the last statement of Lemma 2.1, we observe that $\exp(-D)F = f + H$ for some $H \in \mathcal{E}''$. We can rewrite formula (2.8) as follows:

$$(\Gamma_1 - Q)\, H = Qf. \tag{2.9}$$

Since the operator Q maps $\mathcal{E}$ to $\mathcal{E}''$ and Γ_1 is invertible on $\mathcal{E}''$, the operator $\Gamma_1^{-1}Q$ is well defined on $\mathcal{E}$ and raises the filtration by one, we obtain from (2.9) that

$$\left(1 - \Gamma_1^{-1}Q\right) H = \Gamma_1^{-1}Qf. \tag{2.10}$$

The operator $1 - \Gamma_1^{-1}Q$ is invertible and its inverse is given by the convergent series

$$\left(1 - \Gamma_1^{-1}Q\right)^{-1} = \sum_{r=0}^{\infty} \left(\Gamma_1^{-1}Q\right)^r.$$

We have

$$F = \mathrm{e}^D(f+H) = \mathrm{e}^D\left(f + \left(\sum_{r=0}^{\infty}\left(\Gamma_1^{-1}Q\right)^r\right)\Gamma_1^{-1}Qf\right) =$$

$$\mathrm{e}^D\left(\sum_{r=0}^{\infty}\left(\Gamma_1^{-1}Q\right)^r\right) f = \mathrm{e}^D\left(1 - \Gamma_1^{-1}Q\right)^{-1} f.$$

Combining these arguments we arrive at the following theorem.

Theorem 2.3. *Given the standard star product with separation of variables on a pseudo-Kähler manifold* (M, ω_{-1})*, a coordinate chart* U *on* M*, and a function* $f \in C^\infty(U)[[\nu]]$*, then the total symbol* $F = \tau(L_f)$ *of the left star multiplication operator by* f *is given by the following explicit formula,*

$$F = \mathrm{e}^D\left(1 - \Gamma_1^{-1}Q\right)^{-1} f. \tag{2.11}$$

Now, to find the star product $f * g$, one has to calculate the total symbol F of the operator L_f using formula (2.11), recover L_f from F, and apply it to g, $f * g = L_f g$.

One can use the same formula (2.11) to express the total symbol of the left multiplication operator L_f of the star product with separation of variables $*_\omega$ corresponding to an arbitrary formal deformation ω of the pseudo-Kähler form ω_{-1}. To this end one has to modify the operators Γ_r and D as follows. On a contractible coordinate chart U find a formal potential $\Phi = \frac{1}{\nu}\Phi_{-1} + \Phi_0 + \ldots$ of the form ω and set

$$G_{k_1 \ldots k_r \bar{l}} := \frac{\partial^{r+1}\Phi}{\partial z^{k_1} \ldots \partial z^{k_r} \partial \bar{z}^l}.$$

Then $G_{k_1 \ldots k_r \bar{l}} = \frac{1}{\nu} g_{k_1 \ldots k_r \bar{l}} + \ldots$. Denote the inverse of $G_{k\bar{l}}$ by $G^{\bar{l}k} = \nu g^{\bar{l}k} + \ldots$. Now modify Γ_r and D (retaining the same notations) as follows:

$$\Gamma_r = G_{k_1 \ldots k_r \bar{l}} G^{\bar{l}k} \zeta_k \frac{\partial^r}{\partial \zeta_{k_1} \ldots \partial \zeta_{k_r}} \text{ and } D = G^{\bar{l}k} \zeta_k \frac{\partial}{\partial \bar{z}^l}.$$

The Euler operator Γ_1 will not change. Define the operator Q by the same formula (2.7) with the modified Γ_r and D. Observe that we get the old operators Γ_r, D, and Q for the trivial deformation $\omega = \frac{1}{\nu}\omega_{-1}$. One can show along the same lines that formula (2.11) with the modified operators D and Q will be given by a convergent series in the topology induced by the filtration on $\mathcal{E}$ and will define the total symbol of the left star multiplication operator L_f with respect to the star product $*_\omega$.

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Quantization of Whitney functions

by M.J. Pflaum, H. Posthuma, and X. Tang

Abstract

We propose to study deformation quantizations of Whitney functions. To this end, we extend the notion of a deformation quantization to algebras of Whitney functions over a singular set, and show the existence of a deformation quantization of Whitney functions over a closed subset of a symplectic manifold. Under the assumption that the underlying symplectic manifold is analytic and the singular subset subanalytic, we determine that the Hochschild and cyclic homology of the deformed algebra of Whitney functions over the subanalytic subset coincide with the Whitney–de Rham cohomology. Finally, we note how an algebraic index theorem for Whitney functions can be derived.

Dedicated to the memory of our friend and collaborator Nikolai Neumaier

Introduction

In physics, many interesting systems are described mathematically by phase spaces with singularities such as for example the moduli spaces of flat connections on a Riemann surface. The study of such singular phase spaces raises a very interesting question in mathematical physics. How does one quantize a singular Poisson manifold? In his seminal paper [Kon], Kontsevich completely solved the problem of constructing deformation quantizations of Poisson manifolds by his famous formality theorem. However, the problem of proving a general existence theorem for deformation quantizations over singular spaces is still open 15 years later (see [BoHePf, HeIyPf] for progress in this direction).

One of the key difficulties in the quantization theory of singular phase spaces spaces is that the algebra of smooth functions over a space with singularities appears to be complicated to study since certain crucial results such as a de Rham Theorem or a Hochschild–Kostant–Rosenberg type theorem do in general not hold true in the presence of singularities.

In this paper, we propose to replace the algebra of smooth functions by the so-called Whitney functions, and discuss some examples of quantizations of Whitney functions.

Let M be a smooth manifold, and $X \subset M$ be a closed subset of M. A Whitney function on X, roughly speaking, is the (infinite) jet of a smooth function f on M at the subset X. We denote the algebra of Whitney functions on X by $\mathcal{E}^\infty(X)$. A Whitney–Poisson structure on X is a Poisson structure on $\mathcal{E}^\infty(X)$, i.e. an antisymmetric bilinear bracket $\{-,-\}$ on $\mathcal{E}^\infty(X)$ which is a derivation in each of its arguments and satisfies the Jacobi-identity. Several interesting questions arise in the study of Whitney–Poisson structures.

1. First observe that if a neighborhood of X in M is equipped with a Poisson bivector Π, then Π naturally defines a Whitney–Poisson structure on X. This construction usually provides various different Whitney–Poisson structures on X, which we will call global Whitney–Poisson structures. In general, is every Whitney–Poisson structure on X a global one? This question is closely related to the existence of a normal form of a Poisson structure near X. We expect to see obstructions for a general X in M, which is probably connected to the singularities of X and the embedding of X in M.

2. Whitney functions naturally factorize to smooth functions on X. In general, a Whitney–Poisson structure does not factorize to a Poisson structure on X by which we mean an antisymmetric and bilinear bracket on $\mathcal{C}^\infty(X)$ which is a derivation in each of its arguments and satisfies the Jacobi-identity. It appears to be an interesting question to describe those Whitney–Poisson structures that do factorize to X. This problem appears to be closely related to the question under which conditions one can embed a singular Poisson variety into a smooth Poisson manifold, see [Egi, Dav, McMil].

In this paper, we propose to study the problem of deformation quantization of Whitney–Poisson structures on X. We will construct a natural deformation quantization of a global Whitney–Poisson structure on X. Moreover, we study such a deformation quantization by computing its Hochschild homology when the global Whitney–Poisson structure is symplectic using the methods developed in [PPT10].

We would like to dedicate this short article to Nicolai Neumaier, who unfortunately passed away in Spring 2010 after a brave and long battle with cancer. Nicolai has been a good friend and excellent collaborator. The idea to study the quantization of Whitney functions goes back to our collaboration in 2004 on deformation quantization of orbifolds [NePfPoTa]. We are picking up this idea as a memory to Nicolai's important contribution to the subject of deformation quantization of singular spaces.

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1 Formal quantizations of Whitney functions

Assume to be given a smooth manifold M, and let $X \subset M$ be a closed subset. Denote by $\mathcal{J}^\infty(X, M) \subset \mathcal{C}^\infty(M)$ the ideal of smooth functions on M which are flat on X, i.e. the space of all $f \in \mathcal{C}^\infty(M)$ such that for every differential operator D on M the restricted function $Df_{|X}$ vanishes. By Whitney's Extension Theorem, the quotient $\mathcal{E}^\infty(X) := \mathcal{C}^\infty(M)/\mathcal{J}^\infty(X, M)$ naturally coincides with the algebra of Whitney functions on X. This implies in particular that $\mathcal{E}^\infty(X) \subset \mathsf{J}^\infty(X)$, where $\mathsf{J}^\infty(X)$ denotes the space of infinite jets over X. Now consider the complex $\Omega(M)$ of differential forms on M. Then the spaces $\Omega^k_{\mathcal{J}^\infty}(X, M) := \mathcal{J}^\infty(X, M) \cdot \Omega^k(M)$ are modules over $\mathcal{C}^\infty(M)$ preserved by the exterior derivative d, which means that $d\big(\Omega^k_{\mathcal{J}^\infty}(X, M)\big) \subset \Omega^{k+1}_{\mathcal{J}^\infty}(X, M)$. One thus obtains a subcomplex $\Omega^\bullet_{\mathcal{J}^\infty}(X, M) \subset \Omega^\bullet(M)$ which we call the complex of differential forms on M which are flat on X. The quotient complex $\Omega^\bullet_{\mathcal{E}^\infty}(X) := \Omega^\bullet(M)/\Omega^\bullet_{\mathcal{J}^\infty}(X, M)$ will be called the complex of Whitney-de Rham forms on X. According to [BrPf], the cohomology of $\Omega^\bullet_{\mathcal{E}^\infty}(X)$ coincides with the singular cohomology (with values in $\mathbb{R}$), if M is an analytic manifold, and $X \subset M$ a subanalytic subset.

Let us now define what we understand by a deformation quantization of Whitney functions.

Definition 1.1. Assume to be given a manifold M, a closed subset $X \subset M$ and a *Whitney–Poisson* structure on X, i.e. a bilinear map $\{-, -\}$ on $\mathcal{E}^\infty(X)$ which satisfies for all $F, G, H \in \mathcal{E}^\infty(X)$ the relations

(P1) $\{F, GH\} = \{F, G\}H + G\{F, H\}$, and

(P2) $\{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} = 0$.

By a *formal deformation quantization* of the algebra $\mathcal{E}^\infty(X)$ or in other words a *star product* on $\mathcal{E}^\infty(X)$ we understand an associative product

$$\star : \mathcal{E}^\infty(X)[[\hbar]] \times \mathcal{E}^\infty(X)[[\hbar]] \to \mathcal{E}^\infty(X)[[\hbar]]$$

on the space $\mathcal{E}^\infty(X)[[\hbar]]$ of formal power series in the variable $\hbar$ with coefficients in $\mathcal{E}^\infty(X)$ such that the following is satisfied:

(DQ0) The product $\star$ is $\mathbb{R}[[\hbar]]$-linear and $\hbar$-adically continuous in each argument.

(DQ1) There exist $\mathbb{R}$-bilinear operators $c_k : \mathcal{E}^\infty(X) \times \mathcal{E}^\infty(X) \to \mathcal{E}^\infty(X)$, $k \in \mathbb{N}$ such that c_0 is the standard commutative product on $\mathcal{E}^\infty(X)$ and such that for all $F, G \in \mathcal{E}^\infty(X)$ there is an expansion of the product $F \star G$ of the form

$$F \star G = \sum_{k\in\mathbb{N}} c_k(F, G)\hbar^k. \tag{1.1}$$

(DQ2) The constant function $1 \in \mathcal{E}^\infty$ satisfies $1 \star F = F \star 1 = F$ for all $F \in \mathcal{E}^\infty(X)$.

(DQ3) The star commutator $[F,G]_\star := F \star G - G \star F$ of two Whitney functions $F, G \in \mathcal{E}^\infty(X)$ satisfies the commutation relation

$$[F,G]_\star = -i\hbar\{F,G\} + o(\hbar^2).$$

If in addition $\star$ is local in the sense that

(DQ4) $\operatorname{supp}(F \star G) \subset \operatorname{supp}(F) \cap \operatorname{supp}(G)$ for all $F, G \in \mathcal{E}^\infty(X)$,

then the star product is called *local*.

Remark 1.2. If (M,Π) is a Poisson manifold, the ideal $\mathcal{J}^\infty(X;M)$ is a even Poisson ideal in $\mathcal{C}^\infty(M)$. This implies that the Poisson bracket on $\mathcal{C}^\infty(M)$ factors to the quotient $\mathcal{E}^\infty(X)$. We denote the inherited Poisson bracket on $\mathcal{E}^\infty(X)$ also by $\{-,-\}$, and call it *global Whitney–Poisson structure*.

Assume now to be given a Poisson manifold (M,Π), a closed subset $X \subset M$, and let $\star$ be a local star product on $\mathcal{C}^\infty(M)$. By Peetre's Theorem one then knows that each of the operators $c_k : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ in the expansion Eq. (1.1) of the star product on $\mathcal{C}^\infty(M)$ is locally bidifferential. But this implies that for every $k \in \mathbb{N}$ the sets $c_k\big(\mathcal{J}^\infty(X,M) \times \mathcal{C}^\infty(M)\big)$ and $c_k\big(\mathcal{C}^\infty(M) \times \mathcal{J}^\infty(X,M)\big)$ are contained in $\mathcal{J}^\infty(X,M)$. This immediately entails the following result.

Proposition 1.3. *Let (M,Π) be a Poisson manifold and $\star$ a local star product on $\mathcal{C}^\infty(M)$. Then for each closed subset $X \subset M$ the subspace $\mathcal{J}^\infty(X,M)[[\hbar]]$ is an ideal in $\big(\mathcal{C}^\infty(M),\star\big)$ which gives rise to an exact sequence of deformed algebras*

$$0 \to \big(\mathcal{J}^\infty(X,M)[[\hbar]],\star\big) \to \big(\mathcal{C}^\infty(M),\star\big) \to \big(\mathcal{E}^\infty(X),\star\big) \to 0,$$

where the induced star product on $\mathcal{E}^\infty(X)$ is denoted by $\star$ as well.

Remark 1.4. One knows by the work of FEDOSOV [FED] that every symplectic manifold carries a local star product, and by KONTSEVICH [KON] that on every Poisson manifold there exists a local star product. The proceeding proposition then entails that for every closed subset X of a Poisson manifold (M,Π) there exists a deformation quantization of $\mathcal{E}^\infty(X)$ with the induced global Whitney–Poisson structure.

Let us briefly recall Fedosov's approach [FED] for the construction of a deformation quantization over a symplectic manifold (M,ω) and use this to describe the induced star product on $\mathcal{E}^\infty(X)$ with $X \subset M$ closed in more detail. To this end, observe first that each of the tangent spaces T_pM is a linear symplectic space, hence gives rise to the formal Weyl algebra $\mathbb{W}(T_pM)$. As a vector space,

$\mathbb{W}(T_pM)$ coincides with $\widehat{\mathrm{Sym}}(T_p^*M)[[\hbar]]$, the space of formal power series in $\hbar$ with coefficients in the space of Taylor expansions at the origin of smooth functions on T_pM. Note that $\widehat{\mathrm{Sym}}(T_p^*M)$ coincides with the $\mathfrak{m}$-adic completion of the space $\mathrm{Sym}(T_p^*M)$ of polynomial functions on T_pM, where $\mathfrak{m}$ denotes the maximal ideal in $\mathrm{Sym}(T_p^*M)$. In other words this means that $\widehat{\mathrm{Sym}}(T_p^*M)$ can be identified with the product $\prod_{s\in\mathbb{N}} \mathrm{Sym}^s(T_p^*M)$, where $\mathrm{Sym}^s(T_p^*M)$ denotes the space of s-homogenous polynomial functions on T_pM. Hence every element a of $\mathbb{W}(T_pM)$ can be uniquely expressed in the form

$$a = \sum_{s\in\mathbb{N},\, k\in\mathbb{N}} a_{s,k}\hbar^k, \tag{1.2}$$

where the $a_{s,k} \in \mathrm{Sym}^s(T_p^*M)$ are uniquely defined by a. For later purposes note that $\mathbb{W}(T_pM)$ is filtered by the *Fedosov-degree*

$$\deg_{\mathrm{F}}(a) := \min\{s + 2k \mid a_{sk} \neq 0\}, \quad a \in \mathbb{W}(T_pM).$$

Next observe that the Poisson bivector Π on T_pM is linear and can be written in the form

$$\Pi = \sum_{i=1}^{\frac{\dim T_pM}{2}} \Pi_{i1} \otimes \Pi_{i2} \quad \text{with } \Pi_{i1}, \Pi_{i2} \in T_pM,\ i = 1, \cdots, \frac{\dim T_pM}{2}. \tag{1.3}$$

Since the elements of T_pM act as derivations on $\mathrm{Sym}(T_pM)$ one obtains an operator

$$\begin{aligned} \widehat{\Pi} : \mathrm{Sym}(T_pM) \otimes \mathrm{Sym}(T_pM) &\to \mathrm{Sym}(T_pM) \otimes \mathrm{Sym}(T_pM), \\ a \otimes b &\mapsto \sum_{i=1}^{\frac{\dim T_pM}{2}} \Pi_{i1} a \otimes \Pi_{i2} b, \end{aligned} \tag{1.4}$$

which does not depend on the particular representation (1.3). Note that by $\mathbb{C}[[\hbar]]$-linearity and $\mathfrak{m}$-adic continuity, $\widehat{\Pi}$ uniquely extends to an operator

$$\widehat{\Pi} : \widehat{\mathrm{Sym}}(T_pM)[[\hbar]] \otimes \widehat{\mathrm{Sym}}(T_pM)[[\hbar]] \to \widehat{\mathrm{Sym}}(T_pM)[[\hbar]] \otimes \widehat{\mathrm{Sym}}(T_pM)[[\hbar]].$$

The product of two elements $a, b \in \mathbb{W}(T_pM)$ can now be written down. It is the so-called *Moyal–Weyl* product of a and b and is given by

$$a \circ b := \sum \frac{(-i\hbar)^k}{k!} \mu\big(\widehat{\Pi}(a \otimes b)\big). \tag{1.5}$$

One checks easily that $\circ$ is a star product on $\mathbb{W}(T_pM)$.

Denote by $\mathbb{W}(M)$ the bundle of formal Weyl algebras over M, which is the (profinite dimensional) vector bundle over M having fibers $\mathbb{W}(T_pM)$, $p \in M$.

Furthermore, let $\Omega^\bullet\mathbb{W}$ be the sheaf of smooth differential forms with values in the bundle $\mathbb{W}(M)$. Note that both the space $\mathcal{W}(M)$ of smooth sections of $\mathbb{W}(M)$ and the space $\Omega^\bullet\mathbb{W}(M)$ are filtered by the Fedosov-degree. More precisely, the Fedosov filtration $\big(\mathcal{F}^k\mathcal{W}(M)\big)_{k\in\mathbb{N}}$ of $\mathcal{W}(M)$ is given by

$$\mathcal{F}^k\mathcal{W}(M) := \{a \in \mathcal{W}(M) \mid \deg_{\mathrm{F}}(a(p)) \geq k \text{ for all } p \in M\},$$

and similarly for $\Omega^\bullet\mathbb{W}(M)$. Note also that an element $a \in \mathcal{W}(M)$ can be uniquely written in the form (1.2), where the $a_{s,k}$ with $s, k \in \mathbb{N}$ then are smooth sections of the symmetric powers $\mathrm{Sym}^s(T^*M)$. This representation allows us to define the *symbol map* $\sigma : \mathcal{W} \to \mathcal{C}^\infty(M)[[\hbar]]$ by

$$\sigma(a) = \sum_{k\in\mathbb{N}} a_{0,k}\hbar^k \quad \text{for } a \in \mathcal{W}.$$

Next, choose a a symplectic connection ∇ on M, i.e. a connection on M which satisfies $\nabla\omega = 0$. The symplectic connection canonically lifts to a connection

$$\nabla : \Omega^\bullet\mathbb{W}(M) \to \Omega^{\bullet+1}\mathbb{W}(M).$$

By Fedosov's construction, there exists a section $A \in \Omega^1\mathbb{W}(M)$ such that the connection

$$D := \nabla + \frac{i}{\hbar}[-, A] \tag{1.6}$$

is abelian, i.e. satisfies $D \circ D = 0$. The 1-form A is even uniquely determined by the latter property, if one additionally requires that $\deg_{\mathrm{F}}(A) \geq 2$. The connection D defined by such a 1-form A will be called a *Fedosov connection*.

As has been observed by Fedosov [Fed], the space

$$\mathcal{W}_D(M) := \{a \in \mathcal{W}(M) \mid Da = 0\}$$

of flat sections of the Weyl algebras bundle gives rise to a deformation quantization of $\mathcal{C}^\infty(M)$ via the symbol map

$$\sigma : \mathcal{W}(M) \to \mathcal{C}^\infty(M)[[\hbar]], \quad a = \sum_{s\in\mathbb{N},k\in\mathbb{N}} a_{s,k}\hbar^k \mapsto \sum_{k\in\mathbb{N}} a_{0,k}\hbar^k.$$

More precisely, if the 1-form A has been chosen as above, the restriction

$$\sigma_{|\mathcal{W}_D(M)} : \mathcal{W}_D(M) \to \mathcal{C}^\infty(M)[[\hbar]]$$

is a linear isomorphism. Let

$$\mathfrak{q} : \mathcal{C}^\infty(M)[[\hbar]] \to \mathcal{W}_D(M)$$

be its inverse, the so-called *quantization map*. Then there exist uniquely determined differential operators $\mathfrak{q}_k : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ such that

$$\mathfrak{q}(f) = \sum_{k\in\mathbb{N}} \mathfrak{q}_k(f)\hbar^k \quad \text{for all } f \in \mathcal{C}^\infty(M), \tag{1.7}$$

and

$$\star : \mathcal{C}^\infty(M)[[\hbar]] \times \mathcal{C}^\infty(M)[[\hbar]], \quad (f,g) \mapsto \sigma\big(\mathfrak{q}(f) \circ \mathfrak{q}(g)\big)$$

is a star product on $\mathcal{C}^\infty(M)$.

Now observe that the Fedosov connection D leaves the module $\mathcal{J}^\infty(X;M) \cdot \Omega^\bullet(M;\mathbb{W}M)$ invariant. This implies that D factors to the quotient

$$\Omega^\bullet_{\mathcal{E}^\infty}(X;\mathbb{W}M) := \Omega^\bullet(M;\mathbb{W}M)/\mathcal{J}^\infty(X;M) \cdot \Omega^\bullet(M;\mathbb{W}M),$$

and acts on $\mathcal{E}^\infty(X;\mathbb{W}M) := \mathcal{W}(M)/\mathcal{J}^\infty(X;M) \cdot \mathcal{W}(M)$. Moreover, the symbol map σ maps $\mathcal{J}^\infty(X;M) \cdot \mathcal{W}(M)$ to $\mathcal{J}^\infty(X;M)[[\hbar]]$, and $\mathfrak{q}\big(\mathcal{J}^\infty(X;M)[[\hbar]]\big)$ is contained in $\mathcal{J}^\infty(X;M) \cdot \mathcal{W}(M)$, since in the expansion (1.7) the operators $\mathfrak{q}_k$ are all differential operators. Hence σ and $\mathfrak{q}$ factor to $\mathcal{E}^\infty(X;\mathbb{W}M)$ respectively $\mathcal{E}^\infty(X)[[\hbar]]$. This entails the following result.

Theorem 1.5. *Let (M,ω) be a symplectic manifold, D a Fedosov connection on $\Omega^\bullet\mathbb{W}$, and $X \subset M$ a closed subset. Then the space of flat sections*

$$\mathcal{W}_D(X) := \{a \in \mathcal{E}^\infty(X;\mathbb{W}M) \mid Da = 0\}$$

is a subalgebra of $\mathcal{E}^\infty(X;\mathbb{W}M)$, and the symbol map induces an isomorphism of linear spaces $\sigma_X : \mathcal{W}_D(X) \to \mathcal{E}^\infty(X)[[\hbar]]$. Moreover, the unique product $\star$ on $\mathcal{E}^\infty(X)[[\hbar]]$ with respect to which σ_X becomes an isomorphism of algebras is a formal deformation quantization of $\mathcal{E}^\infty(X)$.

2 Hochschild and cyclic homology

The Hochschild homology of algebras of Whitney functions $\mathcal{E}^\infty(X)$ has been computed for a large class of singular subspaces $X \subset M$ in [BrPf]. In particular, it follows from this work that for (locally) subanalytic sets $X \subset M$ with M an analytic manifold the Hochschild homology of $\mathcal{E}^\infty(X)$ is given by

$$HH_\bullet\big(\mathcal{E}^\infty(X)\big) = \Omega^\bullet_{\mathcal{E}^\infty}(X). \tag{2.1}$$

In case (M,ω) is symplectic of dimension $2m$, and $\star$ a star product on $\mathcal{C}^\infty(M)$, the Hochschild homology of the deformed algebra $\big(\mathcal{C}^\infty(M)((\hbar)),\star\big)$ was first computed in [NeTs]. (We extend the star product $\star$ on $\mathcal{C}^\infty(M)[[\hbar]]$ to $\mathcal{C}^\infty(M)((\hbar))$). It is given by

$$HH_\bullet\big(\mathcal{C}^\infty(M)((\hbar))\big) = H^{2m-\bullet}_{\mathrm{dR}}(M,\mathbb{C}((\hbar))). \tag{2.2}$$

If $X \subset M$ now is closed, the natural question arises what the Hochschild homology of the deformed algebra of Whitney functions $\big(\mathcal{E}^\infty(X)((\hbar)), \star\big)$ then is. Observe that via Teleman's localization technique [Tel], the Hochschild and cyclic homology of $\mathcal{E}^\infty(X)$ and $\mathcal{E}^\infty(X)((\star))$ (and also of $\mathcal{C}^\infty(M)$ and $\mathcal{C}^\infty(M)((\hbar))$) can be computed as the sheaf cohomology of the corresponding sheaf complexes for Hochschild and cyclic complexes on X (and on M) as is explained in [BrPf].

We start the computation of the homology groups by first noting that $\mathcal{E}^\infty(X)((\hbar))$ carries a filtration $\big(\mathcal{F}^k_\hbar \mathcal{E}^\infty(X)((\hbar))\big)_{k\in\mathbb{Z}}$ by the $\hbar$-degree. More precisely,

$$\mathcal{F}^k_\hbar \mathcal{E}^\infty(X)((\hbar)) = \{F \in \mathcal{E}^\infty(X)((\hbar)) \mid \deg_\hbar F \geq k\},$$

where the $\hbar$-degree of $F = \sum_{k\in\mathbb{Z}} F_k \hbar^k \in \mathcal{E}^\infty(X)((\hbar))$ with $F_k \in \mathcal{E}^\infty(X)$ is given by

$$\deg_\hbar(F) = \min\{k \in \mathbb{Z} \mid F_k \neq 0\}.$$

The $\hbar$-filtration of $\mathcal{E}^\infty(X)((\hbar))$ induces a filtration $\big(\mathcal{F}^k_\hbar C^\bullet(\mathcal{E}^\infty(X)((\hbar))\big)_{k\in\mathbb{Z}}$ of the Hochschild chain complex $C^\bullet(\mathcal{E}^\infty(X)((\hbar)))$ which then gives rise to a spectral sequence $E^\bullet_{pq}$. Since

$$\mathcal{F}^{k+1}_\hbar \mathcal{E}^\infty(X)((\hbar))/\mathcal{F}^k_\hbar \mathcal{E}^\infty(X)((\hbar))) \cong \mathcal{E}^\infty(X),$$

the E^1-term has to coincide with the Hochschild homology of $\mathcal{E}^\infty(X)$, hence

$$E^1_{pq} = \Omega^q_{\mathcal{E}^\infty}(X). \tag{2.3}$$

Since $\mathcal{E}^\infty(X)$ is the quotient of $\mathcal{C}^\infty(M)$ by the ideal $\mathcal{J}^\infty(X;M)$, it follows from [Bry, Sec. 3] that the differential $d^1_{pq} : \Omega^q_{\mathcal{E}^\infty}(X) \to \Omega^{q-1}_{\mathcal{E}^\infty}(X)$ coincides with the canonical derivative

$$\begin{aligned}\delta : \Omega^q_{\mathcal{E}^\infty}(X) \to \Omega^{q-1}_{\mathcal{E}^\infty}(X), \quad & f_0\, df_1 \wedge \ldots \wedge df_q \mapsto \\ & \sum_{i=1}^q (-1)^{i+1}\{f_o, f_i\} df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_q + \\ & \sum_{1\leq i<j\leq q} (-1)^{i+j} f_o d\{f_i, f_j\} \wedge df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge \widehat{df_j} \wedge \ldots \wedge df_q.\end{aligned}$$

Next let us recall Brylinski's definition of the symplectic Hodge $*$-operator (see [Bry]). Let ν be the volume form $\frac{1}{m!}\omega^m$ over M, and $\Lambda^k\Pi$ the operator

$$\begin{aligned}&\Omega^k M \times \Omega^k M \to \mathcal{C}^\infty(M),\\ &(f_0 df_1 \wedge \ldots \wedge df_k, g_0 dg_1 \wedge \ldots \wedge dg_k) \mapsto f_0 g_0 \big(\Pi \lrcorner\, df_1 \wedge dg_1\big) \cdot \ldots \cdot \big(\Pi \lrcorner\, df_k \wedge dg_k\big).\end{aligned}$$

The symplectic $*$-operator $* : \Omega^k(M) \to \Omega^{2m-k}(M)$ now is uniquely defined by requiring that $\alpha \wedge (*\beta) = \Lambda^k\Pi(\alpha,\beta)\,\nu$ for all $\alpha, \beta \in \Omega^k(M)$. Obviously, $*$ leaves the $\mathcal{J}^\infty(X;M)\cdot\Omega^\bullet(M)$ invariant, hence induces an operator $* : \Omega^k_{\mathcal{E}^\infty}(M) \to \Omega^{2m-k}_{\mathcal{E}^\infty}(M)$

which by the properties of the corresponding operator on $\Omega^\bullet(M)$ satisfies the equality $* \circ * = \mathrm{id}$. By [Bry] it also follows that on $\Omega^k_{\mathcal{E}^\infty}(X)$ the canonical differential δ is equal to $(-1)^{k+1} * d *$. But this implies by [BrPf] that

$$E^2_{pq} = H^{2m-q}(X). \tag{2.4}$$

Under the assumption that X is compact subanalytic, there exists a finite triangulation of X, hence the singular cohomology with values in $\mathbb{R}$, and by [BrPf] the periodic cyclic homology of $\mathcal{E}^\infty(X)$ then have to be finite dimensional. Arguing like in [NeTs], one concludes that under this assumption on X, the spectral sequence degenerates at E^2, and the Hochschild homology of the deformed algebra $\mathcal{E}^\infty(X)[[\hbar]]$ is given by (2.4). Let us show that this holds even in more generality.

For this more refined computation of the Hochschild and cyclic homology of $\mathcal{E}^\infty(X)$, we use a specific quasi-isomorphism implementing the isomorphism (2.2) above. In [PPT10], we have constructed morphisms

$$\Psi^i_{2k} : \mathcal{C}_{2k-i}\left(\mathcal{C}^\infty(M)((\hbar)), \star\right) \to \Omega^i(M)((\hbar)),$$

satisfying the property

$$(-1)^i d \circ \Psi^i_{2k} = \Psi^{i+1}_{2k} \circ b + \Psi^{i+1}_{2k+2} \circ B, \tag{2.5}$$

where b and B are the Hochschild and Connes' B-operator computing cyclic homology. From [PPT09, Thm 2.4], it follows by Eq. (2.5) that the combination $\Psi_i := \sum_{l\geq 0} \Psi^{2m-2l-i}_{2m-2l}$ defines an S-morphism $\Psi_\bullet$ of complexes of sheaves

$$\Psi_\bullet : \mathrm{Tot}_\bullet\left(\mathcal{BC}\left(\mathcal{C}^\infty(M)((\hbar)), \star\right), b + B\right) \to \left(\bigoplus_{l\geq 0} \Omega^{2m-2l-\bullet}(M)((\hbar)), (-1)^{2m-2l-\bullet} d\right),$$

where on the left we have the total sheaf complex of Connes' (b, B)-complex (cf. [Lod, Prop. 2.5.15] for more on S-morphisms).

Proposition 2.1. *Ψ^i_{2k} maps $C_{2k-i}\left(\mathcal{J}^\infty(X, M)((\hbar))\right)$ to $\Omega^i_{\mathcal{J}^\infty}(M)((\hbar))$.*

Proof. The proof is given by two observations: first, since the Fedosov–Taylor series defining the quantization map $\mathfrak{q} : \mathcal{C}^\infty(M) \to \mathcal{W}(M)$ only involves partial derivatives, it will map $\mathcal{J}^\infty(X, M)$ to $\mathcal{J}^\infty(X, M) \cdot \mathcal{W}(M)$. Second, we see from [PPT10] that Ψ^i_{2k} is given by contraction of an explicitly given cyclic cocycle on the formal Weyl algebra acting fiberwise on $\mathbb{W}(M)$, with the Fedosov connection D. From this, the result is obvious. □

Proposition 2.1 proves that the S-morphism $\Psi_\bullet$ descends to define an S-morphism of complexes of sheaves on X

$$\Psi_\bullet : \mathrm{Tot}_\bullet\left(\mathcal{BC}\left(\mathcal{E}^\infty(X)((\hbar)),\star\right), b+B\right) \to \left(\bigoplus_{l\geq 0}\Omega^{2m-2l-\bullet}_{\mathcal{E}^\infty}(X)((\hbar)), (-1)^{2m-2l-\bullet}d\right).$$

Theorem 2.2. *Let (M,ω) be a real analytic symplectic manifold, and $X \subset M$ a subanalytic subset. Then the S-morphism $\Psi_\bullet$ defined above is a quasi-isomorphism, and therefore*

$$\begin{aligned} HH_\bullet(\mathcal{E}^\infty(X)((\hbar)),\star) &= H^{2m-\bullet}(X)((\hbar)),\\ HC_\bullet(\mathcal{E}^\infty(X)((\hbar)),\star) &= \bigoplus_{k\geq 0} H^{2m-\bullet-2k}(X)((\hbar)). \end{aligned}$$

Proof. The proof is essentially a repetition of the arguments [PPT10, Theorem 3.9]. Since Ψ is an S-morphism, it suffices to check that $\Psi^i_{2m} : C_i(\mathcal{E}^\infty(X)((\hbar))) \to \Omega^{2m-i}_{\mathcal{E}^\infty}((\hbar))$ is a quasi-isomorphism. Since Ψ^i_{2m} is a morphism of complexes of sheaves, we only need to check that Ψ^i_{2m} is a quasi-isomorphism on a sufficiently nice local chart of X, which we choose to be the intersection of a Darboux chart U of M with X.

We note that Ψ^i_{2m} is compatible with the $\hbar$-filtrations on the Hochschild complexes $C_i(\mathcal{E}^\infty(X)((\hbar)))$ and $\Omega^{2m-i}_{\mathcal{E}^\infty}((\hbar))$, and therefore induces a natural morphism between the spectral sequences associated to the $\hbar$-filtrations. To prove that $\Psi^\bullet_{2m}$ is a quasi-isomorphism, it suffices to check that $\Psi^\bullet_{2m}$ is a quasi-isomorphism at the E^2-level of the spectral sequences associated to the $\hbar$-filtrations. Over U, the algebra $(\mathcal{C}^\infty(U)((\hbar)),\star)$ can be identified with the standard Weyl algebra. In addition, the E^2-level of the spectral sequence associated to the Hochschild complex of $(\mathcal{C}^\infty(U)((\hbar)),\star)$ is the Poisson homology complex $(\Omega^\bullet(U)((\hbar)),\delta)$. Similarly, the E^2-level associated to $(\mathcal{E}^\infty(X)((\hbar)),\star)$ is again the Poisson homology complex $(\Omega^\bullet_{\mathcal{E}^\infty}(X)((\hbar)),\delta)$. Under this identification, Ψ^i_{2m} becomes the symplectic Hodge star operator, which is an isomorphism between the Poisson homology and the de Rham cohomology in (2.4). □

Remark 2.3. Theorem 2.2 has a natural generalization to deformation quantizations of global Whitney–Poisson structures on X using the method in [Dol], i.e.

$$\begin{aligned} HH_\bullet(\mathcal{E}^\infty(X)((\hbar)),\star) &= H^\pi_\bullet(X)((\hbar)),\\ HP_\bullet(\mathcal{E}^\infty(X)((\hbar)),\star) &= H_\bullet(X)((\hbar)), \end{aligned}$$

where $H^\pi_\bullet(X)((\hbar))$ is the Poisson homology of (X,π). We leave the details to diligent readers.

Remark 2.4. It is easy to see that the so-called "algebraic index theorem" [NeTs] descends to the level of Whitney functions: consider the morphism

$$\mu : C_\bullet\left(\mathcal{E}^\infty(X)[[\hbar]], \star\right) \to \Omega^\bullet_{\mathcal{E}^\infty}(X)$$

given by

$$\mu(f_0 \otimes \ldots \otimes f_k) := f_0 df_1 \wedge \ldots \wedge df_k|_{\hbar=0},$$

where $\mathcal{E}^\infty(X)[[\hbar]]$ is viewed as an algebra over $\mathbb{C}$. This map sends the Hochschild differential b to zero and intertwines B with the Whitney–de Rham operator d. The previously defined quasi-isomorphism Ψ naturally extends to define a chain morphism

$$\Psi : \mathrm{Tot}_\bullet\left(\mathcal{BC}(\mathcal{E}^\infty(X)[[\hbar]], \star)\right) \longrightarrow \bigoplus_{l\geq 0} \Omega^{2m-2l-\bullet}_{\mathcal{E}^\infty}(X)((\hbar)).$$

The algebraic index theorem gives the defect of the map μ to agree with the morphism Ψ:

Theorem 2.5. *Under the assumptions of Thm. 2.2 the following diagram commutes after taking homology:*

$$\begin{array}{ccc} \mathrm{Tot}_\bullet\left(\mathcal{BC}(\mathcal{E}^\infty(X)[[\hbar]], \star)\right) & \xrightarrow{\mu} & \bigoplus_{l\geq 0} \Omega^{2m-2l-\bullet}_{\mathcal{E}^\infty}(X) \\ & \searrow^{\Psi} & \downarrow{\scriptstyle \wedge \hat{A}(M) e^{-\Omega/2\pi\sqrt{-1}\hbar}} \\ & & \bigoplus_{l\geq 0} \Omega^{2m-2l-\bullet}_{\mathcal{E}^\infty}(X)((\hbar)) \end{array}.$$

Hereby, $\hat{A}(M)$ is the standard $\hat{A}$-class of M associated to the symplectic structure, and Ω is the characteristic class of the star product $\star$ on M.

As a consequence, the following equality holds true in $H^\bullet(X)((\hbar))$:

$$\Psi(a) = \left([\hat{A}(M)] \cup [e^{\sqrt{-1}\Omega/2\pi\hbar}]\right) \cup \mu(a),$$

for all $a = a_0 \otimes \ldots \otimes a_k \in C_k(\mathcal{E}^\infty(X), \star)$.

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Why Do we See a Classical World?

by Hartmann Römer

Abstract

From a general abstract system theoretical perspective, a quantum-like system description in the spirit of a generalized Quantum Theory may appear to be simpler and more natural than a classically inspired description. We investigate the reasons why we nevertheless conceive ourselves embedded into a classically structured world. Categorial, physical and pragmatic reasons are proposed as explanations.

To Nikolai, who was always open for discussing such matters.

1 Introduction

The underlying world views of classical and quantum physics are quite different. For contrasting purposes and neglecting intermediate positions they might be characterized as follows:

The world of classical physics is a realistic world of facts, which exist independently of their observation and are registered but not created by the act of measurement. On the other hand, the world of quantum theory is a world of potentialities, which, by the act of measurement, are elevated to a factual status as measurement results. As compared to classical physics, the role of the observer is not only a receptive, registering but an active and in part creative one. Indeed, the violation of Bell's inequalities [1] strongly suggests an exclusion of local realism in the spirit of classical physics and the Kochen-Specker theorem [2, 3] is an obstruction for any realistic hidden variable theory with non-contextual observables.

In our everyday world we are used and inclined to consider the classical world view as the view of common sense, whereas quantum physics looks like a rather extravagant view, admittedly imposed by experimental facts but emerging only lately and being mainly confined to the notoriously strange microphysical world.

In this note, we shall present evidence that quantum features of the world are much more widespread and natural than suggested by current common sense, in fact to such an extent that one may wonder about the reasons for the strong favoring of the classical view.

For what follows it is essential to realize that the world is not directly given to us as such but only as and as far as it appears to us on our inner screen. (Using a common philosophical term we refer to this as to the *phenomenal character of the world*.) Probably, almost everybody will subscribe to this apparently trivial statement, but, taken seriously, it leads to far reaching consequences. The question is about the relationship between the phenomenal and the "real" world. Naive realism asserts that the world essentially appears to us as it really is. In the terminology of Thomas Metzinger [4] naïve realism employs a *transparent model*: We are modelling creatures, creating representations of the outer world, of our body and also higher order representations of our cognitive system. A model is called *opaque*, if it is recognizable as a representation and *transparent* (invisible), if its representational character is not manifest and if, hence, the representation is identified with the represented entity.
A reflection about the foundations of Quantum Theory and physics in general must contain an investigation of the the prerequisites given by the basics of the human mode of existence and cognition, which are prior to any physical theory or act of measurement. It is safe to say that the classical world view is closer to the strong assumption of naïve realism than the quantum view, which, attributing an active constitutive role to the observer, is more aware of the phenomenal character of the world and, in a way, more cautious.

Caution and methodological prudence are no logically cogent reason for a widespread "ontophobic" attitude of contemporary philosophy, an abstention from any kind of ontological commitment in favor of phenomenal, existential, language or discourse analytical approaches. Later on we shall see that our cognitive system strongly urges if not compels us to build at least tentative ontological scenarios, for instance classically realistically inspired ones as for some interpretations of Bohmian mechanics [5], or scenarios of quantum type.
Early on from the advent of quantum mechanics Niels Bohr was convinced that the quantum theoretical figure of complementarity was of universal significance far beyond the realm of physics. Speculation along this line never ceased [6, 7]. In particular Wolfgang Pauli pointed out the possibly universal importance of quantum like entanglement [8, 9]. The idea of quantum reality gained unfortunate popularity in esoteric circles but it was also followed in a serious and formally well controlled way [10]. Indeed, a quantum analogue structure may be suspected to be realized, whenever the order of successive observations/measurements matters.
A world of strict quantum like constitution would be a world of potentialities. It would show a strongly phenomenal character, because it would be an appearing world whenever a measurement result becomes factual for an observer. From a less observer centered point of view and using a philosophical term, such a world might also be called a *worlding* (German: "weltend") world.

Assuming that the significance of quantum like structural features beyond the realm of physics in the narrow sense were a plain direct effect of quantum physics would amount to an extreme physical reductionism of very low plausibility. Rather one should look for structural isomorphisms with quantum physics. In general, formal work on wider applicability of quantum theory sought to employ the full quantum theoretical formalism to non physical situations. An alternative is the isolation and formalization of a conceptual core of quantum theory followed by an investigation of the extended applicability of the resulting generalized scheme. This has been undertaken under the name of *"Weak Quantum Theory"* or *"Generalized Quantum Theory"* [11, 12, 13], which we are going the describe in the next section.

2 Generalized Quantum Theory

Weak Quantum Theory [11, 12] arose from an axiomatic formulation of physical quantum theory by leaving out all features which seemed to be special for physical systems. The term "Weak Quantum Theory" was chosen because the resulting system of axioms is weaker than quantum physics. It is of course stronger in as much as it has a wider range of applicability. In order to avoid misunderstandings we now prefer the term "Generalized Quantum Theory" (GQT). In order to make this presentation reasonably self sustained we here repeat a short account the vital structural features of GQT to which we can refer in the sequel. For recent developments and applications see [13].
The following notions are taken over from quantum physics:

System: A system is anything which can be (imagined to be) isolated from the rest of the world and be subject to an investigation. A system can be as general as an object or a school of art together with all persons involved in production and interpretation. Unlike the situation in, e.g., classical mechanics the identification of a system is not always a trivial procedure but sometimes a creative act. In many cases it is possible to define *subsystems* inside a system

State: A system must have the capacity to reside in different states without losing its identity as a system. One may differentiate between *pure states*, which correspond to maximal possible knowledge of the system and *mixed states* corresponding to incomplete knowledge.

Observable: An observable corresponds to a feature of a system, which can be investigated in a more or less meaningful way. *Global observables* pertain to the system as a whole, *local observables* pertain to subsystems. In the above mentioned example systems, observables may correspond to esthetic investigations for systems of (schools of) art.

Measurement: Doing a measurement of an observable A means performing the investigation which belongs to the observable A and arriving at a result a, which can claim factual validity. What factual validity means depends on the system: Validity of a measurement result for a system of physics, internal conviction for self observation, consensus for groups of human beings. The result of the measurement of A will in general depend on the state z of the system before the measurement but will not be completely determined by it.

Moreover, to every observable A we associate its *spectrum*, a set $\operatorname{Spec} A$, which is just the set of all possible measurement results of A. Immediately after a measurement of an observable A with result a in $\operatorname{Spec} A$, the system will be in an *eigenstate* z_a of the observable A with *eigenvalue* a. The eigenstate z_a is a state, for which an immediate repetition of the measurement of the same observable A will again yield the same result a with certainty, and after this repeated measurement the system will still be in the same state z_a. This property, which is also crucial in quantum physics justifies the terminology "eigenstate of an observable A" for z_a and "eigenvalue" for the result a. We emphasize that this is an idealized description of a measurement process abstracting from its detailed temporal structure.
Two observables A and B are called *complementary*, if the corresponding measurements are not interchangeable. This means that the state of the system depends on the order in which the measurement results, say a and b, were obtained. If the last measurement was a measurement of A, the system will end up in an eigenstate z_a of A, and if the last measurement was a measurement of B, an eigenstate z_b will result eventually. For complementary observables A and B there will be at least some eigenvalue, say a, of one of the observables for which no common eigenstate z_{ab} of both observables exists. This means that it is not generally possible to ascribe sharp values to the complementary observables A and B, although both of them may be equally important for the description of the system. This is the essence of quantum theoretical complementarity which is well defined also for GQT.
Non complementary observables, for which the order of measurement does not matter, are called *compatible*. After the measurement of compatible observables A and B with results a and b, the system will be in the same common eigenstate z_{ab} of A and B irrespective of the order in which the measurements were performed.
Entanglement can also be defined in the framework of Generalized Quantum Theory [11, 12, 13, 14]. It may and will show up under the following conditions:

1. Subsystems can be identified within the system such that local observables pertaining to different subsystems are compatible.

2. There is a global observable of the total system, which is complementary to local observables of the subsystems.

3. The system is in an *entangled state* for instance in an eigenstate of the above mentioned global observable and not an eigenstate of the local observables.

Given these conditions, the measured values of the local observables will be uncertain because of the complementarity of the global and the local observables. However, so-called *entanglement correlations* will be observed between the measured values of the local observables pertaining to different subsystems. These correlations are non local and instantaneous.
Comparing Generalized with full physical quantum theory the following vital differences are worth noticing:

- In GQT there is no quantity like Planck's constant controlling the degree of complementarity of observables. Thus, contrary to physical quantum theory, where quantum effects are essentially restricted to the microscopic regime, macroscopic quantum like effects in GQT are to be expected.

- At least in its minimal version described here, GQT contains no direct reference to time or dynamics.

- In its minimal version GQT does not ascribe quantified probabilities to the outcomes of measurements of an observable A in a given state z. Indeed, to give just one example, for esthetic observables quantified probabilities seem to be inappropriate from the outset. What rather remains are modal logical qualifications like "impossible", "possible" and "certain". Related to the absence of quantified observables, the set of states in GQT is in general not modelled by a linear Hilbert space. Moreover, no addition of observables (operationally difficult to access even in quantum physics) is defined in GQT.

- Related to this, GQT in its minimal form provides no basis for the derivation of inequalities of Bell's type for measurement probabilities, which allow for the conclusion that the indeterminacies of measurement values are of an intrinsic ontic nature rather than an epistemic lack of knowledge. In many (but not all) applications of GQT indeterminacies may be epistemic and due to incomplete knowledge of the full state or uncontrollable perturbations by outside influences or by the process of measurement. Notice that complementarity in the sense of GQT may even occur in coarse grained classical dynamical systems [15, 7].

For some applications (see, e.g.,[16, 17, 18, 19]) one may want to enrich the above described minimal scheme of GQT by adding further structure, e.g., an underlying Hilbert space structure for the states.

We should stress here that for very general systems like the above mentioned schools of art, observables are not so directly given by the system and read off from it like many mechanical observables. On the contrary, as already suggested by the name of an "observable", the identification of an observable may be a highly creative act of the observer, which will be essentially determined by his horizon of questions and expectations. This marks a decidedly epistemic trait of the notion of observables in GQT even more so than in quantum physics. Moreover, the horizon of the observer will change, not the least as a result of his previous observations adding to the open and dynamical character of the set of observables. What has just been said about observables also applies to *partitioning* a system into subsystems. In fact, partitioning is achieved by means of *partition observables* whose different values differentiate between the subsystems. In general, subsystems do not preexist in a naïve way but are in a sense created in the constitutive act of their identification.
Quantum like phenomena like complementarity in the sense of GQT may be expected whenever "measurement" operations change system states and are not commutable. Such situations should abound in cognitive science and in everyday life. They apply in a paradigmatic way to the human mind as seen from a first person perspective, because the state of mind will invariably be changed by the very act of its conscious realization. Human communities provide another important field of possible applications of GQT. Detailed empirical investigations of quantum features in psychological systems have been performed for bistable perception [16, 17, 18], decision processes, semantic networks, learning and order effects in questionnaires [19]. (See [20] for further information.)
From the general system theoretic point of view adopted in our account of GQT and also from everyday experience, classical as opposed to quantum like systems should be a rather special and rare case. They correspond to systems without complementarities: All measurement operations commute without limitation and reveal an underlying objective reality essentially untouched by the measurements. This is a very strong assumption and a quantal world view in the sense of GQT looks quite natural and suggested by ontological parsimony. The natural and to some extent even a priori character of quantum structure is clearly pointed out by M. Bitbol. (See [21] and references therein.) Asking for the reasons why nevertheless a classical world view is widely favored seems to be a legitimate task.

3 Fundamentals of the Mode of Human Existence

Any reflection about the phenomenal character of the world requires a detailed analysis of the mode of human existence as a conscious being. This has been a main subject of philosophy since the second half of the 19th century in particular of its the phenomenological line. Of course, in this study we can in no way do justice to the vast body of work and thought done along this line associated to prominent names like Franz Brentano, Edmund Husserl, Martin Heidegger or Jean-Paul Sartre. For a deep and comprehensive account see [22]. For our purposes, it must suffice to point out a few constitutive characteristic basics of human existence emerging from its analysis:

a) The figure of oppositeness

world as an observer, set apart from and to some extent opposed to the object of his attention. Ernst Tugendhat [23, 24, 25] from the position of analytic philosophy refers to this basic human existential as to the *"egocentricity"* of man as an "I-sayer". In quantum physics the separation between observer and observed system is known as the Heisenberg cut, which is movable but not removable. In our more general framework we shall talk about the *epistemic cut*: Every cognition of a form accessible to us is the cognition of someone about something. The location of the epistemic cut may change depending on whether attention is directed to an object outside or introspectively inside to the own state of mind, but the epistemic cut never disappears altogether.

b) Temporality

Man's mode of existence is inescapably time-bounded. The world appears to us not in the form of a simultaneous panoramic picture but rather in the form of a movie: A narrow window of a "now" is shifted over our reality giving a free direct view only over an ever-changing small part of it. This internal mental time is of a type denoted by Mc Taggart [26] as an *A-Time*, which is characterized by the existence of a privileged instance of a "now" and by its directedness towards a future. In strong contrast to this, the outer time of physics is what Mc Taggart calls a *B-Time*, a scale time without privileged points and not necessarily directed. For the physical origin of time directedness see [27]. More about the difficult problem of the relationship between inner and outer time in the framework of GQT may be found in [28] and [29]. On an increasingly fundamental level of physics, proceeding from Newtonian Mechanics to Special and General Relativity Theory, physical B-Time shows a tendency to become more and more similar to space and eventually to fade away as a fundamental notion if quantum effects of space-time and very strong gravitational fields are considered. (See [28] and references therein.) However, internal A-Time persists and leaves deep traces in thermodynamics via the close relationship between the thermodynamic time arrow [27] and the so-called psychological time arrow and, as we shall see in a moment also in Quantum Theory. The two basic existentials to be mentioned next are closely related to temporality.

c) Facticity

We conceive ourselves as living in a world of facts. The feeling of certainty of a visual perception and the immediate presence in introspection all carry an inexorable imprint of facticity. The "now" is located in the heart of both temporality and facticity. Facts underly Boolean Logic.

d) Causality and freedom

Causality and freedom of action are both offshoots of the same common root of a developed temporality unfolded into past, presence and future. Rather than being in an exclusively contradictory relationship they rely on each other, because freedom is only possible if actions have foreseeable consequences and causality can only be seen if there is freedom in the choice of causes and initial conditions.

e) Agentivity

In our existence we experience ourself as agents, who actively steer the focus of their attention and their bodily motions. Planning, worrying and procuring are our future directed activities and attitudes. In this context it is also worth remembering that "factum" literally means "made".

f) Emotionality

This study is centered around the cognitional activity of man. Nevertheless, it should be kept in mind that emotions color all our perceptions and cognitions. We are continuously assessing and judging. Emotions guide our will and intentions, are constitutive for our personality and lie at the basis of our creativity.

We already saw that (Generalized) Quantum theory, more than classical theory, takes into account the phenomenal character of our world. So, we should ask ourselves, whether the basic categorical existentials enumerated above are reflected in the structure of GQT. This, indeed, turns out to be true to a large extent:

a) The structures of *oppositeness and epistemic cut* are deeply rooted in the distinction between system and observer as well as in the central role attributed to measurement. Observables neither exclusively pertain to the observer nor to the observed system but could be said to be located astride of the epistemic cut.

b) *Temporality* leaves a subtle trace in the vital importance of the (temporal) order of measurements. If observables A and B can be composed, their composition AB means A *after* B. In addition, the facticity of measurement results, mentioned under point c), enters via the "now" of human A-Time.

c) *Facticity* is strongly present in the factual validity of measurement results. In a quantum picture of the world, a quantum state before measurement describes a world of potentialities or, more precisely, of timelessly extended simultaneity rather than factual localization in a "now". From this point of view, every completed measurement corresponds to an inroad of a classical world into a quantum world.

d) and e) become apparent in GQT in the planning and execution of experiments, and in the choice of observables to be measured. They may also be formalized in dynamical equations of motion.

f) Beyond its general great importance, *emotionality* does not play any special role in GQT, which is essentially a theory of cognition. Moreover, and for good reasons, science strives for emotional neutrality. However, systems of GQT may possess emotional observables concerning e.g. mood, contention, pleasantness, esthetic or moral value. Such variables pertain to the cognitive, assessing component of emotions, which after all is almost never missing.

The above-mentioned categorical existentials are to some extent suggestive of a classical world view. *Evolutionary epistemology* [30, 31] asserts that our cognitional system, which is based on these existentials adaptively arose by Darwinian evolution: mutation and selection. Comparison with other forms of life and with older pre-lingual stages of man shows beyond any doubt that an evolution indeed occurred. It is also clear that our cognitional system should not jeopardize our chances of survival. On the other hand, one should not overlook some problematic features of evolutionary epistemology, at least in its most popular interpretation:

- The environment, to which adaptation of the cognitional system has to proceed is normally conceived as being of classical type, often even identified with a classical physical system. Quantum notions are usually not assumed to be relevant. This classical environment is normally considered to be rigid and not subject to evolution, at least as long as cultural evolution does not become topical. Evolution time is identified with a directed physical time of B-type in the sense of Mc Taggart [26]. In addition, evolutionary epistemology often relies on a strong classical background materialism and reductionism. This implies the danger of a gross underestimation of the phenomenal character of our world. The world view of classical physics arises from a particular modelization of the world. As already mentioned in the Introduction, this is not completely illegitimate as a tentative ontological scenario. However, in a naïve realistic world view this model has become completely transparent and a certain degree of opaqueness seems to be desirable.

- Even if we take the correctness of the central hypotheses of evolutionary epistemology for granted, the survival success of the evolved cognitive system in no way guarantees the ontological validity of the emerging culture dependent world view, let alone of reductive classical materialism. On the contrary, there are many examples, in particular in cultural history demonstrating that the evolutionally more viable view is not necessary the more correct one.

4 Excursus: Language

Language is an inseparable part of our human psychic endowment. So, we should not be surprised to find the basic existentials of the previous section in human language. We shall demonstrate this for 1) Facticity, 2) Temporality and 3) Agentivity:
1) Facticity
Facticity is reflected in what is called the *propositional character of language* [23, 24, 25]: A normal uttering in human language is either a clause of statement or question. The former directly claims facticity, and the latter asks about facticity. The only exceptions are exclamations and imperative sentences. Both are archaic and syntactically isolated. Imperatives are typically the most simple forms of the verb.
2) Temporality
Temporality is met in human languages in various forms

- It is manifest in the threefold temporal sequentiality of language in sounds, words and sentences.

- Reference to time is expressed in the verb in many ways. *Tenses* express temporal location with respect to the speaker (ex: "He wrote") and sometimes also with respect to the reported action (ex: "He had written"). *Modes of action* are related to the lexical meaning of a verb and describe the temporal form of the action (durative, ingressive, iterative, punctual,...) and *aspects*, which are of key importance e.g. in Slavic languages, are forms of the verb allowing to express whether the speaker wants to report on the action as ongoing or as a completed entity [32]. (English ex: "He was writing a letter" vs "He wrote a letter")

3) Agentivity

The default attitude whether a speaker understands himself primarily as (a) an acting or as (b) an experiencing being differs between various languages. It has several linguistic reflexes which show a tendency to be correlated:

- Most European languages favor attitude (a). For these languages the main distinction is between tenses, which is morphologically most clearly expressed is the distinction between past and non-past (present/future), because it coincides with the distinction between "non influenciable" and "influenciable". For attitude (b) the main distinction tends to be between future and non future (presence/past), which corresponds to the distinction between invisible and visible. Eskimo languages are an example for this state of affairs [33].

- European peoples normally conceive the future as approaching us from the front and receding to the past which lies behind us. This is in line with an active attitude (a), which considers the future as something to be faced and influenced. The converse view, in accordance with attitude (b), for which the invisible future approaches from the back side and turns into the visible presence and past in front of us has been observed in Babylonian [34] and Aymara [35]. For instance, in Babylonian future literally means "lying in the back" and past "lying in front". Aymara speakers point backwards when referring to the future.

- The difference between the active attitude (a) and the receptive attitude (b) may also be mirrored in a preference for a *accusativic* and *ergativic* [36, 33] sentence structure. Let us briefly explain this: Intransitive verbs (ex: "to sit") have only one participant, the *subject* (S) (ex: "Peter (S) is sitting"). The subject normally stands in the most simple unmarked case, the *nominative*. Transitive verbs (ex: "to hit") have (at least) two participants, the *actor* (A) and the *experiencer* (E) (ex: "Peter (A) hits the ball (E)"). Almost all European languages except Basque employ an accusativic sentence structure for transitive verbs: The actor (A) of a transitive verb stands in the nominative case just like the subject (S) of the intransitive verb, whereas the experiencer (E) stands in a different case, the *accusative*. (In English, where nominative and accusative are morphologically differentiated only for pronouns, both (S) and (A) stand before the verb and (E) behind the verb.) This parallel treatment of (S) and (A) signals an active attitude placing the actor in a privileged primary position.
  Basque and many languages outside Europe (Caucasian languages, Eskimo languages, Maya languages, Australian aboriginal languages, Chukotian languages,...) choose a different sentence construction for transitive verbs: The syntactic position of (E) runs in parallel with (S), whereas (A) stands in a

different case called *ergative*. Here, the pivotal position is occupied by the receptive experiencer (E). The ergativic sentence construction is somewhat similar to the passive construction in European languages (ex: "The ball (E) is hit by Peter (A)"). However, the European passive only arises by an additional transformation of an active sentence and the accusative construction is the default. In ergative languages the ergative structure is the default. (Indeed, many ergative languages have an "antipassive" transformation yielding an analogue of the normal sentence construction of accusative languages.) Let us finally mention that many languages (e.g. Georgian and Sumerian) have what is called an *split ergative* structure: Depending on the tense of the transitive verb an accusative or ergative construction is applied. Not surprisingly, the ergative sentence structure is favored in the past tense, because an action in the past cannot be really performed but only reported or imagined.

5 Why Classical?

We have argued that in many respects a quantum like world view seems to be more natural and ontologically parsimonious. Moreover, our introspective world as well as much of our outside world, at least on closer inspection, makes a quantum like impression. In what follows we shall give (A) categorial, (B) physical and (C) pragmatic reasons for our strong inclination to conceive ourselves as living in a classical world. None of them is completely cogent. After all, by a special intellectual effort, man has proved to be capable to device a quantum-like world view and even to get acquainted to it to some extent. But taking all these reasons together, our predilection for a classical world view becomes almost irresistible, at least for everyday life.

A) We already mentioned that the basic categorical existentials of section 3 rather suggest a classical world view. This in particular applies to the existencial of facticity. Our world, as we experience it, is inescapably fact like, which is also reflected in the propositional character of our language. From our very nature we have a deeply rooted tendency to be naïve realists unhesitatingly taking the representations on our internal screen as the real world. Metzinger [4] asserts that transparent models are evolutionally favored. In fact, in view of an approaching predator it would be a waste of time and energy for life saving reaction to realize the representational character of its appearance on our inner stage. On a higher level, we are naturally inclined to ontologize what on closer scrutiny could only be granted a phenomenal status. This predilection for ontological scenarios is an inseparable part of our mental endowment and of our culture. We already pointed out that an ontophobic ascetism may be barren. Ontologization is invaluable for understanding and orientation in our world, as long as some degree of fluidity is preserved, which sometimes allows us to look behind the screen and to correct inappropriate one-sidedness, petrifaction and sclerotization.

B) The macroscopic validity of Classical Mechanics is often invoked as the reason for the classical appearance of the world. In the macroscopic regime, to which Classical mechanics applies, quantum uncertainties are normally invisible because of the smallness of Planck's constant $\hbar$. Moreover, by the *quantum Zeno effect* [37], repeated measuring and monitoring of the system will prevent an uncontrolled growth of uncertainties. From a fundamental point of view, the macroscopic classical limit of Quantum Theory and the measurement process as an interaction with a macroscopic measurement device are not completely understood in quantum physical terms, at least not for individual systems, rather than ensembles. Decoherence theory [38] goes an important step in this direction. It explains how normal unitary time evolution of pure states of macroscopic systems coupled to an environment leads to states which, by local measurements on the system, are indistinguishable from mixed states. The decoherence time needed to reach such states quickly decreases with the sizes of the system and the environment and is typically very small. What is not described by decoherence theory is the collapse of the wave function, the transition from potentiality to measured facticity, which does not correspond to a unitary evolution in time. Indeed, for individual physical systems instead of ensembles there is so far no description of the collapse in terms of normal unmodified Quantum Theory. This may be interpreted as a hint that measurement is not exclusively to be understood as a physical process but as an act of cognition, which is, of course, accompanied by a physical process on a physical substrate but not to be identified with this physical process. In fact, no clear physical criterium seems to be in sight qualifying a physical process as a measurement process or as an act of cognition. This remark about a possible non physical but cognitive nature of the measurement process applies to GQT even more than to quantum physics. A physical analysis of a measurement process is important even if it does not capture all its cognitional aspects. The situation presents itself as follows: The requirement of the possibility of cognition is of course logically prior to any kind of physics and physical measurement. The result of an investigation of the physical process accompanying an act of cognition and measurement must be consistent with this possibility. The Quantum Theory of the measurement process meets this requirement very well. The measurement process is described by a quantum theoretical system containing the measured system S, the measuring device M and possibly some environment E. An entangled state evolves by unitary time evolution, which, by reduction to the measuring device M yields a mixed state of M reproducing exactly the probabilities of measurement results for S predicted by Quantum Theory for a state ρ of S before measurement. The same probabilities are also obtained by applying the measurement of S in the state ρ' of S arising by decoherence theory after reducing an evolved entangled state ρ'' of $S + M(+E)$ to S.

Given the macroscopic validity of Classical mechanics, we should not forget that Classical Mechanics only describes a narrow and highly idealized sector of the world in which we find ourselves living. As already mentioned, other important parts of it, including our inner and social world, are rather quantum-like constituted in the sense of GQT. So, the hint to Classical Mechanics does not really answer our original question but rather rephrases it in the form: Why do we attribute so much importance to Classical Mechanics in the formation of our world model? A categorial reason for this inclination has already been given under (A). There are other logically not completely unrelated reasons for favoring a classical world view:

C1) Man in his temporal mode of existence has good reasons to keep to the more stable and reliable features of his physical and social environment. In the material world, a bow spanned and pointed the same way must produce the same shot and a leap done with the same force must carry over the same distance. The necessary stability of a human society is based on a common stock of accepted facts and values and a collection of compatible observables and of histories whose consistency [39] is generally acknowledged. A cultural habitat of (floating) islands of stability is woven as a result of continuous collective work. (This comparison comes from a visit of the Uru-Chipaya tribe, who really lives on floating islands on lake Titicaca built from reed and continuously enlarged and repaired also by incorporation of waste.) The subtle and impressive building of classical natural science is a monumental example of probably the largest consistent structure of our time. Historiography and belief systems build other islands. Hans Primas [40] talks about *partially Boolean Systems.* Our cultural activity tries to extend them as much as possible. Consistency between different islands and sometimes even inside the islands cannot always be achieved, if complementarity is really a general constitutive feature of the world. For the sake of cohesion of society it is natural not to stress but rather to suppress such inconsistencies and anomalies. All this leads to the stabilization of a world view of predominantly classical type.

C2) All kind of information is factual, even information about Quantum Theory. In our life we are swamped with (hard) facts, which peremptorily call for attention, respect and action. The inevitability of death is a particularly grim example of impending factuality. The possibility to store and accumulate facts as documents further adds to their overwhelming dominance.

C3) In a world of surprises and unpredictability man tends to explain uncertainties by lack of knowledge or understanding. This suggests a classical background model of the world, which is difficult empirically to tell apart from a quantum-like model. The key paradigm of unpredictability is the autonomous behavior of personal beings. Quite naturally in earlier stages of mankind animistic world models prevailed and soothing and reconciliating strategies were largely employed to influence potentially dangerous or helpful personal instances. Even for the rather quantum-like internal world intuitions and dreams were widely interpreted as messages from outside intelligences. The development proceeded in the direction of successively substituting personal agents by "natural" ones, which promised a higher degree of control and understanding. A culmination of this development is marked by the success of deterministic Classical Mechanics together with a program of replacing all spiritual aspects of the world by physical reductionism. In addition, classical logic seemed to imply a classical world view. (In fact, also Quantum Theory can be formulated with classical logic.)

Finally, we should mention that also in GQT a quantum Zeno effect [37, 16, 17, 18] strengthens the facticity of measurement results, which can be stabilized and held fixed by continuous observation and sufficiently frequent repetition of a measurement.

6 Concluding Remarks

Although many factors, including our categorical framework, urge us to adopt a classical world view, this tendency is not an inescapable fate. Man at least has the capability to reflect on his categorial endowment, to question it and to try a glimpse behind this curtain.

We already mentioned several times that large parts of our world are organized in a quantum like way, even if a classical background model prevents us from acknowledging this explicitly and suggests alternative terminologies and explanations. The human mind and its products, the internal and social world of human beings are quantum reservations.

The simultaneous presence of alternatives in a quantum state has an enormous creative potential, which may very well be active in such highly creative processes like formation of concepts, identification of systems, detection of observables and also in social empathy and cultural activities like poesy and fine arts. The notion of *implicate order* developed by D. Bohm and B. Hiley [41, 5] is closely related to this creative potential. It would be surprising if evolution had not made use of it, and the work on the development of a quantum computer is an endeavor to exploit it even technically.

Moreover, the quantum theory of measurement teaches us that measurement/cognition are realized by means of quantum entanglement correlations.

There is another reason that the limitations imposed on us by the framework of our categorical existentials are not unsurmountable: Mankind is continuously striving to transcend its own categorical framework. In fact, the very term of "existence" literally means "stepping out". This tendency is already prepared in the phylogeny of man and repeated in its ontogeny. The temporality of simple animals strictly confines them to a narrow "now". The unfolding of temporalty into present, past and future is an act of emancipation. The possibility to re-present other instances of time enormously widens the temporal screen. Planning, worrying and freedom of action now become possible. Language enables symbolic representations and an emancipation from blunt facts in a mode of contrafactuality, in which the space of possibilities can be freely explored. Under this perspective, the emergence of quantum theory may be interpreted as a late highlight in this emancipatory process.

Man also rebels against the limitation by oppositeness and the epistemic cut trying to see himself integrated and secured in an all-comprising world. Seeking mystic unity [23, 24, 25] or strict mechanistic reductionism can be seen to stand for two opposite extremal attempts to overcome the structure of an individuum confronted to its world. Both of them tend to neglect the phenomenal character of the world, which is taken into account in a balanced and subtle way by quantum theory.

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Some naturally defined star products for Kähler manifolds

by Martin Schlichenmaier

Abstract

We give for the Kähler manifold case an overview of the constructions of some naturally defined star products. In particular, the Berezin-Toeplitz, Berezin, Geometric Quantization, Bordemann-Waldmann, and Karbegov standard star product are introduced. With the exception of the Geometric Quantization case they are of separation of variables type. The classifying Karabegov forms and the Deligne-Fedosov classes are given. Besides the Bordemann-Waldmann star product they are all equivalent.

1 Introduction

One of the mathematical basis of quantization is the passage from the commutative world (i.e. the functions on the phase space manifold, also called classical observables) to the non-commutative world (i.e. non-commutative objects, the quantum observables associated to the classical observables). There exists different methods to achieve this. In *operator quantization* one assigns to the classical observables operators acting on a certain Hilbert space. In *deformation quantization* one deforms the point-wise commutative product of functions into a non-commutative product. In "first order" the direction of the deformation is given by the Poisson structure which governs the classical situation. It turns out that this can only be done on the level of formal power series over the algebra of functions. Such a product is called a star product.

In this article we give an overview of certain naturally defined star products in the case that our "phase-space manifold" is a Kähler manifold. There are constructions and classifications of star products in the symplectic and even in the more general Poisson case. And as a Kähler form is a symplectic form they fall into this classification. But we have an additional complex structure and are searching for star products respecting it in a certain sense. These will be the star products of separation of variables type as introduced by Karabegov [15], resp. Wick or anti-Wick type as considered by Bordemann and Waldmann [6]. We will give their definition below. Both constructions are quite different. Karabegov uses local constructions which globalize. Bordemann and Waldmann modified

Fedosov's approach accordingly to the Kähler setting. One of the important contributions of Nikolai Neumaier to the field was that he generalizes the construction of Bordemann-Waldmann and showed that there is a 1:1 correspondence of both constructions [23].

In this article we will first introduce the notion of a star product of separation of variables type, discuss the Karabegov construction and make some comments on the Bordemann-Waldmann construction. These methods work for arbitrary Kähler manifolds (even for pseudo-Kähler manifolds). Next, for quantizable compact Kähler manifolds (i.e. Kähler manifolds admitting a quantum line bundle) we explain the construction of the Berezin-Toeplitz star product $\star_{BT}$. With the help of the Berezin transform a dual and opposite star product to the Berezin-Toeplitz will be given, the Berezin star product $\star_B$. In addition, as another naturally defined star product the star product of geometric quantization $\star_{GQ}$ (which is not of separation of variable type) shows up. They are all equivalent, we will give the equivalence transformation. Moreover, we have the star product given by the Bordemann-Waldmann construction $\star_{BW}$ and Karabegov standard star product $\star_K$. We will give their Deligne-Fedosov class and their Karabegov forms. The Deligne-Fedosov form classifies star products up to equivalence. In contrast, the Karabegov form classifies star products of separation of variables type up to identity not only up to equivalence. The Karabegov standard star product has the same Deligne-Fedosov class as $\star_{BT}$. Hence, it is equivalent. The star product $\star_{BW}$ (at least in its original construction) has a different Deligne-Fedosov class. See Section 7 for detailed results.

The intention of this review is to stay rather short. No proofs are given, also there are only a limited number of references. For a more detailed exposition, see the review [35]. For more details of the Berezin-Toeplitz quantization scheme, also see [33], [34].

There is other interesting work of Nikolai Neumaier together with Michael Müller-Bahns on invariant star products on Kähler manifolds and quotients, which is not be covered here. Let me just mention them [24], [25].

It is a pleasure for me to acknowledge inspiring discussions with Pierre Schapira on the microlocal approach to symplectic geometry and to deformation quantization.

2 Geometric setup – star products

Let (M, ω) be a pseudo-Kähler manifold. This means M is a complex manifold and ω, the pseudo-Kähler form, is a non-degenerate closed $(1,1)$-form. If ω is a positive form then (M, ω) is a honest Kähler manifold. Despite the fact, that here we are only interested in the Kähler case, we will need this more general setting for relating different star products in the Karabegov construction.

Denote by $C^\infty(M)$ the algebra of complex-valued (arbitrary often) differentiable functions with associative product given by point-wise multiplication. Ignoring the complex structure of M, our pseudo-Kähler form ω is a symplectic form. A Lie algebra structure is introduced on $C^\infty(M)$ via the *Poisson bracket* $\{.,.\}$. We recall its definition: First we assign to every $f \in C^\infty(M)$ its *Hamiltonian vector field* X_f, and then to every pair of functions f and g the *Poisson bracket* $\{.,.\}$ via

$$\omega(X_f,\cdot) = df(\cdot), \qquad \{f,g\} := \omega(X_f,X_g)\ . \tag{2.1}$$

In this way $C^\infty(M)$ becomes a *Poisson algebra.*

As we will need it further down let me give the definition of a quantizable Kähler manifold already here. For a given Kähler manifold a *quantum line bundle* for (M,ω) is a triple (L,h,∇), where L is a holomorphic line bundle, h a Hermitian metric on L, and ∇ a connection compatible with the metric h and the complex structure, such that the (pre)quantum condition

$$\begin{gathered}\mathrm{curv}_{L,\nabla}(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = -\,\mathrm{i}\,\omega(X,Y),\\ \text{in other words}\quad \mathrm{curv}_{L,\nabla} = -\,\mathrm{i}\,\omega\ ,\end{gathered} \tag{2.2}$$

is fulfilled. If there exists such a quantum line bundle for (M,ω) then M is called quantizable. Not all Kähler manifolds are quantizable. Exactly those compact Kähler manifolds are quantizable which can be embedded as complex manifolds (but not necessarily as Kähler manifolds) into some projective space.

For our Poisson algebra of smooth functions on the manifold M, a *star product* for M is an associative product $\star$ on $\mathcal{A} := C^\infty(M)[[\nu]]$, the space of formal power series with coefficients from $C^\infty(M)$, such that for $f,g \in C^\infty(M)$

1. $f \star g = f \cdot g \mod \nu$,
2. $(f \star g - g \star f)/\nu = -\mathrm{i}\{f,g\} \mod \nu$.

The star product of two functions f and g can be expressed as

$$f \star g = \sum_{k=0}^{\infty} \nu^k C_k(f,g), \qquad C_k(f,g) \in C^\infty(M), \tag{2.3}$$

and is extended $\mathbb{C}[[\nu]]$-bilinearly. It is called differential (or local) if the $C_k(\,,\,)$ are bidifferential operators with respect to their entries. If nothing else is said one requires $1 \star f = f \star 1 = f$, which is also called "null on constants".

Two star products $\star$ and $\star'$ for the same Poisson structure are called *equivalent* if and only if there exists a formal series of linear operators

$$B = \sum_{i=0}^{\infty} B_i \nu^i, \qquad B_i : C^\infty(M) \to C^\infty(M), \tag{2.4}$$

with $B_0 = id$ such that $B(f)\star' B(g) = B(f \star g)$.

To every equivalence class of a differential star product its *Deligne-Fedosov class* can be assigned. It is a formal de-Rham class of the form

$$cl(\star) \in \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \mathrm{H}^2_{dR}(M,\mathbb{C})[[\nu]]). \tag{2.5}$$

This assignment gives a 1:1 correspondence between equivalence classes of star products and such formal forms.

The notion of deformation quantization was around quite some time. See e.g. Berezin [2],[4], Moyal [22], Weyl [37], etc. Finally, the notion was formalized in [1]. See [12] for historical remarks. In the symplectic case different existence proofs, from different perspectives, were given by DeWilde-Lecomte [11], Omori-Maeda-Yoshioka [26], and Fedosov [13]. The general Poisson case was settled by Kontsevich [21].

In the pseudo-Kähler case we might look for star products adapted to the complex structure. Karabegov [15] introduced the notion of star products with *separation of variables type* for differential star products. Equivalently, Bordemann and Waldmann [6] introduced star products of Wick, and anti-Wick type respectively. There are two different conventions. In Karabegov's original definition a star product is of separation of variables type if in $C_k(.,.)$ for $k \geq 1$ the first argument is only differentiated in anti-holomorphic and the second argument in holomorphic directions. For clarification we call this convention *separation of variables (anti-Wick) type* and call a star product of *separation of variables (Wick) type* if the role of the variables is switched, i.e. in $C_k(.,.)$ for $k \geq 1$ the first argument is only differentiated in holomorphic and the second argument in anti-holomorphic directions. Unfortunately, we cannot simply retreat to one these conventions, as we really have to deal in the following with naturally defined star products and relations between them, which are of separation of variables type of both conventions.

3 Star product of separation of variables type

3.1 The Karabegov construction

Let (M, ω_{-1}) be a pseudo-Kähler manifold. We will explain the construction of Karabegov of star products of separation of variables type (anti-Wick convention), see [15, 16]. In this context it is convenient to denote the pseudo-Kähler form ω by ω_{-1}. We will switch freely between these two conventions.

A formal form

$$\widehat{\omega} = (1/\nu)\omega_{-1} + \omega_0 + \nu\,\omega_1 + \ldots \tag{3.1}$$

is called a formal deformation of the form $(1/\nu)\omega_{-1}$ if the forms ω_r, $r \geq 0$, are closed but not necessarily nondegenerate (1,1)-forms on M. Karabegov showed that to every such $\widehat{\omega}$ there exists a star product $\star$. Moreover he showed that all deformation quantizations with separation of variables on the pseudo-Kähler manifold (M, ω_{-1}) are bijectively parameterized by the formal deformations of the form $(1/\nu)\omega_{-1}$. By definition the *Karabegov form* of the star product $\star$ is $kf(\star) := \widehat{\omega}$, i.e. it is taken to be the $\widehat{\omega}$ defining $\star$. Karabegov calls the unique star product $\star_K$ with classifying Karabegov form $(1/\nu)\omega_{-1}$ *the standard star product.*

Let me sketch the principle idea of his construction. First, assume that we have such a star product $(\mathcal{A} := C^\infty(M)[[\nu]], \star)$. Then for $f, g \in \mathcal{A}$ the operators of left and right multiplication L_f, R_g are given by $L_f g = f \star g = R_g f$. The associativity of the star-product $\star$ is equivalent to the fact that L_f commutes with R_g for all $f, g \in \mathcal{A}$. If a star product is differential then L_f, R_g are formal differential operators. Now Karabegov constructs his star product associated to the deformation $\widehat{\omega}$ in the following way. First he chooses on every contractible coordinate chart $U \subset M$ (with holomorphic coordinates $\{z_k\}$) its formal potential

$$\widehat{\Phi} = (1/\nu)\Phi_{-1} + \Phi_0 + \nu\Phi_1 + \ldots, \qquad \widehat{\omega} = i\partial\bar{\partial}\widehat{\Phi}. \tag{3.2}$$

Then the construction is done in such a way that the left (right) multiplication operators $L_{\partial\widehat{\Phi}/\partial z_k}$ $(R_{\partial\widehat{\Phi}/\partial \bar{z}_l})$ on U are realized as formal differential operators

$$L_{\partial\widehat{\Phi}/\partial z_k} = \partial\widehat{\Phi}/\partial z_k + \partial/\partial z_k, \quad \text{and} \quad R_{\partial\widehat{\Phi}/\partial \bar{z}_l} = \partial\widehat{\Phi}/\partial \bar{z}_l + \partial/\partial \bar{z}_l. \tag{3.3}$$

The set $\mathcal{L}(U)$ of all left multiplication operators on U is completely described as the set of all formal differential operators commuting with the point-wise multiplication operators by antiholomorphic coordinates $R_{\bar{z}_l} = \bar{z}_l$ and the operators $R_{\partial\widehat{\Phi}/\partial \bar{z}_l}$. From the knowledge of $\mathcal{L}(U)$ the star product on U can be reconstructed. This follows from the simple fact that $L_g(1) = g$ and $L_f(L_g)(1) = f \star g$. The operator corresponding to the left multiplication with the (formal) function g can recursively (in the ν-degree) be calculated from the fact that it commutes with the operators $R_{\partial\widehat{\Phi}/\partial \bar{z}_l}$. The local star-products agree on the intersections of the charts and define the global star-product $\star$ on M. See the original work of Karabegov [15] for these statements.

In [18], [19] Karabegov gave a more direct construction of the star product $\star_K$ with Karabegov form $(1/\nu)\omega$.

3.2 Karabegov's formal Berezin transform

Given a pseudo-Kähler manifold (M, ω_{-1}). In the frame of his construction and classification Karabegov assigned to each star products $\star$ with the separation of variables property the formal *Berezin transform* $I_\star$. It is the unique formal differential operator on M such that for any open subset $U \subset M$, antiholomorphic functions a and holomorphic functions b on U the relation

$$a \star b = I(b \cdot a) = I(b \star a), \tag{3.4}$$

holds true. The last equality is automatic and is due to the fact, that by the separation of variables property $b \star a$ is the point-wise product $b \cdot a$. He shows

$$I = \sum_{i=0}^{\infty} I_i \, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M), \quad I_0 = id, \quad I_1 = \Delta. \tag{3.5}$$

Karabegov's classification gives for a fixed pseudo-Kähler manifold a 1:1 correspondence between (1) the set of star products with separation of variables type in Karabegov convention and (2) the set of formal deformations (3.1) of ω_{-1}. Moreover, the formal Berezin transform $I_\star$ determines the $\star$ uniquely.

3.3 Dual and opposite star products

Given for the pseudo-Kähler manifold (M, ω_{-1}) a star product $\star$ of separation of variables type (anti-Wick) then Karabegov defined with the help of $I = I_\star$ the following associated star products. First the *dual* star-product $\tilde{\star}$ on M is defined for $f, g \in \mathcal{A}$ by the formula

$$f \,\tilde{\star}\, g = I^{-1}(I(g) \star I(f)). \tag{3.6}$$

It is a star-product with separation of variables (anti-Wick) but now on the pseudo-Kähler manifold $(M, -\omega_{-1})$. Denote by $\tilde{\omega} = -(1/\nu)\omega_{-1} + \tilde{\omega}_0 + \nu\tilde{\omega}_1 + \ldots$ the formal form parameterizing the star-product $\tilde{\star}$. By definition $\tilde{\omega} = kf(\tilde{\star})$. Its formal Berezin transform equals I^{-1}, and thus the dual to $\tilde{\star}$ is again $\star$.

Given a star product, the opposite star product is obtained

$$f \star^{op} g = g \star f \tag{3.7}$$

by switching the arguments. Of course the sign of the Poisson bracket is changed and we obtain a star product for $(M, -\omega_{-1})$. Moreover, it switches anti-Wick with Wick type.

Finally, we take the opposite of the dual star-product, $\star' = \tilde{\star}^{op}$, given by

$$f \star' g = g \,\tilde{\star}\, f = I^{-1}(I(f) \star I(g)). \tag{3.8}$$

It defines a deformation quantization with separation of variables on M, but now of Wick type. The pseudo-Kähler manifold will again (M, ω_{-1}). Indeed the formal Berezin transform I establishes an equivalence of the deformation quantizations $(\mathcal{A}, \star)$ and $(\mathcal{A}, \star')$.

If $\star$ is star product of anti-Wick type with $kf(\star) = \widehat{\omega}$ then its Deligne-Fedosov class calculates as

$$cl(\star) = \frac{1}{\mathrm{i}}([\widehat{\omega}] - \frac{\delta}{2}). \tag{3.9}$$

See [20, Eq. 2.2], which corrects a sign error in [16]. Here [..] denotes the de-Rham class of the forms and δ is the canonical class of the manifold, i.e. the first Chern class of the canonical holomorphic line bundle K_M, resp. $\delta := c_1(K_M)$. Recall that K_M is the n^{th} exterior power of the holomorphic bundle of 1-differentials. Furthermore, we have for the opposite star product $cl(\star^{op}) = -cl(\star)$.

For the standard star product $\star_K$ given by the Karabegov form $\widehat{\omega} = (1/\nu)\omega_{-1}$ we obtain

$$cl(\star_K) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega_{-1}] - \frac{\delta}{2}). \tag{3.10}$$

In the following we will calculate Karabegov forms of star products of separation of variables type with respect to both conventions, Wick and anti-Wick. But to obtain a 1:1 correspondence we have to fix one convention. Here we refer to the anti-Wick type product. If $\star$ is of Wick type we set

$$kf(\star) := kf(\star^{op}), \tag{3.11}$$

which is a star product of separation of variables (anti-Wick) type but now for the pseudo-Kähler manifold $(M, -\omega)$.

3.4 Bordemann and Waldmann construction

Bordemann and Waldmann [6] gave another construction of a star product of separation of variables (Wick) type for a general (pseudo)Kähler manifold. It is a modification of Fedosov's geometric existence proof. They showed that the fibre-wise Weyl product used by Fedosov could be substituted by the fibre-wise Wick product. Using a modified Fedosov connection a star product $\star_{BW}$ of Wick type is obtained. Karabegov calculated its Karabegov form as $-(1/\nu)\omega$, see Karabegov [17]. Recall that by our convention this is the Karabegov form of the opposite $\star_{BW}^{op}$. Its Deligne class class calculates as

$$cl(\star_{BW}) = -cl(\star_{BW}^{op}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \frac{\delta}{2}). \tag{3.12}$$

Later Neumaier [23] was able to show that each star product of separation of variables type can be obtained by the Bordemann-Waldmann construction by adding a formal closed $(1,1)$ form as parameter in the construction.

Remark 3.1. In fact, Karabegov in [17] changed the set-up and conventions of Bordemann-Waldmann by constructing via their method a star product which is of anti-Wick type (in contrast to the original Wick type). He obtained as classifying Karabegov form $(1/\nu)\omega$ and hence the standard star product $\star_K$ with Deligne-Fedosov class (3.10) as the Bordemann-Waldmann star product in Karabegov's normalisation. By taking the opposite in the Bordemann-Waldmann construction one obtains the Karabegov modification but now with respect to the pseudo-Kähler form $-\omega$.

3.5 Reshetikhin and Takhtajan construction

Reshetikhin and Takhtajan [28] presented another general method. It is based on formal Laplace expansions of formal integrals related to the star product. The coefficients of the star product can be expressed with the help of partition functions of a restricted set of locally oriented graphs (Feynman diagrams) fulfilling some additional conditions and equipped with additional data. For details see [28], and some more remarks in [35, Section 9.2]. This approach should be compared with the Kontsevich approach in the Poisson case which also uses graphs [21].

4 The Berezin - Toeplitz star product

4.1 Toeplitz operators

For the rest of the article our manifold will be a **compact** and **quantizable** Kähler manifold (M,ω), $\omega=\omega_{-1}$, with quantum line bundle (L,h,∇). We consider all positive tensor powers of the quantum line bundle: $(L^m,h^{(m)},\nabla^{(m)})$, here $L^m:=L^{\otimes m}$ and $h^{(m)}$ and $\nabla^{(m)}$ are naturally extended. Let the Liouville form $\Omega=\frac{1}{n!}\omega^{\wedge n}$ be the volume form on M and set for the product and the norm on the space $\Gamma_\infty(M,L^m)$ of global C^∞-sections

$$(4.1)\qquad \langle\varphi,\psi\rangle:=\int_M h^{(m)}(\varphi,\psi)\,\Omega\ ,\qquad ||\varphi||:=\sqrt{\langle\varphi,\varphi\rangle}\ .$$

Let $\mathrm{L}^2(M,L^m)$ be the L^2-completed space with respect to this norm. Furthermore, let $\Gamma_{hol}(M,L^m)$ be the (finite-dimensional) subspace corresponding to the global holomorphic sections, and

$$(4.2)\qquad \Pi^{(m)}:\mathrm{L}^2(M,L^m)\to\Gamma_{hol}(M,L^m)$$

the orthogonal projection.

For a function $f\in C^\infty(M)$ the associated *Toeplitz operator* $T_f^{(m)}$ *(of level m)* is defined as

$$(4.3)\qquad T_f^{(m)}:=\Pi^{(m)}\,(f\cdot):\quad \Gamma_{hol}(M,L^m)\to\Gamma_{hol}(M,L^m)\ .$$

In words: One takes a holomorphic section s and multiplies it with the differentiable function f. The resulting section $f \cdot s$ will only be differentiable. To obtain a holomorphic section, one has to project it back on the subspace of holomorphic sections.

The linear map

$$(4.4) \qquad T^{(m)} : C^\infty(M) \to \mathrm{End}\big(\Gamma_{hol}(M, L^m)\big), \quad f \to T_f^{(m)} = \Pi^{(m)}(f\cdot)\ , m \in \mathbb{N}_0$$

is the *Toeplitz* or *Berezin-Toeplitz quantization map (of level m)*. The *Berezin-Toeplitz (BT) quantization* is the map

$$(4.5) \qquad C^\infty(M) \to \prod_{m\in\mathbb{N}_0} \mathrm{End}(\Gamma_{hol}(M, L^{(m)})), \qquad f \to (T_f^{(m)})_{m\in\mathbb{N}_0}.$$

Let for $f \in C^\infty(M)$ by $|f|_\infty$ the sup-norm of f on M and $||T_f^{(m)}||$ the operator norm with respect to the norm (4.1) on $\Gamma_{hol}(M, L^m)$.

Theorem 4.1. *[Bordemann, Meinrenken, Schlichenmaier] [5]*
(a) For every $f \in C^\infty(M)$ there exists a $C > 0$ such that

$$(4.6) \qquad |f|_\infty - \frac{C}{m} \quad \le \quad ||T_f^{(m)}|| \quad \le \quad |f|_\infty\ .$$

In particular, $\lim_{m\to\infty} ||T_f^{(m)}|| = |f|_\infty$.
(b) For every $f, g \in C^\infty(M)$

$$(4.7) \qquad ||m\,\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| \quad = \quad O(\frac{1}{m})\ .$$

(c) For every $f, g \in C^\infty(M)$

$$(4.8) \qquad ||T_f^{(m)} T_g^{(m)} - T_{f\cdot g}^{(m)}|| \quad = \quad O(\frac{1}{m})\ .$$

4.2 Star Product

Based on the Toeplitz operators and in generalization of the Theorem 4.1 we obtained

Theorem 4.2. *[5],[29],[30],[31],[20] There exists a unique differential star product*

$$(4.9) \qquad f \star_{BT} g = \sum \nu^k C_k(f, g)$$

such that

$$(4.10) \qquad T_f^{(m)} T_g^{(m)} \sim \sum_{k=0}^{\infty} \left(\frac{1}{m}\right)^k T_{C_k(f,g)}^{(m)}.$$

This star product is of separation of variables type (Wick) with classifying Deligne-Fedosov class cl and Karabegov form kf

$$cl(\star_{BT}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2}), \qquad kf(\star_{BT}) = \frac{-1}{\nu}\omega + \omega_{can}. \tag{4.11}$$

First, the asymptotic expansion in (4.10) has to be understood in a strong operator norm sense. Second, recall the definition of the canonical class δ as the first Chern class of the canonical bundle K_M. If we take in K_M the fiber metric coming from the Liouville form Ω then this defines a unique connection and further a unique curvature $(1,1)$-form ω_{can}. In our sign conventions we have $\delta = [\omega_{can}]$, and the formula for $cl(\star_{BT})$ follows as this class is equal to $-cl(\star_{BT}^{op})$ which by (3.9) can be calculated from $kf(\star_{BT})$.

Remark 4.3. It is possible to incorporate an auxiliary hermitian line (or even vector) bundle in the whole set-up. In this way it is possible to do quantization with meta-plectic correction, see [35, Rem. 3.7].

4.3 Geometric Quantisation

Kostant and Souriau introduced the operators of geometric quantization in this geometric setting. In our compact Kähler setting and if one chooses the *Kähler polarization* for the passage of prequantization to quantization then Tuynman lemma [36] gives the following relation between the operators of geometric quantization and Toeplitz quantization

$$Q_f^{(m)} = \mathrm{i} \cdot T^{(m)}_{f - \frac{1}{2m}\Delta f}, \tag{4.12}$$

where Δ is the Laplacian with respect to the Kähler metric given by ω. As a consequence the operators $Q_f^{(m)}$ and $T_f^{(m)}$ have the same asymptotic behavior for $m \to \infty$.

Using Theorem 4.1, Theorem 4.2 and the Tuynman relation (4.12) one can show that there exists a star product $\star_{GQ}$ given by asymptotic expansion of the product of geometric quantization operators. The star product $\star_{GQ}$ is equivalent to $\star_{BT}$, via the equivalence transformation $B(f) := (id - \nu\frac{\Delta}{2})f$. In particular, it has the same Deligne-Fedosov class. But it is not of separation of variables type, see [31].

5 The Berezin transform

Recall that we are in the quantizable compact Kähler case. In the Karabegov construction to every star product $\star$ a unique formal Berezin transform $I_\star$ was assigned. But to understand the relations between the different star products better we will need the geometric Berezin transform. For its definition we first have to introduce coherent vectors and covariant symbols.

5.1 The disc bundle

Without restriction we might assume that our quantum line bundle L is very ample. This means it has enough global holomorphic sections to embed the manifold into projective space. If not then at least by the quantum condition the line bundle L will be positive and a certain positive tensor power will be very ample. This tensor power will be a quantum line bundle for a rescaled Kähler form.

We pass to the dual line bundle $(U, k) := (L^*, h^{-1})$ with dual metric k. Inside the total space U, we consider the circle bundle

$$Q := \{\lambda \in U \mid k(\lambda, \lambda) = 1\},$$

and denote by $\tau : Q \to M$ (or $\tau : U \to M$) the projections to the base manifold M.

The bundle Q is a contact manifold, i.e. there is a 1-form ν such that $\mu = \frac{1}{2\pi}\tau^*\Omega \wedge \nu$ is a volume form on Q. Denote by $\mathrm{L}^2(Q, \mu)$ the corresponding L^2-space on Q. Let $\mathcal{H}$ be the space of (differentiable) functions on Q which can be extended to holomorphic functions on the disc bundle (i.e. to the "interior" of the circle bundle), and $\mathcal{H}^{(m)}$ the subspace of $\mathcal{H}$ consisting of m-homogeneous functions on Q. Here m-homogeneous means $\psi(c\lambda) = c^m\psi(\lambda)$. We introduce the following (orthogonal) projectors: the *Szegö projector*

$$\Pi : \mathrm{L}^2(Q, \mu) \to \mathcal{H}, \tag{5.1}$$

and its components the *Bergman projectors*

$$\hat{\Pi}^{(m)} : \mathrm{L}^2(Q, \mu) \to \mathcal{H}^{(m)}. \tag{5.2}$$

The bundle Q is a S^1-bundle, and the L^m are associated line bundles. The sections of $L^m = U^{-m}$ are identified with those functions ψ on Q which are homogeneous of degree m. This identification is given on the level of the L^2 spaces by the map

$$\gamma_m : \mathrm{L}^2(M, L^m) \to \mathrm{L}^2(Q, \mu), \quad s \mapsto \psi_s \quad \text{where} \tag{5.3}$$

$$\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))). \tag{5.4}$$

Restricted to the holomorphic sections we obtain the unitary isomorphism

$$\gamma_m : \Gamma_{hol}(M, L^m) \cong \mathcal{H}^{(m)}. \tag{5.5}$$

5.2 Coherent vectors

If we fix in the relation (5.4) $\alpha \in U \setminus 0$ and vary the sections s we obtain a linear evaluation functional. The *coherent vector (of level m)* associated to the point $\alpha \in U \setminus 0$ is the element $e_\alpha^{(m)}$ of $\Gamma_{hol}(M, L^m)$ with

$$\langle e_\alpha^{(m)}, s\rangle = \psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))) \tag{5.6}$$

for all $s \in \Gamma_{hol}(M, L^m)$. A direct verification shows $e_{c\alpha}^{(m)} = \bar{c}^m \cdot e_\alpha^{(m)}$ for $c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Moreover, as the bundle is very ample we get $e_\alpha^{(m)} \neq 0$.

Hence, the *coherent state (of level m)* associated to $x \in M$ as projective class

$$\mathrm{e}_x^{(m)} := [e_\alpha^{(m)}] \in \mathbb{P}(\Gamma_{hol}(M, L^m)), \qquad \alpha \in \tau^{-1}(x), \alpha \neq 0. \tag{5.7}$$

is well-defined.

Remark 5.1. This coordinate independent version of Berezin's original definition of coherent vectors and states and extensions to line bundles were given by Rawnsley [27]. It plays an important role in the work of Cahen, Gutt, and Rawnsley on the quantization of Kähler manifolds [7, 8, 9, 10], via Berezin's covariant symbols. In those works the coherent vectors are parameterized by the elements of $L \setminus 0$. The definition here uses the points of the total space of the dual bundle U. It has the advantage that one can consider all tensor powers of L together on an equal footing.

5.3 Covariant Berezin symbol

For an operator $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ its *covariant Berezin symbol* $\sigma^{(m)}(A)$ *(of level m)* is defined as the function

$$\sigma^{(m)}(A) : M \to \mathbb{C}, \quad x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e_\alpha^{(m)}, A e_\alpha^{(m)} \rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}, \quad \alpha \in \tau^{-1}(x) \setminus \{0\}. \tag{5.8}$$

5.4 Definition of the Berezin transform

Definition 5.2. The map

$$I^{(m)} : C^\infty(M) \to C^\infty(M), \qquad f \mapsto I^{(m)}(f) := \sigma^{(m)}(T_f^{(m)}), \tag{5.9}$$

obtained by starting with a function $f \in C^\infty(M)$, taking its Toeplitz operator $T_f^{(m)}$, and then calculating the covariant symbol is called the (geometric) *Berezin transform (of level m)*.

Theorem 5.3. *[20] Given $x \in M$ then the Berezin transform $I^{(m)}(f)$ has a complete asymptotic expansion in powers of $1/m$ as $m \to \infty$*

$$I^{(m)}(f)(x) \quad \sim \quad \sum_{i=0}^{\infty} I_i(f)(x) \frac{1}{m^i} \ , \tag{5.10}$$

where $I_i : C^\infty(M) \to C^\infty(M)$ are linear maps given by differential operators, uniformly defined for all $x \in M$. Furthermore, $I_0(f) = f, \quad I_1(f) = \Delta f$.

Here Δ is the Laplacian with respect to the metric given by the Kähler form ω.

5.5 Bergman kernel

Recall from above the Bergman projectors (5.2). They have smooth integral kernels, the *Bergman kernels* $\mathcal{B}_m(\alpha,\beta)$ defined on $Q \times Q$, i.e.

$$\hat{\Pi}^{(m)}(\psi)(\alpha) = \int_Q \mathcal{B}_m(\alpha,\beta)\psi(\beta)\mu(\beta). \tag{5.11}$$

The Bergman kernels can be expressed with the help of the coherent vectors.

$$\mathcal{B}_m(\alpha,\beta) = \langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle. \tag{5.12}$$

For the proofs of these properties see [20], or [32].

Let $x \in M$ and choose $\alpha \in Q$ with $\tau(\alpha) = x$ then the function

$$u_m(x) := \mathcal{B}_m(\alpha,\alpha) = \langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle, \tag{5.13}$$

is well-defined on M.

6 Berezin transform and star products

6.1 Identification of the BT star product

In [20] it was shown that the BT star product $\star_{BT}$ is the opposite of the dual of the star product $\star$ associated to the geometric Berezin transform introduced in the last section. To identify $\star$ we will give its classifying Karabegov form $\widehat{\omega}$. Zelditch [40] proved that the function u_m (5.13) has a complete asymptotic expansion in powers of $1/m$. In detail he showed

$$u_m(x) \sim m^n \sum_{k=0}^{\infty} \frac{1}{m^k}\, b_k(x), \quad b_0 = 1. \tag{6.1}$$

If we replace in the expansion $1/m$ by the formal variable ν we obtain a formal function s defined by

$$e^s(x) = \sum_{k=0}^{\infty} \nu^k\, b_k(x). \tag{6.2}$$

Now take as formal potential (3.2)

$$\widehat{\Phi} = \frac{1}{\nu}\Phi_{-1} + s,$$

where Φ_{-1} is the local Kähler potential of the Kähler form $\omega = \omega_{-1}$. Then $\widehat{\omega} = \mathrm{i}\,\partial\bar{\partial}\widehat{\Phi}$. It might also be written in the form

$$\widehat{\omega} = \frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\,\partial\bar{\partial}\log \mathcal{B}_m(\alpha,\alpha)). \tag{6.3}$$

We use for the replacement of $1/m$ by the formal variable ν the symbol $\mathbb{F}$.

6.2 The Berezin star products for arbitrary Kähler manifolds

We will introduce for general quantizable compact Kähler manifolds the Berezin star product. We extract from the asymptotic expansion of the Berezin transform (5.10) the formal expression

$$I = \sum_{i=0}^{\infty} I_i \, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M), \tag{6.4}$$

as a *formal Berezin transform*, and set

$$f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g)). \tag{6.5}$$

As $I_0 = id$ this $\star_B$ is a star product for our Kähler manifold, which we call the *Berezin star product.* Obviously, the formal map I gives the equivalence transformation to $\star_{BT}$. Hence, the Deligne-Fedosov classes will be the same. It will be of separation of variables type (but now of anti-Wick type). We showed in [20] that $I = I_\star$ with star product given by the form (6.3). We can rewrite (6.5) as

$$f \star_{BT} g := I^{-1}(I(f) \star_B I(g)). \tag{6.6}$$

and get exactly the relation (3.8). Hence, $\star = \star_B$ and both star products $\star_B$ and $\star_{BT}$ are dual and opposite to each other.

6.3 The original Berezin star product

Under very restrictive conditions on the manifold it is possible to construct the *Berezin star product* with the help of the covariant symbol map. This was done by Berezin himself [2],[3] and later by Cahen, Gutt, and Rawnsley [7][8][9][10] for more examples. We will indicate this in the following.

Denote by $\mathcal{A}^{(m)} \le C^\infty(M)$, the subspace of functions which appear as level m covariant symbols of operators. From the surjectivity of the Toeplitz map one concludes that the covariant symbol map is injective, see [35, Prop.6.5]. Hence, for the symbols $\sigma^{(m)}(A)$ and $\sigma^{(m)}(B)$ the operators A and B are uniquely fixed. We define a deformed product by

$$\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B). \tag{6.7}$$

Now $\star_{(m)}$ defines on $\mathcal{A}^{(m)}$ an associative and noncommutative product.

The crucial problem is how to relate different levels m to define for all possible symbols a unique product not depending on m. In certain special situations like those studied by Berezin, and Cahen, Gutt and Rawnsley the subspaces are nested into each other and the union $\mathcal{A} = \bigcup_{m \in \mathbb{N}} \mathcal{A}^{(m)}$ is a dense subalgebra of $C^\infty(M)$. A detailed analysis shows that in this case a star product is given. The star product will coincide with the star product $\star_B$ introduced above.

7 Summary of naturally defined star products

By the presented techniques we obtained for quantizable compact Kähler manifolds three different naturally defined star products $\star_{BT}$, $\star_{GQ}$, and $\star_B$. All three are equivalent and have classifying Deligne-Fedosov class

$$cl(\star_{BT}) = cl(\star_B) = cl(\star_{GQ}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2}). \tag{7.1}$$

But all three are distinct. In fact $\star_{BT}$ is of separation of variables type (Wick-type), $\star_B$ is of separation of variables type (anti-Wick-type), and $\star_{GQ}$ neither. For their Karabegov forms we obtained

$$kf(\star_{BT}) = \frac{-1}{\nu}\omega + \omega_{can}. \qquad kf(\star_B) = \frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\,\partial\overline{\partial}\log u_m). \tag{7.2}$$

The function u_m was introduced above as the function on M obtained by evaluating the Bergman kernel along the diagonal in $Q \times Q$.

In addition we have the Bordemann-Waldmann [6] star product which exists for every Kähler manifold. It is of Wick-type. Its Karabegov form [17] is given by $kf(\star_{BW}) = kf(\star_{BW}^{opp}) = -(1/\nu)\,\omega$ and it has Deligne Fedosov class

$$cl(\star_{BW}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \frac{\delta}{2}). \tag{7.3}$$

Hence, it will be only equivalent to the star products above if the canonical class of the manifold will be trivial. For compact Riemann surfaces this will exactly be the case if it is a torus.

Another star product is the standard star product (of anti-Wick type) of Karabegov $\star_K$ with Karabegov form $kf(\star_K) = (1/\nu)\,\omega$. It can be also obtained in a modified Bordemann - Waldmann approach by an anti-Wick Fedosov type construction. Via the formula (3.9) its Deligne-Fedosov class $cl(\star_K)$ calculates to (7.1). Hence, it is equivalent to the above three star products.

I like to point out, that the Berezin transform, resp. the defining Karabegov form can be used to calculate the coefficients of these naturally defined star products. This can be done either directly or with the help of the certain type of graphs (in the latter case see the work of Gammelgaard [14] and Hua Xu [38],[39]). See [35, Section 8.4, 9.] for an overview on these techniques.

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A very short presentation of deformation quantization, some of its developments in the past two decades, and conjectural perspectives[1]

by Daniel Sternheimer

Abstract

Deformation quantization is the main paradigm for Flato's "deformation philosophy" on how to interpret the emergence of new physical theories. It gives a framework in which quantization can be understood as a deformation of the classical (commutative) composition law of observables, functions on phase space (manifolds with a Poisson bracket). We sketch its formulation and significant examples showing the essence of deformation quantization, and its relations with usual quantization. We then indicate some developments and avatars in the past two decades, during Neumaier's active scientific life. We end with the presentation of a multifaceted framework in which Anti de Sitter (space and/or symmetry) would be quantized, with conjectural implications in cosmology and to a deformations-based possible space-time origin of elementary particle symmetries.

1 Introduction: the deformation philosophy

However seducing the idea may be, the notion of "Theory of Everything" is to me unphysical. In physics, sometimes knowingly but often not (because one simply ignores a more elaborate reality that has yet to be discovered), one makes approximations in order to have as manageable a theory (or model) as possible, or simply to try and describe the reality known at the time. In other words, physical theories have their domain of applicability defined e.g. by the relevant distances, velocities, energies, etc. involved. However in physics the passages from one domain (of distances, etc.) to another do not happen in an uncontrolled way: experimental phenomena appear that cause a paradox and contradict accepted

[1]Based on a talk at the Neumaier Commemorative Colloquium, Mulhouse, 16 June 2011

theories. Eventually a new fundamental constant enters and the formalism is modified: the attached structures (symmetries, observables, states, etc.) *deform* the initial structure to a new structure which in the limit, when the new parameter goes to zero, "contracts" to the previous formalism.

A first example of that phenomenon can be traced back to the antiquity, when it was gradually realized that the earth is not flat, and in mathematics to the nineteenth century with Riemann surface theory. However the main developments happened a century later, in particular with the seminal analytic geometry works of Kodaira and Spencer [KS58] (and their lesser known interpretation by Grothendieck [Gr61], where one can see in watermark his "EGA" that started a couple of years later). These deep geometric works were in some sense "linearized" in the theory of deformations of algebras by Gerstenhaber [Ge64]. The realization that deformations are fundamental in the development of physics happened a couple of years later in France, when it was noticed that the passage from the Galilean invariance of Newtonian mechanics $(SO(3) \cdot \mathbb{R}^3 \cdot \mathbb{R}^4)$ is deformed, in the Gerstenhaber sense [Ge64], to the Poincaré group $(SO(3,1) \cdot \mathbb{R}^4)$ of special relativity. In spite of the fact that the composition law of symbols of pseudodifferential operators, essential in the Atiyah–Singer index theorem developed at that time (to the exposition of which I took part in Paris in 1963/64), was in effect a deformation of their abelian product, it took another ten years or so to develop the tools which enabled us to make explicit, rigorous and convincing, what was in the back of the mind of many: quantum mechanics is a deformation of classical mechanics. That developed into what became known as *deformation quantization* and its manifold avatars and more generally into the realization that quantization is deformation. This stumbling block being removed, the paramount importance of deformations in theoretical physics became clear [Fl82], giving rise to "Flato's deformation philosophy".

2 The essence of deformation quantization

For a quasi-complete (not in the topological vector spaces sense!) overview of the "state of the art" at the turn of the millennium, see e.g. [DS01], including references therein. Shorter later presentations can be found in e.g. [St08] and [St11] and references therein. To make the presentation slightly self-contained, we mention here some main points, indicating in particular that deformation quantization *is* quantization, without the sometimes Procrustean bed of a Hilbert space formulation – something that may seem heretic to those bred within the "Copenhagen interpretation" taken restrictively,

2.1 The founding papers in the 70's and around

2.1.1 Classical mechanics

In its Hamiltonian formulation, classical mechanics is based on a phase space, a symplectic or more generally Poisson (see e.g. [BFFLS]) manifold W on which a function H (the Hamiltonian) expresses the dynamics of the system considered: W is a differentiable manifold on which is defined a skewsymmetric contravariant tensor π (not necessarily nondegenerate, and which can be expressed in local coordinates as $\pi = \sum_{i,j=1}^{2n} \pi^{ij}\partial_i \wedge \partial_j$) such that $\{F, G\} = \pi(dF \wedge dG)$ (in local coordinates, $\{F, G\} = \sum_{i,j} \pi^{ij}\partial_i F \wedge \partial_j G$) is a Poisson bracket P, i.e. the bracket $\{\cdot, \cdot\}\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a bilinear map which is skewsymmetric ($\{F, G\} = -\{G, F\}$) and satisfies the Jacobi identity $\{\{F, G\}, H\}+\{\{G, H\}, F\}+\{\{H, F\}, G\} = 0$ and the Leibniz rule $\{FG, H\} = \{F, H\}G + F\{G, H\}$.

For symplectic manifolds the 2-tensor π is everywhere nondegenerate (it then has an inverse, a closed 2-form ω; closedness is equivalent to the Jacobi identity). The simplest example is $W = \mathbb{R}^{2n}$ with coordinates (q_α, p_α), $\alpha = 1, \ldots, n$, e.g. m particles in 3-space with $n = 3m$. The motion of a particle in 3-space is invariant under the Galileo group $SO(3) \cdot \mathbb{R}^3 \cdot \mathbb{R}^4$ (space rotations, velocity translations and space-time translations, respectively). The Poisson bracket of two classical observables (functions F and G) has then the well known expression $\{F, G\} = \sum_\alpha \frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial G}{\partial q_\alpha}\frac{\partial F}{\partial p_\alpha}$. The Hamiltonian is a real-valued function $H(q, p)$ on phase space and Hamilton's equations of motion for an observable F (e.g. a coordinate) are $\frac{dF}{dt} \equiv \dot{F} = \{H, F\}$. An important example of Poisson manifold that is not symplectic is the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. Any Poisson manifold is "foliated" by symplectic leaves (in general, of variable even dimension).

2.1.2 Quantum mechanics

The idea of quantization arose around 1900 when Planck proposed the quantum hypothesis: the energy of light is not emitted continuously but in quanta proportional to its frequency. Einstein's 1905 theory of the photoelectric effect (which was the reason for which he eventually was awarded in 1923 the 1922 Nobel prize in physics) builds on that idea, as well as Bohr's 1913 model for an atom with "quantized" orbits for electrons around the nucleus. But the real beginning of quantum mechanics dates from the mid twenties when Louis de Broglie suggested in 1923 what he called "wave mechanics" based on the (somewhat schizophrenic) idea that waves and particles are two aspects of the same physical reality. The idea was shortly afterward confirmed with the discovery of electron diffraction by crystals in 1927 by Davisson and Germer, and Louis de Broglie was awarded the 1929 Nobel Prize in physics. A couple of years after de Broglie, Schrödinger, Heisenberg, Weyl and eventually Bohr (inter alia) "translated into German"[2] de Broglie's idea and developed the quantum mechanics that we know, with for observables operators in a space introduced some years before by Hilbert, and the "Copenhagen" probabilistic interpretation that comes with it – which until now a number of eminent (and less eminent) physicists are not entirely happy with.

In the traditional quantization of a classical system $(\mathbb{R}^{2n}, \{\cdot,\cdot\}, H)$ we take a Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n) \ni \psi$ in which acts a "quantized" Hamiltonian $\hat{H}$, the energy levels of which are defined by an eigenvalue equation $\hat{H}\psi = \lambda\psi$. An essential ingredient is the von Neumann representation of the canonical commutation relations (CCR) for which, defining the operators $\hat{q}_\alpha(f)(q) = q_\alpha f(q)$ and $\hat{p}_\beta(f)(q) = -i\hbar\frac{\partial f(q)}{\partial q_\beta}$ for f differentiable in $\mathcal{H}$, we have (CCR) $[\hat{p}_\alpha, \hat{q}_\beta] = i\hbar\delta_{\alpha\beta}I$ $(\alpha, \beta = 1, ..., n)$. We say that the couple $(\hat{q}, \hat{p})$ "quantizes" the coordinates (q, p). A polynomial classical Hamiltonian H is quantized once chosen an operator ordering [AW70], e.g. the (Weyl) complete symmetrization of $\hat{p}$ and $\hat{q}$. In general the quantization on $\mathbb{R}^{2n}$ of a function $H(q, p)$ with inverse Fourier transform $\tilde{H}(\xi, \eta)$ can be given by (that formula is already in Weyl [We27] when the weight is $\varpi = 1$):

$$H \mapsto \hat{H} = \Omega_\varpi(H) = \int_{\mathbb{R}^{2n}} \tilde{H}(\xi, \eta)\exp(i(\hat{p}.\xi + \hat{q}.\eta)/\hbar)\varpi(\xi, \eta)d^n\xi d^n\eta. \tag{2.1}$$

The weight ϖ is, in most orderings, the exponential of a second order polynomial ($\simeq \xi\eta$ or $\simeq \xi^2 \pm \eta^2$ for the main orderings).

The map Ω_1 given by (2.1) has an inverse, obtained by Wigner [Wi32] which (when the expression is defined) can be written as:

$$H = (2\pi\hbar)^{-n}\mathrm{Tr}[\Omega_1(H)\exp((\xi.\hat{p} + \eta.\hat{q})/i\hbar)] \tag{2.2}$$

The commutator of two operators comes then from what is now called the Moyal bracket of the corresponding classical observables:

$$M(u_1, u_2) = \nu^{-1}\sinh(\nu P)(u_1, u_2) = P(u_1, u_2) + \sum_{r=1}^{\infty} \frac{\nu^{2r}}{(2r+1)!}P^{2r+1}(u_1, u_2) \tag{2.3}$$

where P denotes the Poisson bracket and the "deformation parameter" is $\nu = i\frac{\hbar}{2}$ while the product of operators comes from:

$$u_1 \star_M u_2 = \exp(\nu P)(u_1, u_2) = u_1 u_2 + \sum_{r=1}^{\infty} \frac{\nu^r}{r!}P^r(u_1, u_2) \tag{2.4}$$

the two being related by $M(u_1, u_2) = \frac{1}{i\hbar}(u_1 \star_M u_2 - u_2 \star_M u_1)$. We recognize in the right-hand sides of (2.3) and (2.4) the formulas for deformations of algebras

[2]Compare with Goethe's quote: *Die Mathematiker sind eine Art Franzosen. Spricht man zu ihnen, so übersetzen sie alles in ihre eigene Sprache, und so wird es alsobald etwas ganz anderes.* (Mathematicians are a kind of Frenchmen. Whenever you talk to them, they translate everything into their own language, and right away it becomes something completely different.)

(in this case, Lie with bracket P and associative with usual product of functions, resp.) in the sense of Gerstenhaber [Ge64] where the Chevalley–Eilenberg (resp. Hochschild) cochains giving the deformation are (resp.) $\frac{P^{2r+1}}{(2r+1)!}$ and $\frac{P^r}{r!}$.

2.1.3 Deformation quantization

In the early 70's, having in mind the deformation philosophy, we started to study 1-differentiable deformations (in the sense of Gerstenhaber) of the Lie algebra of classical observables ("functions" on phase space) endowed with the Poisson bracket [FLS75]. Then Jacques Vey [Ve75], inspired by our works, showed the existence of differentiable deformations of the Poisson bracket Lie algebra $A = C^\infty(W)$ (of functions on a symplectic manifold W with vanishing 3rd Betti number b_3). Doing so he rediscovered the Moyal bracket [Mo49] and the Hochschild cohomology of the associative algebra structure on A, which turned out to follow from [HKR62], both of which (like most physicists and mathematicians at the time) he was unaware of. That in turn triggered our development of what people now call "the founding papers" [BFFLS] and deformation quantization.

In keeping with the style of this presentation in which we insist on the conceptual aspect and refer to the bibliography (and references therein) for a little more details, we shall briefly give some characteristic features, including physical examples, showing that we can indeed say in confidence that *quantization is deformation*, deformation being of course a more general notion (cf. e.g. deformations within the Lie algebra category).

Let W be a differentiable manifold (of finite, or possibly infinite, dimension). We assume given on W a Poisson bracket P.

Definition 2.1. A *star product* on W is a deformation of an associative algebra of functions, e.g. $A = C^\infty(W)$, of the form $\star = \sum_{n=0}^{\infty} \nu^n C_n$ with $C_0(u,v) = uv$, $C_1(u,v) - C_1(v,u) = 2P(u,v)$, $u, v \in A$, the C_n being bidifferential operators (locally of finite order). A star-product is *strongly closed* if it satisfies a trace condition, $\int_W (u \star v - v \star u)\, dx = 0$ where dx is a volume element on W.

In deformation quantization, quantization is not performed with a drastic change in the nature of observables, but understood as a deformation of the abelian composition law of these observables, piloted by the Poisson bracket. It is *more* than "a reformulation of the problem of quantizing a classical mechanical system" [DN01], it *is* quantization, in spite of what two brilliant scientists wrote recently [GW08], after very nice words on the approach (viewed mostly from a 'stringy' perspective): "deformation quantization is not quantization. [...] It does not lead to a natural Hilbert space H on which the deformed algebra acts." In other words, the "original sin" of deformation quantization would be that (in the general case) we do not need the sacrosanct Hilbert space required by the Copenhagen interpretation. Nevertheless deformation quantization is in line with a prophetic general statement by Dirac [Di49], which applies to many situations in physics:

Two points of view may be mathematically equivalent, and you may think for that reason if you understand one of them you need not bother about the other and can neglect it. But it may be that one point of view may suggest a future development which another point does not suggest, and although in their present state the two points of view are equivalent they may lead to different possibilities for the future. Therefore, I think that we cannot afford to neglect any possible point of view for looking at Quantum Mechanics and in particular its relation to Classical Mechanics.

What Dirac had then in mind is certainly the quantization of constrained systems which he developed shortly afterward [Di50]. In that case the "canonical quantization" mentioned in Section 2.1.2 cannot be used. Dirac's "by hand" method is a typical example of quantizing symplectic manifolds (with second class constraints) and Poisson manifolds (with first class constraints) (see e.g. [Li75]).

Dirac's statement applies even better to deformation quantization, which (see below) works for any (finite dimensional) Poisson manifold. When there are Weyl (2.1) or Wigner (2.2) maps between an algebra of functions and an algebra of operators, the two formalisms are (more or less) equivalent. If that is not the case, deformation quantization is basically what is left. Without the Hilbert space restrictive frame there may sometimes be too much freedom in deformation quantization, depending on how it is performed (e.g. with star-spectral equations), and some "auxiliary conditions" are then needed (as noted also by Gukov, private communication), but in my view that is preferable to the restrictive approach of geometric quantization, which proved to be very powerful in representation theory of Lie groups (in particular solvable) but with which only few observables could be quantized. These "auxiliary conditions" were in fact "built in" the examples treated in [BFFLS]. Indeed, these follow from the closed formulas which we obtained (in examples, with some effort and a little bit of luck) for the analog of the unitary evolution operator, the *star exponential* $\mathrm{Exp}_*(\frac{tH}{i\hbar}) = \sum_{r\geq 0} \frac{1}{r!}(\frac{t}{i\hbar})^r H^{*r}$ (where $2\nu = i\hbar$ and H^{*r} denotes the r^{th} star power of H). It is a singular object, in particular it does not belong to the quantized algebra $(A[[\nu]], *)$ but to $(A[[\nu, \nu^{-1}]], *)$. *Spectrum and states* are given by a "spectral" (Fourier-Stieltjes in the time t) decomposition of the star-exponential.

Examples. In order to show that a star-product provides an *autonomous* quantization of a manifold M we treated in [BFFLS] a number of examples. For the harmonic oscillator $H = \frac{1}{2}(p^2 + q^2)$, with the Moyal product on $\mathbb{R}^{2n}$, we obtain $\mathrm{Exp}_*(\frac{tH}{i\hbar}) = (\cos(\frac{t}{2}))^{-1} \exp(\frac{2H}{i\hbar} \tan(\frac{t}{2})) = \sum_{k=0}^{\infty} \exp(-i(k + \frac{n}{2})t)\pi_k^n$ where π_k^n can be expressed as a function of H. As expected the energy levels of H are $E_k = \hbar(k + \frac{n}{2})$. With normal ordering, $E_k = k\hbar$. While $E_0 \to \infty$ for $n \to \infty$ in Moyal ordering, $E_0 \equiv 0$ in normal ordering, thus preferred in Field Theory.

For such a formal series of formal series to be well-defined as a formal series, we need that the coefficients of the resulting powers of $\hbar$ be finite. That requirement is here a "built in" auxiliary condition, expressed in the closed formulas obtained, well defined as distributions in both p, q and t.

That one-parameter group (with parameter t) can be completed to what we call a *star representation* of (a 2-fold covering of) the Lie group $\mathrm{SL}(2, \mathbb{R})$, corresponding to the metaplectic representation. In retrospect we were somewhat lucky because what corresponds to the trace of operators is the integral over phase space ($\mathbb{R}^{2n}$ here) of the corresponding functions (or distributions), which is the analog of the character of the representation. It is one of the tools which permit a comparison with usual representation theory and is often singular at the origin in irreducible representations (e.g. as shown by Harish Chandra, for semi-simple Lie groups). That requires caution in computing the star exponential. But in the case of the harmonic oscillator, the difficulty is masked by the fact that the corresponding representation of the Lie algebra $\mathfrak{sl}(2)$ generated by (p^2+q^2, p^2-q^2, pq) is integrable to a double covering of $\mathrm{SL}(2, \mathbb{R})$ and decomposes into a sum, usually denoted by $D(\frac{1}{4}) \oplus D(\frac{3}{4})$: the singularities at the origin cancel each other for the two components. This made possible the computation of the above closed formula for the star exponential of the compact generator H, and provided implicitly the required auxiliary conditions, that do not appear when trying to compute directly the "star spectrum" of H with an equation of the type $H * \pi_k = E_k \pi_k$.

Other standard examples can be quantized in an autonomous manner by choosing adapted star products, e.g. the angular momentum with spectrum $k(k + (n-2))\hbar^2$ for the Casimir element of $\mathfrak{so}(n)$ and the hydrogen atom with $H = \frac{1}{2}p^2 - |q|^{-1}$ on $M = T^*S^3$, $E = \frac{1}{2}(k+1)^{-2}\hbar^{-2}$ for the discrete spectrum, and $E \in \mathbb{R}^+$ for the continuous spectrum; etc.

2.2 Some of the main progresses in the 80's

The "founding papers" created among many a strong interest in the new notion. The idea was "in the back of the mind" of most of those who dealt with quantum mechanics [one even wrote us, asking to be quoted for that reason, but we did not know how to refer to the back of the mind of that person!]. But the Hilbert space formalism created a "quantum jump" (from classical observables to operators in Hilbert space) that was hard to express. It was only when we dared the unconventional approach – may be indirectly related to de Broglie's idea that particles and waves are two manifestations of the same physical reality – to look at quantization as a deformation of the same algebra from commutative to noncommutative, that (as Commissaire Maigret says when he discovers "whodunit": "But of course!") what is for us the essence of quantization became clear.

The notion of *equivalence* of star products is the standard one for deformations, a formal series of linear maps (here necessarily differential operators) intertwining two star products. By equivalence any star product can be brought to one for which the function 1 is still a unit (that follows from a general result of Gerstenhaber [GS88] on deformations leaving a subalgebra invariant). The *existence* of star products was shown in increasing generality, first in the "founding papers", then for symplectic manifolds with Betti number $b_3 = 0$, eventually for all symplectic manifolds in the 80's, and later for Poisson manifolds. We shall briefly come back to these issues in the general context of Section 3.1. For some more details and references see e.g. [DS01].

Of particular interest are the notions of invariance and of covariance of star products, which occupy a significant part of the first "founding paper" and shortly afterward gave rise to the interesting notion of "star representations" (without operators) of Lie groups and algebras.

2.3 "Metamorphoses" in the 80's, quantum groups and noncommutative geometry

Starting from different premises, important developments appeared in the 80's which, a posteriori and though additional notions are involved, can be considered as metamorphoses or avatars of deformation quantization. For the record we mention here very briefly the main two.

The first is the notion of what are now called quantized enveloping algebras of Lie groups, which appeared in works by the Faddeev school in Leningrad around 1980, in relation with the inverse problem for two-dimensional integrable models in quantum field theory. A few years later Drinfeld made the notion more systematic as deformations of Hopf algebras and popularized it under the name "quantum groups" together with their "dual" notion of star products on functions on Lie groups. (It is a dual in the sense of topological vector spaces duality, as shown by us in the 90's, see e.g. the review part of [BGGS].) Several books have been dedicated to the subject and its applications, with thousands of references.

A second major development is the advent, from the beginning of the 80's and motivated by his seminal works on operator algebras in the 70's, of Alain Connes' noncommutative geometry [Co94] of which (as we showed in a joint paper) star products algebras (having a trace) constitute another example. That is since then a very active frontier domain of mathematics with a variety of applications, including in physics, and a dedicated journal.

3 Some highlights of the last two decades

We mention here very briefly a few highlights, in particular those related to the published works of Neumaier (in good journals, between 1998 and 2010). Again this section is not meant to be exhaustive. Slightly less concise presentations with more references can be found in [DS01] and later reviews.

3.1 Existence and Classification (symplectic and Poisson manifolds)

The question of existence of star products was first solved in the 80's for symplectic manifolds [DL83], based on the idea of gluing local Moyal star-products defined on Darboux charts, using algebraic and cohomological tools. Together with their classification, that became understood geometrically only in the 90's.

For symplectic manifolds the idea is to "glue" together Moyal products on symplectic (Darboux) charts: Since the Moyal product on a chart is unique (up to equivalence) that is always possible on the intersection of two charts, but problems occur on the intersection of 3 charts, except when $b_3 = 0$ (as we mentioned in Section 2.3). The idea (which has received a number of formulations, some quite sophisticated) is essentially to use, instead of the original manifold W, a bundle of Weyl algebras (CCR) on W, obtain a global section and project it on W. That was used in two (related) forms. A "local" form (on sizeable charts) was developed by a Japanese group [OMY] and shortly afterward an alternative to the original construction of [DL83] was developed by its authors [DL92].

In 1985, motivated by index theory and independently of the previous proofs, Fedosov (who unfortunately died recently) announced a purely geometrical construction of star-products on a symplectic manifold, also based on Weyl algebras, but that second method was noticed only when a complete version was published in an international journal [Fe94]. The beautiful algorithmic construction of Fedosov (in which the Maurer-Cartan equation has a crucial role) not only provides a novel proof of existence, it also gives a much better understanding of deformation quantization, paving the way to further major developments. The construction and its context were developed in a book by Fedosov and outlined in many papers, including in our review [DS01] where many references can be found.

The relations between the "Russian" (Fedosov) approach and that of the "Belgian" team were creatively expressed in his own language by a famous Belgian mathematician (with Russian wife) in an interesting paper published in Gelfand's journal [De95]. Though (unfairly) it ignores the above mentioned Japanese approach, the paper deserves to be better known, in particular in view of this milenium's developments (see below) using languages of gerbes, stacks, etc.

The classification of equivalence classes of star products for *symplectic* manifolds can be obtained from there. More concretely it follows from the fact (Nest and Tsygan [NT95]) that any differentiable star product $\star$ on a symplectic manifold M is equivalent to a star product constructed with Fedosov's algorithm. This permits to define the characteristic class of a star product as the class of Weyl curvature $H^2_{dR}(M)[[\nu]]$ (formal series in de Rham cohomology) associated to any Fedosov star-product equivalent to $\star$. As a consequence, one gets a parametrization of the equivalence classes of star-products on (M, ω) by elements in $H^2_{dR}(M)[[\nu]]$. That parametrization of equivalence classes of star-products has also been made explicit by Bertelson, Cahen and Gutt [BCG97]. They are in (1–1) correspondence with formal deformations of the symplectic form,

The case of Poisson manifolds was harder to deal with (except for regular Poisson manifolds, for which all the symplectic leaves have the same dimension). Kontsevich first showed [Ko96] that what was then his "formality conjecture" implies that any Poisson manifold can be formally quantized, giving strong evidence for the conjecture to be true. A year later he came with a seminal work [Ko97] that has been extensively rewritten and developed by many, including himself (e.g. [Ko99] after Tamarkin came with an operadic approach [Hi03]). These papers contain many important notions and results, far beyond existence and classification (in (1–1) correspondence with formal deformations of a Poisson tensor).

Super-Remark. Supersymmetry became popular in the 70's. [Incidentally a first example of the super Poincaré algebra can be found in [FH70] where the spinorial translations, an $\mathbb{R}^4$ Lie algebra supplementing the $\mathbb{R}^4$ algebra of vector space-time translations, were at some point multiplied by an anticommuting operator denoted by F, in effect producing the super Poincaré algebra. Both Wess and Zumino told me much later that they did not know of the paper.] A natural consequence of the introduction of supersymmetry was to develop, especially in the 80's and 90's, "supersymmetric quantum mechanics" (see e.g. an excellent review in [CKS95]). The extension of deformation quantization to supermanifolds was thus a natural thing to do. We mentioned the issue in some early papers but it is only at the end of the 90's that a number of scientists (including Neumaier) considered it in a more precise way (see e.g. [Bo00]). So far that did not go beyond adapting Fedosov's construction to the context of supermanifolds. [Bordemann informed me recently that at the time he put on the problem a good student, who unfortunately left science shortly after.] A nice "warm up exercise" for a good graduate student would thus be to start with treating the supersymmetric harmonic oscillator in deformation quantization.

3.2 More general context (varieties and singular spaces, Lie groupoids and algebroids, field theory, etc.)

In the past decade the geometrical context of deformation quantization (DQ) has been extended from manifolds to algebraic geometry and a variety of more general structures, real or complex, often singular in some sense. DQ became also more used in physics, unfortunately not (yet) so much in developing rigorously new examples of "autonomous quantization" (when needed without Hilbert space but with appropriate auxiliary conditions) but often, at best at the "physics level of rigor", in the strings framework and in field theory on "noncommutative space-time" (an approach developed in particular in works by Grosse, Rivasseau, and coworkers, that can be found on arXiv). Here also these works are too numerous to be detailed, or even all mentioned, in such a short overview. Some are quoted in a recent review [St11] (and references therein). We shall be satisfied with indicating a few directions closer to the original formulation of DQ.

Already in the 90's (see e.g. [NT95] and later works) Nest and Tsygan had extended the initial framework of DQ in various directions, in particular various kinds of index theorems (these had earlier brought to DQ Fedosov, and Connes from a different point of view, see e.g. [Co94] and references therein). With other coworkers they obtained a variety of sophisticated extensions (see e.g. a couple of the more recent [DTT09, BGNT11]), somewhat in the spirit of [De95], using gerbes, algebroid stacks, etc., and the connection with formality crucial in Kontsevich [Ko96, Ko97]. Listing all these works and their content would be a long review paper in itself, so we shall stop here.

At this stage it should be clear that index theorems and DQ are intimately related. In fact this should have been clear to us from the start. Indeed in 1963/64 I participated (together with Louis Boutet de Monvel and others) in [CS64], where my part was the multiplicative property of the analytic index of elliptic operators. But neither I until we had completed our papers [BFFLS], nor Boutet de Monvel (see his footnote in the obituary for Moshé Flato in Gazette des Mathématiciens, No 81, 1999) until much later (nor even Flato who attended part of the Seminar), realized that, like Mr. Jourdain speaking prose, we were then dealing with star-products of symbols! The initial index theorem was soon extended (see Atiyah's exposé in [CS64]) to manifolds with boundary. Natural (nontrivial) extensions are then to cones and manifolds with corners, and to algebraic varieties and to singular spaces.

The former gave rise to what is often called the "Melrose *b*-calculus" and its generalizations, see e.g. [LP05, MR11], the connection of which with DQ has not been much studied. The same can be said of the related sophisticated new approach to geometric quantization (in particular using higher structures such as gerbes) developed in past years by Mathai and coworkers (see e.g. [BMW12]).

It is important to remember that when one goes beyond the differentiable framework (see e.g. (2.3.2.2) in [St11]), the situation can change significantly. E.g. the Harrison cohomology does not vanish in general, permitting nontrivial abelian deformations, which could be of interest to quantize Nambu brackets (replacing the usual product with an abelian deformation in the defining matrices) in a more direct manner than what was done in [DFST]. The cases of complex analytic manifolds and of algebraic varieties present many intricacies, as indicated in [St11]. Specific examples of interest can be seen in [FK07] and [Fr09] (on the *closure* of minimal coadjoint orbits, a situation which resisted "conventional" treatments as mentioned in [GW08]). The treatment of the Berezin-Toeplitz quantization reviewed in [Sc10] is also relevant to geometric objects in the complex context.

3.3 Some words on highlights in two related "avatars" (quantum groups and quantized spaces)

A transition to the next section is provided by this "tachyonic" overview of what can be considered (at least from a chronological point of view) two major avatars of DQ. Many frontier works continue to appear in both. We shall here be satisfied with mentioning their existence and give a minimum of relevant references. We have mentioned before the relation with quantum groups, presented e.g. in [BGGS]. Both "avatars" show up also in recent reviews such as [St08, St11]. A natural (highly nontrivial) combination by Nest and Tsygan (to be posted soon) of all these aspects of quantization is the simultaneous quantization, in a compatible way, of all three notions involved in Hamiltonian actions of Poisson-Lie groups on Poisson manifolds.

Among the many works in the framework of noncommutative (NC) geometry we mention here only the NC manifolds of Connes and coworkers, developed in the Riemannian context. That approach can be seen as "dual" of the quantum groups algebra approach (in the same sense as, for a commutative topological algebra, its "spectrum" is the Gelfand dual). They study in particular noncommutative spheres (of low dimensions) cf. e.g. [CD03]. These are realized as "spectral triples" $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is some algebra acting on a Hilbert space $\mathcal{H}$ and D is a "Dirac operator" with compact resolvent such that $[D, a]$ extends to a bounded operator on $\mathcal{H}$. The idea is to generalize Riemannian geometry to the noncommutative setting.

In Section 4.2.5 we briefly indicate *how* a similar approach can be developed, mutatis mutandi (we [BCSV] are in the Lorentzian framework, not the Euclidean version), in the case of an hyperbolic sphere AdS, and *what* it may conjecturally be good for. The main part of next Section deals with new ideas around symmetries and their quantization. Hopefully that will explain *why* we would like to promote such a potentially revolutionary general framework,

4 Conjectural perspectives around quantized Anti de Sitter (AdS)

4.1 Deforming symmetries: Poincaré to anti de Sitter

As we mentioned in the Introduction, in 1964, shortly after Flato's arrival in Paris and 7 years before Neumaier was born, appeared the founding paper by Gerstenhaber [Ge64] on deformation theory. It became gradually clear to many that the idea of deformation is crucial in physics, first from the symmetries point of view, but eventually also as expressed in what I call "Flato's deformation philosophy" [Fl82].

Practically everybody concurs that two major breakthroughs revolutionized physics: relativity (special and general) and quantum mechanics. Both occurred in the first half of the twentieth century and, in spite of many advances, reconciling them is not yet achieved. We have seen that, in the second half of last century, both were interpreted as deformations of the mathematical frameworks associated with previously accepted theories. In this short paper we concentrated so far on the latter aspect, quantization. We shall now try and consider both as simultaneously as possible, from the point of view of the deformation philosophy.

Dealing with symmetries of particles, from the point of view of particle interpretation, for massless particles at least [AFFS], since we want that the momentum of particles be bounded below, among the only two possible choices in the Lie group category, the natural deformation of the Poincaré group is not the de Sitter group $SO(1,4)$ but the anti de Sitter (AdS) group $SO(2,3)$ (or some covering of it) and its most degenerate (lowest weight) representations. Flat Minkowski space-time is then deformed to a (negatively curved) AdS universe. That has, since the end of the seventies, given rise to many papers (in part mentioned in [St08] and references therein). These deal with what we call "singleton physics". In particular *massless particles are composite*, not only kinematically (from the point of view of symmetries) but also dynamically for the photon (now the only truly massless particle), in a manner compatible with quantum electrodynamics à la Dyson [FF88]. Later it was shown, combining the $U(2)$ symmetry of electroweak interactions with flavor symmetry, that leptons may also be considered [Frø00] as initially massless composites of singletons, massified by interactions with 5 pairs of Higgs-like particles; such a model predicts the existence of two new mesons ("flavor analogs" to the W and Z mesons of electroweak theory), albeit with a large mass difference due to the large mass differences between the three lepton generations – unless, somewhat like for massive neutrinos, the "physical" bosons are linear combinations of those appearing in the theory. An important question is then, in such a space-time based approach, how to deal with hadrons (heavier strongly interacting particles). The very ambitious "deformation framework" that we sketch in the remainder of the paper might, as a by product, provide answers to such a question.

4.2 Some bold mathematical ideas around quantized AdS at root of unity and "affinizations"

4.2.1 qAdS symmetry at root of unity

It is a known fact among specialists that quantum groups at root of unity have special properties. In particular it has been observed in 1993 [FHT93] that the quantized AdS group at even root of unity has finite-dimensional unitary irreducible representations (UIRs). The fact was later rediscovered and somewhat extended in several papers, in particular [Sta98]. As we mention in the introduction of [BCSV], one is then tempted to consider Minkowski space-time and the Poincaré group as limits of these qAdS counterparts when $q\rho \to -0$, where $q = \mathrm{e}^{i\theta}$ is the quantum group deformation parameter and ρ the curvature of AdS space.

A natural idea is then to try and use such UIRs as possible substitutes to the unitary symmetries of particle spectroscopy. Quantum groups, being deformations of the Hopf algebras associated with Lie groups and enveloping algebras, can be expected to behave nicely with respect to tensor product. That is probably true of the generic case but the case of even root of unity seems to be special. In particular already for what they call the finite-dimensional quantum group associated to $\mathfrak{sl}(2)$, when the quantum parameter q is a $2p^{\text{th}}$ root of unity, it has been recently shown in [KS11] that strange things may happen, showing in particular that the category of representations cannot be braided; the case of higher ranks seems hopelessly wilder[3].

Nevertheless this does not mean that for physical applications these symmetries cannot be considered. On the contrary, with some luck, the situation could turn out to be better, since "nice" behavior could be restricted to a few exceptional cases for qAdS. That is a difference between the approaches of mathematicians and of physicists in many contexts: mathematicians (even when they start 'in petto' with specific examples) tend to study general cases, while physicists care mainly for the particular cases needed for their models or theory, which usually means low ranks and low dimensions. The idea would then be to try such (so far hypothetic) very special qAdS representations as alternatives for the (compact) "internal symmetries" (unitary groups) empirically (and successfully, see below) used for more than 60 years to classify elementary particles. The advantage of such an approach is that it would give conceptual foundation to an alternative to the empirically introduced symmetries. The price to pay is that it requires hard mathematics, which only now might become within reach.

4.2.2 "Superized" and/or affine qAdS

The above idea is however possibly too naive, in particular since one cannot at present be satisfied with simple particle spectroscopy as one was in the sixties. Since for hadrons (strongly interacting particles) half-integer spins occur, a natural idea is to complete $\mathfrak{so}(2,3) = \mathfrak{sp}(\mathbb{R}^4)$ to its graded extension by adding (like for the Wess-Zumino super-Poincaré algebra or implicitly earlier in [FH70]) four spinorial translations whose anticommutators give $\mathfrak{sp}(\mathbb{R}^4)$. That is what happens when realizing the latter (as two coupled harmonic oscillators) with generators that are quadratic polynomials in 4 variables p_1, p_2, q_1, q_2 (see e.g. [Frø82]). The quantized versions of such superalgebras were and still are studied (see e.g. [AB97, CW12]). It would however be interesting to study further what happens for superized qAdS

[3] I thank M. Jimbo for drawing my attention to these facts.

at even root of unity, in particular not only for the representations themselves but also for their tensor products.

More importantly, the dynamics of the various interactions should now be part of the picture. That requires at some point to take into account singularities. A standard mathematical way to tackle such a situation, in the spirit of Hironaka[4] is to "blow up" the singularity by introducing extra dimensions. [That is also one of the original motivations for string theory.]

Dealing with Lie groups that process is "affinization", passing to loop groups (mappings from a closed string S^1 to the original Lie group) or affine (simple) Lie algebras, or some central extension of these. These structures can in turn be quantized and their representations studied, which is an active topic in modern mathematics with many ramifications (see e.g. recent papers such as [HJ12, JRZ10]).

4.2.3 Generalized affinizations

One may be even bolder, in (at least) two directions which are so far largely virgin territory from the mathematical point of view. A first direction is to generalize loop groups to mappings from a higher dimensional manifold or variety M to a Lie group G. For example one may take for M a K_3 surface or a Calabi-Yau (complex) 3-fold, very popular in string theory, and for G the AdS group. The mathematical problems involved appear to be very hard. Trying to "quantize" such structures would be even harder. While it could very well be hopeless to try and develop a generic theory for such structures and their representations, which is what mathematicians tend to be interested in, it might be that here also, in particular when some form of discretization is possible (which, in a different context, is what 't Hooft very recently did [tH12]), specific cases could be manageable. That is what is of interest to physicists. It seems to be an experimental epistemological fact that often problems suggested by Nature turn out to be more seminal than problems imagined "out of the blue" by mathematicians.

4.2.4 Generalized deformations

Another general framework is to generalize the notion of deformation, beyond the theory of Gerstenhaber and even beyond multiparameter deformations, which are a natural extension that has been considered for quantum groups in the past two decades. In a couple of not so known papers [Pi97, Na98], G. Pinczon and his student F. Nadaud considered deformations in which the "deformation parameter" acts on the algebra to the left, to the right, or both ways, with interesting consequences.

But one can go even further and try to replace the scalar (complex) formal deformation parameter (an element of the group algebra over $\mathbb{C}$ of the trivial

[4] In a way also in the spirit of e.g. Cauchy in the 19$^{\text{th}}$ century, with notions such as the principal value of a divergent integral, which is a distribution in the sense of L. Schwartz.

group) with something more general. Remaining in the abelian context, for multiparameter deformations, the "parameter" can be viewed as an element of the group algebra of $\mathbb{Z}/n\mathbb{Z}$, which is the center of $SU(n)$. The theory of such deformations goes along the same lines as the "G-deformations" of algebras considered by Gerstenhaber.

Going further, this time in the nonabelian direction, one can first consider quaternionic deformation parameters, which do not seem to have been really studied. There have of course been numerous attempts to develop quaternionic quantum mechanics, and books have been written on the subject, including by leading scientists. The approach belongs to what Ray Streater (see his web site) calls "lost causes of physics". Nevertheless considering deformation quantization with deformations of algebras on the field of quaternions may have at least some mathematical interest, and developing a theory of quantum groups with such deformations may lead to interesting results.

More generally we could take as "parameter" elements of the group algebra of a finite group, e.g. the symmetric group S_n which is the Weyl group of $SU(n)$ and carries much of the information for its representation theory. That does not seem to have ever been considered; it is not even clear that the theory is still governed by some cohomology. Dealing with such deformations, starting from (the Hopf algebras associated with) the Poincaré or better AdS groups, would certainly bring interesting new mathematics. In combination with a generalization of "G-deformations" at root of unity and some "affinizations", that might eventually provide a more fundamental approach to the symmetries and dynamics of particle physics, as we indicate in Section 4.3 below.

4.2.5 Quantized AdS space and conjectural cosmological consequences

In [BCSV] we showed how to build "quantized hyperbolic spheres". More precisely we build (with a universal deformation formula [BGGS]) a (closed [Co94]) star product using an oscillatory integral, on a 1-dimensional extension $\mathcal{R}_0$ of the Heisenberg group (naturally endowed with a left invariant symplectic structure) and a Dirac operator D on the space $\mathcal{H}$ of a regular representation of $\mathcal{R}_0$. The star product endows the space $\mathcal{A}^\infty$ of smooth vectors in $\mathcal{H}$ with a noncommutative Fréchet algebra structure. We get in this way a noncommutative spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ à la Connes, but in a Lorentzian context, which induces on (an open $\mathcal{R}_0$ orbit $\mathcal{M}_0$ in) AdS space-time a pseudo-Riemannian deformation triple similar (except for the compactness of the resolvent) to the triples developed for quantized spheres by Connes et al. (see e.g. [CD03]). This "quantized AdS space" has an horizon which permits to consider it as a black hole (similar to the BTZ black holes [BHTZ], which exist for all AdS_n when $n \geq 3$).

For q an even root of unity, since the corresponding quantum AdS group has finite dimensional UIRs, such a quantized AdS black hole can be considered as "q-compact" in a sense to be made precise. As we mention in [BCSV, St07], at least in some regions of our universe, our Minkowski space-time could be, at very small distances, both deformed to anti de Sitter and quantized, to qAdS. These regions would appear as black holes which might be found at the edge of our expanding universe, a kind of "stem cells" of the initial singularity dispersed at the Big Bang. From these (that is so far mere speculation) might emerge matter, possibly first some kind of singletons that couple and become massified by interaction with dark matter and/or dark energy. Such a scheme could be responsible, at very large distances, for the observed positive cosmological constant – and might bring us a bit closer to quantizing gravity, the Holy Grail of modern physics.

4.3 Conjectural space-time origin of internal symmetries

4.3.1 On the connection between external and internal symmetries

In the mid-sixties, in view of the fundamental role of relativity in physics, a natural question was to know whether there was some connection between "external symmetries" (in particular the Poincaré group) and the empirically discovered "internal symmetries". We co-organized a CNRS Colloque on the topic in April 1966. At that time the "internal symmetries" were mainly the $SU(3)$ of the "eightfold way" and some subalgebras. Later color $SU(3)$ was introduced, followed with QCD to express the dynamics, 'grand unified' symmetries (e.g. $SU(5)$), and eventually the 'standard model'. See e.g. a short presentation in [Ra10].

The representations of the internal symmetries gave "nice boxes" into which one could fit many newly discovered elementary particles, and predict new ones that were later found (which eventually contributed to bring to Stockholm a most influential theoretician). As we explain in [St07], the prevailing trend (in spite of our objections that "it ain't necessarily so" due to mathematical problems) became that there is no connection except direct sum. In contradistinction with atomic or molecular spectroscopy where the (known) dynamics dictate the symmetry (e.g. a crystalline structure breaks the rotational symmetry), in this case the dynamics were eventually "invented" to fit the empirical symmetries (after the latter, e.g. $SU(3)$, changed somewhat their interpretation – but that is another story).

There is a part of self-fulfilling prophecy in the interpretations of raw experimental data that by default are made in the framework of the detailed and so far successful construct which constitutes the "standard model." The leaders of experimental groups are of course aware of the problem (private communication from Gerard 't Hooft), but no caveat is publicized. It could be desirable to apply to such physics recent developments in information theory such as those developed in a different context in [Re11], in order to make as much as possible model-independent the experimental data.

4.3.2 Is it necessarily so?

It eventually dawned on me that the problem of connection between symmetries, especially in the somewhat restrictive context of Lie algebras, could be a false problem. Namely, in line with our deformation philosophy, it would be quite natural that the "internal symmetries" of interacting particles *emerge* from the Poincaré symmetry of free particles via some process of (possibly generalized) deformation. We express this as follows:

Conjecture 4.1. The Deformation Conjecture. *Internal symmetries of elementary particles emerge from their relativistic counterparts by some form of deformation (possibly generalized, including quantization), along with "superization" and maybe a kind of "affinization".*

Internal symmetries, especially in the modern form of the standard model which so far fits so well the spectroscopy of elementary particles, can be seen as an ultimate paradigm of quantum mechanics. Their relativistic counterparts are of geometric origin. One of the hopes of modern developments is to reconcile both and quantize gravity. That is in particular the case of the 'strings framework' (as say some, e.g. David Gross) and of the approach of Alain Connes and coworkers to the standard model via noncommutative geometry (which very recently found a 20% correction to its previous prediction for the mass of the Higgs boson, fitting the mass of the boson discovered at LHC).

What we are saying here is that perhaps if and when, with a lot of work and a little bit of luck, some of the mathematical avenues sketched above are successful and can be made to fit the data coming from experiments (which may have to be re-examined step by step), the seeked reconciliation will in a way be a by-product. In any case the mathematical problems are worthy of attack – and can be expected to prove their worth by hitting back!

And if (in a generation or more) one avenue can be shown to fit experimental data, so much the better; one of the advantages on the experimental side is that the reconstruction of the puzzle can be achieved with the available tools, without need for more expensive ones which society can no more give us.

4.3.3 A tentative road map

The mathematical problems listed in Section 4.2 (which looks a bit like a mail order catalog) may be treated independently. But significant progress in most of them can take a long time and, as usual in research, new problems are bound to pop up. Nevertheless, since as we explained for physical applications we need only some special cases, albeit treated in much more details than a pure mathematician would be tempted to do, we shall now indicate a few directions which, with some luck, might produce within finite time the beginning of a foundation of internal symmetries on quantized relativistic symmetries, which is what we suggest here.

At first one could study some of these finite-dimensional UIRs of qAdS for q an even (possibly 6^{th} since we have 3 generations) root of unity, in particular their tensor products and whether one has there something like the singletons for AdS. Then we could try and see what can be said of their affinizations and of the representations of the latter. Another direction could be to look at such deformations over the quaternions, which might have "built in" at least some of the present internal symmetries. A related direction would be to check what can be said, in the same spirit, of generalized deformations with "parameter" in the group algebra of $\mathbb{Z}/n\mathbb{Z}$ or S_n, in particular for $n = 3$. We leave further problems to the imagination of the readers who would have the patience to read the various parts of this unusual paper.

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