MULTIPOINT LAX OPERATOR ALGEBRAS. ALMOST-GRADED STRUCTURE AND CENTRAL EXTENSIONS

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Abstract. Recently, Lax operator algebras appeared as a new class of higher genus current type algebras. Based on I. Krichever's theory of Lax operators on algebraic curves they were introduced by I. Krichever and O. Sheinman. These algebras are almost-graded Lie algebras of currents on Riemann surfaces with marked points (in-points, out-points, and Tyurin points). In a previous joint article of the author with Sheinman the local cocycles and associated almost-graded central extensions are classified in the case of one in-point and one out-point. It was shown that the almost-graded extension is essentially unique. In this article the general case of Lax operator algebras corresponding to several in- and out-points is considered. In a first step it is shown that they are almost-graded. The grading is given by the splitting of the marked points which are non-Tyurin points into in- and out-points. Next, classification results both for local and bounded cocycles are shown. The uniqueness theorem for almost-graded central extensions follows. For this generalization additional techniques are needed which are presented in this article.

Contents

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1. Introduction

Lax operator algebras are a recently introduced new class of current type Lie algebras. In their full generality they were introduced by Krichever and Sheinman in [10]. There the concept of Lax operators on algebraic curves, as considered by Krichever in [5], was generalized to $\mathfrak{g}$-valued Lax operators, where $\mathfrak{g}$ is a classical complex Lie algebra. Krichever [5] extended the conventional Lax operator representation with a rational parameter to the case of algebraic curves of arbitrary genus. Such generalizations of Lax operators appear in many fields. They are closely related to integrable systems (Krichever-Novikov equations on elliptic curves, elliptic Calogero-Moser systems, Baker-Akhieser functions), see [5], [6]. Another important application appears in the context of moduli spaces of bundles. In particular, they are related to Tyurin's result on the classification of framed semi-stable holomorphic vector bundles on algebraic curves [30]. The classification uses *Tyurin parameters* of such bundles, consisting of points γ_s ($s = 1, \ldots, ng$), and associated elements $\alpha_s \in \mathbb{P}^{n-1}(C)$ (where g denotes the genus of the Riemann surface Σ, and n corresponds to the rank of the bundle). In the following I will not make any reference to these applications. Beside the above mentioned work the reader might refer to Sheinman [27], [28] for more background in the case of integrable systems.

Here I will concentrate on the mathematical structure of these algebras. Lax operator algebras are infinite dimensional Lie algebras of geometric origin and are interesting mathematical objects. In contrast to the classical genus zero algebras, appearing in Conformal Field Theory, they are not graded anymore. In this article we will introduce an almost-graded structure (see Definition 3.1) for them. Such an almost-grading will be an indispensable tool. A crucial task for such infinite dimensional Lie algebras is the construction and classification of central extensions. This is done in the article. We will concentrate on such central extensions for which the almost-grading can be extended.

In certain respect the Lax operator algebras can be considered as generalizations of the higher-genus Krichever-Novikov type current and affine algebras, see [7], [24], [25], [22], [17], [19]. They themselves are generalizations of the classical affine Lie algebras as e.g. introduced by Kac [3], [4] and Moody [11].

This article extends the results on the two-point case (see the next paragraph for its definition) to the multi-point case. As far as the almost-grading in the two-point case is concerned, see Krichever and Sheinman [10]. For the central extensions in the two-point case see the joint work of the author with Sheinman [23].

To describe the obtained results we first have to give a rough description of the setup. Full details will be given in Section 2. Let $\mathfrak{g}$ be one of the classical Lie algebras [1] $\mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(n)$ over $\mathbb{C}$ and Σ a compact Riemann surface. Let A be a finite set of points of Σ divided into two disjoint non-empty subsets I and O. Furthermore, let W be another finite set of points (called weak singular points). Our Lax operator algebra consists of meromorphic functions $\Sigma \to \mathfrak{g}$, holomorphic outside of $W \cup A$ with possibly poles of order 1 (resp. of order 2 for $\mathfrak{sp}(n)$) at the points in W and certain additional conditions, depending on $\mathfrak{g}$, on the Laurent series expansion there (see e.g. (2.6)). It turns out [10] that due to the additional condition this set of matrix-valued functions closes to a Lie algebra $\overline{\mathfrak{g}}$ under the point-wise commutator. In case that $W = \emptyset$ then $\overline{\mathfrak{g}}$ will be the Krichever-Novikov type current algebra (associated to this special finite-dimensional Lie algebras). They were extensively studied by Krichever and Novikov, Sheinman, and Schlichenmaier see e.g. [7], [24], [25], [26], [17], [22], [19]. It has to be pointed out that the Krichever-Novikov type algebras can be defined for all finite-dimensional Lie algebras $\mathfrak{g}$.

If furthermore, the genus of the Riemann surface is zero and A consists only of two points, which we might assume to be $\{0\}$ and $\{\infty\}$, then the algebras will be the usual classical current algebras. These classical algebras are graded algebras. Such a grading is used e.g. to introduce highest weight representations, Verma modules, Fock spaces, and to classify these representations. Unfortunately, the algebras which we consider here will not be graded. But they admit an almost-grading, see Definition 3.1. As was realized by Krichever and Novikov [7] for most applications it is a valuable replacement for the grading. They also gave a method how to introduce it for the two-point algebras of Krichever-Novikov type.

For the multi-point case of the Krichever-Novikov type algebras such an almost-grading was given by the author [16], [15], [19], [20], [21], see also [12]. The crucial point is that the almost-grading will depend on the splitting of A into I and O. Different splittings will give different almost-gradings. Hence, the multi-point case is more involved than he two-point case.

For the Krichever-Novikov current algebra $\overline{\mathfrak{g}}$ the grading comes from the grading of the function algebra (to be found in the above cited works of the author). This is due to the fact, that they are tensor products. If $W \neq \emptyset$ the Lax operator algebras are not tensor products anymore and their almost-grading has to be constructed directly. This has been done in the two point case by Krichever and Sheinman [10].

Our first result in this article is to introduce an almost-grading of $\overline{\mathfrak{g}}$ for the arbitrary multi-point case. As mentioned above, it will depend in an essential way on the splitting of A into $I \cup O$. This is done in Section 3. The construction is much more involved than in the two-point case.

[1] As far as G_2 is concerned see the recent preprint of Sheinman [29], and the remark at the end of the introduction.

Our second goal is to study central extensions $\widehat{\mathfrak{g}}$ of the Lax operator algebras $\overline{\mathfrak{g}}$. It is well-known that central extensions are given by Lie algebra two-cocycles of $\overline{\mathfrak{g}}$ with values in the trivial module $\mathbb{C}$. Equivalence classes of central extensions are in 1:1 correspondence to the elements of the Lie algebra cohomology space $\mathrm{H}^2(\overline{\mathfrak{g}},\mathbb{C})$. Whereas for the classical current algebras associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$ the extension class will be unique this is not the case anymore for higher genus and even for genus zero in the multi-point case. But we are interested only in central extensions $\widehat{\mathfrak{g}}$ which allow us to extend the almost-grading of $\overline{\mathfrak{g}}$. This reduces the possibilities. The condition for the cocycle defining the central extension will be that it is *local* (see (5.13)) with respect to the almost-grading given by the splitting $A = I \cup O$. Hence, which cocycles will be local will depend on the splitting as well.

If $\mathfrak{g}$ is simple then the space of local cohomology classes for $\overline{\mathfrak{g}}$ will be one-dimensional. For $\mathfrak{gl}(n)$ we have to add another natural property for the cocycle meaning that it is invariant under the action of the vector field algebra $\mathcal{L}$ (see (5.3)). In this case the space of local and $\mathcal{L}$-invariant cocycle classes will be two-dimensional.

The action of the vector field algebra $\mathcal{L}$ on $\overline{\mathfrak{g}}$ is given in terms of a certain connection $\nabla^{(\omega)}$, see Section 4.2. With the help of the connection we can define geometric cocycles

$$\gamma_{1,\omega,C}(L,L') = \frac{1}{2\pi \mathrm{i}}\int_C \mathrm{tr}(L\cdot\nabla^{(\omega)}L'), \tag{1.1}$$

$$\gamma_{2,\omega,C}(L,L') = \frac{1}{2\pi \mathrm{i}}\int_C \mathrm{tr}(L)\cdot\mathrm{tr}(\nabla^{(\omega)}L'), \tag{1.2}$$

where C is an arbitrary cycle on Σ avoiding the points of possible singularities. The cocycle $\gamma_{2,\omega,C}$ will only be different from zero in the $\mathfrak{gl}(n)$ case.

Special integration paths are circles C_i around the points in I, resp. around the points in O, and a path C_S separating the points in I form the points in O.

Our main result is Theorem 6.7 about uniqueness of local cocycles classes and that the cocycles are given by integrating along C_S. The proof presented in Section 6 is based on Theorem 6.4 which gives the classification of bounded (from above) cohomology classes (see (5.12)). The bounded cohomology classes constitute a subspace of dimension N, (resp. $2N$ for $\mathfrak{gl}(n)$) where $N = \#I$ and the integration is done over the $C_i, i = 1,\dots,N$.

The proof of Theorem 6.4 is given in Section 7 and Section 8. We use recursive techniques as developed in [18] and [19]. Using the boundedness and $\mathcal{L}$-invariance we show that such a cocycle is given by its values at pairs of homogeneous elements for which the sum of their degrees is equal to zero. Furthermore, we show that an $\mathcal{L}$-invariant and bounded cocycle will be uniquely fixed by a certain finite number of such cocycle values. A more detailed analysis shows that the cocycles are of the form claimed. In Section 8 we show that in the simple Lie algebra case in each bounded cohomology class there is a representing cocycle which is $\mathcal{L}$-invariant. For this we use the internal structure of the Lie algebra $\overline{\mathfrak{g}}$ related to the root system of the underlying finite dimensional simple Lie algebra $\mathfrak{g}$, and the almost-gradedness of $\overline{\mathfrak{g}}$. Recall that in the classical case $\mathfrak{g}\otimes\mathbb{C}[z,z^{-1}]$ the algebra is graded. In this very special case the chain of arguments gets simpler and is similar to the arguments of Garland [1].

As already mentioned above, in joint work with Sheinman [23] the two-point case was considered. This article extends the result to the multi-point case. Unfortunately, it is not an application of the results of the two-point case. In this more general context the proofs have to be done anew. (The two-point case will finally be a special case.) Only at few places references to proofs in [23] can be made.

I like to thank Oleg Sheinman for extensive discussions which were very helpful during writing this article. After I finished this work he succeeded [29] to give a definition of a Lax operator algebra for the exceptional Lie algebra G_2 in such a way, that all properties and statements presented here will also be true in this case. Hence, there is now another element in the list of Lax operator algebra associated to simple Lie algebras.

2. The algebras

2.1. Lax operator algebras.

Let $\mathfrak{g}$ be one of the classical matrix algebras $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$, $\mathfrak{so}(n)$, $\mathfrak{sp}(2n)$, or $\mathfrak{s}(n)$, where the latter denotes the algebra of scalar matrices. Our algebras will consist of certain $\mathfrak{g}$-valued meromorphic functions, forms, etc, defined on Riemann surfaces with additional structures (marked points, vectors associated to this points, ...).

To become more precise, let Σ be a compact Riemann surface of genus g (g arbitrary) and A a finite subset of points in Σ divided into two non-empty disjoint subsets

$$I := \{P_1, P_2, \ldots, P_N\}, \qquad O := \{Q_1, Q_2, \ldots, Q_M\} \tag{2.1}$$

with $\#A = N + M$. The points in I are called incoming-points the points in O outgoing-points.

To define Lax operator algebras we have to fix some additional data. Fix $K \in \mathbb{N}_0$ and a collection of points

$$W := \{\gamma_s \in \Sigma \setminus A \mid s = 1, \ldots, K\}. \tag{2.2}$$

We assign to every point γ_s a vector $\alpha_s \in \mathbb{C}^n$ (resp. from $\mathbb{C}^{2n}$ for $\mathfrak{sp}(2n)$). The system

$$\mathcal{T} := \{(\gamma_s, \alpha_s) \in \Sigma \times \mathbb{C}^n \mid s = 1, \ldots, K\} \tag{2.3}$$

is called *Tyurin data*. We will be more general than in our earlier joint paper [23] with Sheinman, not only in respect that we allow for A more than two points also that our K is not bound to be $n \cdot g$. Even $K = 0$ is allowed. In the latter case the Tyurin data will be empty.

Remark. For $K = n \cdot g$ and for generic values of (γ_s, α_s) with $\alpha_s \neq 0$ the tuples of pairs $(\gamma_s, [\alpha_s])$ with $[\alpha_s] \in \mathbb{P}^{n-1}(\mathbb{C})$ parameterize framed semi-stable rank n and degree $n\,g$ holomorphic vector bundles as shown by Tyurin [30]. Hence, the name Tyurin data.

We fix local coordinates z_l, $l = 1, \ldots, N$ centered at the points $P_l \in I$ and w_s centered at γ_s, $s = 1, \ldots, K$. In fact nothing will dependent on the choice of w_s. This is essentially also true for z_l. Only its first jet will be used to normalize certain basis elements uniquely.

We consider $\mathfrak{g}$-valued meromorphic functions

$$L : \Sigma \to \mathfrak{g}, \tag{2.4}$$

which are holomorphic outside $W \cup A$, have at most poles of order one (resp. of order two for $\mathfrak{sp}(2n)$) at the points in W, and fulfill certain conditions at W depending on $\mathcal{T}$, A, and $\mathfrak{g}$. These conditions will be described in the following. The singularities at W are called *weak singularities*. These objects were introduced by Krichever [5] for $\mathfrak{gl}(n)$ in the context of Lax operators for algebraic curves, and further generalized by Krichever and Sheinman in [10]. The conditions are exactly the same as in [23]. But for the convenience of the reader we recall them here.

For $\mathfrak{gl}(n)$ the conditions are as follows. For $s = 1, \ldots, K$ we require that there exist $\beta_s \in \mathbb{C}^n$ and $\kappa_s \in \mathbb{C}$ such that the function L has the following expansion at $\gamma_s \in W$

$$L(w_s) = \frac{L_{s,-1}}{w_s} + L_{s,0} + \sum_{k>0} L_{s,k} w_s^k, \tag{2.5}$$

with

$$L_{s,-1} = \alpha_s \beta_s^t, \quad \mathrm{tr}(L_{s,-1}) = \beta_s^t \alpha_s = 0, \quad L_{s,0}\, \alpha_s = \kappa_s \alpha_s. \tag{2.6}$$

In particular, if $L_{s,-1}$ is non-vanishing then it is a rank 1 matrix, and if $\alpha_s \neq 0$ then it is an eigenvector of $L_{s,0}$.

The requirements (2.6) are independent of the chosen coordinates w_s and the set of all such functions constitute an associative algebra under the point-wise matrix multiplication, see [10]. The proof transfers without changes to the multi-point case. For the convenience of the reader and for illustration we will nevertheless recall the proof in an appendix to this article. We denote this algebra by $\overline{\mathfrak{gl}}(n)$. Of course, it will depend on the Riemann surface Σ, the finite set of points A, and the Tyurin data T. As there should be no confusion, we prefer to avoid cumbersome notation and will just use $\overline{\mathfrak{gl}}(n)$. The same we do for the other Lie algebras.

Note that if one of the $\alpha_s = 0$ then the conditions at the point γ_s correspond to the fact, that L has to be holomorphic there. We can erase the point from the Tyurin data. Also if $\alpha_s \neq 0$ and $\lambda \in \mathbb{C}, \lambda \neq 0$ then α and $\lambda\alpha$ induce the same conditions at the point γ_s. Hence only the projective vector $[\alpha_s] \in \mathbb{P}^{n-1}(\mathbb{C})$ plays a role.

The splitting $\mathfrak{gl}(n) = \mathfrak{s}(n) \oplus \mathfrak{sl}(n)$ given by

$$X \mapsto \left(\frac{\mathrm{tr}(X)}{n} I_n \ , \ X - \frac{\mathrm{tr}(X)}{n} I_n \right), \tag{2.7}$$

where I_n is the $n \times n$-unit matrix, induces a corresponding splitting for the Lax operator algebra $\overline{\mathfrak{gl}}(n)$:

$$\overline{\mathfrak{gl}}(n) = \overline{\mathfrak{s}}(n) \oplus \overline{\mathfrak{sl}}(n). \tag{2.8}$$

For $\overline{\mathfrak{sl}}(n)$ the only additional condition is that in (2.5) all matrices $L_{s,k}$ are trace-less. The conditions (2.6) remain unchanged.

For $\overline{\mathfrak{s}}(n)$ all matrices in (2.5) are scalar matrices. This implies that the corresponding $L_{s,-1}$ vanish. In particular, the elements of $\overline{\mathfrak{s}}(n)$ are holomorphic at W. Also $L_{s,0}$, as a scalar matrix, has every α_s as eigenvector. This means that beside the holomorphicity there are no further conditions. And we get $\overline{\mathfrak{s}}(n) \cong \mathcal{A}$, where $\mathcal{A}$ be the (associative) algebra of meromorphic functions on Σ holomorphic outside of A. This is the (multi-point) Krichever-Novikov type function algebra. It will be discussed further down in Section 3.2.

In the case of $\mathfrak{so}(n)$ we require that all $L_{s,k}$ in (2.5) are skew-symmetric. In particular, they are trace-less. Following [10] the set-up has to be slightly modified. First only those Tyurin parameters α_s are allowed which satisfy $\alpha_s^t\alpha_s = 0$. Then the first requirement in (2.6) is changed to obtain

$$L_{s,-1} = \alpha_s\beta_s^t - \beta_s\alpha_s^t, \quad \operatorname{tr}(L_{s,-1}) = \beta_s^t\alpha_s = 0, \quad L_{s,0}\,\alpha_s = \kappa_s\alpha_s. \tag{2.9}$$

For $\mathfrak{sp}(2n)$ we consider a symplectic form $\hat{\sigma}$ for $\mathbb{C}^{2n}$ given by a non-degenerate skew-symmetric matrix σ. The Lie algebra $\mathfrak{sp}(2n)$ is the Lie algebra of matrices X such that $X^t\sigma + \sigma X = 0$. The condition $\operatorname{tr}(X) = 0$ will be automatic. At the weak singularities we have the expansion

$$L(z_s) = \frac{L_{s,-2}}{w_s^2} + \frac{L_{s,-1}}{w_s} + L_{s,0} + L_{s,1}w_s + \sum_{k>1} L_{s,k}w_s^k. \tag{2.10}$$

The condition (2.6) is modified as follows (see [10]): there exist $\beta_s \in \mathbb{C}^{2n}$, $\nu_s, \kappa_s \in \mathbb{C}$ such that

$$L_{s,-2} = \nu_s\alpha_s\alpha_s^t\sigma, \quad L_{s,-1} = (\alpha_s\beta_s^t + \beta_s\alpha_s^t)\sigma, \quad {\beta_s}^t\sigma\alpha_s = 0, \quad L_{s,0}\,\alpha_s = \kappa_s\alpha_s. \tag{2.11}$$

Moreover, we require

$$\alpha_s^t\sigma L_{s,1}\alpha_s = 0. \tag{2.12}$$

Again under the point-wise matrix commutator the set of such maps constitute a Lie algebra.

Theorem 2.1. *Let $\overline{\mathfrak{g}}$ be the space of Lax operators associated to $\mathfrak{g}$, one of the above introduced finite-dimensional classical Lie algebras. Then $\overline{\mathfrak{g}}$ is a Lie algebra under the point-wise matrix commutator. For $\overline{\mathfrak{g}} = \overline{\mathfrak{gl}}(n)$ it is an associative algebra under point-wise matrix multiplication.*

The proof in [10] extends without problems to the multi-point situation (see the appendix for an example).

These Lie algebras are called *Lax operator algebras*.

2.2. **Krichever-Novikov algebras of current type.**

Let $\mathcal{A}$ be the (associative) algebra of meromorphic functions on Σ holomorphic outside of A. Let $\mathfrak{g}$ be an arbitrary finite-dimensional Lie algebra. On the tensor product $\mathfrak{g} \otimes \mathcal{A}$ a Lie algebra structure is given by

$$[x \otimes f, y \otimes g] := [x, y] \otimes (f \cdot g), \quad x, y \in \mathfrak{g},\ f, g \in \mathcal{A}. \tag{2.13}$$

The elements of this Lie algebra can be considered as the set of those meromorphic maps $\Sigma \to \mathfrak{g}$, which are holomorphic outside of A. These algebras are called *(multi-point) Krichever Novikov algebras of current type*, see [7], [8], [9], [24], [25], [17], [19].

If the genus of the surface is zero and if A consists of two points, the Krichever-Novikov current algebras are the classical current (or loop algebra) $\mathfrak{g} \otimes \mathbb{C}[z^{-1}, z]$.

In the case that in the defining data of the Lax operator algebra there are no weak singularities, resp. all $\alpha_s = 0$, then for the $\mathfrak{g}$-valued meromorphic functions the requirements reduce to the condition that they are holomorphic outside of A. Hence, we obtain (for these $\mathfrak{g}$) the Krichever-Novikov current type algebra. But note that not for all finite-dimensional $\mathfrak{g}$ we have an extension of the notion Krichever-Novikov current to a Lax operator algebra.

3. The almost-graded structure

3.1. **The statements.**

For the construction of certain important representations of infinite dimensional Lie algebras (Fock space representations, Verma modules, etc.) a graded structure is usually assumed and heavily used. The algebras we are considering for higher genus, or even for genus zero with many marked points were poles are allowed, cannot be nontrivially graded. As realized by Krichever and Novikov [7] a weaker concept, an almost-grading, will be enough to allow to do the above mentioned constructions.

Definition 3.1. A Lie algebra V will be called *almost-graded* (over $\mathbb{Z}$) if there exists finite-dimensional subspaces V_m and constants $S_1, S_2 \in \mathbb{Z}$ such that

(1) $V = \bigoplus_{m\in\mathbb{Z}} V_m$,
(2) $\dim V_m < \infty, \quad \forall m \in \mathbb{Z}$,
(3) $[V_n, V_m] \quad\subseteq\quad \sum_{h=n+m+S_1}^{n+m+S_2} V_h$.

If there exists an R such that $\dim V_m \le R$ for all m it is called *strongly almost-graded*.

Accordingly, an almost-grading can be defined for associative algebras and for modules over almost-graded algebras.

We will introduce for our multi-point Lax operator in the following such a (strong) almost-graded structure. The almost-grading will be induced by the splitting of our set A into I and O. Recall that $I = \{P_1, P_2, \dots, P_N\}$. In the Krichever Novikov function, vector field, and current algebra case this was done by Krichever and Novikov [7] for the two-point situation. In the two-point Lax operator algebra it was done by Krichever and Sheinman [10]. In the two-point case there is only one splitting possible. This is in contrast to the multi-point case which turns out to be more difficult. The multi-point Krichever-Novikov algebras of different types were done by Schlichenmaier [16],[15]. We will recall it in Section 3.2.

In Section 3.3 we will single out for each $m \in \mathbb{Z}$ a subspace $\overline{\mathfrak{g}}_m$ of $\overline{\mathfrak{g}}$, called *(quasi-)homogeneous* subspace of degree m. The degree is essentially related to the order of the elements of $\overline{\mathfrak{g}}$ at the points in I. We will show

Theorem 3.2. *Induced by the splitting $A = I \cup O$ the (multi-point) Lax operator algebra $\overline{\mathfrak{g}}$ becomes a (strongly) almost-graded Lie algebra*

$$\overline{\mathfrak{g}} = \bigoplus_{m\in\mathbb{Z}} \overline{\mathfrak{g}}_m, \qquad \dim \overline{\mathfrak{g}}_m = N \cdot \dim \mathfrak{g}$$
$$[\overline{\mathfrak{g}}_m, \overline{\mathfrak{g}}_n] \quad\subseteq\quad \bigoplus_{h=n+m}^{n+m+S} \overline{\mathfrak{g}}_h, \tag{3.1}$$

with a constant S independent of n and m.

In addition we will show

Proposition 3.3. *Let X be an element of $\mathfrak{g}$. For each (m,s), $m \in \mathbb{Z}$ and $s = 1,\dots,N$ there is a unique element $X_{m,s}$ in $\overline{\mathfrak{g}}_m$ such that locally in the neighbourhood of the point $P_p \in I$ we have*

$$X_{m,s|}(z_p) = X z_p^m \cdot \delta_s^p + O(z_p^{m+1}), \quad \forall p = 1,\dots,N.^2 \tag{3.2}$$

Proposition 3.4. *Let $\{X^u \mid u = 1,\dots,\dim\mathfrak{g}\}$ be a basis of the finite dimensional Lie algebra $\mathfrak{g}$. Then*

$$\mathcal{B}_m := \{X^u_{m,p},\ u = 1,\dots,\dim\mathfrak{g},\ p = 1,\dots,N\} \tag{3.3}$$

is a basis of $\overline{\mathfrak{g}}_m$, and $\mathcal{B} = \cup_{m\in\mathbb{Z}} \mathcal{B}_m$ is a basis of $\overline{\mathfrak{g}}$.

Proof. By (3.1) we know that $\dim\overline{\mathfrak{g}}_m = N \cdot \dim\mathfrak{g}$. The elements in $\mathcal{B}_m$ are pairwise different. Hence, we have $\#\mathcal{B}_m = N \cdot \dim\mathfrak{g}$ elements $\{X^u_{m,p}\}$ in $\overline{\mathfrak{g}}_m$. For being a basis it suffices to show that they are linearly independent. Take $\sum_u\sum_p \alpha^u_{m,p}X^u_{m,p} = 0$ a linear combination of zero. We consider the local expansions at the point P_s, for $s = 1,\dots,N$. From (3.2) we obtain

$$0 = (\sum_u \alpha^u_{m,s}X^u)z_s^m + O(z_s^{m+1}).$$

Hence $0 = \sum_u \alpha^u_{m,s}X^u$. As the X^u are a basis of $\mathfrak{g}$ this implies that $a^u_{m,s} = 0$ for all u, s. That $\mathcal{B}$ is a basis of the full $\overline{\mathfrak{g}}$ follows from the direct sum decomposition in (3.1). □

It is very convenient to introduce the associated filtration

$$\overline{\mathfrak{g}}_{(k)} := \bigoplus_{m\ge k}\overline{\mathfrak{g}}_m, \qquad \overline{\mathfrak{g}}_{(k)} \subseteq \overline{\mathfrak{g}}_{(k')}, \quad k \ge k'. \tag{3.4}$$

Proposition 3.5.

(a) $\overline{\mathfrak{g}} = \bigcup_{m\in\mathbb{Z}}\overline{\mathfrak{g}}_{(m)}$,

(b) $[\overline{\mathfrak{g}}_{(k)}, \overline{\mathfrak{g}}_{(m)}] \subseteq \overline{\mathfrak{g}}_{(k+m)}$,

(c) $\overline{\mathfrak{g}}_{(m)}/\overline{\mathfrak{g}}_{(m+1)} \cong \overline{\mathfrak{g}}_m$.

(d) The equivalence classes of the elements of the set $\mathcal{B}_m$ (see (3.3)) constitute a basis for the quotient space $\overline{\mathfrak{g}}_{(m)}/\overline{\mathfrak{g}}_{(m+1)}$.

Proof. Equation (3.1) implies directly (a), (b), and (c). Part (d) follows from Proposition 3.4. □

There is another filtration.

$$\overline{\mathfrak{g}}'_{(m)} := \{L \in \overline{\mathfrak{g}} \mid \mathrm{ord}_{P_s}(L) \ge m,\ s = 1,\dots,N\}. \tag{3.5}$$

Note that the elements L are meromorphic maps from Σ to $\mathfrak{g}$, hence it makes sense to talk about the orders of the component functions with respect to a basis. The minimum of these orders is meant in (3.5).

[2]The symbol δ_s^p denotes the Kronecker delta, which is equal to 1 if $s = p$, otherwise 0.

Proposition 3.6.
(a) $\overline{\mathfrak{g}} = \bigcup_{m\in\mathbb{Z}} \overline{\mathfrak{g}}'_{(m)}$.
(b) The two filtrations coincide, i.e.

$$\overline{\mathfrak{g}}_{(m)} = \overline{\mathfrak{g}}'_{(m)}, \qquad \forall m \in \mathbb{Z}.$$

Proof. Let $L \in \overline{\mathfrak{g}}$, then as $\mathfrak{g}$-valued meromorphic functions the pole orders of the component functions at the points P_s are individually bounded. As there are only finitely many, there is a bound k for the pole order, hence $L \in \overline{\mathfrak{g}}_{(-k)}$. This shows (a) and consequently that $(\overline{\mathfrak{g}}'_{(m)})$ is a filtration.
By Proposition 3.4 we know that $\mathcal{B}$ is a basis of $\overline{\mathfrak{g}}$. Let $L \in \overline{\mathfrak{g}}'_{(m)}$. Every element of $L \in \overline{\mathfrak{g}}$ will be a finite linear combination of the basis elements. The elements of $\mathcal{B}_k$ have exact order k and are linearly independent. Moreover, with respect to a fixed basis element of the finite dimensional Lie algebra we have N basis elements in $\mathcal{B}_k$ with orders given by (3.2). Hence the individual orders at the points P_s cannot increase with non-trivial linear combinations. Hence only $k \geq m$ can appear in the combination. This shows $L \in \overline{\mathfrak{g}}_{(m)}$. Vice versa, obviously all elements from $\mathcal{B}_k$ for $k \geq m$ lie in the set (3.4). Hence, we have equality. □

The second description of the filtration has the big advantage, that it is very naturally defined. The only data which enters is the splitting of the points A into $I \cup O$. Hence, it is canonically given by I. In contrast, it will turn out that in the multi-point case if $\#O > 1$ there might be some choices necessary to fix $\overline{\mathfrak{g}}_m$, like numbering the points in O, resp. even some different rules for the points in O. But via Proposition 3.6 we know that the induced filtration (3.4) will not depend on any of these choices.

Here we have to remark that we supplied above a proof of Proposition 3.6. But it was based on results (i.e. Theorem 3.2 and Proposition 3.3) which we only will prove in Section 3.3. Our starting point there will be the filtration $\overline{\mathfrak{g}}'_{(m)}$, hence we cannot assume equality from the very beginning.

We have the very important fact

Proposition 3.7. *Let $X_{k,s}$ and $Y_{m,p}$ be the elements in $\overline{\mathfrak{g}}_k$ and $\overline{\mathfrak{g}}_m$ corresponding to $X, Y \in \mathfrak{g}$ respectively then*

$$[X_{k,s}, Y_{m,p}] = [X,Y]_{k+m,s}\delta_s^p + L, \tag{3.6}$$

with $[X,Y]$ the bracket in $\mathfrak{g}$ and $L \in \overline{\mathfrak{g}}_{(k+m+1)}$.

Proof. Using for $X_{k,s}$ and $Y_{m,p}$ the expression (3.2) we obtain

$$[X_{k,s}, Y_{m,p}]_{|}(z_t) = [X,Y] z_s^{k+m}\delta_t^p\delta_t^s + O(z_t^{k+m+1}),$$

for every t. Hence, the element

$$[X_{k,s}, Y_{m,p}] - ([X,Y])_{k+m,s}\delta_s^p$$

has at all points in I an order $\geq k+m+1$. With (3.5) and Proposition 3.6 we obtain that it lies in $\overline{\mathfrak{g}}_{(k+m+1)}$, which is the claim. □

3.2. **The function algebra $\mathcal{A}$ and the vector field algebra $\mathcal{L}$.**

Before we supply the proofs of the statements in Section 3.1 we want to introduce those Krichever-Novikov type algebras which are of relevance in the following. We start with the Krichever-Novikov function algebra $\mathcal{A}$ and the Krichever-Novikov vector field algebra $\mathcal{L}$. Both algebras are almost-graded algebras

$$\mathcal{A} = \bigoplus_{m\in\mathbb{Z}} \mathcal{A}_m, \qquad \mathcal{L} = \bigoplus_{m\in\mathbb{Z}} \mathcal{L}_m, \tag{3.7}$$

where the almost-grading is induced by the same splitting of A into $I \cup O$ as used for defining the Lax operator algebras. Recall that $I = \{P_1, \dots, P_N\}$ and $O = \{Q_1, \dots, Q_M\}$.

Let $\mathcal{A}$, respectively $\mathcal{L}$, be the space of meromorphic functions, respectively of meromorphic vector fields on Σ, holomorphic on $\Sigma \setminus A$. In particular, they are holomorphic also at the points in W. Obviously, $\mathcal{A}$ is an associative algebra under the product of functions and $\mathcal{L}$ is a Lie algebra under the Lie bracket of vector fields. In the two point case their almost-graded structure was introduced by Krichever and Novikov [7]. In the multi-point case they were given by Schlichenmaier [15], [16]. The results will be described in the following.

The homogeneous spaces $\mathcal{A}_m$ have as basis the set of functions $\{A_{m,s}, s = 1, \dots, N\}$ given by the conditions

$$\operatorname{ord}_{P_i}(A_{m,s}) = (n+1) - \delta_i^s, \quad i = 1, \dots, N, \tag{3.8}$$

and certain compensating conditions at the points in O to make it unique up to multiplication with a scalar. For example, in case that $\#O = M = 1$ and the genus is either 0, or ≥ 2, and the points are in generic position, then the condition is (with the exception for finitely many m)

$$\operatorname{ord}_{Q_M}(A_{m,s}) = -N \cdot (n+1) - g + 1. \tag{3.9}$$

To make it unique we require for the local expansion at the P_s (with respect to the chosen local coordinate z_s)

$$A_{n,s|}(z_s) = z_s^n + O(z_s^{n+1}). \tag{3.10}$$

For the vector field algebra $\mathcal{L}_m$ we have the basis $\{e_{m,s} \mid s = 1, \dots, N\}$, where the elements $e_{m,s}$ are given by the condition

$$\operatorname{ord}_{P_i}(e_{m,s}) = (n+2) - \delta_i^s, \quad i = 1, \dots, N, \tag{3.11}$$

and corresponding compensating conditions at the points in O to make it unique up to multiplication with a scalar. In exactly the same special situation as above the condition is

$$\operatorname{ord}_{Q_M}(e_{m,s}) = -N \cdot (n+2) - 3(g-1). \tag{3.12}$$

The local expansion at P_s is

$$e_{n,s|}(z_s) = (z_s^{n+1} + O(z_s^{n+2}))\frac{d}{dz_s}. \tag{3.13}$$

There are constants S_1 and S_2 (not depending on m,n) such that

$$\mathcal{A}_k\cdot\mathcal{A}_m\subseteq\bigoplus_{h=k+m}^{k+m+S_1}\mathcal{A}_h,\qquad [\mathcal{L}_k,\mathcal{L}_m]\subseteq\bigoplus_{h=k+m}^{k+m+S_2}\mathcal{L}_h. \tag{3.14}$$

This says that we have almost-gradedness. In what follows we will need the fine structure of the almost-grading

$$A_{k,s}\cdot A_{m,t}=A_{k+m,s}\,\delta_s^t+Y,\qquad Y\in\sum_{h=k+m+1}^{k+m+S_1}\mathcal{A}_h, \tag{3.15}$$

$$[e_{k,s},e_{m,t}]=(m-k)\,e_{k+m,s}\,\delta_s^t+Z,\qquad Z\in\sum_{h=k+m+1}^{k+m+S_2}\mathcal{L}_h. \tag{3.16}$$

Again we have the induced filtrations $\mathcal{A}_{(m)}$ and $\mathcal{L}_{(m)}$.

The elements of the Lie algebra $\mathcal{L}$ act on $\mathcal{A}$ as derivations. This makes the space $\mathcal{A}$ an almost-graded module over $\mathcal{L}$. In particular, we have

$$e_{k,s}\,.\,A_{m,r}=mA_{k+m}\,\delta_s^r+U,\qquad U\in\sum_{h=k+m+1}^{k+m+S_3}\mathcal{A}_h, \tag{3.17}$$

with a constant S_3 not depending on k and m.

Induced by the almost-grading of $\mathcal{A}=\oplus_m\mathcal{A}_m$ we get an almost-grading for the Krichever-Novikov type algebra of current type by setting

$$\mathfrak{g}\otimes\mathcal{A}=\bigoplus_{m\in\mathbb{Z}}(\mathfrak{g}\otimes\mathcal{A})_m\quad\text{with}\quad(\mathfrak{g}\otimes\mathcal{A})_m:=\mathfrak{g}\otimes\mathcal{A}_m,\ \forall m\in\mathbb{Z}. \tag{3.18}$$

3.3. **The proofs.**

Readers being in a hurry, or readers only interested in the results may skip this rather technical section (involving Riemann-Roch type arguments) during a first reading and jump directly to Section 4.

Recall the definition

$$\overline{\mathfrak{g}}'_{(m)}:=\{L\in\overline{\mathfrak{g}}\mid \mathrm{ord}_{P_s}(L)\ge m,\ s=1,\ldots,N\} \tag{3.19}$$

of the filtration. We will only deal with this filtration in this section, hence for notational reason we will drop the $'$ in the following. Finally, the primed and unprimed will coincide.

Proposition 3.8. *Given $X\in\mathfrak{g}$, $X\ne 0$, $s=1,\ldots,N$, $m\in\mathbb{Z}$ then there exists at least one $X_{m,s}$ such that*

$$X_{m,s_|}(z_p)=X\,z_s^m\,\delta_p^s+O(z_p^{m+1}). \tag{3.20}$$

The proof is based on the theorem of Riemann-Roch. The technique will be used all-over in this section. Hence, we will introduce some notation, before we proceed with the proof. For any $m\in\mathbb{Z}$ we will consider certain divisors

$$D_m=(D_m)_I+D_W+(D_m)_O. \tag{3.21}$$

Where

$$\begin{aligned}(D_m)_I &= -m\sum_{s=1}^{N} P_s,\\ (D_m)_O &= \sum_{s=1}^{M} a_{s,m} Q_s, \quad a_{s,m}\in\mathbb{Z} \\ D_W &= \epsilon\sum_{s=1}^{K}\gamma_s, \quad \epsilon=1,\ \text{for } \mathfrak{gl}(n),\mathfrak{sl}(n),\mathfrak{so}(n), \quad \epsilon=2,\ \text{for } \overline{\mathfrak{sp}}(n).\end{aligned} \tag{3.22}$$

Recall that the genus of Σ is g. Denote by $\mathcal{K}$ a canonical divisor. Set $L(D)$ the space consisting of meromorphic functions u on Σ for which we have for their divisors $(u)\ge -D$. Riemann-Roch says

$$\dim L(D) - \dim L(\mathcal{K}-D) = \deg D - g + 1. \tag{3.23}$$

In particular, we have

$$\dim L(D) \ge \deg D - g + 1. \tag{3.24}$$

We have several cases which we will need in the following

1. If $\deg D \ge 2g-1$ then we have equality in (3.24).
2. If D is a generic divisor then also for $g \le \deg D \le 2g-2$ we have equality.
3. If $D\ge 0$ and D is generic we have $\dim L(D)=1$ for $0\le \deg D\le g-1$.
4. If $D \not\ge 0$ (meaning that there is at least one point in the support of D with negative multiplicity) and D is generic we have $\dim L(D)=0$ for $0\le\deg D\le g-1$.
5. For $g=0$ every divisor is generic and we have equality in (3.24) as long as the right hand side is ≥ 0, i.e. $\dim L(D)=\max(0,\deg D+1)$.

See e.g. [13] for informations on divisors, Riemann-Roch and their applications, see also [2].

In case that $u=(u_1,u_2,\dots,u_r)$ is a vector valued function we define $L(D)$ to be the vector space of vector valued functions with $(u)\ge -D$. This means that $(u_i)\ge -D$ for all $i=1,\dots,r$. Now all dimension formulas have to be multiplied by r:

$$\dim L(D) \ge r(\deg D - g + 1). \tag{3.25}$$

We apply this to our Lax operator algebra $\overline{\mathfrak{g}}$ by considering the component functions u_i, $i=1,\dots,r=\dim\mathfrak{g}$ with respect to a fixed basis. We set

$$L'(D) := \{u\in L(D) \mid u \text{ gives an element of } \overline{\mathfrak{g}}\} \subseteq L(D). \tag{3.26}$$

In $L'(D_m)$ we have to take into account that at the weak singular points γ_s we have H additional linear conditions for the elements of the solution space $L(D_m)$ to be fulfilled. They are formulated in terms of the corresponding α_s for some finite part of the Laurent series. In total this are finitely many conditions. In case that the α_s are generic they will exactly compensate for the possible poles at γ_s [10]. But for the moment we still consider them to be arbitrary.

By the very definition of the filtration we always have

$$\overline{\mathfrak{g}}_{(m)} = L'((D_m)_I) \quad \text{and} \quad \overline{\mathfrak{g}}_{(m)} \geq L'(D_m). \tag{3.27}$$

Proof. (Proposition 3.8) We start with a divisor D_m by choosing the part $(D_m)_O = T$ such that the degree of the divisors D_m and $D_m - \sum P_i$ is still big enough such that for both the case (1) of the Riemann-Roch equality (3.25) is true and that $\dim L(D) = l \geq r(N+1)+H$. Hence, after applying the H linear conditions we have $\dim L'(D) \geq r(N+1)$. Let P_s be a fixed point from I. We consider

$$D'_m = D_m - \sum_{i=1}^{N} P_i, \quad D''_m = D'_m + P_s. \tag{3.28}$$

This yields

$$\dim L'(D'_m) = l - rN, \qquad \dim L'(D''_m) = l - rN + r. \tag{3.29}$$

The element in $L'(D'_m)$ have orders $\geq (m+1)$ at all points in I. The elements in $L'(D''_m)$ have orders $\geq (m+1)$ at all points P_i, $i \neq s$ and orders $\geq m$ at P_s. From the dimension formula (3.29) we conclude that there exists r elements which have exact order m at P_s and orders $\geq (m+1)$ at the other points in I. This says that there is for every basis element X^u in the Lie algebra $\mathfrak{g}$ an element $X^u_{m,s} \in \overline{\mathfrak{g}}$ which has exact order m at the point P_s and order higher than m at the other points in I and can be written there as required in (3.20). By linearity we get the statement for all $X \in \mathfrak{g}$. □

Remark. **1.** By modifying the divisor T in its degree we can even show that there exists elements such that the orders of $X_{m,s}$ at the points P_p, $p \neq s$ are equal to $m+1$.
2. We remark that for this proof no genericity arguments, neither with respect to the points A and W, nor with respect to the parameter α_s were used. Hence, the statement is true for all situations.
3. In the very definition of $X_{m,s}$ the local coordinate z_s enters. In fact it only depends on the first order jet of the coordinate, two different elements will just differ by a rescaling.
4. The elements $X_{m,s}$ are highly non-unique. For introducing the almost-grading we will have to make them essentially unique by trying to find a divisor T as small as possible but such that the statement is still true. Further down, we will come back to this.

Proposition 3.9. *Let $X^u, u = 1, \ldots, \dim \mathfrak{g}$ be a basis of $\mathfrak{g}$ and*

$$X^u_{m,s}, \quad u = 1, \ldots, \dim \mathfrak{g}, \quad s = 1, \ldots, N, \quad m \in \mathbb{Z} \tag{3.30}$$

any fixed set of elements chosen according to Proposition 3.8 then
(a) These elements are linearly independent.
(b) The set of classes $[X^u_{m,s}]$, $u = 1, \ldots, \dim \mathfrak{g}$, $s = 1, \ldots, N$ will constitute a basis of the quotient $\overline{\mathfrak{g}}_{(m)}/\overline{\mathfrak{g}}_{(m+1)}$.
(c) $\dim \overline{\mathfrak{g}}_{(m)}/\overline{\mathfrak{g}}_{(m+1)} = N \cdot \dim \mathfrak{g}$.
(d) The classes of the elements $X^u_{m,s}$ will not depend on the elements chosen.

Proof. By the local expansion it follows like in the proof of Proposition 3.4 that the elements (3.30) are linearly independent, hence (a). Furthermore, by ignoring higher orders, i.e. elements from $\overline{\mathfrak{g}}_{(m+1)}$ they stay linearly independent. Hence (b), and (c) follows. Part (d) is true by the very definition of the elements. □

Given $X \in \mathfrak{g}$ we will denote for the moment by $X_{m,s}$ any element fulfilling the conditions in Proposition 3.8.

As the proof of Proposition 3.7 stays also valid for these elements we have

Proposition 3.10. *The algebra $\overline{\mathfrak{g}}$ is a filtered algebra with respect to the introduced filtration $(\overline{\mathfrak{g}}_{(m)})$ i.e.*

$$[\overline{\mathfrak{g}}_{(m)}, \overline{\mathfrak{g}}_{(k)}] \subseteq \overline{\mathfrak{g}}_{(m+k)}. \tag{3.31}$$

Moreover,

$$[X_{k,s}, Y_{m,p}] = [X, Y]_{k+m,s}\delta_s^p + L, \quad L \in \overline{\mathfrak{g}}_{(m+k+1)}. \tag{3.32}$$

Our next goal is to introduce the homogeneous subspaces $\overline{\mathfrak{g}}_m$. A too naive method would be to take the linear span of a fixed set of elements (3.30) for $\overline{\mathfrak{g}}_m$. The condition of almost-gradedness with respect to the lower bound would be fulfilled by $m + k$, but not necessarily for the upper bound. To fix this we have to place more strict conditions on the pole orders at O, and we have to specify the divisor $(D_m)_O$ in a coherent manner (with respect to m). By our recipe the elements will become essentially unique in the generic situation at least for nearly all m. For non-generic α_s it might be necessary to modify the prescription for individual component functions. But all these modifications will change only the upper bound by a constant.

Remark. Before we advance we recall that for $\mathcal{A}$ and for the usual Krichever-Novikov current algebra $\mathfrak{g} \otimes \mathcal{A}$ we have an almost-graded structure.
1. As explained in Section 2 we have the direct sum decomposition (2.8). Moreover, $\overline{\mathfrak{s}}(n) \cong \mathcal{A}$. Hence the scalar part is almost-graded and fulfills Theorem 3.2 and Proposition 3.3. If we show the statements for $\overline{\mathfrak{sl}}(n)$ then it will follow for $\overline{\mathfrak{gl}}(n)$. Hence, it is enough to consider in the following the case of $\mathfrak{g}$ simple.
2. Moreover, if the Tyurin data is empty (or all $\alpha_s = 0$) then our Lax operator algebras reduce to the Krichever-Novikov current algebras. For those we have the statements. Hence, it is enough to consider Lax operator algebras with non-empty Tyurin data. The reader might ask why we make such a different treatment. In fact, for non-empty Tyurin data the proof will need less case distinctions.

We will now give the general description for the **generic situation** for $\mathfrak{g}$ simple, and proof the claim about almost-gradedness in detail. For the non-generic situation we will show where things have to be modified.

Recall that for the divisor D_m we had the decomposition (3.21). The terms $(D_m)_I$ and D_W stay as above. For $(D_m)_O$ we require

$$(D_m)_O = \sum_{i=1}^{M} (a_i m + b_{m,i})\, Q_i, \tag{3.33}$$

with $a_i, b_{m,i} \in \mathbb{Q}$ such that $a_i m + b_{m,i} \in \mathbb{Z}$, $a_i > 0$ and that there exists a B such that $|b_{m,i}| < B, \forall m \in \mathbb{Z}, i = 1, \ldots, M$. Furthermore,

$$\sum_{i=1}^{M} a_i = N, \qquad \sum_{i=1}^{M} b_{m,i} = N + g - 1, \qquad (D_{m+1})_O > (D_m)_O. \tag{3.34}$$

For the degrees we calculate

$$\deg((D_m)_O) = m \cdot N + (N + g - 1), \quad \deg((D_{m+1})_O) = \deg((D_m)_O) + N. \tag{3.35}$$

Example. **1.** For $M = 1$ we have the unique solution

$$(D_m)_O = (N \cdot m + (N + g - 1)\, Q_M. \tag{3.36}$$

2. For $N \geq M$ the prescription

$$(D_m)_O = (m + 1) \sum_{j=1}^{M-1} Q_j + \big((N - M + 1)(m + 1) + g - 1\big) Q_M \tag{3.37}$$

will do. Apart from the D_W the corresponding divisor D_m was introduced in [15] where the almost-grading in case of multi-point Krichever-Novikov algebras and tensors has been considered for the first time (see also [12]).
3. In [15] also prescriptions for the case $N < M$ were given. We will not reproduce it here.

Hence in all cases we can find such divisors.
Now we set

$$\overline{\mathfrak{g}}_m \ := \ \{L \in \overline{\mathfrak{g}} \ |(L) \geq -D_m\}. \tag{3.38}$$

Proposition 3.11.
(a) $\dim \overline{\mathfrak{g}}_m = N \dim \mathfrak{g}$.
(b) A basis of $\overline{\mathfrak{g}}_m$ is given by elements $X^u_{m,s}$, $u = 1, \ldots, \dim \mathfrak{g}$, $s = 1, \ldots, N$ fulfilling the conditions

$$X^u_{m,s_|}(z_p) = X^u z_s^m\, \delta_p^s + O(z_p^{m+1}). \tag{3.39}$$

Proof. We set $r := \dim \mathfrak{g}$. First we deal with the generic situation. As explained above at the weak singular points we have exactly as much relations as we get parameters by the poles. Hence for the calculation of $\dim L'(D)$ the contribution of the degree of D_W (which is $\epsilon \cdot K$) will be canceled by the relations (which are $r \cdot \epsilon \cdot K$). Here ϵ is equal to 1 or 2, depending on $\mathfrak{g}$. For the degree of D_m we calculate

$$\deg D_m = g + (N - 1) + \epsilon K \geq g. \tag{3.40}$$

We stay in the region where equality for (3.25) is true and calculate

$$\dim L'(D_m) = \dim L(D_m) - \epsilon r K = rN + r\epsilon K - \epsilon r K = rN. \tag{3.41}$$

As by definition $\overline{\mathfrak{g}}_m = L'(D_m)$ we get (a).
Next we consider $D'_m = D_m - \sum_{i=1}^{N} P_i$. For its degree we calculate $\deg(D_m - \sum_{i=1}^{N} P_i) =$

$g-1+\epsilon K$. As $K\ge 1$ we are still in the domain where we have equality for Riemann-Roch. Hence $\dim L'(D'_m)=0$. Now for $D''_m=D'_m+P_s$ we calculate $\dim L'(D''_m)=r$. This shows that for every basis element X^u of $\mathfrak{g}$ there exists up to multiplication with a scalar a unique element $X^u_{m,s}\in\overline{\mathfrak{g}}_m$ which has the local expansion

$$X^u_{m,s_|}(z_p)=X^u\delta^p_s z_p+O(z_p^{m+1}). \tag{3.42}$$

Hence, (b).
In the non-generic case we have to change the pole orders in the definition of the divisor part $(D_m)_O$ in a minimal way by adding or subtracting finitely many points to reach the situation such that we obtain exactly the dimension formula and existence of the basis of the required type. We have to take care that the number of changes maximally needed will be bounded independent of m. In fact this number is bounded by the number of points Q from O needed to add to the divisor D_m of the generic situation (which is of degree $N+g-1+\epsilon K$) to reach a divisor D'_m with $\deg D'_m\ge 2g-1+H$, where H is the number of relations for the α_s. □

Proposition 3.12.

$$\overline{\mathfrak{g}}=\bigoplus_{m\in\mathbb{Z}}\overline{\mathfrak{g}}_m. \tag{3.43}$$

Proof. The elements $X^u_{m,s}$ introduced as the basis elements in $\overline{\mathfrak{g}}_m$ are elements of the type of Proposition 3.8 with respect to the grading. By Proposition 3.9 they stay linearly independent even if we considered all m's together, as their classes are linearly independent. Hence, the sum on the r.h.s. of (3.43) is a direct sum.
To avoid to take care of special adjustments to be done for the non-generic situations we consider $m\gg 0$ and the divisor

$$E_m:=-(D_m)_I+D_W+(D_m)_O=m\sum_{i=1}^N P_i+D_W+(D_m)_O, \tag{3.44}$$

where $(D_m)_O$ is the divisor used for fixing the basis elements in $\overline{\mathfrak{g}}_m$, see (3.33). For its degree we have

$$\deg E_m=2mN+(N+g-1)+\epsilon K. \tag{3.45}$$

For $m\gg 0$ we are in the region where (3.25) is an equality. Hence, after subtraction the relations we get

$$\dim L'(E_m)=\dim\mathfrak{g}\cdot((2m+1)N)\,. \tag{3.46}$$

The basis elements

$$X^u_{k,s},\qquad u=1,\ldots,\dim\mathfrak{g},\quad s=1,\ldots,N,\quad -m\le k\le m \tag{3.47}$$

are in $L'(E_m)$. This is shown by considering the orders at I and O. For I it is obvious. For O we have to use from (3.34) the fact that $(D_{(k+1)})_O>(D_k)_O$. Hence, $-(D_m)_O$ is a lower bound for the O-part of the divisors for the element (3.47). But these are $(2m+1)\cdot N\cdot\dim\mathfrak{g}$ linearly independent elements. Hence,

$$L'(E_m) = \bigoplus_{k=-m}^{m} \overline{\mathfrak{g}}_k. \tag{3.48}$$

An arbitrary element $L \in \overline{\mathfrak{g}}$ has only finite pole orders at the points in I and O. Hence, there exists an m such that $L \in L'(E_m)$. This is again obvious for the points in I. For the points in O we use that by the conditions for $(D_m)_O$, see (3.33) for all $i = 1, \dots, M$ we have that $a_i > 0$. Hence every pole order at O will be superseded by a $(D_m)_O$ with m suitably big. This shows the claim. □

Proposition 3.13. *There exist a constant S independent of n and m such that*

$$[\overline{\mathfrak{g}}_m, \overline{\mathfrak{g}}_k] \subseteq \bigoplus_{h=m+k}^{m+k+S} \overline{\mathfrak{g}}_h. \tag{3.49}$$

Proof. We will give the proof for the generic case (and $\mathfrak{g}$ simple) first and then point out the modification needed for the general situation. Let $L \in [\overline{\mathfrak{g}}_m, \overline{\mathfrak{g}}_k]$ then

$$(L) \ge -(D_m + D_k)_I - D_W - (D_m + D_k)_O \tag{3.50}$$

(observe that D_W does not redouble here). We consider the divisors D_h. Recall the formula (3.33). As all $a_i > 0$ there exists an h_0 such that $\forall h \ge h_0$ we have

$$(D_h)_O \ge (D_m + D_k)_O \tag{3.51}$$

Hence, there exists also a smallest $h \in \mathbb{Z}$ such that (3.51) is still true. We call this h_{max}. Again by (3.34) $h_{max} \ge m + k$. Now we consider the divisor

$$E_m = (D_m + D_k)_I + D_W + (D_{h_{max}})_O. \tag{3.52}$$

From (3.35) we calculate

$$\deg((D_{hmax})_O) = \deg((D_m + D_k)_O) + (h_{max} - (m + k))N. \tag{3.53}$$

Hence,

$$\deg(E_m) = -(m + k)N + \epsilon K + h_{max} \cdot N + (N + g - 1). \tag{3.54}$$

As $\deg(E_m) \ge g$ and under the assumption of genericity we stay in the region where

$$\dim L'(E_m) = \deg\mathfrak{g} \cdot (h_{max} - (m + k) + 1)N. \tag{3.55}$$

As in the proof of Proposition 3.12 we get that the elements (3.47) for $m + k \le h \le h_{max}$ lie in $L'(E_m)$. They are linearly independent, hence

$$L'(E_m) = \bigoplus_{h=n+m}^{h_{max}} \overline{\mathfrak{g}}_h. \tag{3.56}$$

By (3.50) the L, we started with, lies also in $L'(E_m)$ and consequently also on the right hand side of (3.56).

To show almost-grading we have to show that there exists an S (independent of m and k such that $h_{max} = m + k + S$. The relation (3.51) can be rewritten as

$$a_i h + b_{h,i} \geq a_i(m+k) + b_{m,i} + b_{k,i}, \quad \forall i = 1, \ldots, M. \tag{3.57}$$

This rewrites to

$$h \geq (m+k) + \frac{b_{m,i} + b_{k,i} - b_{h,i}}{a_i}, \quad \forall i = 1, \ldots, M. \tag{3.58}$$

The minimal h for which this is true is

$$h_{max} = (m+k) + \min_{i=1,\ldots,M} \lceil \frac{b_{m,i} + b_{k,i} - b_{h,i}}{a_i} \rceil, \tag{3.59}$$

where for any real number x the $\lceil x \rceil$ denotes the smallest integer $\geq x$. As our $|b_{m,i}|$ are bounded uniformly by B the 3.term in (3.59) will be uniformly bounded by a constant S too. Hence, we get almost-grading. In the case of non-generic points and α_s's the divisors at O have to be modified by finitely many modifications. Hence the constant S has to be adapted by adding a finite constant to it. But still everything remains almost-graded. □

From the proof we can even calculate h_{max} if needed. As an example we give

Corollary 3.14. *In the generic simple Lie algebra case for $N \geq M$ with the standard prescription (3.37) we have* $h_{max} = n + m + S$ *with*

$$S = \begin{cases} 0, & g = 0, \ N = M = 1, \\ 1, & g = 0, \ M > 1, \\ 1, & g = 1 \\ 1 + \lceil \frac{g-1}{N-M+1} \rceil, & g \geq 2. \end{cases} \tag{3.60}$$

Proof. For the standard prescription we have

$$\begin{gathered} a_i = 1, \ i = 1, \ldots, M-1, \quad a_M = N - M + 1, \\ b_i = b_{m,i} = 1, \ i = 1, \ldots, M-1, \quad b_M = b_{m,M} = N - M + g. \end{gathered} \tag{3.61}$$

Hence,

$$S = \max_{i=1,\ldots,M} \lceil \frac{b_i}{a_i} \rceil. \tag{3.62}$$

which yields the result. □

Now we are ready to collect the results of Propositions 3.11, 3.12 and 3.13. The statements are exactly the statements both of Theorem 3.2 and Proposition 3.3. All statements of Section 3.1 are now shown to be true. In particular, now we know that both filtrations (3.5) and (3.4) coincide. Hence, (3.4) is also canonically defined by the splitting of A into I and O.

A Lie algebra $\mathcal{V}$ is called *perfect* if $\mathcal{V} = [\mathcal{V}, \mathcal{V}]$. Simple Lie algebras are of course perfect. The usual Krichever-Novikov current algebras $\overline{\mathfrak{g}}$ for $\mathfrak{g}$ simple are perfect too [19, Prop. 3.2]. Lax operator algebras are not necessarily perfect (at least we do not have a proof of it). Lemma 3.15 below might be considered as a weak analog of that property.

Lemma 3.15. *Let $\mathfrak{g}$ be simple and $y \in \overline{\mathfrak{g}}$ then for every $m \in \mathbb{Z}$ there exists finitely many elements $y^{(s,1)}, y^{(s,2)} \in \overline{\mathfrak{g}}$, $i = 1, \dots, l = l(m)$ such that*

$$y - \sum_{s=1}^{l} [y^{(s,1)}, y^{(s,2)}] \quad \in \quad \overline{\mathfrak{g}}_m. \tag{3.63}$$

Proof. Let y be an element of $\overline{\mathfrak{g}}$. Hence there exists a k such that $y \in \overline{\mathfrak{g}}_{(k)}$, but $y \neq \overline{\mathfrak{g}}_{(k+1)}$. In particular there exists for every point P_i elements $X^i_{k,i}$ such that

$$y - \sum_{i=1}^{N} X^i_{k,i} \in \overline{\mathfrak{g}}_{(k+1)}, \tag{3.64}$$

where $X^i_{k,i} = (X^i)_{k,i}$ is the element corresponding to $X^i \in \mathfrak{g}$. As $\mathfrak{g}$ is perfect we have $X^i = [Y^i, Z^i]$ with elements $Y^i, Z^i \in \mathfrak{g}$. We calculate

$$X^i_{k,i} = [Y^i_{0,i}, Z^i_{k,i}] + y^i, \quad y^{(i)} \in \overline{\mathfrak{g}}_{(k+1)}. \tag{3.65}$$

In total

$$y^{(k)} = y - \sum_{i=1}^{N} [Y^i_{0,i}, Z^i_{k,i}] \in \overline{\mathfrak{g}}_{(k+1)}. \tag{3.66}$$

Using the same again for $y^{(k)}$ etc., we can approximate y to every finite order by sums of commutators. □

4. Module structure

4.1. Lax operator algebras as modules over $\mathcal{A}$.

The space $\overline{\mathfrak{g}}$ is an $\mathcal{A}$-module with respect to the point-wise multiplication. Obviously, the relations (2.5), (2.6), (2.9), (2.11), are not disturbed.

Proposition 4.1.
(a) The Lax operator algebra $\overline{\mathfrak{g}}$ is an almost-graded module over $\mathcal{A}$, i.e. there exists a constant S_4 (not depending on k and m) such that

$$\mathcal{A}_k \cdot \overline{\mathfrak{g}}_m \subseteq \bigoplus_{h=k+m}^{k+m+S_4} \overline{\mathfrak{g}}_h. \tag{4.1}$$

(b) For $X \in \mathfrak{g}$

$$A_{m,s} \cdot X_{n,p} = X_{m+n,s}\, \delta^s_p + L, \qquad L \in \overline{\mathfrak{g}}_{(m+n+1)}. \tag{4.2}$$

Proof. We consider the orders of the elements in I and O. As in the proof of Proposition 3.13 the existence of a constant S_4 follows so that (4.1) is true. Hence (a).
We study the lowest order term of $A_{m,s} \cdot X_{n,r}$ at the points $P_i \in I$. Using (3.20), (3.8),(3.10) we see that if $s \neq r$ then $A_{m,s} \cdot X_{n,r} \in \overline{\mathfrak{g}}_{(m+n+1)}$ as all orders are $\geq n + m + 1$. The same is true for $s = r$ for the element $A_{m,s} \cdot X_{n,s} - X_{m+n,s}$. Hence the claim. □

Warning: in general we do not have $A_{m,s}\cdot X_{0,s}=X_{m,s}$ as the orders at O do not coincide. Also, $A_{m,s}\cdot X$ does not necessarily belong to $\overline{\mathfrak{g}}$.

4.2. **Lax operator algebras as modules over $\mathcal{L}$.**

Next we introduce an action of $\mathcal{L}$ on $\overline{\mathfrak{g}}$. This is done with the help of a certain connection $\nabla^{(\omega)}$ following the lines of [5], [6], [10] with the modification made in [23]. The connection form ω is a $\mathfrak{g}$-valued meromorphic 1-form, holomorphic outside I, O and W, and has a certain prescribed behavior at the points in W. For $\gamma_s\in W$ with $\alpha_s=0$ the requirement is that ω is also regular there. For the points γ_s with $\alpha_s\neq 0$ it is required that it has an expansion of the form

$$\omega(z_s)=\left(\frac{\omega_{s,-1}}{z_s}+\omega_{s,0}+\omega_{s,1}+\sum_{k>1}\omega_{s,k}z_s^k\right)dz_s. \tag{4.3}$$

For $\mathfrak{gl}(n)$: there exist $\tilde\beta_s\in\mathbb{C}^n$ and $\tilde\kappa_s\in\mathbb{C}$ such that

$$\omega_{s,-1}=\alpha_s\tilde\beta_s^t,\quad \omega_{s,0}\,\alpha_s=\tilde\kappa_s\alpha_s,\quad \operatorname{tr}(\omega_{s,-1})=\tilde\beta_s^t\alpha_s=1. \tag{4.4}$$

For $\mathfrak{so}(n)$: there exist $\tilde\beta_s\in\mathbb{C}^n$ and $\tilde\kappa_s\in\mathbb{C}$ such that

$$\omega_{s,-1}=\alpha_s\tilde\beta_s^t-\tilde\beta_s\alpha_s^t,\quad \omega_{s,0}\,\alpha_s=\tilde\kappa_s\alpha_s,\quad \tilde\beta_s^t\alpha_s=1. \tag{4.5}$$

For $\mathfrak{sp}(2n)$: there exist $\tilde\beta_s\in\mathbb{C}^{2n}$, $\tilde\kappa_s\in\mathbb{C}$ such that

$$\omega_{s,-1}=(\alpha_s\tilde\beta_s^t+\tilde\beta_s\alpha_s^t)\sigma,\quad \omega_{s,0}\,\alpha_s=\tilde\kappa_s\alpha_s,\quad \alpha_s^t\sigma\omega_{s,1}\alpha_s=0,\quad \tilde\beta_s^t\sigma\alpha_s=1. \tag{4.6}$$

The existence of nontrivial connection forms fulfilling the listed conditions is proved by Riemann-Roch type argument as Proposition 3.8. We might even require, and actually always will do so, that the connection form is holomorphic at I. Note also that if all $\alpha_s=0$ we could take $\omega=0$.

The connection form ω induces the following connection $\nabla^{(\omega)}$ on $\overline{\mathfrak{g}}$

$$\nabla^{(\omega)}=d+[\omega,.]. \tag{4.7}$$

Let $e\in\mathcal{L}$ be a vector field. In a local coordinate z the connection form and the vector field are represented as $\omega=\tilde\omega dz$ and $e=\tilde e\frac{d}{dz}$ with a local function $\tilde e$ and a local matrix valued function $\tilde\omega$. The covariant derivative in direction of e is given by

$$\nabla_e^{(\omega)}=dz(e)\frac{d}{dz}+[\omega(e),.\,]=e.+[\,\tilde\omega\tilde e\,,.\,]=\tilde e\cdot\big(\frac{d}{dz}+[\,\tilde\omega\,,.\,]\big). \tag{4.8}$$

Here the first term ($e.$) corresponds to taking the usual derivative of functions in each matrix element separately, whereas $\tilde e\cdot$ means multiplication with the local function $\tilde e$.

Using the last description we obtain for $L\in\overline{\mathfrak{g}},\ g\in\mathcal{A},\ e,f\in\mathcal{L}$

$$\nabla_e^{(\omega)}(g\cdot L)=(e.g)\cdot L+g\cdot\nabla_e^{(\omega)}L,\qquad \nabla_{g\cdot e}^{(\omega)}L=g\cdot\nabla_e^{(\omega)}L, \tag{4.9}$$

and

$$\nabla_{[e,f]}^{(\omega)}=[\nabla_e^{(\omega)},\nabla_f^{(\omega)}]. \tag{4.10}$$

The proofs of the following statements are completely the same as the proofs for the two-point case presented in [23]. Hence, they are here omitted.

Proposition 4.2.
(a) $\nabla_e^{(\omega)}$ *acts as a derivation on the Lie algebra* $\overline{\mathfrak{g}}$*, i.e.*

$$\nabla_e^{(\omega)}[L,L'] = [\nabla_e^{(\omega)}L,L'] + [L,\nabla_e^{(\omega)}L']. \tag{4.11}$$

(b) The covariant derivative makes $\overline{\mathfrak{g}}$ *to a Lie module over* $\mathcal{L}$*.*
(c) The decomposition $\overline{\mathfrak{gl}}(n) = \overline{\mathfrak{s}}(n) \oplus \overline{\mathfrak{sl}}(n)$ *is a decomposition into* $\mathcal{L}$*-modules, i.e.*

$$\nabla_e^{(\omega)} : \overline{\mathfrak{s}}(n) \to \overline{\mathfrak{s}}(n), \qquad \nabla_e^{(\omega)} : \overline{\mathfrak{sl}}(n) \to \overline{\mathfrak{sl}}(n). \tag{4.12}$$

Moreover, the $\mathcal{L}$*-module* $\overline{\mathfrak{s}}(n)$ *is equivalent to the* $\mathcal{L}$*-module* $\mathcal{A}$*.*

Proposition 4.3.
(a) $\overline{\mathfrak{g}}$ *is an almost-graded* $\mathcal{L}$*-module.*
(b) For the corresponding $\mathcal{L}$*-action we have*

$$\nabla_{e_{k,s}}^{(\omega)} X_{m,r} = m \cdot X_{k+m,s}\,\delta_s^r + L, \quad L \in \overline{\mathfrak{g}}_{(k+m+1)}. \tag{4.13}$$

Proof. (a) By Proposition 4.2 $\overline{\mathfrak{g}}$ is an $\mathcal{L}$-module. It remains to show that there is an upper bound for the order of the elements of the type $n+m+S_5$, with S_5 independent of n and m (but may depend on ω). We write (4.8) for homogeneous elements and obtain

$$\nabla_{e_{k,s}}^{(\omega)} X_{m,r} = e_{k,s} \,.\, X_{m,r} + [\,\tilde\omega\tilde e_{k,s}\,, X_{m,r}]. \tag{4.14}$$

The form ω has fixed orders at I and at O, the action of $\mathcal{L}$ on $\mathcal{A}$ is almost-graded, and the bracket corresponds to the commutator in the almost-graded $\overline{\mathfrak{g}}$. By considering the corresponding bounds for the order of poles at I and O we get such an universal bound.
(b) Locally at P_i, $i=1,\ldots,N$ we have

$$X_{m,r}{}_|(z_i) = X z_i^m \delta_i^r + O(z_i^{m+1}), \qquad e_{k|}(z_i) = z_i^{k+1}\delta_i^k \frac{d}{dz} + O(z_i^{k+2}). \tag{4.15}$$

This implies

$$e_{k,s} \,.\, X_{m,r}(z_i) = mX z_i^{k+m}\delta_i^r\delta_i^s + O(z_i^{k+m+1}), \quad \tilde\omega\tilde e_k(z_i) = B z_i^{k+1} + O(z_i^{k+2}), \tag{4.16}$$

with $B \in \mathfrak{gl}(n)$. Hence

$$[\,\tilde\omega\tilde e_k\,, X_m] = O(z_i^{k+m+1}), \forall i, \tag{4.17}$$

and the second term will only contribute to higher order. It remains the first term in (4.14). If $r \neq s$ then $e_{k,s} \,.\, X_{m,r}(z_i) \in O(z_i^{k+m+1})$ for all i. If $r = s$ then $(e_{k,s} \,.\, X_{m,s} - mX_{m+k,s})(z_i)) \in O(z_i^{k+m+1})$. Hence, (4.13) follows. □

4.3. Module structure over $\mathcal{D}^1$ and the algebra $\mathcal{D}^1_{\mathfrak{g}}$.

The Lie algebra $\mathcal{D}^1$ of meromorphic differential operators on Σ of degree ≤ 1 holomorphic outside of $I \cup O$ is defined as the semi-direct sum of $\mathcal{A}$ and $\mathcal{L}$ with the commutator between them given by the action of $\mathcal{L}$ on A. It is the vector space direct sum $\mathcal{D}^1 = \mathcal{A} \oplus \mathcal{L}$ with Lie bracket

$$[(g,e),(h,f)] := (e.h - f.g, [e,f]). \tag{4.18}$$

In particular

$$[e, h] = e.h. \tag{4.19}$$

It is an almost-graded Lie algebra [18].

Proposition 4.4. *The Lax operator algebras* $\overline{\mathfrak{g}}$ *are almost-graded Lie modules over* $\mathcal{D}^1$ *via*

$$e.L := \nabla_e^{(\omega)} L, \qquad h.L := h \cdot L. \tag{4.20}$$

Proof. As $\overline{\mathfrak{g}}$ is an almost-graded $\mathcal{A}$- and $\mathcal{L}$-module it is enough to show that the relation (4.19) is satisfied. For $e \in \mathcal{L}, h \in \mathcal{A}, L \in \overline{\mathfrak{g}}$ using (4.8) we get

$$e.(h.L) - h.(e.L) = \nabla_e^{(\omega)}(hL) - h\nabla_e^{(\omega)}(L) =$$
$$\tilde{e}\left(\frac{d(hL)}{dz} + [\tilde{\omega}, hL]\right) - h\tilde{e}\left(\frac{dL}{dz} + [\tilde{\omega}, L]\right) = \left(\tilde{e}\frac{dh}{dz}\right) L = (e.h)L = [e, h].L.$$

□

The Lax operator algebra $\overline{\mathfrak{g}}$ is a module over the Lie algebra $\mathcal{L}$ which acts on $\overline{\mathfrak{g}}$ by derivations (according to Proposition 4.2). Proposition 4.2 says that this action of $\mathcal{L}$ on $\overline{\mathfrak{g}}$ is an action by derivations. Hence as above we can consider the semi-direct sum $\mathcal{D}^1_{\mathfrak{g}} = \overline{\mathfrak{g}} \oplus \mathcal{L}$ with Lie product given by

$$[e, L] := e.L = \nabla_e^{(\omega)} L, \tag{4.21}$$

for the mixed pairs. See [19] for the corresponding construction for the classical Krichever-Novikov algebras of affine type.

5. Cocycles

In this section we will study 2-cocycles for the Lie algebra $\overline{\mathfrak{g}}$ with values in $\mathbb{C}$. It is well-known that the corresponding cohomology space $\mathrm{H}^2(\overline{\mathfrak{g}}, \mathbb{C})$ classifies equivalence classes of (one-dimensional) central extensions of $\overline{\mathfrak{g}}$.

For the convenience of the reader we recall that a 2-cocycle for $\overline{\mathfrak{g}}$ is a bilinear form $\gamma : \overline{\mathfrak{g}} \times \overline{\mathfrak{g}} \to \mathbb{C}$ which is (1) antisymmetric and (2) fulfills the condition

$$\gamma([L, L'], L'') + \gamma([L', L''], L) + \gamma([L'', L], L') = 0, \qquad L, L', L'' \in \overline{\mathfrak{g}}\ . \tag{5.1}$$

A 2-cocycle γ is a coboundary if there exists a linear form ϕ on $\overline{\mathfrak{g}}$ such that

$$\gamma(L, L') = \phi([L, L']), \qquad L, L' \in \overline{\mathfrak{g}}. \tag{5.2}$$

Given a 2-cocycle γ for $\overline{\mathfrak{g}}$, the associated central extension $\widehat{\mathfrak{g}}_\gamma$ is given as vector space direct sum $\widehat{\mathfrak{g}}_\gamma = \overline{\mathfrak{g}} \oplus \mathbb{C} \cdot t$ with Lie product

$$[\widehat{L}, \widehat{L'}] = \widehat{[L, L']} + \gamma(L, L') \cdot t, \quad [\widehat{L}, t] = 0, \qquad L, L' \in \overline{\mathfrak{g}}. \tag{5.3}$$

Here we used $\widehat{L} := (L, 0)$ and $t := (0, 1)$. Vice versa, every central extension

$$0 \longrightarrow \mathbb{C} \xrightarrow{\ i_2\ } \widehat{\mathfrak{g}} \xrightarrow{\ p_1\ } \overline{\mathfrak{g}} \longrightarrow 0, \tag{5.4}$$

defines a 2-cocycle $\gamma : \overline{\mathfrak{g}} \to \mathbb{C}$ by choosing a section $s : \overline{\mathfrak{g}} \to \widehat{\mathfrak{g}}$.

Two central extensions $\widehat{\mathfrak{g}}_\gamma$ and $\widehat{\mathfrak{g}}_{\gamma'}$ are equivalent if and only if the defining cocycles γ and γ' are cohomologous.

5.1. **Geometric cocycles.**

Next we introduce geometric 2-cocycles. Let ω be a connection form as introduced in Section 4.2 for defining the connection (4.7). Furthermore, let C be a (not necessarily connected) differentiable cycle on Σ not meeting the sets $A = I \cup O$ and W.

As in the two point situation considered in [23] we define the following bilinear forms on $\overline{\mathfrak{g}}$:

$$\gamma_{1,\omega,C}(L, L') = \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{tr}(L \cdot \nabla^{(\omega)} L'), \qquad L, L' \in \overline{\mathfrak{g}}, \tag{5.5}$$

and

$$\gamma_{2,\omega,C}(L, L') = \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{tr}(L) \cdot \mathrm{tr}(\nabla^{(\omega)} L'), \qquad L, L' \in \overline{\mathfrak{g}}. \tag{5.6}$$

The following propositions and their proofs remain the same as in [23] (of course now to be interpreted in this more general context), and we will not repeat them.

Proposition 5.1. *The bilinear forms $\gamma_{1,\omega,C}$ and $\gamma_{2,\omega,C}$ are cocycles.*

Proposition 5.2.
(a) The cocycle $\gamma_{2,\omega,C}$ does not depend on the choice of the connection form ω.
(b) The cohomology class $[\gamma_{1,\omega,C}]$ does not depend on the choice of the connection form ω. More precisely

$$\gamma_{1,\omega,C}(L, L') - \gamma_{1,\omega',C}(L, L') = \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{tr}\big((\omega - \omega')[L, L']\big). \tag{5.7}$$

As $\gamma_{2,\omega,C}$ does not depend on ω we will drop ω in the notation. Note that $\gamma_{2,C}$ vanishes on $\overline{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(2n)$. But it does not vanish on $\overline{\mathfrak{s}}(n)$, hence not on $\overline{\mathfrak{gl}}(n)$.

5.2. **$\mathcal{L}$-invariant cocycles.**

As explained in Section 4.2 after fixing a connection form ω' the vector field algebra $\mathcal{L}$ operates on $\overline{\mathfrak{g}}$ via the covariant derivative $e \mapsto \nabla_e^{(\omega')}$.

Definition 5.3. A cocycle γ for $\overline{\mathfrak{g}}$ is called *$\mathcal{L}$-invariant* (with respect to ω') if

$$\gamma(\nabla_e^{(\omega')} L, L') + \gamma(L, \nabla_e^{(\omega')} L') = 0, \qquad \forall e \in \mathcal{L}, \quad \forall L, L' \in \overline{\mathfrak{g}}. \tag{5.8}$$

Proposition 5.4. *(a) The cocycle $\gamma_{2,C}$ is $\mathcal{L}$-invariant.*
(b) If $\omega = \omega'$ then the cocycle $\gamma_{1,\omega,C}$ is $\mathcal{L}$-invariant.

The proof is the same as presented in [23] for the two point case.

We call a cohomology class *$\mathcal{L}$-invariant* if it has a representing cocycle which is $\mathcal{L}$-invariant. The reader should be warned that this does not mean that all representing cocycles are $\mathcal{L}$-invariant. On the contrary, see Corollary 6.5. Clearly, the $\mathcal{L}$-invariant classes constitute a subspace of $\mathrm{H}^2(\overline{\mathfrak{g}}, \mathbb{C})$ which we denote by $\mathrm{H}^2_{\mathcal{L}}(\overline{\mathfrak{g}}, \mathbb{C})$.

5.3. Some remarks on the cocycles on $\mathcal{D}^1_{\mathfrak{g}}$.

In the following let $\omega = \omega'$. The property of $\mathcal{L}$-invariance of a cocycle has a deeper meaning. In Section 4.3 we introduced the algebra $\mathcal{D}^1_{\mathfrak{g}}$. The Lax operator algebra $\overline{\mathfrak{g}}$ is a subalgebra of $\mathcal{D}^1_{\mathfrak{g}}$. Given a 2-cocycle γ for $\overline{\mathfrak{g}}$ we might extend it to $\mathcal{D}^1_{\mathfrak{g}}$ as a bilinear form by setting ($L, L' \in \overline{\mathfrak{g}}$, $e, f \in \mathcal{L}$)

$$\tilde{\gamma}(L, L') = \gamma(L, L'), \quad \tilde{\gamma}(e, L) = \tilde{\gamma}(L, e) = 0, \quad \tilde{\gamma}(e, f) = 0. \tag{5.9}$$

Proposition 5.5. *The extended bilinear form $\tilde{\gamma}$ is a cocycle for $\mathcal{D}^1_{\mathfrak{g}}$ if and only if γ is $\mathcal{L}$-invariant.*

Proof. The conditions defining a cocycle are obviously fulfilled for the triples of elements consisting either of currents or of vector fields. The only condition which does not follow automatically from (5.9) for $\tilde{\gamma}$ is

$$\tilde{\gamma}([L, L'], e) + \tilde{\gamma}([L', e], L) + \tilde{\gamma}([e, L], L') = 0. \tag{5.10}$$

Using (4.21) we get that (5.10) is true if an only if

$$\gamma(\nabla_e^{(\omega)} L, L') + \gamma(L, \nabla_e^{(\omega)} L') = 0, \tag{5.11}$$

which is $\mathcal{L}$-invariance. □

5.4. Bounded and local cocycles.

Definition 5.6. Given an almost-graded Lie algebra $\mathcal{V} = \bigoplus_{m\in\mathbb{Z}} \mathcal{V}_m$. A cocycle γ is called *bounded (from above)* if there exists a constant $R_1 \in \mathbb{Z}$ such that

$$\gamma(\mathcal{V}_n, \mathcal{V}_m) \neq 0 \implies n + m \leq R_1. \tag{5.12}$$

Similarly bounded from below is defined.
A cocycle is called *local* if and only if it is bounded from above and below. Equivalently, there exist $R_1, R_2 \in \mathbb{Z}$ such

$$\gamma(\mathcal{V}_n, \mathcal{V}_m) \neq 0 \implies R_2 \leq n + m \leq R_1. \tag{5.13}$$

The almost-grading of $\mathcal{V}$ can be extended from $\mathcal{V}$ to the corresponding central extension $\widehat{\mathcal{V}}_\gamma$ (5.3) by assigning to the central element t a certain degree (e.g. the degree 0) if and only if the defining cocycle for the central extension is local.

We call a cohomology class *bounded (resp. local)* if it contains a bounded (resp. local) representing cocycle. Again, not every representing cocycle of a bounded (resp. local) class is bounded (resp. local). The set of bounded cohomology classes is a subspace of $\mathrm{H}^2(\overline{\mathfrak{g}}, \mathbb{C})$ which we denote by $\mathrm{H}^2_b(\overline{\mathfrak{g}}, \mathbb{C})$. It contains the subspace of local cohomology classes denoted by $\mathrm{H}^2_{loc}(\overline{\mathfrak{g}}, \mathbb{C})$. This space classifies the almost-graded central extensions of $\overline{\mathfrak{g}}$ up to equivalence. Both spaces admit subspaces consisting of those cohomology classes admitting a representing cocycle which is both bounded (resp. local) and $\mathcal{L}$-invariant. The subspaces are denoted by $\mathrm{H}^2_{b,\mathcal{L}}(\overline{\mathfrak{g}}, \mathbb{C})$, resp. $\mathrm{H}^2_{loc,\mathcal{L}}(\overline{\mathfrak{g}}, \mathbb{C})$.

If we consider our geometric cocycles $\gamma_{2,C}$ and $\gamma_{1,\omega,C}$ obtained by integrating over an arbitrary cycle then they will neither be bounded, nor local, nor will they define a bounded or local cohomology class.

Next we will consider special integration paths. Let C_i be positively oriented (deformed) circles around the points P_i in I, $i = 1, \ldots, N$ and C_j^* positively oriented ones around the points Q_j in O, $j = 1, \ldots, M$. The cocycle values of γ if integrated over such cycles can be calculated via residues, e.g.

$$\gamma_{1,\omega,C_i}(L, L') = \mathrm{res}_{P_i}(\mathrm{tr}(L \cdot \nabla^{(\omega)} L')), \quad i = 1, \ldots, N \ . \tag{5.14}$$

Proposition 5.7. *(1) The 1-form* $\mathrm{tr}(L \cdot \nabla^{(\omega)} L')$ *has no poles outside of* $A = I \cup O$.
(2) The 1-form $\mathrm{tr}(L) \cdot \mathrm{tr}(dL')$ *has no poles outside of* $A = I \cup O$.

Proof. For (1) see [10]. For (2) see [23]. □

A cycle C_S is called a separating cycle if it is smooth, positively oriented of multiplicity one, it separates the points in I from the points in O, and it does not meet A or W. It might have multiple components. For our cocycles (5.5), (5.6) we integrate the forms of Proposition 5.7 over closed curves C. By this proposition the integrals will yield the same results if $[C] = [C']$ in $\mathrm{H}^0(\Sigma \setminus A, \mathbb{Z})$. Note that the weak singular points will not show up in this context. In this sense we can write for every separating cycle

$$[C_S] = \sum_{i=1}^{K} [C_i] = -\sum_{j=1}^{M} [C_j^*]. \tag{5.15}$$

The minus sign appears due to the opposite orientation. In particular the cocycle values obtained by integrating over a C_S can be obtained by calculating residues either over the points in I or the points in O.

Theorem 5.8. *Let ω coincides with the connection form ω' associated to the $\mathcal{L}$-action then*
(a) For $i = 1, \ldots, N$ the cocycles γ_{1,ω,C_i} and γ_{2,C_i} with C_i a circle around P_i will be bounded from above and $\mathcal{L}$-invariant.
(b) For $j = 1, \ldots, M$ the cocycles γ_{1,ω,C_j^*} and γ_{2,C_j^*} with C_j^* a circle around Q_j will be bounded from below and $\mathcal{L}$-invariant.
(c) The cocycles γ_{1,ω,C_S} and γ_{2,C_S} with C_S a separating cycle will be local and $\mathcal{L}$-invariant.
(d) In case (a) and (c) the upper bound will be zero.

Proof. The statement about $\mathcal{L}$-invariance follows from Proposition 5.4. In fact only for this $\omega = \omega'$ is needed.
As explained above the cocycle calculation if integrated over C_i (or over C_j^*) reduces to the calculation of residues. Let $L \in \overline{\mathfrak{g}}_n$, $L' \in \overline{\mathfrak{g}}_m$ then $\mathrm{ord}_{P_i}(L) \geq n$ and $\mathrm{ord}_{P_i}(L') \geq m$. As ω is holomorphic at P_i we obtain

$$\mathrm{ord}_{P_i}(dL') \geq m - 1, \quad \mathrm{ord}_{P_i}(\nabla^{(\omega)} L') \geq m - 1 \ .$$

Hence, if $n + m > 0$ neither one of the 1-forms appearing in the cocycle definition has poles at I and consequently no residues. This shows (a).
For (b) we have to consider the orders at the points in O of the basis elements of $\overline{\mathfrak{g}}_m$. By the prescriptions (3.33) and (3.34) and taking into account possible poles of ω at O we find an R_2 such that if $n + m \leq R_2$ the integrands will not have poles anymore. This

shows (b).
Using (5.15) we can obtain the values of the cocycles integrated over C_S either by adding up the values obtained by integration either over I or over O. Hence boundedness from below and from above. Hence, locality.
That zero is an upper bound followed already during the proof. □

6. Classification Results

Recall that we are in the multi-point situation $A = I \cup O$ with $\#I = N$. The C_i, C_j^*, and C_S are the special cycles introduced in Section 5.4. If we will use the word *bounded* for a cocycle we always mean bounded from above if nothing else is said.

Proposition 6.1. *The cocycles* γ_{1,ω,C_i}, $i = 1, \dots, N$ *(and* γ_{2,C_i}, $i = 1, \dots, N$ *for* $\overline{\mathfrak{gl}}(n)$*) are linearly independent.*

Proof. Assume that there is a linear relation

$$0 = \sum_{i=1}^{N} \alpha_i \gamma_{1,\omega,C_i} + \sum_{i=1}^{N} \beta_i \gamma_{2,C_i}, \quad \alpha_i, \beta_i \in \mathbb{C}. \tag{6.1}$$

The last sum will not appear in the simple algebra case. Recall that for a pair $L, L' \in \overline{\mathfrak{g}}$ the above cocycles can be calculated by taking residues

$$0 = \sum_{i=1}^{N} \alpha_i \operatorname{res}_{P_i}(\operatorname{tr}(L \cdot \nabla^{(\omega)} L')) + \sum_{i=1}^{N} \beta_i \operatorname{res}_{P_i}(\operatorname{tr}(L) \cdot \operatorname{tr}(\nabla^{(\omega)} L')). \tag{6.2}$$

In the first sum the Cartan-Killing form is present which is non-degenerated. Hence there exist $X, Y \in \mathfrak{g}$ such that $\operatorname{tr}(XY) \neq 0$ and $\operatorname{tr}(X) = \operatorname{tr}(Y) = 0$. For $k = 1, \dots, N$, using the almost-graded structure and following Proposition 3.3 we take $L = X_{1,k}$ and $L' = Y_{-1,k}$. In the neighbourhood of the point P_l, $l = 1, \dots, N$ we have

$$\begin{gathered} L(z_l) = X z_l \delta_l^k + O(z_l^2), \quad L'(z_l) = Y z_l^{-1} \delta_l^k + O(z_l^0), \\ \nabla^{(\omega)} L'(z_l) = -Y z_l^{-2} \delta_l^k + O(z_l^{-1}), \end{gathered} \tag{6.3}$$

as $\nabla^{(\omega)} L' = dL' + [\omega, L']$. Hence,

$$\operatorname{res}_{P_l}(\operatorname{tr}(L \cdot \nabla^{(\omega)} L')) = -\operatorname{tr}(XY) \delta_l^k. \tag{6.4}$$

As $\operatorname{tr}(X) = 0$ the second sum will vanish anyway and we conclude $\alpha_k = 0$, for all $k = 1, \dots, N$. For the second sum we take $X = Y$ a nonvanishing scalar matrix and chose $L = X_{1,k}$ and $L' = X_{-1,k}$. We obtain $\beta_k = 0$ for all $k = 1, \dots, N$. □

Proposition 6.2. *(*$\overline{\mathfrak{g}} = \overline{\mathfrak{gl}}(n)$*) Let* $\gamma = \sum_{i=1}^{N} \beta_i \gamma_{2,C_i}$ *be a nontrivial linear combination, then it is not a coboundary.*

Proof. Recall from (2.8) that $\overline{\mathfrak{s}}(n)$ is an abelian subalgebra of $\overline{\mathfrak{gl}}(n)$. Hence, every coboundary restricted to it will be identically zero. If we take again as in the previous proof elements $X_{1,k}$ and $X_{-1,k}$ from the scalar subalgebra we obtain as above $\beta_k = 0$. □

Proposition 6.3. *Let $\gamma = \sum_{i=1}^N \alpha_i \gamma_{1,\omega,C_i}$ be a non-trivial linear combination then it is not a coboundary.*

Proof. Assume that γ is a coboundary. This means that there exists a linear form $\phi : \overline{\mathfrak{g}} \to \mathbb{C}$ such that $\forall L, L' \in \overline{\mathfrak{g}}$

$$\gamma(L, L') = \sum_{i=1}^N \alpha_i \operatorname{res}_{P_i} \operatorname{tr}(L \cdot \nabla^{(\omega)} L') = \phi([L, L']). \tag{6.5}$$

Assume that $\gamma \neq 0$, hence one of the coefficients α_k will be non-zero. Take $H \in \mathfrak{h}$ with $\kappa(H, H) \neq 0$, where $\mathfrak{h}$ is the Cartan subalgebra of the simple part of $\mathfrak{g}$ and κ its Cartan-Killing form. Let $H_{0,k} \in \overline{\mathfrak{g}}$ be the element defined by (3.2). In particular, we have $H_{0,k} = H + O(z_k)$. We set[3] $H_{(n,k)} := H_{0,k} \cdot A_{n,k} \in \overline{\mathfrak{g}}$ and hence $H_{(n,k)} = H \cdot A_{n,k} + O(z_k^{n+1})$ in the neighbourhood of the point P_k. Recall that from the local forms (3.2) and (3.8) of our basis elements we have in the neighbourhood of points P_l with $l \neq k$

$$H_{n,k} = O(z_l^{n+1}), \quad A_{n,k} = O(z_l^{n+1}), \quad H_{(n,k)} = O(z_l^{n+1}). \tag{6.6}$$

In the following, let $n \neq 0$. We have

$$\nabla^{(\omega)} H_{(n,k)} = \nabla^{(\omega)} (H_{0,k} \cdot A_{n,k}) = \nabla^{(\omega)} (H_{0,k}) \cdot A_{n,k} + H_{0,k}\, dA_{n,k}. \tag{6.7}$$

The expression $\nabla^{(\omega)} H_{0,k}$ is of nonnegative order, $A_{n,k}$ is of order n, $H_{0,k}$ of order 0 and $dA_{n,k}$ of order $n-1$ at the point P_k. Hence

$$\nabla^{(\omega)} H_{(n,k)} = H_{0,k}\, dA_{n,k} + O(z_k^n) dz_k. \tag{6.8}$$

Now we compute

(6.9)

$$\gamma(H_{(-1,k)}, H_{(1,k)}) = \sum_{i=1}^N \alpha_i \operatorname{res}_{P_i} \operatorname{tr}(H_{(-1,k)} \cdot \nabla^{(\omega)} H_{(1,k)}) = \alpha_k \operatorname{res}_{P_k} \operatorname{tr}(H_{(-1,k)} \cdot \nabla^{(\omega)} H_{(1,k)}).$$

The last equality follows from the fact that by (6.6) we do not have any poles at the points P_l for $l \neq k$. From the above it follows

$$(\alpha_k)^{-1} \gamma(H_{(-1,k)}, H_{(1,k)}) = \operatorname{res}_{P_k} \operatorname{tr}(H_{0,k} A_{-1,k} H_{0,k} dA_{1,k}) = \operatorname{res}_{P_k} \operatorname{tr}(H_{0,k}^2 \frac{dz_k}{z_k}). \tag{6.10}$$

As $H_{0,k}^2 = H^2 + O(z_k)$ we obtain

$$(\alpha_k)^{-1} \gamma(H_{(-1,k)}, H_{(1,k)}) = \operatorname{res}_{P_k} (\operatorname{tr}(H^2) \frac{dz_k}{z_k}) = \operatorname{tr}(H^2) = \beta \cdot \kappa(H, H) \neq 0, \tag{6.11}$$

with a non-vanishing constant β relating the trace form with the Cartan-Killing form. But

$$[H_{(-1,k)}, H_{(1,k)}] = [H_{0,k} A_{-1,k}, H_{0,k} A_{1,k}] = [H_{0,k}, H_{0,k}] A_{-1,k} A_{1,k} = 0. \tag{6.12}$$

The relations (6.11) and (6.12) are in contradiction to (6.5). □

Now we are able to formulate the basic theorem.

[3] Notice that $H_{(n,k)}$ and $H_{n,k}$, in general, are different but coincide up to higher order.

Theorem 6.4.

(a) If $\mathfrak{g}$ is simple (i.e. $\overline{\mathfrak{g}} = \mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(2n)$) then the space of bounded cohomology classes is N-dimensional. If we fix any connection form ω then this space has as basis the classes of γ_{1,ω,C_i}, $i = 1, \dots, N$. Every $\mathcal{L}$-invariant (with respect to the connection ω) bounded cocycle is a linear combination of the γ_{1,ω,C_i}.

(b) For $\overline{\mathfrak{g}} = \overline{\mathfrak{gl}}(n)$ the space of local cohomology classes which are $\mathcal{L}$-invariant having been restricted to the scalar subalgebra is 2N-dimensional. If we fix any connection form ω then the space has as basis the classes of the cocycles γ_{1,ω,C_i} and γ_{2,C_i}, $i = 1, \dots, N$. Every $\mathcal{L}$-invariant local cocycle is a linear combination of the γ_{1,ω,C_i} and γ_{2,C_i}.

Proof of the theorem. Here we only outline the proof. The technicalities are postponed until Sections 7 and 8.

By Propositions 7.8 and 7.10 it follows that $\mathcal{L}$-invariant and bounded cocycles are necessarily linear combinations of the claimed form. This proves the theorem for the cohomology space $\mathrm{H}_{b,\mathcal{L}}(\overline{\mathfrak{g}}, \mathbb{C})$. For the scalar subalgebra we are done since we included the $\mathcal{L}$-invariance into the conditions of the theorem. For semi-simple algebras we have to show that there is an $\mathcal{L}$-invariant representative in each local cohomology class. But by Theorem 8.1 the space $\mathrm{H}_b(\overline{\mathfrak{g}}, \mathbb{C})$ is at most N-dimensional. As by Proposition 6.3 no non-trivial linear combination of the cocycles γ_{1,ω,C_i} is a coboundary, this space is exactly N-dimensional and $[\gamma_{1,\omega,C_i}]$ for $i = 1, \dots, N$ constitute a basis. □

We conclude the following.

Corollary 6.5. *Let $\mathfrak{g}$ be a simple classical Lie algebra and $\overline{\mathfrak{g}}$ the associated Lax operator algebra. Let ω be a fixed connection form. Then in each $[\gamma] \in \mathrm{H}_b(\overline{\mathfrak{g}}, \mathbb{C})$ there exists a unique representative γ' which is bounded and $\mathcal{L}$-invariant (with respect to ω). Moreover, $\gamma' = \sum_{i=1}^{N} a_i \gamma_{1,\omega,C_i}$, with $a_i \in \mathbb{C}$.*

Proposition 6.6.

(a) Let γ be a bounded and $\mathcal{L}$-invariant cocycle which is a coboundary, then $\gamma = 0$.

(b) Let $\mathfrak{g}$ be simple, then the cocycle γ_{1,ω',C_i} is $\mathcal{L}$-invariant with respect to ω, if and only if $\omega = \omega'$.

Proof. (a) By Theorem 6.4 we get $\gamma = \sum_{i=1}^{N}(\alpha_i \gamma_{1,\omega,C_i} + \beta_i \gamma_{2,C_i})$, with all $\beta_i = 0$ for the case $\mathfrak{g}$ is simple. The summands constitute a basis of the cohomology. Hence, γ can only be a coboundary if all coefficients vanish.

(b) As γ_{1,ω,C_i} and γ_{1,ω',C_i} are local and $\mathcal{L}$-invariant with respect to ω their difference $\gamma_{1,\omega,C_i} - \gamma_{1,\omega',C_i}$ is also local and $\mathcal{L}$-invariant. By Proposition 5.2 it is a coboundary. Hence by part (a) $\gamma_{1,\omega,C_i} - \gamma_{1,\omega',C_i} = 0$. The relation (5.7) gives the explicit expression for the left hand side. Assume $\omega \neq \omega'$. Let m be the order of the element

$$\theta = \omega - \omega' = (\theta_m z_i^m + O(z_i^m))dz_i \tag{6.13}$$

at the point P_i. As $\mathfrak{g}$ is simple the trace form $\mathrm{tr}(A \cdot B)$ is nondegenerate and we find

$$\hat{\theta} = \hat{\theta}_{-m-1} z_i^{-m-1} + O(z_i^{-m}), \tag{6.14}$$

such that $\beta = \mathrm{tr}(\theta_m \cdot \hat\theta_{-m-1}) \neq 0$. By Lemma 3.15 we get $\hat\theta = [L, L'] + L''$ with $\mathrm{ord}_{P_i}(L'') \geq -m$. Hence,

$$0 \neq \beta = \mathrm{tr}(\theta_m \cdot \hat\theta_{-m-1}) = \frac{1}{2\pi\,\mathrm{i}} \int_{C_i} \mathrm{tr}\left((\omega - \omega') \cdot ([L, L'] + L'')\right) \tag{6.15}$$
$$= \frac{1}{2\pi\,\mathrm{i}} \int_{C_i} \mathrm{tr}\left((\omega - \omega') \cdot [L, L']\right) = \gamma_{1,\omega,C_i}(L, L') - \gamma_{1,\omega',C_i}(L, L') = 0$$

which is a contradiction. □

After these results which are valid for bounded cocycles we will deduce the corresponding classification theorem for local cocycles. In some sense this is the main theorem of this article. It will show for example that for Lax operator algebras associated to simple Lie algebras there is up to rescaling and equivalence only one non-trivial almost-graded central extension.

Recall the relation for the separating cycle

$$[C_S] = \sum_{i=1}^{N} [C_i] = -\sum_{j=1}^{M} [C_j^*], \tag{6.16}$$

and the corresponding relation for the cocycle obtained by integration.

Theorem 6.7.

(a) If $\mathfrak{g}$ is simple (i.e. $\overline{\mathfrak{g}} = \mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(2n)$) then the space of local cohomology classes is one-dimensional. If we fix any connection form ω then this space will be generated by the class of γ_{1,ω,C_S}. Every $\mathcal{L}$-invariant (with respect to the connection ω) local cocycle is a scalar multiple of γ_{1,ω,C_S}.

(b) For $\overline{\mathfrak{g}} = \overline{\mathfrak{gl}}(n)$ the space of local cohomology classes which are $\mathcal{L}$-invariant having been restricted to the scalar subalgebra is two-dimensional. If we fix any connection form ω then the space will be generated by the classes of the cocycles γ_{1,ω,C_S} and γ_{2,C_S}. Every $\mathcal{L}$-invariant local cocycle is a linear combination of γ_{1,ω,C_S} and γ_{2,C_S}.

Proof. Let γ be a local cocycle. This says it is bounded from above and from below. For simplicity we abbreviate in this proof

$$\gamma_{1,i} := \gamma_{1,\omega,C_i} \quad \gamma_{2,i} := \gamma_{2,C_i}, \quad \gamma_{1,j}^* := \gamma_{1,\omega,C_j^*}, \quad \gamma_{2,j}^* := \gamma_{2,C_j^*}. \tag{6.17}$$

If we switch the role of I and O we get an inverted almost-grading. Every bounded from below cocycle of the original grading, will get bounded from above with respect to the inverted grading. Hence we can employ Theorem 6.4 in both directions and obtain for the same cocycle two representations

$$\gamma = \sum_{i=1}^{N} a_i \gamma_{1,i} + \sum_{i=1}^{N} b_i \gamma_{2,i} = -\sum_{j=1}^{M} a_j^* \gamma_{1,j}^* - \sum_{j=1}^{M} b_j^* \gamma_{2,j}^*, \quad \text{with} \quad a_i, a_j^*, b_i, b_j^* \in \mathbb{C}. \tag{6.18}$$

If either $N = 1$ or $M = 1$ then via (6.16) the cocycle is obtained via integration over a separating cycle. Hence the statement.

Otherwise both $N, M > 1$. By Proposition 6.1 the type (1) and type (2) cocycles are linearly independent, hence can be treated independently also in this context. First consider type (1). Note that from (6.16) we get the relation that

$$\sum_{i=1}^{N} \gamma_{1,i} = -\sum_{j=1}^{M} \gamma^*_{1,j}. \tag{6.19}$$

Hence,

$$\gamma^*_{1,1} = -\sum_{i=1}^{N} \gamma_{1,i} - \sum_{j=2}^{M} \gamma^*_{1,j}. \tag{6.20}$$

From (6.18) we get

$$0 = \sum_{i=1}^{N} a_i \gamma_{1,i} + \sum_{j=1}^{M} a^*_j \gamma^*_{1,j}. \tag{6.21}$$

If we plug (6.20) into this relation we obtain

$$0 = (a_1 - a^*_1) \sum_{i=1}^{N} \gamma_{1,i} + \sum_{i=2}^{N} (a_i - a_1) \gamma_{1,i} + \sum_{j=2}^{M} (a^*_k - a^*_1) \gamma^*_{1,j}. \tag{6.22}$$

Fix a k. We take $X, Y \in \overline{\mathfrak{g}}$ such that $\operatorname{tr}(XY) \neq 0$. By Riemann-Roch there exists $L', L \in \overline{\mathfrak{g}}$ such that around the point P_k we have

$$L(z_k) = X z_k + O(z_k^2), \qquad L'(z_k) = Y z_k^{-1} + O(z_k^0), \tag{6.23}$$

both holomorphic at the points in O and at P_l, $l \neq 1, k$. The elements might have pole orders of sufficiently high degree at P_1 to guarantee existence. The weak singularities will not disturb. By construction

$$\begin{gathered} \gamma_{1,k}(L, L') \neq 0, \qquad \gamma_{1,l}(L, L') = 0, \quad l = 2, \dots, N, l \neq k, \\ \gamma^*_{1,j}(L, L') = 0, \quad j = 1, \dots, M. \end{gathered} \tag{6.24}$$

Hence

$$\sum_{i=1}^{N} \gamma_{1,i}(L, L') = -\sum_{j=1}^{M} \gamma^*_{1,j}(L, L') = 0. \tag{6.25}$$

If we plug (L, L') into (6.22), all terms in (6.22) will vanish, with the only exception

$$0 = (a_k - a_1) \gamma_{1,k}(L, L'). \tag{6.26}$$

This shows $a_k - a_1$ for all k. (In a similar way we get $a^*_j - a^*_1$ for all j.) In particular, our cocycle we started with (resp. the γ_1 part of it) is a multiple of the cocycle obtained by integration over the separating cycle. This was the claim. The proof for the γ_2 part works completely the same if we take $X = Y$ a nonzero scalar matrix. □

As in the bounded case we obtain also for the local case the following corollary.

Corollary 6.8. *Let $\mathfrak{g}$ be a simple classical Lie algebra and $\overline{\mathfrak{g}}$ the associated Lax operator algebra. Let ω be a fixed connection form. Then in each $[\gamma] \in \mathrm{H}_{loc}(\overline{\mathfrak{g}}, \mathbb{C})$ there exists a unique representative γ' which is local and $\mathcal{L}$-invariant (with respect to ω). Moreover, $\gamma' = a\gamma_{1,\omega}$, with $a \in \mathbb{C}$.*

7. Uniqueness of $\mathcal{L}$-invariant cocycles

7.1. **The induction step.**

Recall from Section 3 the almost-graded structure of the Lax operator algebra $\overline{\mathfrak{g}}$ and in particular the decomposition $\overline{\mathfrak{g}} = \oplus_{n\in\mathbb{Z}}\overline{\mathfrak{g}}_n$ into subspaces of homogeneous elements of degree n. Also there the basis $\{L^u_{n,p} \mid u = 1, \ldots, \dim \mathfrak{g}, p = 1, \ldots, N\}$ of the subspace $\overline{\mathfrak{g}}_n$ was introduced (see (3.3)).

Let γ be an $\mathcal{L}$-invariant cocycle for the algebra $\overline{\mathfrak{g}}$ which is bounded from above, i.e. there exists an R (independent of n and m) such that $\gamma(\overline{\mathfrak{g}}_n, \overline{\mathfrak{g}}_m) \neq 0$ implies $n + m \leq R$. Furthermore, we recall that our connection ω needed to define the action of $\mathcal{L}$ on $\overline{\mathfrak{g}}$ is chosen to be holomorphic at the points in I.

For a pair $(L^u_{n,k}, L^v_{m,t})$ of homogeneous elements we call $n + m$ the *level* of the pair. We apply the technique developed in [18]. We will consider cocycle values $\gamma(L^u_{n,k}, L^v_{m,t})$ on pairs of level $l = n + m$ and will make induction over the level. By the boundedness from above, the cocycle values will vanish at all pairs of sufficiently high level. It will turn out that everything will be fixed by the values of the cocycle at level zero. Finally, we will show that the cocycle is a linear combination of the N (resp. $2N$) basic cocycles as claimed in Theorem 6.4.

For a cocycle γ evaluated for pairs of elements of level l we will use the symbol $\equiv$ to denote that the expressions are the same on both sides of an equation involving cocycle values up to values of γ at higher level. This has to be understood in the following strong sense:

$$\sum \alpha^{(n,p,t)}_{(u,v)} \gamma(L^u_{n,p}, L^v_{l-n,t}) \equiv 0, \qquad \alpha^{(n,p,t)}_{(u,v)} \in \mathbb{C} \tag{7.1}$$

means a congruence modulo a linear combination of values of γ at pairs of basis elements of level $l' > l$. The coefficients of that linear combination, as well as the $\alpha^{(n,p,t)}_{(u,v)}$, depend only on the structure of the Lie algebra $\overline{\mathfrak{g}}$ and do not depend on γ.

We will also use the same symbol $\equiv$ for equalities in $\overline{\mathfrak{g}}$ which are true modulo terms of higher degree compared to the terms under consideration.

By the $\mathcal{L}$-invariance we have

$$\gamma(\nabla^{(\omega)}_{e_{k,r}} L^u_{m,p}, L^v_{n,s}) + \gamma(L^u_{m,p}, \nabla^{(\omega)}_{e_{k,r}} L^v_{n,s}) = 0. \tag{7.2}$$

Using the almost-graded structure (4.13) we obtain (up to order $> (k + m + n)$)

$$m\gamma(L^u_{k+m,p}, L^v_{n,s})\, \delta^p_r + n\gamma(L^v_{m,p}, L^v_{n+k,s})\, \delta^s_r \equiv 0, \tag{7.3}$$

valid for all $n, m, k \in \mathbb{Z}$.

If in (7.3) all three indices r, p and s are different then the term on the left hand side vanishes. If $r = p \neq s$ then we obtain

$$m\gamma(L^u_{k+m,p}, L^v_{n,s}) \equiv 0. \tag{7.4}$$

which is true for every m. Hence

$$\gamma(L^u_{l,p}, L^v_{n,s}) \equiv 0, \quad \text{for} \quad p \neq s \ . \tag{7.5}$$

It remains $r = p = s$ and this yields

$$m\gamma(L^u_{k+m,s}, L^v_{n,s}) + n\gamma(L^v_{m,s}, L^v_{n+k,s}) \equiv 0. \tag{7.6}$$

Proposition 7.1. *Let $m + n \neq 0$ then at level $m + n$ we have*

$$\gamma(\overline{\mathfrak{g}}_m, \overline{\mathfrak{g}}_n) \equiv 0. \tag{7.7}$$

Proof. From (7.5) we conclude that only elements with the same second index could contribute in level $m + n$. We put $k = 0$ in (7.6) and obtain

$$(m + n)\gamma(L^u_{m,s}, L^v_{n,s}) \equiv 0, \quad \forall u, v. \tag{7.8}$$

Hence if $(m + n) \neq 0$ the claim follows. □

Proposition 7.2.

$$\gamma(\overline{\mathfrak{g}}_m, \overline{\mathfrak{g}}_0) \equiv 0, \qquad \forall m \in \mathbb{Z}. \tag{7.9}$$

Proof. We evaluate (7.6) for the values $m = 1$ and $n = 0$ and obtain the result. □

Proposition 7.3. *(a) We have $\gamma(\overline{\mathfrak{g}}_n, \overline{\mathfrak{g}}_m) = 0$ if $n + m > 0$, i.e. the cocycle is bounded from above by zero.*
(b) If $\gamma(\overline{\mathfrak{g}}_n, \overline{\mathfrak{g}}_{-n}) = 0$ then the cocycle γ vanishes identically.

Proof. The proof stays word by word the same as in [23]. But as it is one of the central arguments and for the convenience of the reader we repeat the arguments. If $\gamma = 0$ there is nothing to prove. Assume $\gamma \neq 0$. As γ is bounded from above, there will be a minimal upper bound l, such that above l all cocycle values will vanish. Assume that $l > 0$, then by Proposition 7.1 the values at level l are expressions of levels bigger than l. But the cocycle vanishes there. Hence it vanishes at level l too. This is a contradiction which proves (a).

By induction, using again Proposition 7.1 we obtain that if the cocycle vanishes at level 0, it vanishes everywhere. This proves (b). □

Combining Propositions 7.2 and 7.3 we obtain

Corollary 7.4.

$$\gamma(\overline{\mathfrak{g}}_m, \overline{\mathfrak{g}}_0) = 0, \qquad \forall m \geq 0. \tag{7.10}$$

Proposition 7.5.

$$\gamma(L^u_{n,r}, L^v_{-n,s}) = n \cdot \gamma(L^u_{1,r}, L^v_{-1,r})\delta^s_r, \tag{7.11}$$

$$\gamma(L^u_{1,r}, L^v_{-1,s}) = \gamma(L^v_{1,s}, L^u_{-1,r}) \tag{7.12}$$

Proof. Assume $s \neq r$ then all expressions are of positive level and vanishes by Proposition 7.3, hence the statement is true. For $r = s$ we take in (7.6) the values $n = -p$, $m = 1$ and $k = p - 1$. This yields the expression (7.11) up to higher level terms. But as the level is zero, the higher level terms vanish. Setting $n = -1$ in (7.11) we obtain (7.12). □

Independently of the structure of the Lie algebra $\mathfrak{g}$, we obtained the following results for every $\mathcal{L}$-invariant and bounded cocycle γ:

(1) The cocycle is bounded from above by zero.
(2) The cocycle is uniquely given by its values at level zero.
(3) At level zero the cocycle is uniquely fixed by its values $\gamma(L^u_{1,s}, L^v_{-1,s})$, for $u, v = 1, \dots, \dim \mathfrak{g}$ and $s = 1, \dots, N$.
(4) The other cocycle values at level zero are given by $\gamma(L^u_{n,s}, L^v_{-n,r}) = 0$ if $s \neq r$, $\gamma(L^u_{0,s}, L^v_{0,s}) = 0$ and $\gamma(L^u_{n,s}, L^v_{-n,s})$ given by (7.11).

Let $X \in \mathfrak{g}$ then we denote as always by $X_{n,s}$, $s = 1, \dots, N$ the element in $\overline{\mathfrak{g}}$ with leading term $X z_s^n$ at P_s and higher orders at the other points in I. We define for $s = 1, \dots, N$ the maps

$$\psi_{\gamma,s} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \qquad \psi_{\gamma,s}(X, Y) := \gamma(X_{1,s}, Y_{-1,s}). \tag{7.13}$$

Obviously, $\psi_{\gamma,s}$ is a bilinear form on $\mathfrak{g}$.

Proposition 7.6. *(a) $\psi_{\gamma,s}$ is symmetric, i.e. $\psi_{\gamma,s}(X, Y) = \psi_{\gamma,s}(Y, X)$.*
(b) $\psi_{\gamma,s}$ is invariant, i.e.

$$\psi_{\gamma,s}([X, Y], Z) = \psi_{\gamma,s}(X, [Y, Z]). \tag{7.14}$$

Proof. First we have by (7.12)

$$\psi_{\gamma,s}(X, Y) = \gamma(X_{1,s}, Y_{-1,s}) = \gamma(Y_{1,s}, X_{-1,s}) = \psi_\gamma(Y, X).$$

This is the symmetry. Furthermore, using $[X_{1,s}, Y_{0,s}] \equiv [X, Y]_{1,s}$, the fact that the cocycle vanishes for positive level, and by the cocycle condition we obtain

$$\psi_{\gamma,s}([X, Y], Z) = \gamma([X, Y]_{1,s}, Z_{-1,s}) = \gamma([X_{1,s}, Y_{0,s}], Z_{-1,s}) = \\ - \gamma([Y_{0,s}, Z_{-1,s}], X_{1,s}) - \gamma([Z_{-1,s}, X_{1,s}], Y_{0,s}).$$

The last term vanishes due to Corollary 7.4. Hence

$$\psi_{\gamma,s}([X, Y], Z) = \gamma(X_{1,s}, [Y_{0,s}, Z_{-1,s}]) = \gamma(X_{1,s}, [Y, Z]_{-1,s}) = \psi_{\gamma,s}(X, [Y, Z]).$$

□

As the cocycle γ is fixed by the values $\gamma(L^u_{1,s}, L^v_{-1,s})$, $s = 1, \dots, N$ and they are fixed by the bilinear maps $\psi_{\gamma,s}$ we proved:

Theorem 7.7. *Let γ be an $\mathcal{L}$-invariant cocycle for $\overline{\mathfrak{g}}$ which is bounded from above. Then γ it is bounded from above by zero and is completely fixed by the associated symmetric and invariant bilinear forms $\psi_{\gamma,s}$, $s = 1, \dots, N$ on $\mathfrak{g}$ defined via (7.13).*

7.2. **Simple Lie algebras $\mathfrak{g}$.**

By Theorem 7.7 the $\mathcal{L}$-invariant cocycle γ is completely given by fixing the N-tuple $(\psi_{\gamma,1}, \psi_{\gamma,2}, \dots, \psi_{\gamma,N})$ of symmetric invariant bilinear forms $\psi_{\gamma,s}$. For a finite-dimensional simple Lie algebra every such form is a multiple of the Cartan-Killing form κ. Hence the space of bounded cocycles is at most N-dimensional. Our geometric cocycles γ_{1,ω,C_i}, see (5.14), for $i = 1, \dots, N$ are $\mathcal{L}$-invariant and bounded cocycles. They are linearly independent, see Proposition 6.1. Hence, we obtain that every bounded and $\mathcal{L}$-invariant cocycle is a linear combination of the γ_{1,ω,C_i}. Moreover, they are a basis of the space of $\mathcal{L}$-invariant and bounded cocycles. By Proposition 6.3 they stay linearly independent after passing to cohomology and we obtain

Proposition 7.8. *Let* $\mathfrak{g}$ *be simple, then*

$$\dim \mathrm{H}_{b,\mathcal{L}}(\overline{\mathfrak{g}},\mathbb{C}) = N, \tag{7.15}$$

and this cohomology space is generated by the classes of γ_{1,ω,C_i}, $i=1,\ldots,N$.

7.3. The case of $\overline{\mathfrak{gl}}(n)$.

We have the direct decomposition, as Lie algebras, $\overline{\mathfrak{gl}}(n) = \overline{\mathfrak{s}}(n) \oplus \overline{\mathfrak{sl}}(n)$. Let γ be a cocycle of $\overline{\mathfrak{gl}}(n)$ and denote by γ' and γ'' its restriction to $\overline{\mathfrak{s}}(n)$ and $\overline{\mathfrak{sl}}(n)$ respectively. As in [23] we obtain using Lemma 3.15

Proposition 7.9.

$$\gamma(x,y) = 0, \quad \forall x \in \overline{\mathfrak{s}}(n),\ y \in \overline{\mathfrak{sl}}(n). \tag{7.16}$$

Hence we can decompose the cocycle as $\gamma = \gamma' \oplus \gamma''$. If γ is bounded/local and/or $\mathcal{L}$-invariant the same is true for γ' and γ''.

First we consider the algebra $\overline{\mathfrak{s}}(n)$. It is isomorphic to $\mathcal{A}$, the isomorphism is given by

$$L \mapsto \frac{1}{n}\,\mathrm{tr}(L). \tag{7.17}$$

In [18, Thm. 5.7] it was shown that the space of $\mathcal{L}$-invariant cocycles for $\mathcal{A}$ bounded from above is N-dimensional and a basis is given by

$$\gamma_i(f,g) = \frac{1}{2\pi\mathrm{i}}\int_{C_i} f dg = \mathrm{res}_{P_i}(f dg), \quad \mathrm{i} = 1,\ldots,N. \tag{7.18}$$

Note that as $\mathcal{A}$ is abelian there do not exist non-trivial coboundaries. We obtain

$$\gamma'(L,M) = \sum_{i=1}^{N} \alpha_i \,\mathrm{res}_{P_i}(\mathrm{tr}(L)\cdot \mathrm{tr}(dM)) = \sum_{i=1}^{N} \alpha_i \gamma_{2,C_i}(L,M), \tag{7.19}$$

by Definition (5.6).

For the cocycle γ'' of $\overline{\mathfrak{sl}}(n)$ we use Proposition 7.8 and obtain $\gamma'' = \sum_{i=1}^N \beta_i\gamma_{1,\omega,C_i}$. Altogether we showed

Proposition 7.10.

$$\dim \mathrm{H}_{b,\mathcal{L}}(\overline{\mathfrak{gl}}(n),\mathbb{C}) = 2N. \tag{7.20}$$

A basis is given by the classes of γ_{1,ω,C_i} *and* γ_{2,C_i}, $i=1,\ldots,N$.

In this section we showed those parts of Theorem 6.4 which deal with $\mathcal{L}$-invariant cocycles. In fact we showed the complete theorem under the additional assumption that our cohomology classes are $\mathcal{L}$-invariant. For the scalar part this is the best what could be expected. Without $\mathcal{L}$-invariance there will be much more non-trivial cohomology classes for the scalar algebra, see [18] for more information. In the next section we will present a way how to get rid of this condition for the simple Lie algebras.

8. The simple case in general

In this section the Lax operator algebra $\overline{\mathfrak{g}}$ is always based on a finite simple classical Lie algebra. As explained in the previous section if we put $\mathcal{L}$-invariance in the assumption then Theorem 6.4 would have been proved. One way to complete the general proof is to to show that after cohomological changes every bounded cocycle has also a bounded $\mathcal{L}$-invariant representing it. In fact, we will do this. But unfortunately, we do not have a direct proof. Instead, by a quite different approach we will show that for the simple Lie algebra case the space of bounded cohomology classes (of the Lax operator algebras) is at most N-dimensional without assuming $\mathcal{L}$-invariance a priori. Combining this result with the result of the last section that the space of $\mathcal{L}$-invariant bounded cohomology classes is N-dimensional we see that in the simple case each bounded cohomology class is automatically $\mathcal{L}$-invariant. Moreover, we showed there that it has a unique $\mathcal{L}$-invariant representing cocycle which is given as linear combination of γ_{1,ω,C_i}, $i=1,\ldots,N$.

The theorem we are heading for is

Theorem 8.1. *Let $\mathfrak{g}$ be a simple classical Lie algebra over $\mathbb{C}$ and $\overline{\mathfrak{g}}$ the associated Lax operator algebra with its almost-grading. Every bounded cocycle on $\overline{\mathfrak{g}}$ is cohomologous to a distinguished cocycle which is bounded from above by zero. The space of distinguished cocycles is at most N-dimensional.*

Remark. What we will show is the following. Every cocycle bounded from above is cohomologous to a cocycle which is fixed by its value at N special pair of elements in $\overline{\mathfrak{g}}$ (namely by $\gamma(H^\alpha_{1,s}, H^\alpha_{-1,s})$ for one fixed simple root α, see below for the notation). Besides the structure of $\mathfrak{g}$ we only use the almost-gradedness of $\overline{\mathfrak{g}}$ with leading terms given in (8.4).

The presentation is quite similar to [23]. Those proofs which are completely of the same structure will not be repeated here.

First we need to recall some facts about the Chevalley generators of $\mathfrak{g}$. Choose a root space decomposition $\mathfrak{g}=\mathfrak{h}\oplus_{\alpha\in\Delta}\mathfrak{g}^\alpha$. As usual Δ denotes the set of all roots $\alpha\in\mathfrak{h}^*$. Furthermore, let $\{\alpha_1,\alpha_2,\ldots,\alpha_p\}$ be a set of simple roots ($p=\dim\mathfrak{h}$). With respect to this basis, the root system splits into positive and negative roots, Δ_+ and Δ_- respectively. With α a positive root, $-\alpha$ is a negative root and vice versa. For $\alpha\in\Delta$ we have $\dim\mathfrak{g}^\alpha=1$. Certain elements $E^\alpha\in\mathfrak{g}^\alpha$ and $H^\alpha\in\mathfrak{h}$, $\alpha\in\Delta$ can be fixed so that for every positive root α

$$[E^\alpha,E^{-\alpha}]=H^\alpha,\qquad [H^\alpha,E^\alpha]=2E^\alpha,\qquad [H^\alpha,E^{-\alpha}]=-2E^{-\alpha}. \tag{8.1}$$

We use also $H^i:=H^{\alpha_i}$, $i=1,\ldots,p$ for the elements assigned to the simple roots. A vector space basis, the Chevalley basis, of $\mathfrak{g}$ is given by $\{E^\alpha,\alpha\in\Delta;\ H^i, 1\le i\le p\}$.

We denote by $(\ ,\ )$ the inner product on $\mathfrak{h}^*$ induced by the Cartan-Killing form of $\mathfrak{g}$. The following relations hold

$$
\begin{aligned}
[H^\alpha, H^\beta] &= 0, \\
[H^\alpha, E^{\pm\beta}] &= \pm 2\frac{(\beta,\alpha)}{(\beta,\beta)} E^{\pm\alpha}, \\
[H, E^\alpha] &= \alpha(H) E^\alpha, \quad H \in \mathfrak{h}, \\
[E^\alpha, E^\beta] &= \begin{cases} H^\alpha, & \alpha \in \Delta_+,\ \beta = -\alpha, \\ -H^\alpha, & \alpha \in \Delta_-,\ \beta = -\alpha, \\ \pm(r+1)E^{\alpha+\beta}, & \alpha,\ \beta,\ \alpha+\beta \in \Delta, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
\tag{8.2}
$$

Here r is the largest nonnegative integer such that $\alpha - r\beta$ still is a root.

As in the other parts of this article, we denote by $E^\alpha_{n,s}$, $H^\alpha_{n,s}$ the unique elements in $\overline{\mathfrak{g}}_n$ (i.e. of degree n) for which the expansions at P_s start with $E^\alpha z_s^n$ and $H^\alpha z_s^n$ respectively, and at the Points $P_i \in I$, $i \neq s$ it is of higher order.

The following elements form a basis of $\overline{\mathfrak{g}}$:

$$
\{\ E^\alpha_{n,s}, \alpha \in \Delta;\ H^i_{n,s}, 1 \le i \le p \mid\ n \in \mathbb{Z}, s = 1, \dots, N\ \}. \tag{8.3}
$$

The structure equations, up to higher degree terms, are

$$
\begin{aligned}
[H^\alpha_{n,s}, H^\beta_{m,r}] &\equiv 0, \\
[H^\alpha_{n,s}, E^{\pm\beta}_{m,r}] &\equiv \pm 2\frac{(\beta,\alpha)}{(\beta,\beta)} E^{\pm\beta}_{n+m,r}\,\delta^s_r, \\
[H_{n,s}, E^\alpha_{m,r}] &\equiv \alpha(H) E^\alpha_{n+m,r}\,\delta^s_r, \quad H \in \mathfrak{h}, \\
[E^\alpha_{n,s}, E^\beta_{m,r}] &\equiv \begin{cases} H^\alpha_{n+m,s}\,\delta^s_r, & \alpha \in \Delta_+,\ \beta = -\alpha, \\ -H^\alpha_{n+m,s}\,\delta^s_r, & \alpha \in \Delta_-,\ \beta = -\alpha, \\ \pm(r+1)E^{\alpha+\beta}_{n+m,s}\,\delta^s_r,, & \alpha,\ \beta,\ \alpha+\beta \in \Delta, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
\tag{8.4}
$$

Recall that the symbol $\equiv$ denotes equality up to elements of degree higher than the sum of the degrees of the elements under consideration. Here, the elements not written down are elements of degree $> n+m$. Also recall that by the almost-gradedness there exists a S, independent of n and m, such that only elements of degree $\le n+m+S$ appear.

Let γ' be a cocycle for $\overline{\mathfrak{g}}$ which is bounded from above. For the elements in $\mathfrak{g}$ we get

$$
E^{\pm\alpha} = \pm 1/2[H^\alpha, E^{\pm\alpha}], \quad H^i = [E^{\alpha_i}, E^{-\alpha_i}],\ i = 1, \dots, p. \tag{8.5}
$$

Consequently, for $\overline{\mathfrak{g}}$ we obtain

$$
\begin{aligned}
E^{\pm\alpha}_{n,s} &= \pm 1/2[H^\alpha_{0,s}, E^{\pm\alpha}_{n,s}] + Y(n,s,\alpha), \\
H^i_{n,s} &= [E^{\alpha_i}_{0,s}, E^{-\alpha_i}_{n,s}] + Z(n,s,i),\ i = 1, \dots, p.
\end{aligned}
\tag{8.6}
$$

where $Y(n,s,\alpha)$ and $Z(n,s,i)$ are sums of elements of degree between $n+1$ and $n+S$. Fix a number $M \in Z$ such that the cocycle γ' vanishes for all levels $\geq M$. We define a linear map $\Phi : \overline{\mathfrak{g}} \to \mathbb{C}$ by (descending) induction on the degree of the basis elements (8.3). First

$$\Phi(E^{\alpha}_{n,s}) := \Phi(H^{i}_{n,s}) := 0, \qquad \alpha \in \Delta,\ i = 1,\ldots,p,\ s = 1,\ldots,N \quad n \geq M. \tag{8.7}$$

Next we define inductively ($\alpha \in \Delta_+$, $s = 1,\ldots,N$)

$$\begin{aligned} \Phi(E^{\pm\alpha}_{n,s}) &:= \pm 1/2\gamma'(H^{\alpha}_{0,s}, E^{\pm a}_{n,s}) + \Phi(Y(n,s,\pm\alpha)), \\ \Phi(H^{i}_{n,s}) &:= \gamma'(E^{\alpha_i}_{0,s}, E^{-\alpha_i}_{n,s}) + \Phi(Z(n,s,i)). \end{aligned} \tag{8.8}$$

The cocycle $\gamma = \gamma' - \delta\Phi$ is cohomologous to the original cocycle γ'. As γ' is bounded from above, and, by definition, Φ is also bounded from above, the cocycle γ is bounded from above too.

By the construction of Φ we have $\Phi([H^{\alpha}_{0,s}, E^{\pm\alpha}_{n,s}] = \gamma'(H^{\alpha}_{0,s}, E^{\pm\alpha}_{n,s})$ and $\Phi([E^{\alpha_i}_{0,s}, E^{-\alpha_i}_{n,s}]) = \gamma'(E^{\alpha_i}_{0,s}, E^{-\alpha_i}_{n,s})$. Hence

Proposition 8.2.

$$\begin{gathered} \gamma(H^{\alpha}_{0,s}, E^{\pm\alpha}_{n,s}) = 0, \quad \gamma(E^{\alpha_i}_{0,s}, E^{-\alpha_i}_{n,s}) = 0, \\ \alpha \in \Delta_+,\ i = 1,\ldots,p,\ s = 1,\ldots,N, \quad n \in \mathbb{Z}. \end{gathered} \tag{8.9}$$

Definition 8.3. A cocycle γ is called *normalized* if it fulfills (8.9).

By the above construction we showed that every cocycle bounded from above is cohomologous to a normalized one, which is also bounded from above. In the following we assume that our cocycle is already normalized.

Proposition 8.4. *Let α_1 be a fixed simple root, α and β arbitrary roots and γ a normalized cocycle, then for all $s,r = 1,\ldots,N$, $n,m \in \mathbb{Z}$ we have*

$$\begin{aligned} &\gamma(E^{\alpha}_{m,s}, H_{n,r}) \equiv 0, && H \in \mathfrak{h},\ \alpha \in \Delta \\ &\gamma(E^{\alpha}_{m,s}, E^{\beta}_{n,r}) \equiv 0, && \alpha,\beta \in \Delta,\ \beta \neq -\alpha, \\ &\gamma(E^{\alpha}_{m,r}, E^{-\alpha}_{n,s}) \equiv u\gamma(H^{\alpha_1}_{m,r}, H^{\alpha_1}_{n,r})\,\delta^{s}_{r}, && \alpha \in \Delta, \\ &\gamma(H^{\alpha}_{m,r}, H^{\beta}_{n,s}) \equiv t\gamma(H^{\alpha_1}_{m,r}, H^{\alpha_1}_{n,r})\delta^{s}_{r},, && \alpha,\beta \in \Delta_+, \end{aligned} \tag{8.10}$$

with $u,t \in \mathbb{C}$.

$$\gamma(H^{\alpha_1}_{n,r}, H^{\alpha_1}_{0,s}) \equiv 0, \tag{8.11}$$

$$\gamma(H^{\alpha_1}_{n+1,s}, H^{\alpha_1}_{l-(n+1),r}) \equiv \Big(\gamma(H^{\alpha_1}_{n-1,s}, H^{\alpha_1}_{l-(n-1),s}) + 2\gamma(H^{\alpha_1}_{1,s}, H^{\alpha_1)}_{l-1,s}\Big)\,\delta^{s}_{r}. \tag{8.12}$$

For a simple root α_1 and for a level $l \neq 0$ we have

$$\gamma(H^{\alpha_1}_{n,r}, H^{\alpha_1}_{l-n,s}) \equiv 0. \tag{8.13}$$

Proof. In the two point case the statement of the proposition consists of a chain of individual statement which were proved in [23]. In fact, the proofs presented there remain valid if one just adds in all relations there for the Lie algebra elements Y_n the second index to obtain $Y_{n,s}$. By the almost-graded structure, resp. its fine structure (3.6) for the expressions $[Y_{n,s}, Z_{m,r}]$ in the relations only terms involving $s = r$ will contribute on the level under considerations. If $s \neq r$ they will contribute only to higher level. Hence, all relations there can be read with respect to all the second indices the same up to higher level. Hence, the proof is completely analogous. □

Proposition 8.5. *Let γ be a normalized cocycle. Then*
(a) it vanishes for levels greater than zero, i.e.

$$\gamma(\overline{\mathfrak{g}}_n, \overline{\mathfrak{g}}_n) = 0, \quad \text{for } n + m > 0. \tag{8.14}$$

(b) All levels $l < 0$ are fixed by the level zero.

Proof. By the propositions above we showed that the expressions at level l of the cocycle can be reduced to expressions of levels $> l$ and values $\gamma(H^\alpha_{n,r}, H^\alpha_{l-n,r})$. As long as the level is $\neq 0$, by (8.13) also these values can be expressed by higher level. Hence by induction, starting with the upper bound of the cocycle, we obtain that the upper bound for the level of the cocycle values is equal to zero. Also it follows that the values at levels $l < 0$ are fixed by induction by the values at level zero. □

Hence it remains to consider the level zero.

Proposition 8.6. *Let α be a simple root. At level $l = 0$ the cocycle values for $s = 1, \dots, N$ are given by the relations*

$$\gamma(H^\alpha_{n,s}, H^\alpha_{-n,r}) = n \cdot \gamma(H^\alpha_{1,s}, H^\alpha_{-1,s})\, \delta^r_s, \qquad \gamma(H^\alpha_{0,r}, H^\alpha_{0,s}) = 0. \tag{8.15}$$

Proof. If we set the value $l = 0$ in (8.12) we obtain the relation

$$\gamma(H^\alpha_{n+1,s}, H^\alpha_{-(n+1),r}) \equiv \Big(\gamma(H^\alpha_{n-1,s}, H^\alpha_{-(n-1),s}) + 2\gamma(H^\alpha_{1,s}, H^\alpha_{-1,s})\Big) \cdot \delta^s_r. \tag{8.16}$$

As all cocycle values of level $l > 0$ are vanishing we can replace $\equiv$ by $=$. Now the claimed expression follows. □

Proof of Theorem 8.1. After adding a suitable coboundary we might replace the given γ by a normalized one. Using Proposition 8.2, 8.4, and 8.6 everything depends only on the values $\gamma(H^\alpha_{1,s}, H^\alpha_{-1,s})$, $s = 1, \dots, N$ for one (fixed) simple root. This proves that there are at most N linearly independent normalized cocycles. □

Proposition 8.7. *If a normalized cocycle γ is a coboundary then it vanishes identically.*

Proof. As explained above, a normalized cocycle is fixed by the values $\gamma(H^\alpha_{1,s}, H^\alpha_{-1,s})$. We set

$$H^\alpha_{(1,s)} := H^\alpha_{0,s} A_{1,s} \equiv H^\alpha_{1,s}, \quad \text{and} \quad H^\alpha_{(-1,s)} := H^\alpha_{0,s} A_{-1,s} \equiv H^\alpha_{-1,s}. \tag{8.17}$$

Hence

$$[H^\alpha_{(1,s)}, H^\alpha_{(-1,s)}] = [H^\alpha_{0,s}, H^\alpha_{0,s}] A_{1,s} A_{-1,s} = 0. \tag{8.18}$$

As the cocycle vanishes for positive levels, and as $\gamma=\delta\phi$ is assumed to be a coboundary we get

$$\gamma(H^\alpha_{1,s},H^\alpha_{-1,s})=\gamma(H^\alpha_{(1,s)},H^\alpha_{(-1,s)})=\phi([H^\alpha_{(1,s)},H^\alpha_{(-1,s)}])=\phi(0)=0. \tag{8.19}$$

Hence, all cocycle values are zero, as claimed. □

Appendix A. Example $\mathfrak{gl}(n)$

In this appendix we will reproduce as an illustration for the reader the proof that the product of two Lax operators for the algebra $\mathfrak{gl}(n)$ is again a Lax operator. This means that the equations (2.6) are fulfilled for their product. Hence, $\overline{\mathfrak{gl}}(n)$ will be closed under commutator too. This result is due to Krichever and Sheinman [10]. In a similar manner the other cases are treated (but now only for the commutators). Furthermore, it is shown that the connection operators $\nabla_e^{(\omega)}$ act indeed on $\overline{\mathfrak{g}}$. The original proofs (involving partly tedious calculations) can be found in [10], [23], [28].

The singularities at the points in A are not bounded. Hence, they will not create problems and the proofs need only to consider the weak singular points. Consequently, the statements are also true in the multi-point case.

We start with two elements L' and L'' with corresponding expansions (2.5) and examine their product $L=L'L''$. For this we have to consider each point γ_s (with local coordinate w_s) of the weak singularities with $\alpha_s\neq 0$ separately. Taking into account only those parts which might contribute we obtain for L

$$L=\frac{L'_{s,-1}L''_{s,-1}}{w_s^2}+\frac{L'_{s,-1}L''_{s,0}+L'_{s,0}L''_{s,-1}}{w_s^1} + \left(L'_{s,-1}L''_{s,1}+L'_{s,0}L''_{s,0}+L'_{s,1}L''_{s,-1}\right)+O(w_s^1). \tag{A.1}$$

By expanding the first numerator we get

$$L'_{s,-1}L''_{s,-1}=\alpha_s{\beta'_s}^t\alpha_s{\beta''_s}^t=0 \tag{A.2}$$

as ${\beta'_s}^t\alpha_s=0$ by (2.6). Hence, there is no pole of order two appearing.

Next we consider the expression which comes with pole order one.

$$L_{s,-1}=L'_{s,-1}L''_{s,0}+L'_{s,0}L''_{s,-1}=\alpha_s{\beta'_s}^tL''_{s,0}+L'_{s,0}\alpha_s{\beta''_s}^t. \tag{A.3}$$

As by the conditions $L'_{s,0}\alpha_s=\kappa'_s\alpha_s$ we can write

$$L_{s,-1}=\alpha_s\beta_s^t,\qquad \text{with}\quad \beta_s^t={\beta'_s}^tL''_{s,0}+\kappa'_s{\beta''_s}^t. \tag{A.4}$$

For the trace condition we obtain

$$\operatorname{tr}(L_{s,-1})=({\beta'_s}^tL''_{s,0}+\kappa'_s{\beta''_s}^t)\alpha_s=\kappa''_s{\beta'_s}^t\alpha_s+\kappa'_s{\beta''_s}^t\alpha_s=0. \tag{A.5}$$

Hence, we have the required form.

Finally we have to verify that α_s is an eigenvector of $L_{s,0}$. First we note that $L''_{s,-1}\alpha_s=0$ and $L'_{s,0}L''_{s,0}\alpha_s=\kappa'_s\kappa''_s\alpha_s$. Also

$$L'_{s,-1}L''_{s,1}\alpha_s=\alpha_s({\beta'_s}^tL''_{s,1}\alpha_s)=({\beta'_s}^tL''_{s,1}\alpha_s)\alpha_s. \tag{A.6}$$

Hence, indeed α_s is an eigenvector with eigenvalue ${\beta'_s}^t L''_{s,1}\alpha_s + \kappa'_s\kappa''_s$. This shows the claim that $L \in \overline{\mathfrak{gl}}(n)$.

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