Some naturally defined star products
for Kähler manifolds

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Abstract

We give for the Kähler manifold case an overview of the constructions of some naturally defined star products. In particular, the Berezin-Toeplitz, Berezin, Geometric Quantization, Bordemann-Waldmann, and Karbegov standard star product are introduced. With the exception of the Geometric Quantization case they are of separation of variables type. The classifying Karabegov forms and the Deligne-Fedosov classes are given. Besides the Bordemann-Waldmann star product they are all equivalent.

1   Introduction

One of the mathematical basis of quantization is the passage from the commutative world (i.e. the functions on the phase space manifold, also called classical observables) to the non-commutative world (i.e. non-commutative objects, the quantum observables associated to the classical observables). There exists different methods to achieve this. In *operator quantization* one assigns to the classical observables operators acting on a certain Hilbert space. In *deformation quantization* one deforms the point-wise commutative product of functions into a non-commutative product. In "first order" the direction of the deformation is given by the Poisson structure which governs the classical situation. It turns out that this can only be done on the level of formal power series over the algebra of functions. Such a product is called a star product.

In this article we give an overview of certain naturally defined star products in the case that our "phase-space manifold" is a Kähler manifold. There are constructions and classifications of star products in the symplectic and even in the more general Poisson case. And as a Kähler form is a symplectic form they fall into this classification. But we have an additional complex structure and are searching for star products respecting it in a certain sense. These will be the star products of separation of variables type as introduced by Karabegov [15], resp. Wick or anti-Wick type as considered by Bordemann and Waldmann [6]. We will give their definition below. Both constructions are quite different. Karabegov uses local constructions which globalize. Bordemann and Waldmann modified

Fedosov's approach accordingly to the Kähler setting. One of the important contributions of Nikolai Neumaier to the field was that he generalizes the construction of Bordemann-Waldmann and showed that there is a 1:1 correspondence of both constructions [23].

In this article we will first introduce the notion of a star product of separation of variables type, discuss the Karabegov construction and make some comments on the Bordemann-Waldmann construction. These methods work for arbitrary Kähler manifolds (even for pseudo-Kähler manifolds). Next, for quantizable compact Kähler manifolds (i.e. Kähler manifolds admitting a quantum line bundle) we explain the construction of the Berezin-Toeplitz star product $\star_{BT}$. With the help of the Berezin transform a dual and opposite star product to the Berezin-Toeplitz will be given, the Berezin star product $\star_B$. In addition, as another naturally defined star product the star product of geometric quantization $\star_{GQ}$ (which is not of separation of variable type) shows up. They are all equivalent, we will give the equivalence transformation. Moreover, we have the star product given by the Bordemann-Waldmann construction $\star_{BW}$ and Karabegov standard star product $\star_K$. We will give their Deligne-Fedosov class and their Karabegov forms. The Deligne-Fedosov form classifies star products up to equivalence. In contrast, the Karabegov form classifies star products of separation of variables type up to identity not only up to equivalence. The Karabegov standard star product has the same Deligne-Fedosov class as $\star_{BT}$. Hence, it is equivalent. The star product $\star_{BW}$ (at least in its original construction) has a different Deligne-Fedosov class. See Section 7 for detailed results.

The intention of this review is to stay rather short. No proofs are given, also there are only a limited number of references. For a more detailed exposition, see the review [35]. For more details of the Berezin-Toeplitz quantization scheme, also see [33], [34].

There is other interesting work of Nikolai Neumaier together with Michael Müller-Bahns on invariant star products on Kähler manifolds and quotients, which is not be covered here. Let me just mention them [24], [25].

It is a pleasure for me to acknowledge inspiring discussions with Pierre Schapira on the microlocal approach to symplectic geometry and to deformation quantization.

2   Geometric setup – star products

Let (M, ω) be a pseudo-Kähler manifold. This means M is a complex manifold and ω, the pseudo-Kähler form, is a non-degenerate closed $(1, 1)$-form. If ω is a positive form then (M, ω) is a honest Kähler manifold. Despite the fact, that here we are only interested in the Kähler case, we will need this more general setting for relating different star products in the Karabegov construction.

Denote by $C^\infty(M)$ the algebra of complex-valued (arbitrary often) differentiable functions with associative product given by point-wise multiplication. Ignoring the complex structure of M, our pseudo-Kähler form ω is a symplectic form. A Lie algebra structure is introduced on $C^\infty(M)$ via the *Poisson bracket* $\{.,.\}$. We recall its definition: First we assign to every $f \in C^\infty(M)$ its *Hamiltonian vector field* X_f, and then to every pair of functions f and g the *Poisson bracket* $\{.,.\}$ via

$$(2.1) \qquad \omega(X_f, \cdot) = df(\cdot), \qquad \{\, f, g \,\} := \omega(X_f, X_g) \ .$$

In this way $C^\infty(M)$ becomes a *Poisson algebra*.

As we will need it further down let me give the definition of a quantizable Kähler manifold already here. For a given Kähler manifold a *quantum line bundle* for (M, ω) is a triple (L, h, ∇), where L is a holomorphic line bundle, h a Hermitian metric on L, and ∇ a connection compatible with the metric h and the complex structure, such that the (pre)quantum condition

$$(2.2) \qquad \begin{aligned} \operatorname{curv}_{L,\nabla}(X, Y) &:= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = -\operatorname{i}\omega(X, Y), \\ &\text{in other words} \quad \operatorname{curv}_{L,\nabla} = -\operatorname{i}\omega \ , \end{aligned}$$

is fulfilled. If there exists such a quantum line bundle for (M, ω) then M is called quantizable. Not all Kähler manifolds are quantizable. Exactly those compact Kähler manifolds are quantizable which can be embedded as complex manifolds (but not necessarily as Kähler manifolds) into some projective space.

For our Poisson algebra of smooth functions on the manifold M, a *star product* for M is an associative product $\star$ on $\mathcal{A} := C^\infty(M)[[\nu]]$, the space of formal power series with coefficients from $C^\infty(M)$, such that for $f, g \in C^\infty(M)$

1. $f \star g = f \cdot g \mod \nu$,

2. $(f \star g - g \star f)/\nu = -\mathrm{i}\{f, g\} \mod \nu$.

The star product of two functions f and g can be expressed as

$$(2.3) \qquad f \star g = \sum_{k=0}^\infty \nu^k C_k(f, g), \qquad C_k(f, g) \in C^\infty(M),$$

and is extended $\mathbb{C}[[\nu]]$-bilinearly. It is called differential (or local) if the $C_k(\,,\,)$ are bidifferential operators with respect to their entries. If nothing else is said one requires $1 \star f = f \star 1 = f$, which is also called "null on constants".

Two star products $\star$ and $\star'$ for the same Poisson structure are called *equivalent* if and only if there exists a formal series of linear operators

$$(2.4) \qquad B = \sum_{i=0}^\infty B_i \nu^i, \qquad B_i : C^\infty(M) \to C^\infty(M),$$

with $B_0 = id$ such that $\quad B(f) \star' B(g) = B(f \star g).$

To every equivalence class of a differential star product its *Deligne-Fedosov class* can be assigned. It is a formal de-Rham class of the form

$$(2.5) \qquad cl(\star) \in \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \mathrm{H}^2_{dR}(M, \mathbb{C})[[\nu]]).$$

This assignment gives a 1:1 correspondence between equivalence classes of star products and such formal forms.

The notion of deformation quantization was around quite some time. See e.g. Berezin [2],[4], Moyal [22], Weyl [37], etc. Finally, the notion was formalized in [1]. See [12] for historical remarks. In the symplectic case different existence proofs, from different perspectives, were given by DeWilde-Lecomte [11], Omori-Maeda-Yoshioka [26], and Fedosov [13]. The general Poisson case was settled by Kontsevich [21].

In the pseudo-Kähler case we might look for star products adapted to the complex structure. Karabegov [15] introduced the notion of star products with *separation of variables type* for differential star products. Equivalently, Bordemann and Waldmann [6] introduced star products of Wick, and anti-Wick type respectively. There are two different conventions. In Karabegov's original definition a star product is of separation of variables type if in $C_k(.,.)$ for $k \geq 1$ the first argument is only differentiated in anti-holomorphic and the second argument in holomorphic directions. For clarification we call this convention *separation of variables (anti-Wick) type* and call a star product of *separation of variables (Wick) type* if the role of the variables is switched, i.e. in $C_k(.,.)$ for $k \geq 1$ the first argument is only differentiated in holomorphic and the second argument in anti-holomorphic directions. Unfortunately, we cannot simply retreat to one these conventions, as we really have to deal in the following with naturally defined star products and relations between them, which are of separation of variables type of both conventions.

3   Star product of separation of variables type

3.1   The Karabegov construction

Let (M, ω_{-1}) be a pseudo-Kähler manifold. We will explain the construction of Karabegov of star products of separation of variables type (anti-Wick convention), see [15, 16]. In this context it is convenient to denote the pseudo-Kähler form ω by ω_{-1}. We will switch freely between these two conventions.

A formal form

$$(3.1) \qquad \widehat{\omega} = (1/\nu)\omega_{-1} + \omega_0 + \nu\,\omega_1 + \dots$$

is called a formal deformation of the form $(1/\nu)\omega_{-1}$ if the forms ω_r, $r \geq 0$, are closed but not necessarily nondegenerate (1,1)-forms on M. Karabegov showed that to every such $\widehat{\omega}$ there exists a star product $\star$. Moreover he showed that all deformation quantizations with separation of variables on the pseudo-Kähler manifold (M, ω_{-1}) are bijectively parameterized by the formal deformations of the form $(1/\nu)\omega_{-1}$. By definition the *Karabegov form* of the star product $\star$ is $kf(\star) := \widehat{\omega}$, i.e. it is taken to be the $\widehat{\omega}$ defining $\star$. Karabegov calls the unique star product $\star_K$ with classifying Karabegov form $(1/\nu)\omega_{-1}$ *the standard star product.*

Let me sketch the principle idea of his construction. First, assume that we have such a star product $(\mathcal{A} := C^\infty(M)[[\nu]], \star)$. Then for $f, g \in \mathcal{A}$ the operators of left and right multiplication L_f, R_g are given by $L_f g = f \star g = R_g f$. The associativity of the star-product $\star$ is equivalent to the fact that L_f commutes with R_g for all $f, g \in \mathcal{A}$. If a star product is differential then L_f, R_g are formal differential operators. Now Karabegov constructs his star product associated to the deformation $\widehat{\omega}$ in the following way. First he chooses on every contractible coordinate chart $U \subset M$ (with holomorphic coordinates $\{z_k\}$) its formal potential

$$(3.2) \qquad \widehat{\Phi} = (1/\nu)\Phi_{-1} + \Phi_0 + \nu\Phi_1 + \ldots, \qquad \widehat{\omega} = i\partial\bar{\partial}\widehat{\Phi}.$$

Then the construction is done in such a way that the left (right) multiplication operators $L_{\partial\widehat{\Phi}/\partial z_k}$ $(R_{\partial\widehat{\Phi}/\partial \bar{z}_l})$ on U are realized as formal differential operators

$$(3.3) \qquad L_{\partial\widehat{\Phi}/\partial z_k} = \partial\widehat{\Phi}/\partial z_k + \partial/\partial z_k, \quad \text{and} \quad R_{\partial\widehat{\Phi}/\partial \bar{z}_l} = \partial\widehat{\Phi}/\partial \bar{z}_l + \partial/\partial \bar{z}_l.$$

The set $\mathcal{L}(U)$ of all left multiplication operators on U is completely described as the set of all formal differential operators commuting with the point-wise multiplication operators by antiholomorphic coordinates $R_{\bar{z}_l} = \bar{z}_l$ and the operators $R_{\partial\widehat{\Phi}/\partial\bar{z}_l}$. From the knowledge of $\mathcal{L}(U)$ the star product on U can be reconstructed. This follows from the simple fact that $L_g(1) = g$ and $L_f(L_g)(1) = f \star g$. The operator corresponding to the left multiplication with the (formal) function g can recursively (in the ν-degree) be calculated from the fact that it commutes with the operators $R_{\partial\widehat{\Phi}/\partial\bar{z}_l}$. The local star-products agree on the intersections of the charts and define the global star-product $\star$ on M. See the original work of Karabegov [15] for these statements.

In [18], [19] Karabegov gave a more direct construction of the star product $\star_K$ with Karabegov form $(1/\nu)\omega$.

3.2   Karabegov's formal Berezin transform

Given a pseudo-Kähler manifold (M, ω_{-1}). In the frame of his construction and classification Karabegov assigned to each star products $\star$ with the separation of variables property the formal *Berezin transform* $I_\star$. It is the unique formal

differential operator on M such that for any open subset $U \subset M$, antiholomorphic functions a and holomorphic functions b on U the relation

$$(3.4) \qquad\qquad a \star b = I(b \cdot a) = I(b \star a),$$

holds true. The last equality is automatic and is due to the fact, that by the separation of variables property $b \star a$ is the point-wise product $b \cdot a$. He shows

$$(3.5) \qquad I = \sum_{i=0}^{\infty} I_i \, \nu^i, \quad I_i : C^{\infty}(M) \to C^{\infty}(M), \quad I_0 = id, \quad I_1 = \Delta.$$

Karabegov's classification gives for a fixed pseudo-Kähler manifold a 1:1 correspondence between (1) the set of star products with separation of variables type in Karabegov convention and (2) the set of formal deformations (3.1) of ω_{-1}. Moreover, the formal Berezin transform $I_\star$ determines the $\star$ uniquely.

3.3   Dual and opposite star products

Given for the pseudo-Kähler manifold (M, ω_{-1}) a star product $\star$ of separation of variables type (anti-Wick) then Karabegov defined with the help of $I = I_\star$ the following associated star products. First the *dual* star-product $\tilde{\star}$ on M is defined for $f, g \in \mathcal{A}$ by the formula

$$(3.6) \qquad\qquad f \,\tilde{\star}\, g = I^{-1}(I(g) \star I(f)).$$

It is a star-product with separation of variables (anti-Wick) but now on the pseudo-Kähler manifold $(M, -\omega_{-1})$. Denote by $\tilde{\omega} = -(1/\nu)\omega_{-1} + \tilde{\omega}_0 + \nu\tilde{\omega}_1 + \ldots$ the formal form parameterizing the star-product $\tilde{\star}$. By definition $\tilde{\omega} = kf(\tilde{\star})$. Its formal Berezin transform equals I^{-1}, and thus the dual to $\tilde{\star}$ is again $\star$.

Given a star product, the opposite star product is obtained

$$(3.7) \qquad\qquad f \star^{op} g = g \star f$$

by switching the arguments. Of course the sign of the Poisson bracket is changed and we obtain a star product for $(M, -\omega_{-1})$. Moreover, it switches anti-Wick with Wick type.

Finally, we take the opposite of the dual star-product, $\star' = \tilde{\star}^{op}$, given by

$$(3.8) \qquad\qquad f \star' g = g \,\tilde{\star}\, f = I^{-1}(I(f) \star I(g)).$$

It defines a deformation quantization with separation of variables on M, but now of Wick type. The pseudo-Kähler manifold will again (M, ω_{-1}). Indeed the formal Berezin transform I establishes an equivalence of the deformation quantizations $(\mathcal{A}, \star)$ and $(\mathcal{A}, \star')$.

If $\star$ is star product of anti-Wick type with $kf(\star) = \widehat{\omega}$ then its Deligne-Fedosov class calculates as

$$(3.9) \qquad cl(\star) = \frac{1}{i}([\widehat{\omega}] - \frac{\delta}{2}).$$

See [20, Eq. 2.2], which corrects a sign error in [16]. Here $[..]$ denotes the de-Rham class of the forms and δ is the canonical class of the manifold, i.e. the first Chern class of the canonical holomorphic line bundle K_M, resp. $\delta := c_1(K_M)$. Recall that K_M is the nth exterior power of the holomorphic bundle of 1-differentials. Furthermore, we have for the opposite star product $cl(\star^{op}) = -cl(\star)$.

For the standard star product $\star_K$ given by the Karabegov form $\widehat{\omega} = (1/\nu)\omega_{-1}$ we obtain

$$(3.10) \qquad cl(\star_K) = \frac{1}{i}(\frac{1}{\nu}[\omega_{-1}] - \frac{\delta}{2}).$$

In the following we will calculate Karabegov forms of star products of separation of variables type with respect to both conventions, Wick and anti-Wick. But to obtain a 1:1 correspondence we have to fix one convention. Here we refer to the anti-Wick type product. If $\star$ is of Wick type we set

$$(3.11) \qquad kf(\star) := kf(\star^{op}),$$

which is a star product of separation of variables (anti-Wick) type but now for the pseudo-Kähler manifold $(M, -\omega)$.

3.4 Bordemann and Waldmann construction

Bordemann and Waldmann [6] gave another construction of a star product of separation of variables (Wick) type for a general (pseudo)Kähler manifold. It is a modification of Fedosov's geometric existence proof. They showed that the fibre-wise Weyl product used by Fedosov could be substituted by the fibre-wise Wick product. Using a modified Fedosov connection a star product $\star_{BW}$ of Wick type is obtained. Karabegov calculated its Karabegov form as $-(1/\nu)\omega$, see Karabegov [17]. Recall that by our convention this is the Karabegov form of the opposite $\star_{BW}^{op}$. Its Deligne class class calculates as

$$(3.12) \qquad cl(\star_{BW}) = -cl(\star_{BW}^{op}) = \frac{1}{i}(\frac{1}{\nu}[\omega] + \frac{\delta}{2}).$$

Later Neumaier [23] was able to show that each star product of separation of variables type can be obtained by the Bordemann-Waldmann construction by adding a formal closed $(1, 1)$ form as parameter in the construction.

Remark 3.1. In fact, Karabegov in [17] changed the set-up and conventions of Bordemann-Waldmann by constructing via their method a star product which is of anti-Wick type (in contrast to the original Wick type). He obtained as classifying Karabegov form $(1/\nu)\omega$ and hence the standard star product $\star_K$ with Deligne-Fedosov class (3.10) as the Bordemann-Waldmann star product in Karabegov's normalisation. By taking the opposite in the Bordemann-Waldmann construction one obtains the Karabegov modification but now with respect to the pseudo-Kähler form $-\omega$.

3.5   Reshetikhin and Takhtajan construction

Reshetikhin and Takhtajan [28] presented another general method. It is based on formal Laplace expansions of formal integrals related to the star product. The coefficients of the star product can be expressed with the help of partition functions of a restricted set of locally oriented graphs (Feynman diagrams) fulfilling some additional conditions and equipped with additional data. For details see [28], and some more remarks in [35, Section 9.2]. This approach should be compared with the Kontsevich approach in the Poisson case which also uses graphs [21].

4   The Berezin - Toeplitz star product

4.1   Toeplitz operators

For the rest of the article our manifold will be a **compact** and **quantizable** Kähler manifold (M, ω), $\omega = \omega_{-1}$, with quantum line bundle (L, h, ∇). We consider all positive tensor powers of the quantum line bundle: $(L^m, h^{(m)}, \nabla^{(m)})$, here $L^m := L^{\otimes m}$ and $h^{(m)}$ and $\nabla^{(m)}$ are naturally extended. Let the Liouville form $\Omega = \frac{1}{n!}\omega^{\wedge n}$ be the volume form on M and set for the product and the norm on the space $\Gamma_\infty(M, L^m)$ of global C^∞-sections

$$(4.1) \qquad \langle \varphi, \psi \rangle := \int_M h^{(m)}(\varphi, \psi)\, \Omega\ , \qquad ||\varphi|| := \sqrt{\langle \varphi, \varphi \rangle}\ .$$

Let $\mathrm{L}^2(M, L^m)$ be the L^2-completed space with respect to this norm. Furthermore, let $\Gamma_{hol}(M, L^m)$ be the (finite-dimensional) subspace corresponding to the global holomorphic sections, and

$$(4.2) \qquad \Pi^{(m)} : \mathrm{L}^2(M, L^m) \to \Gamma_{hol}(M, L^m)$$

the orthogonal projection.

For a function $f \in C^\infty(M)$ the associated *Toeplitz operator* $T_f^{(m)}$ *(of level m)* is defined as

$$(4.3) \qquad T_f^{(m)} := \Pi^{(m)}\, (f\cdot) : \quad \Gamma_{hol}(M, L^m) \to \Gamma_{hol}(M, L^m)\ .$$

In words: One takes a holomorphic section s and multiplies it with the differentiable function f. The resulting section $f \cdot s$ will only be differentiable. To obtain a holomorphic section, one has to project it back on the subspace of holomorphic sections.

The linear map

$$(4.4) \quad T^{(m)} : C^\infty(M) \to \mathrm{End}\big(\Gamma_{hol}(M, L^m)\big), \quad f \to T_f^{(m)} = \Pi^{(m)}(f \cdot) \,, m \in \mathbb{N}_0$$

is the *Toeplitz* or *Berezin-Toeplitz quantization map (of level m)*. The *Berezin-Toeplitz (BT) quantization* is the map

$$(4.5) \quad C^\infty(M) \to \prod_{m \in \mathbb{N}_0} \mathrm{End}(\Gamma_{hol}(M, L^{(m)})), \qquad f \to (T_f^{(m)})_{m \in \mathbb{N}_0}.$$

Let for $f \in C^\infty(M)$ by $|f|_\infty$ the sup-norm of f on M and $||T_f^{(m)}||$ the operator norm with respect to the norm (4.1) on $\Gamma_{hol}(M, L^m)$.

Theorem 4.1. *[Bordemann, Meinrenken, Schlichenmaier] [5]*
(a) For every $f \in C^\infty(M)$ there exists a $C > 0$ such that

$$(4.6) \qquad |f|_\infty - \frac{C}{m} \quad \leq \quad ||T_f^{(m)}|| \quad \leq \quad |f|_\infty \;.$$

In particular, $\lim_{m \to \infty} ||T_f^{(m)}|| = |f|_\infty$.
(b) For every $f, g \in C^\infty(M)$

$$(4.7) \qquad ||m \,\mathrm{i}\, [T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| \quad = \quad O(\frac{1}{m}) \;.$$

(c) For every $f, g \in C^\infty(M)$

$$(4.8) \qquad ||T_f^{(m)} T_g^{(m)} - T_{f \cdot g}^{(m)}|| \quad = \quad O(\frac{1}{m}) \;.$$

4.2 Star Product

Based on the Toeplitz operators and in generalization of the Theorem 4.1 we obtained

Theorem 4.2. *[5],[29],[30],[31],[20] There exists a unique differential star product*

$$(4.9) \qquad f \star_{BT} g = \sum \nu^k C_k(f, g)$$

such that

$$(4.10) \qquad T_f^{(m)} T_g^{(m)} \sim \sum_{k=0}^\infty \left(\frac{1}{m}\right)^k T_{C_k(f,g)}^{(m)}.$$

This star product is of separation of variables type (Wick) with classifying Deligne-Fedosov class cl and Karabegov form kf

$$(4.11) \qquad cl(\star_{BT}) = \frac{1}{i}(\frac{1}{\nu}[\omega] - \frac{\delta}{2}), \qquad kf(\star_{BT}) = \frac{-1}{\nu}\omega + \omega_{can}.$$

First, the asymptotic expansion in (4.10) has to be understood in a strong operator norm sense. Second, recall the definition of the canonical class δ as the first Chern class of the canonical bundle K_M. If we take in K_M the fiber metric coming from the Liouville form Ω then this defines a unique connection and further a unique curvature $(1,1)$-form ω_{can}. In our sign conventions we have $\delta = [\omega_{can}]$, and the formula for $cl(\star_{BT})$ follows as this class is equal to $-cl(\star_{BT}^{op})$ which by (3.9) can be calculated from $kf(\star_{BT})$.

Remark 4.3. It is possible to incorporate an auxiliary hermitian line (or even vector) bundle in the whole set-up. In this way it is possible to do quantization with meta-plectic correction, see [35, Rem. 3.7].

4.3   Geometric Quantisation

Kostant and Souriau introduced the operators of geometric quantization in this geometric setting. In our compact Kähler setting and if one chooses the *Kähler polarization* for the passage of prequantization to quantization then Tuynman lemma [36] gives the following relation between the operators of geometric quantization and Toeplitz quantization

$$(4.12) \qquad\qquad Q_f^{(m)} = i \cdot T_{f - \frac{1}{2m}\Delta f}^{(m)},$$

where Δ is the Laplacian with respect to the Kähler metric given by ω. As a consequence the operators $Q_f^{(m)}$ and $T_f^{(m)}$ have the same asymptotic behavior for $m \to \infty$.

Using Theorem 4.1, Theorem 4.2 and the Tuynman relation (4.12) one can show that there exists a star product $\star_{GQ}$ given by asymptotic expansion of the product of geometric quantization operators. The star product $\star_{GQ}$ is equivalent to $\star_{BT}$, via the equivalence transformation $B(f) := (id - \nu\frac{\Delta}{2})f$. In particular, it has the same Deligne-Fedosov class. But it is not of separation of variables type, see [31].

5   The Berezin transform

Recall that we are in the quantizable compact Kähler case. In the Karabegov construction to every star product $\star$ a unique formal Berezin transform $I_\star$ was assigned. But to understand the relations between the different star products better we will need the geometric Berezin transform. For its definition we first have to introduce coherent vectors and covariant symbols.

5.1 The disc bundle

Without restriction we might assume that our quantum line bundle L is very ample. This means it has enough global holomorphic sections to embed the manifold into projective space. If not then at least by the quantum condition the line bundle L will be positive and a certain positive tensor power will be very ample. This tensor power will be a quantum line bundle for a rescaled Kähler form.

We pass to the dual line bundle $(U, k) := (L^*, h^{-1})$ with dual metric k. Inside the total space U, we consider the circle bundle

$$Q := \{\lambda \in U \mid k(\lambda, \lambda) = 1\},$$

and denote by $\tau : Q \to M$ (or $\tau : U \to M$) the projections to the base manifold M.

The bundle Q is a contact manifold, i.e. there is a 1-form ν such that $\mu = \frac{1}{2\pi}\tau^*\Omega \wedge \nu$ is a volume form on Q. Denote by $\mathrm{L}^2(Q, \mu)$ the corresponding L^2-space on Q. Let $\mathcal{H}$ be the space of (differentiable) functions on Q which can be extended to holomorphic functions on the disc bundle (i.e. to the "interior" of the circle bundle), and $\mathcal{H}^{(m)}$ the subspace of $\mathcal{H}$ consisting of m-homogeneous functions on Q. Here m-homogeneous means $\psi(c\lambda) = c^m\psi(\lambda)$. We introduce the following (orthogonal) projectors: the *Szegö projector*

(5.1) $$\Pi : \mathrm{L}^2(Q, \mu) \to \mathcal{H},$$

and its components the *Bergman projectors*

(5.2) $$\hat{\Pi}^{(m)} : \mathrm{L}^2(Q, \mu) \to \mathcal{H}^{(m)}.$$

The bundle Q is a S^1-bundle, and the L^m are associated line bundles. The sections of $L^m = U^{-m}$ are identified with those functions ψ on Q which are homogeneous of degree m. This identification is given on the level of the L^2 spaces by the map

(5.3) $$\gamma_m : \mathrm{L}^2(M, L^m) \to \mathrm{L}^2(Q, \mu), \quad s \mapsto \psi_s \quad \text{where}$$

(5.4) $$\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))).$$

Restricted to the holomorphic sections we obtain the unitary isomorphism

(5.5) $$\gamma_m : \Gamma_{hol}(M, L^m) \cong \mathcal{H}^{(m)}.$$

5.2 Coherent vectors

If we fix in the relation (5.4) $\alpha \in U \setminus 0$ and vary the sections s we obtain a linear evaluation functional. The *coherent vector (of level m)* associated to the point $\alpha \in U \setminus 0$ is the element $e_\alpha^{(m)}$ of $\Gamma_{hol}(M, L^m)$ with

(5.6) $$\langle e_\alpha^{(m)}, s \rangle = \psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha)))$$

for all $s \in \Gamma_{hol}(M, L^m)$. A direct verification shows $e_{c\alpha}^{(m)} = \bar{c}^m \cdot e_{\alpha}^{(m)}$ for $c \in \mathbb{C}^* :=$ $\mathbb{C} \setminus \{0\}$. Moreover, as the bundle is very ample we get $e_{\alpha}^{(m)} \neq 0$.

Hence, the *coherent state (of level m)* associated to $x \in M$ as projective class

$$(5.7) \qquad \mathrm{e}_x^{(m)} := [e_{\alpha}^{(m)}] \in \mathbb{P}(\Gamma_{hol}(M, L^m)), \qquad \alpha \in \tau^{-1}(x), \alpha \neq 0.$$

is well-defined.

Remark 5.1. This coordinate independent version of Berezin's original definition of coherent vectors and states and extensions to line bundles were given by Rawnsley [27]. It plays an important role in the work of Cahen, Gutt, and Rawnsley on the quantization of Kähler manifolds [7, 8, 9, 10], via Berezin's covariant symbols. In those works the coherent vectors are parameterized by the elements of $L \setminus 0$. The definition here uses the points of the total space of the dual bundle U. It has the advantage that one can consider all tensor powers of L together on an equal footing.

5.3   Covariant Berezin symbol

For an operator $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ its *covariant Berezin symbol $\sigma^{(m)}(A)$ (of level m)* is defined as the function

$$(5.8) \quad \sigma^{(m)}(A) : M \to \mathbb{C}, \quad x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e_{\alpha}^{(m)}, A e_{\alpha}^{(m)} \rangle}{\langle e_{\alpha}^{(m)}, e_{\alpha}^{(m)} \rangle}, \quad \alpha \in \tau^{-1}(x) \setminus \{0\}.$$

5.4   Definition of the Berezin transform

Definition 5.2. The map

$$(5.9) \qquad I^{(m)} : C^{\infty}(M) \to C^{\infty}(M), \qquad f \mapsto I^{(m)}(f) := \sigma^{(m)}(T_f^{(m)}),$$

obtained by starting with a function $f \in C^{\infty}(M)$, taking its Toeplitz operator $T_f^{(m)}$, and then calculating the covariant symbol is called the (geometric) *Berezin transform (of level m)*.

Theorem 5.3. *[20] Given $x \in M$ then the Berezin transform $I^{(m)}(f)$ has a complete asymptotic expansion in powers of $1/m$ as $m \to \infty$*

$$(5.10) \qquad\qquad I^{(m)}(f)(x) \quad \sim \quad \sum_{i=0}^{\infty} I_i(f)(x) \frac{1}{m^i} \;,$$

where $I_i : C^{\infty}(M) \to C^{\infty}(M)$ are linear maps given by differential operators, uniformly defined for all $x \in M$. Furthermore, $I_0(f) = f, \quad I_1(f) = \Delta f$.

Here Δ is the Laplacian with respect to the metric given by the Kähler form ω.

5.5   Bergman kernel

Recall from above the Bergman projectors (5.2). They have smooth integral kernels, the *Bergman kernels* $\mathcal{B}_m(\alpha, \beta)$ defined on $Q \times Q$, i.e.

$$(5.11) \qquad \hat{\Pi}^{(m)}(\psi)(\alpha) = \int_Q \mathcal{B}_m(\alpha, \beta)\psi(\beta)\mu(\beta).$$

The Bergman kernels can be expressed with the help of the coherent vectors.

$$(5.12) \qquad \mathcal{B}_m(\alpha, \beta) = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle.$$

For the proofs of these properties see [20], or [32].

Let $x \in M$ and choose $\alpha \in Q$ with $\tau(\alpha) = x$ then the function

$$(5.13) \qquad u_m(x) := \mathcal{B}_m(\alpha, \alpha) = \langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle,$$

is well-defined on M.

6   Berezin transform and star products

6.1   Identification of the BT star product

In [20] it was shown that the BT star product $\star_{BT}$ is the opposite of the dual of the star product $\star$ associated to the geometric Berezin transform introduced in the last section. To identify $\star$ we will give its classifying Karabegov form $\widehat{\omega}$. Zelditch [40] proved that the function u_m (5.13) has a complete asymptotic expansion in powers of $1/m$. In detail he showed

$$(6.1) \qquad u_m(x) \sim m^n \sum_{k=0}^{\infty} \frac{1}{m^k} b_k(x), \quad b_0 = 1.$$

If we replace in the expansion $1/m$ by the formal variable ν we obtain a formal function s defined by

$$(6.2) \qquad e^s(x) = \sum_{k=0}^{\infty} \nu^k b_k(x).$$

Now take as formal potential (3.2)

$$\widehat{\Phi} = \frac{1}{\nu}\Phi_{-1} + s,$$

where Φ_{-1} is the local Kähler potential of the Kähler form $\omega = \omega_{-1}$. Then $\widehat{\omega} = i\,\partial\bar{\partial}\widehat{\Phi}$. It might also be written in the form

$$(6.3) \qquad \widehat{\omega} = \frac{1}{\nu}\omega + \mathbb{F}(i\,\partial\bar{\partial}\log\mathcal{B}_m(\alpha, \alpha)).$$

We use for the replacement of $1/m$ by the formal variable ν the symbol $\mathbb{F}$.

6.2   The Berezin star products for arbitrary Kähler manifolds

We will introduce for general quantizable compact Kähler manifolds the Berezin star product. We extract from the asymptotic expansion of the Berezin transform (5.10) the formal expression

$$(6.4) \qquad\qquad I = \sum_{i=0}^{\infty} I_i\, \nu^i, \quad I_i : C^{\infty}(M) \to C^{\infty}(M),$$

as a *formal Berezin transform*, and set

$$(6.5) \qquad\qquad f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g)).$$

As $I_0 = id$ this $\star_B$ is a star product for our Kähler manifold, which we call the *Berezin star product*. Obviously, the formal map I gives the equivalence transformation to $\star_{BT}$. Hence, the Deligne-Fedosov classes will be the same. It will be of separation of variables type (but now of anti-Wick type). We showed in [20] that $I = I_\star$ with star product given by the form (6.3). We can rewrite (6.5) as

$$(6.6) \qquad\qquad f \star_{BT} g := I^{-1}(I(f) \star_B I(g)).$$

and get exactly the relation (3.8). Hence, $\star = \star_B$ and both star products $\star_B$ and $\star_{BT}$ are dual and opposite to each other.

6.3   The original Berezin star product

Under very restrictive conditions on the manifold it is possible to construct the *Berezin star product* with the help of the covariant symbol map. This was done by Berezin himself [2],[3] and later by Cahen, Gutt, and Rawnsley [7][8][9][10] for more examples. We will indicate this in the following.

Denote by $\mathcal{A}^{(m)} \leq C^{\infty}(M)$, the subspace of functions which appear as level m covariant symbols of operators. From the surjectivity of the Toeplitz map one concludes that the covariant symbol map is injective, see [35, Prop.6.5]. Hence, for the symbols $\sigma^{(m)}(A)$ and $\sigma^{(m)}(B)$ the operators A and B are uniquely fixed. We define a deformed product by

$$(6.7) \qquad\qquad \sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B).$$

Now $\star_{(m)}$ defines on $\mathcal{A}^{(m)}$ an associative and noncommutative product.

The crucial problem is how to relate different levels m to define for all possible symbols a unique product not depending on m. In certain special situations like those studied by Berezin, and Cahen, Gutt and Rawnsley the subspaces are nested into each other and the union $\mathcal{A} = \bigcup_{m \in \mathbb{N}} \mathcal{A}^{(m)}$ is a dense subalgebra of $C^{\infty}(M)$. A detailed analysis shows that in this case a star product is given. The star product will coincide with the star product $\star_B$ introduced above.

7 Summary of naturally defined star products

By the presented techniques we obtained for quantizable compact Kähler manifolds three different naturally defined star products $\star_{BT}$, $\star_{GQ}$, and $\star_B$. All three are equivalent and have classifying Deligne-Fedosov class

$$(7.1) \qquad cl(\star_{BT}) = cl(\star_B) = cl(\star_{GQ}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2}).$$

But all three are distinct. In fact $\star_{BT}$ is of separation of variables type (Wick-type), $\star_B$ is of separation of variables type (anti-Wick-type), and $\star_{GQ}$ neither. For their Karabegov forms we obtained

$$(7.2) \qquad kf(\star_{BT}) = \frac{-1}{\nu}\omega + \omega_{can}. \qquad kf(\star_B) = \frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\,\partial\overline{\partial}\log u_m).$$

The function u_m was introduced above as the function on M obtained by evaluating the Bergman kernel along the diagonal in $Q \times Q$.

In addition we have the Bordemann-Waldmann [6] star product which exists for every Kähler manifold. It is of Wick-type. Its Karabegov form [17] is given by $kf(\star_{BW}) = kf(\star_{BW}^{opp}) = -(1/\nu)\,\omega$ and it has Deligne Fedosov class

$$(7.3) \qquad cl(\star_{BW}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \frac{\delta}{2}).$$

Hence, it will be only equivalent to the star products above if the canonical class of the manifold will be trivial. For compact Riemann surfaces this will exactly be the case if it is a torus.

Another star product is the standard star product (of anti-Wick type) of Karabegov $\star_K$ with Karabegov form $kf(\star_K) = (1/\nu)\,\omega$. It can be also obtained in a modified Bordemann - Waldmann approach by an anti-Wick Fedosov type construction. Via the formula (3.9) its Deligne-Fedosov class $cl(\star_K)$ calculates to (7.1). Hence, it is equivalent to the above three star products.

I like to point out, that the Berezin transform, resp. the defining Karabegov form can be used to calculate the coefficients of these naturally defined star products. This can be done either directly or with the help of the certain type of graphs (in the latter case see the work of Gammelgaard [14] and Hua Xu [38],[39]). See [35, Section 8.4, 9.] for an overview on these techniques.

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