Geometry and Quantization

Lectures presented

at the School Geoquant

September 2009

University of Luxembourg

Grand-Duchy of Luxembourg

Edited by

Martin Schlichenmaier
Armen Sergeev
Oleg Sheinman

Third International School on Geometry and Quantization (Geoquant)
University of Luxembourg, August 31 – September 5, 2009,
Luxembourg City, Grand-Duchy of Luxembourg

Scientific and Local Organizing Committee

Martin Schlichenmaier (Luxembourg) (contact)
Joachim Hilgert (Paderborn, Germany)
Ryoichi Kobayashi (Nagoya, Japan)
Armen Sergeev (Moscow, Russia)
Oleg Sheinman (Moscow, Russia)
Tilmann Wurzbacher (Metz, France)
Yael Fregier (Luxembourg)
Laurent Hofer (Luxembourg)
Marie-Amelie Lawn (Luxembourg)
Martin Olbrich (Luxembourg)
Norbert Poncin (Luxembourg)
Boris Shoikhet (Luxembourg)
Anton Thalmaier (Luxembourg)

Sponsors

University of Luxembourg, Mathematics Research Unit

University Paul Verlaine, LMAM, Metz, France

DFG-IRTG *Geometry and Analysis of Symmetries*, Universities of Paderborn and Metz

ESF Research Networking Programme *Harmonic and Complex Analysis and its Application* (HCAA)

ESF Research Networking Programme *Interaction of lowdimensional Topology and Geometry with Physics* (ITGP)

Steklov Mathematical Institute, Moscow, Russia

Nagoya University, Nagoya, Japan

Fonds National de la Recherche Luxembourg (FNR) (support of the conference week)

Foreword

In September 2009 the *Third International School and Conference on Geometry and Quantization* took place at the University of Luxembourg.

The scientific topics discussed were

- algebraic-geometric and complex-analytic-geometric aspects of quantization,
- geometric quantization and moduli space problems,
- asymptotic geometric analysis,
- infinite-dimensional geometry,
- relations with modern theoretical physics.

The activity lasted for two weeks. The first week was a school with several lecture courses aiming at the newcomer to the field. The second week was a scientific conference.

This volume of the **Travaux Mathématiques** is an outcome of the school. Its aim is to give also to those who could not participate an introduction to some of the hot topics of ongoing research in the field. Furthermore, it was the desire of the participants to have some written material of the courses available. We asked the lecturers whether they would be able to produce such a write-up related to their lectures. We are happy that most of them could manage to contribute. Additionally, the contribution of Ma and Marinescu supplements in an ideal way the material presented at the school

The contribution of *Armen Sergeev* deals with the quantization of universal Teichmüller space. First the complex geometry of the universal Teichmüller space is described. It is realized as an open subset of the complex Banach space of holomorphic quadratic differentials on the unit disc. The quotient of the diffeomorphism group of the circle modulo Möbius transformations is treated as a regular part of it. Based on this its quantization is considered. For the regular part the quantization is obtained by embedding it into a suitable Hilbert–Schmidt-Siegel disc. This method, however, does not apply to the full universal Teichmüller space. For its quantization he uses an approach similar to the "quantized calculus" of Connes and Sullivan.

Johannes Huebschmann considers stratified Kähler spaces. Such a space is a stratified symplectic space together with a complex analytic structure which is compatible with the stratified symplectic structure. In particular, each stratum is a Kähler manifold in an obvious manner. Holomorphic quantization on a stratified Kähler space then yields a costratified Hilbert space, a quantum object having the classical singularities as its shadow. Given a Kähler manifold with an hamiltonian action of a compact Lie group that also preserves the complex structure, reduction after quantization coincides with quantization after reduction in the sense that not only the reduced and unreduced quantum phase spaces correspond but the invariant unreduced and reduced quantum observables as well. He illustrates his approach with a quantum (lattice) gauge theory which incorporates certain classical singularities. He explicitly calculates energy eigenvalues and tunneling probabilities between different strata.

Tatsuya Tate deals with asymptotic analysis over complex polytopes. More precisely, he presents results on its relation to representation theory of compact Lie groups and asymptotic formulas for sections of line bundles over toric Kähler manifolds. In the frame of his presentation an important tool is the lattice path counting functions which he introduce and for which he shows asymptotic formulas.

For Kähler manifolds the Berezin-Toeplitz operator and the Berezin-Toeplitz deformation quantization are naturally adapted quantization schemes. Hence, it is not surprising that they appeared both in the school and in the conference. It is the goal of the contribution of *Martin Schlichenmaier* to give an introduction to the basic definitions and to the results concentrating on the case of compact quantizable Kähler manifolds. The results on the correct semi-classical limit and the existence of the Berezin-Toeplitz star product are presented. Coherent states, co- and covariant Berezin symbols, and the Berezin transform are introduced. For the asymptotic expansion of the Berezin transform the off-diagonal asymptotic expansion of the Bergman kernel places a crucial role (which was obtained by Schlichenmaier and Karabegov).

In the contribution of *Xiaonan Ma* and *George Marinescu* the authors give a review on their recent results of the off-diagonal asymptotic expansion of the Bergman kernel and their application to the Toeplitz operators and Berezin-Toeplitz quantization. Everything is done in the presence of an auxiliary vector bundle. Explicit formulas for the coefficients with small indices in the asymptotic expansions in terms of the metrics involved are given. The results go beyond the Kähler case as they can be employed to the compact symplectic situation by choosing an almost-complex structure and an associated Spin^c structure. Moreover, they apply also to compact Kähler (and symplectic) orbifolds and certain noncompact complete Hermitian manifolds.

Toeplitz operators play an important role in Andersen's approach to Topological Quantum Field Theory (TQFT). In the contribution of *Jørgen E. Andersen* and *Jakob L. Blaavand* this is explained. A review of asymptotic results on Toeplitz operators is given. For this purpose also the differential geometric construction of the Hitchin connection on a prequantizable compact symplectic manifold is reviewed. The asymptotic results relating the Hitchin connection and Toeplitz operators, is applied in the special case of the moduli space of flat $\mathrm{SU}(n)$-connections on a surface, and the proof of the asymptotic faithfulness of the $\mathrm{SU}(n)$ quantum representations of the mapping class group is discussed. Furthermore, the formal Hitchin connections and formal trivializations of these are studied. These fit together to produce a Berezin–Toeplitz star product, which is independent of the complex structure. Explicit examples of the objects in the case of the abelian moduli space are given and an approach to curve operators in the TQFT associated to abelian Chern–Simons theory is presented.

Dmitry Talalaev in his contribution considers the very important spectral curve method in the theory of integrable systems. First, he recalls the classical set-up and then he introduces its quantum theoretical counter-part. Quite a number of such quantized systems are discussed in his contribution (Calogero-Moser, Gaudin, Hitchin, etc.). He closes with a discussion of the relation to the geometric Langlands correspondence.

The organizers of the school thank all lecturers for contributing to the success of the activity in an essential manner. Furthermore, we thank the participants for their active role and for their very positive feedback. Last, but not least, we thank the following institutions for financial support: ESF research networking programme *Harmonic and Complex Analysis and its Application* (HCAA); ESF research networking programme *Interaction of Low-Dimensional Topology and Geometry with Physics* (ITGP); Mathematics Institute of the University of Paderborn, in the frame of the DFG-IRTG *Geometry and Analysis of Symmetries*; LMAM of the University Paul Verlaine, Metz; Steklov Mathematical Institute in Moscow; Mathematical Institute of the University of Nagoya. The University of Luxembourg with its Mathematical Research Unit RMATH contributed both via financial support and by a lot of engagement to the success of the school and conference. Furthermore, we thank the Fonds National de la Recherche Luxembourg (FNR) for its support of the accompanying conference GEOQUANT.

Martin Schlichenmaier (for the organisers)

Travaux mathématiques, Volume 19 (2011), 7–26,

Quantization of universal Teichmüller space

by Armen Sergeev

Abstract

In the first part of the paper we describe the complex geometry of the universal Teichmüller space $\mathcal{T}$ which may be realized as an open subset in the complex Banach space of holomorphic quadratic differentials in the unit disc. The quotient $\mathcal{S}$ of the diffeomorphism group of the circle modulo Möbius transformations is treated as a regular part of $\mathcal{T}$. In the second part we consider the quantization of universal Teichmüller space $\mathcal{T}$. We explain first how to quantize the regular part $\mathcal{S}$ by embedding it into a Hilbert–Schmidt Siegel disc. This quantization method, however, does not apply to the whole universal Teichmüller space $\mathcal{T}$. For its quantization we use an approach, similar to the "quantized calculus" of Connes and Sullivan.

1 Introduction

The universal Teichmüller space $\mathcal{T}$, introduced by Ahlfors and Bers, plays a key role in the theory of quasiconformal maps and Riemann surfaces. It can be defined as the space of quasisymmetric homeomorphisms of the unit circle S^1 (i.e. homeomorphisms of S^1, extending to quasiconformal maps of the unit disc Δ) modulo Möbius transformations. The space $\mathcal{T}$ has a natural complex structure, induced by embedding of $\mathcal{T}$ into the complex Banach space $B_2(\Delta)$ of holomorphic quadratic differentials in the unit disc Δ. It also contains all classical Teichmüller spaces $T(G)$, where G is a Fuchsian group, as complex submanifolds. The space $\mathcal{S} := \text{Diff}_+(S^1)/\text{Möb}(S^1)$ of normalized diffeomorphisms of the circle may be considered as a "regular" part of $\mathcal{T}$.

Our motivation to study $\mathcal{T}$ comes from the string theory. Physicists have noticed that the space $\Omega_d := C_0^\infty(S^1, \mathbb{R}^d)$ of smooth loops in the d-dimensional vector space $\mathbb{R}^d$ may be identified with the phase space of the theory of smooth bosonic closed strings. By this identification the standard symplectic form (of type "$dp \wedge dq$") on the phase space translates into a natural symplectic form ω on Ω_d. This form has a remarkable property that it can be extended to the Sobolev completion of Ω_d, coinciding with the space $V_d := H_0^{1/2}(S^1, \mathbb{R}^d)$ of half-differentiable vector-functions on S^1. Moreover, V_d is the largest space among all Sobolev spaces $H_0^s(S^1, \mathbb{R}^d)$ on which ω can be correctly defined. In other words, V_d is a natural phase space, "chosen" by the form ω itself. From that point of view, it seems more reasonable to consider V_d as the phase space of bosonic string theory, rather than Ω_d. In these lectures we set $d = 1$ for simplicity and study the space $V := V_1 = H_0^{1/2}(S^1, \mathbb{R})$.

According to Nag–Sullivan [7], there is a natural group, attached to the space $V = H_0^{1/2}(S^1, \mathbb{R})$, namely the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of the circle. Again one can say that the space V itself chooses the "right" group to be acted on. The group $\mathrm{QS}(S^1)$ acts on V by reparametrization of loops and this action is symplectic with respect to the form ω. The universal Teichmüller space $\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$ can be identified by this action with the space of complex structures on V which can be obtained from a reference complex structure by the action of reparametrization group $\mathrm{QS}(S^1)$. It is well known that such a space plays a crucial role in quantization which is the main subject of the second part of our lectures.

In these lectures we try to define what is the quantum counterpart of the space $\mathcal{T}$, provided with the action of the group $\mathrm{QS}(S^1)$. In order to explain the arising difficulties we consider first an analogous problem for the regular part $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$ of $\mathcal{T}$, provided with the action of the group $\mathrm{Diff}_+(S^1)$. This space can be quantized, using an embedding of $\mathcal{S}$ into the Hilbert-Schmidt Siegel disc $\mathcal{D}_{\mathrm{HS}}$. Under this embedding the diffeomorphism group $\mathrm{Diff}_+(S^1)$ is realized as a subgroup of the Hilbert–Schmidt symplectic group $\mathrm{Sp}_{\mathrm{HS}}(V)$, acting on the Siegel disc by operator fractional-linear transformations. There is a holomorphic Fock bundle $\mathcal{F}$ over $\mathcal{D}_{\mathrm{HS}}$, provided with a projective action of $\mathrm{Sp}_{\mathrm{HS}}(V)$, which covers its action on $\mathcal{D}_{\mathrm{HS}}$. The infinitesimal version of this action is a projective representation of the Hilbert–Schmidt symplectic Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in the fibre F_0 of the Fock bundle $\mathcal{F}$. This defines the Dirac quantization of the Siegel disc $\mathcal{D}_{\mathrm{HS}}$. Its restriction to $\mathcal{S}$ gives a projective representation of the Lie algebra $\mathrm{Vect}(S^1)$ of the group $\mathrm{Diff}_+(S^1)$ in the Fock space F_0 which defines the Dirac quantization of the space $\mathcal{S}$.

However, the described quantization procedure does not apply to the whole universal Teichmüller space $\mathcal{T}$. By this reason we choose another approach to this problem, based on Connes quantization. Briefly, the idea is the following. The $\mathrm{QS}(S^1)$-action on the Sobolev space V, mentioned above, cannot be differentiated in the classical sense (in particular, there is no Lie algebra, associated to $\mathrm{QS}(S^1)$). However, one can define a quantized infinitesimal version of this action by associating with any quasisymmetric homeomorphism $f \in \mathrm{QS}(S^1)$ a quantum differential $d^q f$ which is an integral operator on V with kernel, given essentially by the finite-difference derivative of f. In these terms the quantization of $\mathcal{T}$ is given by a representation of the algebra of derivations of V, generated by quantum differentials $d^q f$, in the Fock space F_0.

2 Universal Teichmüller space

2.1 Definition of universal Teichmüller space

2.1.1 Quasiconformal maps

Let $w : D \to w(D)$ be a homeomorphism of the domain $D \subset \overline{\mathbb{C}}$ in the extended complex plane (Riemann sphere) $\overline{\mathbb{C}}$ onto domain $w(D) \subset \overline{\mathbb{C}}$ which has locally integrable derivatives (in generalized sense).

Definition 2.1. The homeomorphism w is called *quasiconformal* if there exist a function $\mu \in L^\infty(D)$ with norm $\|\mu\|_\infty =: k < 1$ such that the following *Beltrami equation*

$$w_{\bar{z}} = \mu w_z \tag{2.1}$$

is satisfied for almost all $z \in D$. The function μ is called the *Beltrami differential* of w and the constant k is often indicated in the name of k-*quasiconformal* maps.

Remark 2.1. For $k = 0$ the equation (2.1) reduces to the Cauchy–Riemann equation and so determines a conformal map $w : D \to w(D)$. Such a map sends infinitesimally small circles, centered at a point $z \in D$, again into infinitesimally small circles, centered at $w(z)$. While in the case of a general smooth quasiconformal map w such a map sends infinitesimally small circles, centered at $z \in D$, into infinitesimally small ellipses, centered at $w(z)$, with eccentricity (the ratio of the large axis to the small one) being uniformly bounded (w.r. to $z \in D$) by a common constant $K < \infty$. This constant K is related to the above constant $k = \|\mu\|_\infty$ by the formula

$$K = \frac{1+k}{1-k} \geq 1 \ .$$

The least possible constant K is called the *maximal dilatation* of w and is also sometimes indicated in the name of K-*quasiconformal* maps.

Remark 2.2. The term "Beltrami differential" for μ is motivated by the behavior of μ under conformal changes of variable. Namely, according to (2.1), the function μ should transform under a conformal change $z \mapsto f(z)$ as

$$\mu\left(f(z)\right) = \mu(z)\,\frac{\overline{f'(z)}}{f'(z)},$$

i.e. as a $(-1, 1)$-differential.

Remark 2.3. Quasiconformal maps $w : D \to D$ form a group, i.e. the composition of quasiconformal maps and the inverse of a quasiconformal map are again quasiconformal.

Theorem 2.1 (uniqueness theorem)**.** *Suppose that quasiconformal maps $w_1, w_2 : D \to D'$ satisfy the same Beltrami equation in D (i.e. have the same Beltrami differential in D). Then the maps*

$$w_1 \circ w_2^{-1} \quad \text{and} \quad w_2 \circ w_1^{-1}$$

are conformal. The composition $f \circ w$ of a quasiconformal map $w : D \to D'$ with a conformal map $f : D' \to D''$ satisfy the same Beltrami equation in D as w.

Remark 2.4. A quasiconformal map $w : D \to D'$ is always extended to a homeomorphism $w : \overline{D} \to \overline{D'}$ of the closures which is Hölder-continuous up to the boundary.

Theorem 2.2 (existence theorem)**.** *For any function $\mu \in L^\infty(\overline{\mathbb{C}})$ with $\|\mu\|_\infty < 1$ there exists a solution w of the Beltrami equation in $\overline{\mathbb{C}}$. Any other solution $\tilde{w}$ of this equation has the form $\tilde{w} = w \circ f$ where f is a fractional-linear transform.*

Remark 2.5. In Theorem 2.2 we have restricted ourselves to the case $D = \overline{\mathbb{C}}$ since the case of a general domain $D \subset \overline{\mathbb{C}}$ is easily reduced to the case of the extended complex plane. Indeed, given a Beltrami differential $\mu \in L^\infty(D)$ with norm $\|\mu\|_\infty < 1$ we can always extend it (e.g. by zero outside D) to the whole $\overline{\mathbb{C}}$, preserving the inequality $\|\mu\|_\infty < 1$, and then apply the above theorem to get a solution of Beltrami equation in $\overline{\mathbb{C}}$. Its restriction to D yields a solution of Beltrami equation in D, defined up to conformal maps, according to the uniqueness theorem.

2.1.2 Quasisymmetric homeomorphisms

Definition 2.2. A homeomorphism $f : S^1 \to S^1$ of the unit circle S^1, preserving its orientation, is called *quasisymmetric* if it extends to a quasiconformal homeomorphism $w : \Delta \to \Delta$ of the unit disc Δ. The set of all quasisymmetric homeomorphisms of S^1 is a group, denoted by $\mathrm{QS}(S^1)$.

Definition 2.3. The *universal Teichmüller space* $\mathcal{T}$ is the quotient

$$\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$$

where $\mathrm{Möb}(S^1)$ denotes the Möbius group of fractional-linear automorphisms of the unit disc Δ, restricted to the unit circle S^1.

Remark 2.6. One can avoid taking the quotient by Möbius group in the definition of $\mathcal{T}$ by considering only *normalized* quasisymmetric homeomorphisms, leaving three fixed points in the circle, say $\pm 1, i$, invariant.

Remark 2.7. Any orientation-preserving diffeomorphism in $\mathrm{Diff}_+(S^1)$ extends to a diffeomorphism of the closed unit disc $\overline{\Delta}$ which is quasiconformal, according to Remark 2.1. So $\mathrm{Diff}_+(S^1) \subset \mathrm{QS}(S^1)$, and we have the following chain of embeddings

$$\text{Möb}(S^1) \subset \mathrm{Diff}_+(S^1) \subset \mathrm{QS}(S^1) \subset \mathrm{Homeo}_+(S^1)\ .$$

Hence,

$$\mathcal{S} := \mathrm{Diff}_+(S^1)/\text{Möb}(S^1) \hookrightarrow \mathcal{T} = \mathrm{QS}(S^1)/\text{Möb}(S^1).$$

The space $\mathcal{S}$ can be otherwise defined as the space of normalized diffeomorphisms of S^1 and will be considered as a "regular" part of $\mathcal{T}$.

Since quasisymmetric homeomorphisms of S^1 were defined via quasiconformal maps of Δ, i.e. in terms of solutions of Beltrami equation in Δ, one can expect that there should be a definition of $\mathcal{T}$ directly in terms of Beltrami differentials.

Denote by $B(\Delta)$ the set of Beltrami differentials in the unit disc Δ. It can be identified (as a set) with the unit ball in the complex Banach space $L^\infty(\Delta)$. Given a Beltrami differential $\mu \in B(\Delta)$, we can extend it to a Beltrami differential $\check{\mu}$ on the extended complex plane $\overline{\mathbb{C}}$ by setting $\check{\mu}$ equal to zero outside the unit disc Δ. Then we can apply the existence Theorem 2.2 for quasiconformal maps on the extended complex plane $\overline{\mathbb{C}}$ and obtain a normalized quasiconformal homeomorphism w^μ, satisfying Beltrami equation (2.1) on $\overline{\mathbb{C}}$ with potential $\check{\mu}$. This homeomorphism is conformal on the exterior $\Delta_- := \overline{\mathbb{C}} \setminus \overline{\Delta}$ of the closed unit disc $\overline{\Delta}$ on $\overline{\mathbb{C}}$ and fixes the points $\pm 1, -i$.

Introduce an equivalence relation between Beltrami differentials in Δ by identifying two Beltrami differentials μ and ν for which the corresponding conformal maps coincide: $w^\mu|_{\Delta_-} \equiv w^\nu|_{\Delta_-}$. The universal Teichmüller space $\mathcal{T}$ coincides with the quotient

$$\mathcal{T} = B(\Delta)/\sim$$

of the space $B(\Delta)$ of Beltrami differentials modulo the introduced equivalence relation.

2.2 Complex structure of universal Teichmüller space

We introduce a complex structure on the universal Teichmüller space $\mathcal{T}$, using its embedding into the space of holomorphic quadratic differentials.

Consider an arbitrary point $[\mu]$ of $\mathcal{T}$, represented by the quasiconformal map w^μ. Its restriction to Δ_- is a conformal map so we can take its Schwarzian $S(w^\mu|_{\Delta_-})$.

Digression 1. Recall that the *Schwarzian* of a conformal map f is defined by

$$S(f) := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

A characteristic property of Schwarzian is its invariance under fractional-linear maps

$$S\left(\frac{af+b}{cf+d}\right) = S(f).$$

By taking the Schwarzian $S(w^\mu|_{\Delta_-})$, we get a holomorphic quadratic differential in the disc Δ_- (the latter fact follows from the transformation properties of Beltrami differentials, prescribed by Beltrami equation (2.1)). Moreover, the image of this map does not depend on the choice of Beltrami differential μ in the class $[\mu]$. Composing this map with a standard fractional-linear isomorphism $\Delta_- \to \Delta$, we obtain an embedding

$$\Psi : \mathcal{T} \longrightarrow B_2(\Delta)\ , \quad [\mu] \longmapsto \psi(\mu), \tag{2.2}$$

having its image in the space $B_2(\Delta)$ of holomorphic quadratic differentials in the unit disc Δ.

The space $B_2(\Delta)$ of holomorphic quadratic differentials in Δ is a complex Banach space, provided with a natural hyperbolic norm, given by

$$\|\psi\|_2 := \sup_{z\in\Delta}(1-|z|^2)^2|\psi(z)|$$

for a quadratic differential ψ. It can be proved (cf. [5]) that $\|\psi[\mu]\|_2 \le 6$ for any Beltrami differential $\mu \in B(\Delta)$.

The constructed map $\Psi : \mathcal{T} \to B_2(\Delta)$, called the *Bers embedding*, is a homeomorphism of $\mathcal{T}$ onto an open bounded connected contractible subset in $B_2(\Delta)$, containing the ball of radius $1/2$, centered at the origin (cf. [5]).

Using the constructed embedding (2.2), we can introduce a complex structure on the universal Teichmüller space $\mathcal{T}$ by pulling it back from the complex Banach space $B_2(\Delta)$. It provides $\mathcal{T}$ with the structure of a complex Banach manifold.

2.3 Classical Teichmüller spaces

The universal Teichmüller space $\mathcal{T}$ contains all classical Teichmüller spaces $T(G)$ as complex submanifolds. In particular, it is true for all Teichmüller spaces of compact Riemann surfaces of genus g. This property motivates the use of the term "universal" in the name of $\mathcal{T}$.

Let X be a compact Riemann surface of genus $g > 1$, uniformized by the unit disc Δ. Such a surface can be represented as the quotient

$$X = \Delta/G$$

where G is a discrete (Fuchsian) subgroup of $\text{Möb}(\Delta)$.

Definition 2.4. A quasisymmetric homeomorphism $f : S^1 \to S^1$ is called *G-invariant* if

$$f \circ g \circ f^{-1} \in \text{Möb}(S^1) \text{ for any } g \in G \iff fGf^{-1} \subset \text{Möb}(S^1).$$

Denote by $\text{QS}(S^1)^G$ the subgroup of G-invariant quasisymmetric homeomorphisms in $\text{QS}(S^1)$. Then by definition

$$T(G) := \text{QS}(S^1)^G/\text{Möb}(S^1).$$

The universal Teichmüller space $\mathcal{T}$ itself corresponds to the Fuchsian group $G = \{1\}$.

Remark 2.8. According to definition of $T(G)$, due to Teichmüller, the space $T(G)$ parameterizes different complex structures on the Riemann surface X/Δ which can be obtained from the original complex structure by a quasiconformal deformation.

2.4 Grassmann realization

2.4.1 Sobolev space of half-differentiable functions

Definition 2.5. The *Sobolev space of half-differentiable functions* on S^1 is a Hilbert space

$$V := H_0^{1/2}(S^1, \mathbb{R}),$$

consisting of functions $f \in L_0^2(S^1, \mathbb{R})$ with zero average over the circle, which have Fourier decompositions

$$f(z) = \sum_{k \neq 0} f_k z^k \ , \quad f_k = \bar{f}_{-k} \ , \quad z = e^{i\theta} \ ,$$

and a finite Sobolev norm

$$\|f\|_{1/2}^2 = \sum_{k \neq 0} |k||f_k|^2 = 2 \sum_{k>0} k|f_k|^2 < \infty \ . \qquad (2.3)$$

Properties of $V = H_0^{1/2}(S^1, \mathbb{R})$:

1. *Symplectic structure*: define a 2-form ω on V by the formula

$$\omega(\xi, \eta) = 2 \, \text{Im} \sum_{k>0} k \xi_k \bar{\eta}_k$$

for vectors $\xi, \eta \in V$ with Fourier series

$$\xi(z) = \sum_{k \neq 0} \xi_k z^k, \quad \eta(z) = \sum_{k \neq 0} \eta_k z^k.$$

This form, which is correctly defined on V due to condition (2.3), determines a symplectic form on V. Moreover, $H_0^{1/2}(S^1, \mathbb{R})$ is the largest Hilbert space in the scale of Sobolev spaces $H_0^s(S^1, \mathbb{R})$, $s \in \mathbb{R}$, on which this form is correctly defined.

2. *Complex structure*: the Sobolev space V has a complex structure J^0, defined by the formula

$$\xi(z) = \sum_{k \neq 0} \xi_k z^k \longmapsto (J^0 \xi)(z) = -i \sum_{k>0} \xi_k z^k + i \sum_{k<0} \xi_k z^k$$

for a vector $\xi(z) = \sum_{k \neq 0} \xi_k z^k \in V$.

3. *Riemannian metric*: the introduced symplectic and complex structures on V are compatible with each other in the sense that they generate together a Riemannian metric, defined by

$$g^0(\xi, \eta) = \omega(\xi, J^0 \eta) = 2 \operatorname{Re} \sum_{k>0} k \xi_k \bar{\eta}_k.$$

In other words, V has the structure of a Kähler Hilbert space.

4. *Complexification* of V, equal to

$$V^{\mathbb{C}} = H_0^{1/2}(S^1, \mathbb{C}),$$

is a complex Hilbert space with a Kähler metric, given by the Hermitian extension of the Riemannian metric g^0 on V to $V^{\mathbb{C}}$. The space $V^{\mathbb{C}}$ is decomposed into the direct sum

$$V^{\mathbb{C}} = W_+ \oplus W_-$$

of $(\mp i)$-eigenspaces of the complex structure operator $J^0 \in \operatorname{End} V^{\mathbb{C}}$. More explicitly,

$$W_+ = \{f \in V^{\mathbb{C}} : \ f(z) = \sum_{k>0} f_k z^k\}, \quad W_- = \{f \in V^{\mathbb{C}} : \ f(z) = \sum_{k<0} f_k z^k\}.$$

This splitting is orthogonal with respect to Hermitian inner product on $V^{\mathbb{C}}$.

2.4.2 QS-action on the Sobolev space V

With any homeomorphism $h : S^1 \to S^1$, preserving the orientation, we can associate a "change-of-variable" operator

$$T_h : L^2_0(S^1, \mathbb{R}) \to L^2_0(S^1, \mathbb{R}),$$

defined by

$$T_h(\xi) := \xi \circ h - \frac{1}{2\pi} \int_0^{2\pi} \xi\left(h(\theta)\right) d\theta.$$

This operator has the following remarkable property.

Theorem 2.3 (Nag–Sullivan [7]). *(i) The operator T_h acts on V, i.e. $T_h : V \to V$, if and only if $h \in QS(S^1)$.*

(ii) The operator T_h with $h \in QS(S^1)$ acts symplectically on V, i.e. it preserves symplectic form ω. Moreover, its complex-linear extension to $V^{\mathbb{C}}$ preserves the subspace W_+ if and only if $h \in$ Möb(S^1). In the latter case, T_h acts as a unitary operator on W_+.

Remark 2.9. We have pointed out in the previous subsection that the Sobolev space V is the largest Hilbert space in the scale of Sobolev spaces, on which the form ω is correctly defined. In other words, this space is "chosen" by symplectic form ω itself. According to Theorem 2.3, the space V also "chooses" the reparametrization group $QS(S^1)$ in the sense that it is the largest reparametrization group, leaving V invariant. So we get a natural phase space (V, ω) together with a natural group $QS(S^1)$ of its canonical transformations.

According to Theorem 2.3, we have an embedding

$$\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1) \longrightarrow \mathrm{Sp}(V)/\mathrm{U}(W_+). \tag{2.4}$$

Here, $\mathrm{Sp}(V)$ is the symplectic group of V, consisting of bounded linear symplectic operators on V, and $\mathrm{U}(W_+)$ is its subgroup, consisting of unitary operators (i.e. the operators, whose complex-linear extensions to $V^{\mathbb{C}}$ preserve the subspace W_+).

Digression 2. Recall the definition of symplectic group $\mathrm{Sp}(V)$. In terms of decomposition

$$V^{\mathbb{C}} = W_+ \oplus W_-$$

any linear operator $A : V^{\mathbb{C}} \to V^{\mathbb{C}}$ can be written in the block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Such an operator belongs to the symplectic group $\mathrm{Sp}(V)$ if it has the form

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

with components, satisfying the relations

$$\bar{a}^t a - b^t \bar{b} = 1, \quad \bar{a}^t b = b^t \bar{a}$$

where a^t, b^t denote the transposed operators $a^t : W_- \to W_-$, $b^t : W_- \to W_+$. The unitary group $\mathrm{U}(W_+)$ is embedded into $\mathrm{Sp}(V)$ as a subgroup, consisting of diagonal block matrices of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} .$$

The space

$$\mathcal{J}(V) := \mathrm{Sp}(V)/\mathrm{U}(W_+)$$

on the right hand side of (2.4), can be identified with the space of complex structures on V, compatible with ω. Indeed, any such structure, given by a linear operator J on V with $J^2 = -I$, determines a decomposition

$$V^{\mathbb{C}} = W \oplus \overline{W} \tag{2.5}$$

of $V^{\mathbb{C}}$ into the direct sum of $(\pm i)$-eigenspaces, isotropic with respect to ω. Conversely, any decomposition (2.5) of the space $V^{\mathbb{C}}$ into the direct sum of isotropic subspaces determines a complex structure J on $V^{\mathbb{C}}$, equal to iI on W and $-iI$ on $\overline{W}$, which is compatible with ω. Moreover, a complex structure J, obtained from a reference complex structure J^0 by the action of an element A of $\mathrm{Sp}(V)$, is equivalent to J^0 if and only if $A \in \mathrm{U}(W_+)$. Hence,

$$\mathrm{Sp}(V)/\mathrm{U}(W_+) = \mathcal{J}(V) .$$

The space on the right can be, in its turn, identified with the *Siegel disc* $\mathcal{D}$, defined as the set

$$\mathcal{D} = \{Z : W_+ \to W_- \text{ is a symmetric bounded linear operator with } \bar{Z}Z < I\}.$$

The symmetricity of Z means that $Z^t = Z$ and the condition $\bar{Z}Z < I$ means that symmetric operator $I - \bar{Z}Z$ is positive definite. In order to identify $\mathcal{J}(V)$ with $\mathcal{D}$, consider the action of the group $\mathrm{Sp}(V)$ on $\mathcal{D}$, given by fractional-linear transformations $A : \mathcal{D} \to \mathcal{D}$ of the form

$$Z \longmapsto (\bar{a}Z + \bar{b})(bZ + a)^{-1}$$

where $A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{Sp}(V)$. The isotropy subgroup at $Z = 0$ coincides with the set of operators $A \in \mathrm{Sp}(V)$ such that $b = 0$, i.e. with $\mathrm{U}(W_+)$. So the space

$$\mathcal{J}(V) = \mathrm{Sp}(V)/\mathrm{U}(W_+)$$

can be identified with the Siegel disc $\mathcal{D}$.

It can be proved (cf. [7]) that the constructed embedding of universal Teichmüller space $\mathcal{T}$ into the Siegel disc $\mathcal{D} = \mathrm{Sp}(V)/\mathrm{U}(W_+)$ is an equivariant holomorphic map of Banach manifolds.

Restriction of this map to the regular part $\mathcal{S}$ of universal Teichmüller space yields an embedding

$$\mathcal{S} \hookrightarrow \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+) \tag{2.6}$$

where the *Hilbert–Schmidt subgroup* $\mathrm{Sp}_{\mathrm{HS}}(V)$ of $\mathrm{Sp}(V)$ consists of bounded linear operators $A \in \mathrm{Sp}(V)$, having block representations of the form

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

where b is a Hilbert–Schmidt operator.

Digression 3. Recall that a linear bounded operator $T : H_1 \to H_2$ from a Hilbert space H_1 to a Hilbert space H_2 is called Hilbert–Schmidt if there exists an orthonormal basis $\{e_i\}$ in H_1 such that the Hilbert–Schmidt norm

$$\|T\|_2 := \left(\sum_{i=0}^{\infty} \|Te_i\|_{H_2}^2 \right)^{1/2}$$

is finite. If this is true for some orthonormal basis $\{e_i\}$ in H_1 then it is true for any orthonormal basis in H_1 and the value of the norm $\|T\|_2$ does not depend on the choice of this basis.

We identify, as above, the right hand side of (2.6) with a subspace $\mathcal{J}_{\mathrm{HS}}(V)$ of the space $\mathcal{J}(V)$ of compatible complex structures on V. As before, the space $\mathcal{J}_{\mathrm{HS}}(V)$ of Hilbert–Schmidt complex structures on V can be realized as a *Hilbert–Schmidt Siegel disc*

$$\mathcal{D}_{\mathrm{HS}} = \{Z : W_+ \to W_- \text{ is a symmetric Hilbert–Schmidt operator with } \bar{Z}Z < I\}.$$

The embedding of $\mathcal{S}$ into the Hilbert–Schmidt Siegel disc $\mathcal{D}_{\mathrm{HS}}$ is an equivariant holomorphic map of Banach manifolds.

3 Quantization of Universal Teichmüller Space

3.1 Dirac quantization

3.1.1 Definition

We start by recalling a general definition of quantization of finite-dimensional classical systems, due to Dirac. A *classical system* is given by a pair $(M, \mathcal{A})$ where M is the phase space of the system and $\mathcal{A}$ is its algebra of observables.

The *phase space* M is a smooth symplectic manifold of even dimension $2n$, provided with symplectic 2-form ω. Locally, it is equivalent to the standard model, given by symplectic vector space $M_0 := \mathbb{R}^{2n}$ together with standard symplectic form ω_0, given in canonical coordinates (p_i, q_i), $i = 1, \dots, n$, on $\mathbb{R}^{2n}$ by

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i .$$

The *algebra of observables* $\mathcal{A}$ is a Lie subalgebra of the Lie algebra $C^\infty(M, \mathbb{R})$ of smooth real-valued functions on the phase space M, provided with the Poisson bracket, determined by symplectic 2-form ω. In particular, in the case of standard model $M_0 = (\mathbb{R}^{2n}, \omega_0)$ one can take for $\mathcal{A}$ the Heisenberg algebra, generated by coordinate functions p_i, q_i, $i = 1, \dots, n$, and 1, satisfying the commutation relations

$$\{p_i, p_j\} = \{q_i, q_j\} = 0,$$
$$\{p_i, q_j\} = \delta_{ij} \quad \text{for } i, j = 1, \dots, n.$$

Remark 3.1. One of usual ways to produce algebras of observables is to consider a Lie group Γ of symplectomorphisms of a symplectic manifold (M, ω) and take for $\mathcal{A}$ its Lie algebra $\mathrm{Lie}(\Gamma)$, consisting of Hamiltonian vector fields X_f on M. If M is simply connected then $\mathcal{A}$ can be identified with the dual algebra of functions f, generating Hamiltonian vector fields from $\mathrm{Lie}(\Gamma)$.

Definition 3.1. The *Dirac quantization* of a classical system $(M, \mathcal{A})$ is an irreducible linear representation

$$r : \mathcal{A} \longrightarrow \mathrm{End}^* H$$

of the algebra of observables $\mathcal{A}$ in the space of linear self-adjoint operators, acting on a complex Hilbert space H, called the *quantization space*. The map r should satisfy the condition

$$r(\{f, g\}) = \frac{1}{i} \left(r(f) r(g) - r(g) r(f) \right)$$

for any $f, g \in \mathcal{A}$. We also impose on r the following normalization condition: $r(1) = I$.

Remark 3.2. For complexified algebras of observables $\mathcal{A}^{\mathbb{C}}$ or, more generally, complex involutive Lie algebras of observables (i.e. Lie algebras with conjugation) their Dirac quantization is given by an irreducible Lie-algebra representation

$$r : \mathcal{A}^{\mathbb{C}} \longrightarrow \operatorname{End} H \ ,$$

satisfying the normalization condition and the conjugation law: $r(\bar{f}) = r(f)^*$ for any $f \in \mathcal{A}$.

Remark 3.3. We are going to apply this definition of quantization to infinite-dimensional classical systems, in which both the phase space and algebra of observables are infinite-dimensional. For infinite-dimensional algebras of observables it is more natural to look for their projective representations. Using such a representation for an original algebra $\mathcal{A}$, we can construct the quantization of the extended system $(M, \tilde{\mathcal{A}})$ with $\tilde{\mathcal{A}}$ being a suitable central extension of $\mathcal{A}$.

3.1.2 Statement of the problem

We shall explain first how to quantize the regular part of universal Teichmüller space $\mathcal{T}$, represented by the classical system

$$(\mathcal{S}, \operatorname{Vect}(S^1))$$

where $\mathcal{S} = \operatorname{Diff}_+(S^1)/\text{Möb}(S^1)$ and $\operatorname{Vect}(S^1)$ is the Lie algebra of $\operatorname{Diff}_+(S^1)$, consisting of smooth vector fields on S^1.

To quantize this system, we first enlarge it to an extended system, using the embedding $\mathcal{S} \hookrightarrow \mathcal{J}_{\text{HS}}(V)$ from Subsection 2.4.2. This extended system is given by

$$(\mathcal{J}_{\text{HS}}(V), \operatorname{sp}_{\text{HS}}(V))$$

where $\operatorname{sp}_{\text{HS}}(V)$ is the Lie algebra of $\operatorname{Sp}_{\text{HS}}(V)$.

3.2 Quantization of $\mathcal{S}$

3.2.1 Fock space

Fix a compatible complex structure $J \in \mathcal{J}(V)$, generating a decomposition

$$V^{\mathbb{C}} = W \oplus \overline{W} \tag{3.1}$$

of $V^{\mathbb{C}}$ into the direct sum of $\pm i$-eigenspaces of J and provide $V^{\mathbb{C}}$ with a Hermitian inner product

$$< z, w >_J := \omega(z, Jw),$$

determined by J and symplectic form ω.

The Fock space $F(V^{\mathbb{C}}, J)$ is the completion of the algebra of symmetric polynomials on W with respect to a natural norm, generated by $< \cdot, \cdot >_J$. In more detail, denote by $S(W)$ the algebra of symmetric polynomials in variables $z \in W$. This algebra is provided with an inner product, generated by $< \cdot, \cdot >_J$. By definition, this inner product on monomials of the same degree is equal to

$$< z_1 \cdot \ldots \cdot z_n, z'_1 \cdot \ldots \cdot z'_n >_J = \sum_{\{i_1,\ldots,i_n\}} < z_1, z'_{i_1} >_J \cdot \ldots \cdot < z_n, z'_{i_n} >_J$$

where the summation is taken over all permutations $\{i_1, \ldots, i_n\}$ of the set $\{1, \ldots, n\}$. The inner product of monomials of different degrees is set to zero. The constructed inner product is extended to the whole algebra $S(W)$ by linearity. The completion $\widehat{S(W)}$ of $S(W)$ with respect to the introduced norm is called the *Fock space* of $V^{\mathbb{C}}$ with respect to complex structure J:

$$F_J = F(V^{\mathbb{C}}, J) := \widehat{S(W)}.$$

If $\{w_n\}$, $n = 1, 2, \ldots$, is an orthonormal basis of W one can take for an orthonormal basis of F_J a family of homogeneous polynomials of the form

$$P_K(z) = \frac{1}{\sqrt{k!}} < z, w_1 >_J^{k_1} \cdot \ldots \cdot < z, w_n >_J^{k_n}, \quad z \in W, \tag{3.2}$$

where $K = (k_1, \ldots, k_n, 0, \ldots)$, $k_i \in \mathbb{N} \cup 0$, and $k! = k_1! \cdot \cdots \cdot k_n!$.

3.2.2 Symplectic group action on Fock spaces

We unify different Fock spaces F_J with $J \in \mathcal{J}_{\mathrm{HS}}(V)$ into a single *Fock bundle*

$$\mathcal{F} := \bigcup_{J \in \mathcal{J}_{\mathrm{HS}}(V)} F_J \longrightarrow \mathcal{J}_{\mathrm{HS}}(V) = \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+).$$

Theorem 3.1 (Shale–Berezin)**.** *The Fock bundle*

$$\mathcal{F} \longrightarrow \mathcal{J}_{HS}(V)$$

is a holomorphic Hermitian Hilbert-space bundle. The group $Sp_{HS}(V)$ acts projectively on $\mathcal{F}$ by unitary transformations and this action covers the natural action of $Sp_{HS}(V)$ on $\mathcal{J}_{HS}(V)$ by left translations.

The infinitesimal version of this action yields a projective representation of symplectic Hilbert–Schmidt algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in the Fock space $F_0 = F(V^{\mathbb{C}}, J^0)$, i.e. a quantization of the system

$$\left(\mathcal{J}_{\mathrm{HS}}, \widetilde{\mathrm{sp}_{\mathrm{HS}}(V)}\right)$$

where $\widetilde{\mathrm{sp}_{\mathrm{HS}}(V)}$ is a central extension of Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$.

The restriction of the constructed Fock bundle $\mathcal{F}$ to the submanifold $\mathcal{S} \subset \mathcal{J}_{\mathrm{HS}}$ is a holomorphic Hermitian Hilbert-space bundle

$$\mathcal{F}_{\mathcal{S}} := \bigcup_{J \in \mathcal{S}} F_J \longrightarrow \mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{M\ddot{o}b}(S^1)$$

together with a projective unitary action of $\mathrm{Diff}_+(S^1)$, covering its action on $\mathcal{S}$ by left translations. The infinitesimal version of this action generates a projective unitary representation of the Lie algebra $\mathrm{Vect}(S^1)$ in the Fock space F_0, i.e. a quantization of the system

$$(\mathcal{S}, \mathrm{vir})$$

where vir is the *Virasoro algebra*, being a central extension of Lie algebra $\mathrm{Vect}(S^1)$.

3.3 Quantization of $\mathcal{T}$

3.3.1 Dirac versus Connes quantization

To quantize $\mathcal{S}$, we have used the fact that the symplectic group $\mathrm{Sp}_{\mathrm{HS}}(V)$ acts on the Fock bundle $\mathcal{F} \to \mathcal{J}_{\mathrm{HS}}(V)$. For the whole Teichmüller space $\mathcal{T}$ we still have the embedding

$$\mathcal{T} \longrightarrow \mathcal{J}(V) = \mathrm{Sp}(V)/\mathrm{U}(W_+)$$

but we cannot construct an $\mathrm{Sp}(V)$-action on $\mathcal{F}$, covering its action on $\mathcal{J}(V)$. This is forbidden by Shale–Berezin theorem. So we employ another approach for the quantization of $\mathcal{T}$, using Connes' definition of quantization.

Recall that in Dirac's approach we quantize a classical system $(M, \mathcal{A})$, consisting of the phase space M and the algebra of observables $\mathcal{A}$ which is a Lie algebra, consisting of smooth functions on M. The quantization of this system is given by a representation r of $\mathcal{A}$ in a Hilbert space H, sending the Poisson bracket $\{f, g\}$ of functions $f, g \in \mathcal{A}$ into the commutator $\frac{1}{i}[r(f), r(g)]$ of the corresponding operators. In Connes' approach the algebra of observables $\mathfrak{A}$ is an associative involutive algebra, provided with an exterior differential d. Its quantization is, by definition, a representation π of $\mathfrak{A}$ in a Hilbert space H, sending the differential df of a function $f \in \mathfrak{A}$ into the commutator $[S, \pi(f)]$ of the operator $\pi(f)$ with a self-adjoint symmetry operator S with $S^2 = I$.

In the following table we compare Connes and Dirac approaches to quantization:

	Dirac approach	Connes approach
Classical system	$(M, \mathcal{A})$ where: M – phase space $\mathcal{A}$ – involutive Lie algebra of observables	$(M, \mathfrak{A})$ where: M – phase space $\mathfrak{A}$ – involutive associative algebra of observables with differential d
Quantization	representation $r\colon \mathcal{A} \to \operatorname{End} H$, sending $\{f, g\} \mapsto \frac{1}{i}[r(f), r(g)]$	representation $\pi\colon \mathfrak{A} \to \operatorname{End} H$, sending $df \mapsto [S, \pi(f)]$, where $S = S^*$, $S^2 = I$

Remark 3.4. We can reformulate the Connes definition in terms of Lie algebras by switching to the algebra of derivations of associative algebra of observables $\mathfrak{A}$. Recall that the Lie algebra $\mathrm{Der}(\mathfrak{A})$ of derivations of $\mathfrak{A}$ consists of linear maps $\mathfrak{A} \to \mathfrak{A}$, satisfying the Leibnitz rule. The Connes quantization means in these terms the construction of an irreducible representation of $\mathrm{Der}(\mathfrak{A})$ in the space $\operatorname{End} H$, considered as a Lie algebra with a Lie bracket, given by commutator.

Remark 3.5. If all observables are smooth functions on M, both approaches are equivalent to each other. Indeed, the differential df of a smooth observable f is symplectically dual to the Hamiltonian vector field X_f which establishes a relation between the associative algebra $\mathfrak{A} \ni f$ of functions f on M and the Lie algebra $\mathcal{A} \ni X_f$ of Hamiltonian vector fields X_f. A symmetry operator S is determined by a polarization $H = H_+ \oplus H_-$ of the quantization space H and related to the complex structure J (determined by the same polarization) by a simple formula $S = iJ$.

In the case when the algebra of observables $\mathcal{A}$ contains non-smooth functions, the Dirac approach formally cannot be applied. In Connes approach the differential df of a non-smooth observable $f \in \mathfrak{A}$ is also not defined but its quantum analogue

$$d^q f := [S, \pi(f)]$$

may still have sense, as it is demonstrated by the example in the next subsection.

3.3.2 Example

Suppose that $\mathfrak{A}$ is the algebra $L^\infty(S^1,\mathbb{C})$ of bounded functions on the circle S^1. Any function $f \in \mathfrak{A}$ determines a bounded multiplication operator in the Hilbert space $H = L^2(S^1,\mathbb{C})$ by the formula

$$M_f : v \in H \longmapsto fv \in H.$$

A symmetry operator S in H is given by the *Hilbert transform* $S : L^2(S^1,\mathbb{C}) \to L^2(S^1,\mathbb{C})$:

$$(Sf)(e^{i\varphi}) = \frac{1}{2\pi} P.V. \int_0^{2\pi} K(\varphi,\psi) f(e^{i\psi}) d\psi$$

where the integral is taken in the principal value sense and the kernel is given by

$$K(\varphi,\psi) = 1 + i \cot \frac{\varphi - \psi}{2} \ . \tag{3.3}$$

Note that for φ, close to ψ, this kernel behaves asymptotically like $2/(\varphi - \psi)$.

The differential *df* of a general observable $f \in \mathfrak{A}$ is not defined in the classical sense but its quantum analogue

$$d^q f := [S, M_f]$$

is a bounded operator in H. Moreover, $d^q f$ for $f \in H$ is a Hilbert–Schmidt operator, given by

$$d^q f(v)(e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} k(\varphi,\psi) v(e^{i\psi}) d\psi \tag{3.4}$$

with kernel

$$k(\varphi,\psi) := K(\varphi,\psi)(f(\varphi) - f(\psi)) \ ,$$

where $K(\varphi,\psi)$ is defined by (3.3). The kernel $k(\varphi,\psi)$ for φ, close to ψ, behaves asymptotically like

$$\frac{f(e^{i\varphi}) - f(e^{i\psi})}{\varphi - \psi}.$$

Using this fact, it can be checked that the quasiclassical limit of $d^q f$, arranged by taking the limit $\varphi \to \psi$, coincides (up to a constant) with the multiplication operator $v \mapsto f'v$. So the quantization means in this case simply the replacement of the derivative by its finite-difference analogue.

3.3.3 Quantization of the universal Teichmüller space

We apply these ideas to the universal Teichmüller space $\mathcal{T}$. In Subsection 2.4.2 we have defined a natural action of the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of S^1 on the Sobolev space V. As we have remarked, this action does not admit the differentiation, so classically there is no Lie algebra, associated with $\mathrm{QS}(S^1)$. In other words, there is no classical algebra of observables, associated to $\mathcal{T}$. (The situation is similar to that in the example above.) However, we shall construct a *quantum algebra of observables*, associated to $\mathcal{T}$.

For that we define a quantum infinitesimal version of $\mathrm{QS}(S^1)$-action on V, given by the integral operator $d^q f$, defined by formula (3.4). Then we extend this operator $d^q f$ to the Fock space F_0 by defining it first on elements of the basis (3.2) of F_0 with the help of Leibnitz rule, and then extending to the whole symmetric algebra $S(W_+)$ by linearity. The completion of the obtained operator yields an operator $d^q f$ on F_0. The operators $d^q f$ with $f \in \mathrm{QS}(S^1)$, constructed in this way, generate a *quantum Lie algebra* $\mathrm{Der}^q(\mathrm{QS})$, associated with $\mathcal{T}$. We consider it as a quantum Lie algebra of observables, associated with $\mathcal{T}$. We can also consider the constructed Lie algebra $\mathrm{Der}^q(\mathrm{QS})$ as a replacement of the (non-existing) classical Lie algebra of the group $\mathrm{QS}(S^1)$.

Compare now the main steps of Connes quantization of $\mathcal{T}$ with the analogous steps in Dirac quantization of $\mathcal{J}_{\mathrm{HS}}$.

In the case of $\mathcal{J}_{\mathrm{HS}}$:

1. we start with the $\mathrm{Sp}_{\mathrm{HS}}(V)$-action on $\mathcal{J}_{\mathrm{HS}}$;

2. then, using Shale theorem, extend this action to a projective unitary action of $\mathrm{Sp}_{\mathrm{HS}}(V)$ on Fock spaces $F(V, J)$;

3. an infinitesimal version of this action yields a projective unitary representation of symplectic Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in the Fock space F_0.

In the case of $\mathcal{T}$:

1. we have an action of $\mathrm{QS}(S^1)$ on the space V; however, in contrast with Dirac quantization of $\mathcal{J}_{\mathrm{HS}}$, the step (2) in case of $\mathcal{T}$ is impossible, since by Shale theorem we cannot extend the action of $\mathrm{QS}(S^1)$ to Fock spaces $F(V, S)$;

2. we define instead a quantized infinitesimal action of $\mathrm{QS}(S^1)$ on V, given by quantum differentials $d^q f$;

3. extending operators $d^q f$ to the Fock space F_0, we obtain a quantum Lie algebra $\mathrm{Der}^q(\mathrm{QS})$, generated by extended operators $d^q f$ on F_0.

Conclusion. The Connes quantization of the universal Teichmüller space $\mathcal{T}$ consists of two steps:

1. The first step ("the first quantization") is the construction of quantized infinitesimal $\mathrm{QS}(S^1)$-action on V, given by quantum differentials $d^q f$ with $f \in \mathrm{QS}(S^1)$.

2. The second step ("the second quantization") is the extension of quantum differentials $d^q f$ to the Fock space F_0. The extended operators $d^q f$ with $f \in \mathrm{QS}(S^1)$ generate the quantum algebra of observables $\mathrm{Der}^q(\mathrm{QS})$, associated with $\mathcal{T}$.

Note that the correspondence principle for the constructed Connes quantization of $\mathcal{T}$ means that this quantization, being restricted to $\mathcal{S}$, coincides with Dirac quantization of $\mathcal{S}$.

4 Acknowledgements

While preparing this paper, the author was partly supported by the RFBR grants 08-01-00014, 09-02-91349, by the program of Support of Scientific Schools (grant NSH-7675.2010.1), and by the Scientific Program of RAS "Nonlinear Dynamics".

References

[1] Ahlfors L., Lectures on quasiconformal mappings, Van Nostrand, Princeton, 1966.

[2] Berezin F.A., Method of second quantization, Academic Press, N.Y., 1966.

[3] Bowick M.J., Rajeev S.G., The holomorphic geometry of closed bosonic string theory and Diff S^1/S^1, *Nucl. Phys.* **B293** (1987), 348-384.

[4] Connes A., Geométrie non commutative, Intereditions, Paris, 1990.

[5] Lehto O., Univalent functions and Teichmüller spaces, Springer Verlag, Berlin, 1987.

[6] Nag S., The complex analytic theory of Teichmüller spaces, Wiley Interscience, New York, 1988.

[7] Nag S., Sullivan D., Teichmüller theory and the universal period mapping via quantum calculus and the $H^{1/2}$ space on the circle, *Osaka J. Math.* **32** (1995), 1-34.

[8] Pressley A., Segal G., Loop groups, Clarendon Press, Oxford, 1986.

[9] Segal G., Unitary representations of some infinite dimensional groups, *Comm. Math. Phys.* **80** (1981), 301-342.

[10] Shale D., Linear symmetries of free boson field, *Trans. Amer. Math. Soc.* **103** (1962), 149-167.

Armen Sergeev
Steklov Mathematical Institute, Moscow, Russia
sergeev@mi.ras.ru

Travaux mathématiques, Volume 19 (2011), 27–63, © Université du Luxembourg

Singular Poisson-Kähler geometry of stratified Kähler spaces and quantization

by Johannes Huebschmann

Abstract

In the presence of classical phase space singularities the standard methods are insufficient to attack the problem of quantization. In certain situations the difficulties can be overcome by means of Kähler quantization on *stratified Kähler spaces*. Such a space is a stratified symplectic space together with a complex analytic structure which is compatible with the stratified symplectic structure; in particular each stratum is a Kähler manifold in an obvious fashion. Holomorphic quantization on a stratified Kähler space then yields a costratified Hilbert space, a quantum object having the classical singularities as its shadow. Given a Kähler manifold with a hamiltonian action of a compact Lie group that also preserves the complex structure, reduction after quantization coincides with quantization after reduction in the sense that not only the reduced and unreduced quantum phase spaces correspond but the invariant unreduced and reduced quantum observables as well.

Contents

1 Introduction

In the presence of classical phase space singularities the standard methods are insufficient to attack the problem of quantization. Ordinary Schrödinger quantization leads to a Hilbert space whose elements are classes of L^2-functions, and incorporating singularities here directly seems at present out of sight since we do not know how to handle the singularities in terms of classes of functions. However, Hilbert spaces of holomorphic functions are typically spaces whose points are ordinary functions rather than classes of functions, and we know well how we can understand singularities in terms of ordinary functions. We will show here that, in certain situations, by means of a suitable Kähler quantization procedure on *stratified Kähler spaces*, we can overcome the difficulties at the quantum level arising from classical phase space singularities. A stratified Kähler space is a stratified symplectic space endowed with a complex analytic structure which is compatible with the stratified symplectic structure; in particular each stratum is a Kähler manifold in an obvious fashion. Kähler quantization then yields a Hilbert space whose points are holomorphic functions (or more generally holomorphic sections of a holomorphic line bundle); the resulting quantum Hilbert space actually acquires more structure which, in turn, has the classical singularities as its shadow, as we will explain shortly.

Examples of stratified Kähler spaces abound: Symplectic reduction, applied to Kähler manifolds, yields a particular class of examples; this includes adjoint and generalized adjoint quotients of complex semisimple Lie groups which, in turn, underly certain lattice gauge theories. Other examples come from certain moduli spaces of holomorphic vector bundles on a Riemann surface and variants thereof; in physics language, these are spaces of conformal blocks. Still other examples arise from the closures of holomorphic nilpotent orbits. Symplectic reduction carries a Kähler manifold to a stratified Kähler space in such a way that the sheaf of germs of polarized functions coincides with the ordinary sheaf of germs of holomorphic functions. Projectivization of the closures of holomorphic nilpotent orbits yields exotic stratified Kähler structures on complex projective spaces and on certain complex projective varieties including complex projective quadrics. Other physical examples are reduced spaces relative to a constant value of angular momentum.

In the presence of singularities, a naive approach to quantization might consist in restriction of the quantization problem to a smooth open dense part, the "top stratum". However this naive procedure may lead to a loss of information and in fact to inconsistent results. To explore the potential impact of classical phase space singularities on quantum problems, we developed the notion of *costratified Hilbert space*. This is the appropriate quantum state space over a stratified space; a costratified Hilbert space consists of a system of Hilbert spaces, one for each stratum which arises from quantization on the closure of that stratum, the stratification is reflected in certain bounded linear operators between these Hilbert spaces reversing the partial ordering among the strata, and the linear operators are compatible with the quantizations. The notion of costratified Hilbert space is, perhaps, the *quantum structure having the classical singularities as its shadow*. Within the framework of holomorphic quantization, a suitable quantization procedure on stratified Kähler spaces leads to costratified Hilbert spaces. Given a Kähler manifold with a hamiltonian action of a compact Lie group that also preserves the complex structure, reduction after quantization then coincides with quantization after reduction in the sense that not only the reduced and unreduced quantum phase spaces correspond but the invariant unreduced and reduced quantum observables as well.

We illustrate the approach with a certain concrete model: In a particular case, we describe a quantum (lattice) gauge theory which incorporates certain classical singularities. The reduced phase space is a stratified Kähler space; we make explicit the requisite singular holomorphic quantization procedure and spell out the resulting costratified Hilbert space. In particular, certain tunneling probabilities between the strata emerge, we will explain how the energy eigenstates can be determined, and we will explore the corresponding expectation values of the orthoprojectors onto the subspaces associated with the strata in the strong and weak coupling approximations.

The physics described in the present lecture notes was worked out in research collaboration with my physics friends G. Rudoph and M. Schmidt [28], [29]. I am much indebted to them for having taught me the relevant physics.

2 Physical systems with classical phase space singularities

2.1 An example of a classical phase space singularity

In $\mathbb{R}^3$ with coordinates x, y, r, consider the semicone N given by the equation $x^2 + y^2 = r^2$ and the inequality $r \geq 0$. We refer to this semicone as the *exotic* plane with a single vertex. The semicone N is the classical reduced *phase space* of a single particle moving in ordinary affine space of dimension ≥ 2 with angular momentum zero. In Section 7 below we will actually justify this claim. The reduced Poisson algebra $(C^\infty N, \{\,\cdot\,,\,\cdot\,\})$ may be described in the following fashion: Let x and y be the ordinary coordinate functions in the plane, and consider the algebra $C^\infty N$ of smooth functions in the variables x, y, r subject to the relation $x^2 + y^2 = r^2$. Define the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on this algebra by

$$\{x, y\} = 2r,\ \{x, r\} = 2y,\ \{y, r\} = -2x,$$

and endow N with the complex structure having $z = x + iy$ as holomorphic coordinate. The Poisson bracket is then *defined at the vertex* as well, away from the vertex the Poisson structure is an ordinary *symplectic* Poisson structure, and the complex structure does *not* "see" the vertex. At the vertex, the radius function r is *not* a smooth function of the variables x and y. Thus the vertex is a singular point for the Poisson structure whereas it is *not* a singular point for the complex analytic structure. The Poisson and complex analytic structure combine to a "stratified Kähler structure". Below we will explain what this means.

2.2 Lattice gauge theory

Let K be a compact Lie group, let $\mathfrak{k}$ denote its Lie algebra, and let $K^{\mathbb{C}}$ be the complexification of K. Endow $\mathfrak{k}$ with an invariant inner product. The polar decomposition of the complex group $K^{\mathbb{C}}$ and the inner product on $\mathfrak{k}$ induce a diffeomorphism

$$\mathrm{T}^*K \cong \mathrm{T}K \longrightarrow K \times \mathfrak{k} \longrightarrow K^{\mathbb{C}} \qquad (2.1)$$

in such a way that the complex structure on $K^{\mathbb{C}}$ and the cotangent bundle symplectic structure on T^*K combine to K-bi-invariant Kähler structure. When we then build a lattice gauge theory from a configuration space Q which is the product $Q = K^\ell$ of ℓ copies of K, we arrive at the (unreduced) momentum phase space

$$\mathrm{T}^*Q = \mathrm{T}^*K^\ell \cong (K^{\mathbb{C}})^\ell,$$

and reduction modulo the K-symmetry given by conjugation leads to a reduced phase space of the kind

$$\mathrm{T}^*K^\ell\big/\!\big/K \cong (K^{\mathbb{C}})^\ell\big/\!\big/K^{\mathbb{C}}$$

which necessarily involves singularities in a sense to be made precise, however. Here $\mathrm{T}^*K^\ell\big/\!\big/K$ denotes the symplectic quotient whereas $(K^{\mathbb{C}})^\ell\big/\!\big/K^{\mathbb{C}}$ refers to the complex algebraic quotient (geometric invariant theory quotient). The special case $\ell = 1$, that of a single spatial plaquette—a quotient of the kind $K^{\mathbb{C}}\big/\!\big/K^{\mathbb{C}}$ is referred to in the literature as an *adjoint quotient*—, is mathematically already very attractive and presents a host of problems which we have elaborated upon in [28]. To explain how in this particular case the structure of the reduced phase space can be unravelled, following [28], we proceed as follows:

Pick a maximal torus T of K, denote the rank of T by r, and let W be the Weyl group of T in K. Then, as a space, T^*T is diffeomorphic to the complexification $T^{\mathbb{C}}$ of the torus T and $T^{\mathbb{C}}$, in turn, amounts to a product $(\mathbb{C}^*)^r$ of r copies of the space $\mathbb{C}^*$ of non-zero complex numbers. Moreover, the reduced phase space $\mathcal{P}$ comes down to the space $\mathrm{T}^*T\big/W \cong (\mathbb{C}^*)^r\big/W$ of W-orbits in $(\mathbb{C}^*)^r$ relative to the action of the Weyl group W.

Viewed as the orbit space $\mathrm{T}^*T\big/W$, via singular Marsden-Weinstein reduction, the reduced phase space $\mathcal{P}$ inherits a stratified symplectic structure. That is to say: (i) The algebra $C^\infty(T^{\mathbb{C}})^W$ of ordinary smooth W-invariant functions on $T^{\mathbb{C}}$ inherits a Poisson bracket and thus furnishes a Poisson algebra of continuous functions on $\mathcal{P}$; (ii) for each stratum, the Poisson structure yields an ordinary symplectic Poisson structure on that stratum; and (iii) the restriction mapping from $C^\infty(T^{\mathbb{C}})^W$ to the algebra of ordinary smooth functions on that stratum is a Poisson map.

Viewed as the orbit space $T^{\mathbb{C}}\big/W$, the reduced phase space $\mathcal{P}$ acquires a complex analytic structure. The complex analytic structure and the Poisson structure combine to a *stratified Kähler structure* on $\mathcal{P}$ [20], [24], [25]. The precise meaning of the term "stratified Kähler structure" is that the Poisson structure satisfies (ii) and (iii) above and that the Poisson and complex structures satisfy the additional compatibility condition that, for each stratum, necessarily a complex manifold, the symplectic and complex structures on that stratum combine to an ordinary Kähler structure.

In Section 12 below we will discuss a model that originates, in the hamiltonian approach, from lattice gauge theory with respect to the group K. The (classical unreduced) Hamiltonian $H\colon \mathrm{T}^*K \to \mathbb{R}$ of this model is given by

$$H(x, Y) = -\frac{1}{2}|Y|^2 + \frac{\nu}{2}\left(3 - \operatorname{Re}\operatorname{tr}(x)\right),\ x \in K,\ Y \in \mathfrak{k}\,. \tag{2.2}$$

Here $\nu = 1/g^2$, where g is the coupling constant, the notation $|\cdot|$ refers to the norm defined by the inner product on $\mathfrak{k}$, and the trace refers to some representation of K; below we will suppose K to be realized as a closed subgroup of some unitary group. Moreover, the lattice spacing is here set equal to 1. The Hamiltonian H is manifestly gauge invariant.

2.3 The canoe

We will now explore the following special case:

$$K = \mathrm{SU}(2),\ K^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C}),\ W \cong \mathbb{Z}/2.$$

A maximal torus T in SU(2) is simply a copy of the circle group S^1, the space $\mathrm{T}^*T \cong T^{\mathbb{C}}$ is a copy of the space $\mathbb{C}^*$ of non-zero complex numbers, and the W-invariant holomorphic map

$$f\colon \mathbb{C}^* \longrightarrow \mathbb{C},\ f(z) = z + z^{-1} \tag{2.3}$$

induces a complex analytic isomorphism $\mathcal{P} \longrightarrow \mathbb{C}$ from the reduced space

$$\mathcal{P} = \mathrm{T}^*K//K \cong \mathrm{T}^*T/W \cong \mathbb{C}^*/W$$

onto a single copy $\mathbb{C}$ of the complex line.

Remark. More generally, for $K = \mathrm{SU}(n)$, complex analytically, $\mathrm{T}^*K//K$ comes down to $(n-1)$-dimensional complex affine space $\mathbb{C}^{n-1}$. Indeed, $K^{\mathbb{C}} = \mathrm{SL}(n,\mathbb{C})$, having $(\mathbb{C}^*)^{n-1}$ as a maximal complex torus. Realize this torus as the subspace of $(\mathbb{C}^*)^n$ which consists of all $(z_1,\dots,z_n)$ such that $z_1\dots z_n = 1$. Then the elementary symmetric functions $\sigma_1,\dots,\sigma_{n-1}$ yield the map

$$(\sigma_1,\dots,\sigma_{n-1})\colon (\mathbb{C}^*)^{n-1} \longrightarrow \mathbb{C}^{n-1},$$
$$\mathbf{z} = (z_1,\dots,z_n) \longmapsto (\sigma_1(\mathbf{z}),\dots,\sigma_{n-1}(\mathbf{z}))$$

which, in turn, induces the complex analytic isomorphism

$$\mathrm{SL}(n,\mathbb{C})//\mathrm{SL}(n,\mathbb{C}) \cong (\mathbb{C}^*)^{n-1}/W \cong \mathbb{C}^{n-1}$$

from the quotient onto a copy of $\mathbb{C}^{n-1}$. We note that, more generally, when K is a general connected and simply connected Lie group of rank r (say), in view of an observation of Steinberg's [44], the fundamental characters $\chi_1,\dots,\chi_r$ of $K^{\mathbb{C}}$ furnish a map from $K^{\mathbb{C}}$ onto r-dimensional complex affine space $\mathbb{A}^r$ which identifies the complex adjoint quotient $K^{\mathbb{C}}//K^{\mathbb{C}}$ with $\mathbb{A}^r$. As a *stratified Kähler space*, the quotient has considerably more structure, though. We explain this in the sequel for the special case under consideration.

Thus we return to the special case $K = \mathrm{SU}(2)$: In view of the realization of the complex analytic structure via the holomorphic map $f\colon \mathbb{C}^* \to \mathbb{C}$ given by $f(z) = z + z^{-1}$ spelled out above, complex analytically, the quotient $\mathcal{P}$ is just a copy $\mathbb{C}$ of the complex line, and we will take $Z = z + z^{-1}$ as a holomorphic coordinate on the quotient. On the other hand, in terms of the notation

$$z = x + iy,\ Z = X + iY,\ r^2 = x^2 + y^2,$$
$$X = x + \frac{x}{r^2},\ Y = y - \frac{y}{r^2},\ \tau = \frac{y^2}{r^2},$$

the real structure admits the following description: In the case at hand, the algebra written above as $C^\infty(T^{\mathbb{C}})^W$ comes down the algebra $C^\infty(\mathcal{P})$ of continuous functions on $\mathcal{P} \cong \mathbb{C}$ which are smooth functions in three variables (say) X, Y, τ, subject to certain relations; the notation $C^\infty(\mathcal{P})$ is common for such an algebra of continuous functions even though the elements of this algebra are not necessarily ordinary smooth functions. To explain the precise structure of the algebra $C^\infty(\mathcal{P})$, consider ordinary real 3-space with coordinates X, Y, τ and, in this 3-space, let C be the real semi-algebraic set given by

$$Y^2 = (X^2 + Y^2 + 4(\tau - 1))\tau, \quad \tau \geq 0.$$

As a space, C can be identified with $\mathcal{P}$. Further, a real analytic change of coordinates, spelled out in Section 7 of [25], actually identifies C with the familiar *canoe*. The algebra $C^\infty(\mathcal{P})$ is that of Whitney-smooth functions on C, that is, continuous functions on C that are restrictions of smooth functions in the variables X, Y, τ or, equivalently, smooth functions in the variables X, Y, τ, where two functions are identified whenever they coincide on C. The Poisson brackets on $C^\infty(\mathcal{P})$ are determined by the formulas

$$\begin{aligned}\{X, Y\} &= X^2 + Y^2 + 4(2\tau - 1),\\ \{X, \tau\} &= 2(1 - \tau)Y,\\ \{Y, \tau\} &= 2\tau X.\end{aligned}$$

On the subalgebra of $C^\infty(\mathcal{P})$ which consists of real polynomial functions in the variables X, Y, τ, the relation

$$Y^2 = (X^2 + Y^2 + 4(\tau - 1))\tau$$

is defining. The resulting *stratified Kähler* structure on $\mathcal{P} \cong \mathbb{C}$ is *singular* at $-2 \in \mathbb{C}$ and $2 \in \mathbb{C}$, that is, the Poisson structure *vanishes* at either of these two points. Further, at $-2 \in \mathbb{C}$ and $2 \in \mathbb{C}$, the function τ is *not* an ordinary smooth function of the variables X and Y, viz.

$$\tau = \frac{1}{2}\sqrt{Y^2 + \frac{(X^2 + Y^2 - 4)^2}{16}} - \frac{X^2 + Y^2 - 4}{8},$$

whereas away from $-2 \in \mathbb{C}$ and $2 \in \mathbb{C}$, the Poisson structure is an ordinary symplectic Poisson structure. This makes explicit, in the case at hand, the singular character of the reduced space $\mathcal{P}$ as a stratified Kähler space which, as a complex analytic space, is just a copy of $\mathbb{C}$, though and, as such, has *no* singularities, i. e. is an ordinary complex manifold.

For later reference, we will now describe the stratification of the reduced configuration space $\mathcal{X} \cong T/W$ and that of the reduced phase space $\mathcal{P} \cong (T \times \mathfrak{t})/W$. The stratifications we will use arise from the W-orbit type decompositions: We will not make precise the notion of stratification and that of stratified space, see e. g. [10].

The torus T amounts to the complex unit circle and its Lie algebra $\mathfrak{t}$ to the imaginary axis. The Weyl group $W = S_2$ acts on T by complex conjugation and on $\mathfrak{t}$ by reflection. Hence the reduced configuration space $\mathcal{X} \cong T/W$ is homeomorphic to the closed interval $[-\mathbf{1}, \mathbf{1}]$ and the reduced phase space $\mathcal{P} \cong (T \times \mathfrak{t})/W$ to the well-known canoe, see Figure 1.

Let

$$\mathcal{X}_+ = \{\mathbf{1}\},\ \mathcal{X}_- = \{-\mathbf{1}\},\ \mathcal{X}_0 = \mathcal{X}_+ \cup \mathcal{X}_- = \{-\mathbf{1}, \mathbf{1}\},\ \mathcal{X}_1 =]-\mathbf{1}, \mathbf{1}[$$

so that the orbit type decomposition of $\mathcal{X}$ relative to the W-action has the form $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_0$. The "piece" $\mathcal{X}_1$ (the open interval) is connected; it is the "top" stratum, the open, connected and dense stratum. In particular, the restriction to the pre-image of $\mathcal{X}_1$ of the orbit projection is a W-covering projection. The lower stratum $\mathcal{X}_0$ decomposes into the two connected components $\mathcal{X}_+$ and $\mathcal{X}_-$; the single point in $\mathcal{X}_+$ arises from a fixed point of the W-action, and the same is true of $\mathcal{X}_-$. Likewise the orbit type decomposition of $\mathcal{P}$ relative to the W-action has the form $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_0$. Here the "piece" $\mathcal{P}_1$ is the "top" stratum, i. e. the open, connected and dense stratum which is here 2-dimensional. As before, the restriction to the pre-image of $\mathcal{P}_1$ of the orbit projection is a W-covering projection. Further, $\mathcal{P}_0$ decomposes into two connected components $\mathcal{P}_0 = \mathcal{P}_+ \cup \mathcal{P}_-$, each containing a vertex of the canoe; each such vertex arises from a fixed point of the W-action. Under the identification of $\mathcal{P}$ with the complex line $\mathbb{C}$ described previously, the two vertices of the canoe correspond to the points 2 and -2 of $\mathbb{C}$ so that

$$\mathcal{P}_+ = \{2\} \subseteq \mathbb{C},\ \mathcal{P}_- = \{-2\} \subseteq \mathbb{C},\ \mathcal{P}_1 = \mathbb{C} \setminus \mathcal{P}_0 = \mathbb{C} \setminus \{2, -2\}.$$

A closer look reveals that we can see that decomposition of $\mathbb{C}$ as arising from hyperbolic cosine, viewed as a holomorphic function: The two points 2 and -2 are the focal points of the corresponding families of ellipses and hyperbolas in $\mathbb{C}$. Two of these ellipses and two of these hyperbolas are in fact indicated in Figure 1. We will come back to the stratifications in Sections 10 and 13 below.

Remark 2.1. In the case under discussion ($K = \mathrm{SU}(2)$), as a stratified symplectic space, $\mathcal{P}$ is isomorphic to the reduced phase space of a spherical pendulum, reduced at vertical angular momentum 0 (whence the pendulum is constrained to move in a plane), see [8].

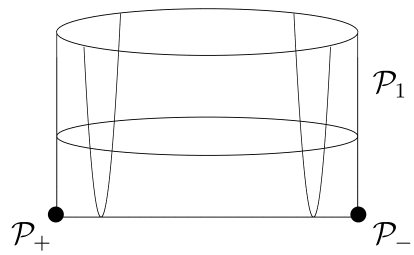


Figure 1: The reduced phase space $\mathcal{P}$ for $K = \mathrm{SU}(2)$.

3 Stratified Kähler spaces

In the presence of singularities, restricting quantization to a smooth open dense stratum, sometimes referred to as "top stratum", can result in a loss of information and may in fact lead to *inconsistent* results. To develop a satisfactory notion of Kähler quantization in the presence of singularities, on the classical level, we isolated a notion of "Kähler space with singularities"; we refer to such a space as a *stratified Kähler space*. Ordinary *Kähler quantization* may then be extended to a *quantization scheme over stratified Kähler spaces*.

We will now explain the concept of a *stratified Kähler space*. In [20] we introduced a general notion of stratified Kähler space and that of complex analytic stratified Kähler space as a special case. We do not know whether the two notions really differ. For the present paper, the notion of complex analytic stratified Kähler space suffices. To simplify the terminology somewhat, "stratified Kähler space" will always mean "complex analytic stratified Kähler space".

We recall first that, given a stratified space N, a *stratified symplectic structure on* N is a Poisson algebra $(C^\infty N, \{\,\cdot\,,\,\cdot\,\})$ of continuous functions on N which, on each stratum, amounts to an ordinary smooth symplectic Poisson algebra. The functions in $C^\infty N$ are not necessarily ordinary smooth functions. Restriction of the functions in $C^\infty N$ to a stratum is required to yield the compactly supported functions on that stratum, and these suffice to generate a symplectic Poisson algebra on the stratum.

Next we recall that a *complex analytic space* (in the sense of GRAUERT) is a topological space X, together with a sheaf of rings $\mathcal{O}_X$, having the following property: The space X can be covered by open sets Y, each of which embeds into the polydisc U in some $\mathbb{C}^n$ (the number n may vary as U varies) as the zero set of a finite system of holomorphic functions $f_1, \ldots, f_q$ defined on U, such that the restriction $\mathcal{O}_Y$ of the sheaf $\mathcal{O}_X$ to Y is isomorphic as a sheaf to the quotient sheaf $\mathcal{O}_U\big/(f_1, \ldots, f_q)$; here $\mathcal{O}_U$ is the sheaf of germs of holomorphic functions on U. The sheaf $\mathcal{O}_X$ is then referred to as the *sheaf of holomorphic functions on* X. See [11] for a development of the general theory of complex analytic spaces.

Definition 3.1. A *stratified Kähler space* consists of a complex analytic space N, together with
(i) a complex analytic stratification (a not necessarily proper refinement of the standard complex analytic stratification), and with
(ii) a stratified symplectic structure $(C^\infty N, \{\,\cdot\,,\,\cdot\,\})$ which is compatible with the complex analytic structure

The two structures being *compatible* means the following:
(i) For each point q of N and each holomorphic function f defined on an open neighborhood U of q, there is an open neighborhood V of q with $V \subset U$ such that, on V, f is the restriction of a function in $C^\infty(N)$;
(ii) on each stratum, the symplectic structure determined by the symplectic Poisson structure (on that stratum) combines with the complex analytic structure to a Kähler structure.

Example 1: The *exotic plane*, endowed with the structure explained in Subection 2.1 above, is a stratified Kähler space. Here the radius function r is *not* an ordinary smooth function of the variables x and y. Thus the stratified symplectic structure cannot be given in terms of ordinary smooth functions of the variables x and y.

This example generalizes to an entire class of examples: The *closure of a holomorphic nilpotent orbit* (in a hermitian Lie algebra) inherits a stratified Kähler structure [20]. Angular momentum zero reduced spaces are special cases thereof; see Section 7 below for details.

Projectivization of the closure of a holomorphic nilpotent orbit yields what we call an *exotic projective variety*. This includes complex quadrics, Severi and Scorza varieties and their *secant* varieties [20], [22]. In physics, spaces of this kind arise as reduced classical phase spaces for systems of harmonic oscillators with zero angular momentum and constant energy. We shall explain some of the details in Section 7 below.

Example 2: Moduli spaces of semistable holomorphic vector bundles or, more generally, moduli spaces of semistable principal bundles on a non-singular complex projective curve carry stratified Kähler structures [20]. These spaces arise as moduli spaces of homomorphisms or more generally twisted homomorphisms from fundamental groups of surfaces to compact connected Lie groups as well. In conformal field theory, they occur as spaces of *conformal blocks*. The construction of the moduli spaces as complex projective varieties goes back to [37] and [42]; see [43] for an exposition of the general theory. Atiyah and Bott [6] initiated another approach to the study of these moduli spaces by identifying them with moduli spaces of projectively flat constant central curvature connections on principal bundles over Riemann surfaces, which they analyzed by methods of gauge theory. In particular, by applying the method of symplectic reduction to the action of the infinite-dimensional group of gauge transformations on the infinite-dimensional symplectic manifold of all connections on a principal bundle, they showed that an invariant inner product on the Lie algebra of the Lie group in question induces a natural symplectic structure on a certain smooth open stratum which, together with the complex analytic structure, turns that stratum into an ordinary Kähler manifold. This infinite-dimensional approach to moduli spaces has roots in quantum field theory. Thereafter a finite-dimensional construction of the moduli space as a symplectic quotient arising from an ordinary finite-dimensional Hamiltonian G-space for a compact Lie group G was developed; see [17], [18] and the literature there; this construction exhibits the moduli space as a stratified symplectic space. The stratified Kähler structure mentioned above combines the complex analytic structure with the stratified symplectic structure; it includes the Kähler manifold structure on the open and dense stratum.

An important special case is that of the moduli space of semistable rank 2 degree zero vector bundles with trivial determinant on a curve of genus 2. As a space, this is just ordinary complex projective 3-space, but the stratified symplectic structure involves more functions than just ordinary smooth functions. The complement of the space of stable vector bundles is a *Kummer surface*. See [16], [18] and the literature there.

Any ordinary Kähler manifold is plainly a stratified Kähler space. This kind of example generalizes in the following fashion: For a Lie group K, we will denote its Lie algebra by $\mathfrak{k}$ and the dual thereof by $\mathfrak{k}^*$. The next result says that, roughly speaking, Kähler reduction, applied to an ordinary Kähler manifold, yields a stratified Kähler structure on the reduced space.

Theorem 3.2 ([20]). *Let N be a Kähler manifold, acted upon holomorphically by a complex Lie group G such that the action, restricted to a compact real form K of G, preserves the Kähler structure and is hamiltonian, with momentum mapping $\mu\colon N \to \mathfrak{k}^*$. Then the reduced space $N_0 = \mu^{-1}(0)\big/K$ inherits a stratified Kähler structure.*

For intelligibility, we explain briefly how the structure on the reduced space N_0 arises. Details may be found in [20]: Define $C^\infty(N_0)$ to be the quotient algebra $C^\infty(N)^K\big/I^K$, that is, the algebra $C^\infty(N)^K$ of smooth K-invariant functions on N, modulo the ideal I^K of functions in $C^\infty(N)^K$ that vanish on the zero locus $\mu^{-1}(0)$. The ordinary smooth symplectic Poisson structure $\{\,\cdot\,,\,\cdot\,\}$ on $C^\infty(N)$ is K-invariant and hence induces a Poisson structure on the algebra $C^\infty(N)^K$ of smooth K-invariant functions on N. Furthermore, Noether's theorem entails that the ideal I^K is a Poisson ideal, that is to say, given $f \in C^\infty(N_0)^K$ and $h \in I^K$, the function $\{f, h\}$ is in I^K as well. Consequently the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ descends to a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_0$ on $C^\infty(N_0)$. Relative to the orbit type stratification, the Poisson algebra $(C^\infty N_0, \{\,\cdot\,,\,\cdot\,\}_0)$ turns N_0 into a stratified symplectic space.

The inclusion of $\mu^{-1}(0)$ into N passes to a homeomorphism from N_0 onto the categorical G-quotient $N\big/\!\big/G$ of N in the category of complex analytic varieties. The stratified symplectic structure combines with the complex analytic structure on $N\big/\!\big/G$ to a stratified Kähler structure. When N is complex algebraic, the complex algebraic G-quotient coincides with the complex analytic G-quotient.

Thus, in view of Theorem 3.2, examples of stratified Kähler spaces abound.

Example 3: Adjoint quotients of complex reductive Lie groups, see (2.2) above.

Remark 3.3. In [6], Atiyah and Bott raised the issue of *determining the singularities* of moduli spaces of semistable holomorphic vector bundles or, more generally, of moduli spaces of semistable principal bundles on a non-singular complex projective curve. The stratified Kähler structure which we isolated on a moduli space of this kind, as explained in Example 2 above, actually determines the singularity structure; in particular, near any point, the structure may be understood in terms of a suitable local model. The appropriate notion of singularity is that of singularity in the sense of stratified Kähler spaces; this notion depends on the entire structure, not just on the complex analytic structure. Indeed, the examples spelled out above (the exotic plane with a single vertex, the exotic plane with two vertices, the 3-dimensional complex projective space with the Kummer surface as singular locus, etc.) show that a point of a stratified Kähler space may well be a singular point without being a complex analytic singularity.

4 Quantum theory and classical singularities

According to Dirac, the *correspondence* between a classical theory and its quantum counterpart should be based on an analogy between their mathematical structures. An interesting issue is then that of the role of singularities in quantum problems. Singularities are known to arise in classical phase spaces. For example, in the hamiltonian picture of a theory, reduction modulo gauge symmetries leads in general to singularities on the classical level. This leads to the question what the significance of singularities on the quantum side might be. Can we ignore them, or is there a quantum structure which has the classical singularities as its shadow? As far as know, one of the first papers in this topic is that of Emmrich and Römer [9]. This paper indicates that wave functions may "congregate" near a *singular* point, which goes counter to the sometimes quoted statement that *singular points in a quantum problem are a set of measure zero so cannot possibly be important.* In a similar vein, Asorey et al observed that vacuum nodes correspond to the chiral gauge orbits of reducible gauge fields with non-trivial magnetic monopole components [4]. It is also noteworthy that in classical mechanics and in classical field theories singularities in the solution spaces are the *rule rather than the exception*. This is in particular true for Yang-Mills theories and for Einstein's gravitational theory where singularities occur even at some of the most interesting and physically relevant solutions, namely at the symmetric ones. It is still not understood what role these singularities might have in quantum gravity. See, for example, Arms, Marsden and Moncrief [2], [3] and the literature there.

5 Correspondence principle and Lie-Rinehart algebras

To make sense of the *correspondence principle* in certain *singular* situations, one needs a tool which, for the stratified symplectic Poisson algebra on a stratified symplectic space, serves as a *replacement* for the tangent bundle of a smooth symplectic manifold. This replacement is provided by an appropriate *Lie-Rinehart algebra*. This Lie-Rinehart algebra yields in particular a satisfactory generalization of the Lie algebra of smooth vector fields in the smooth case. This enables us to put *flesh on the bones of Dirac's correspondence principle in certain singular situations*.

A *Lie-Rinehart algebra* consists of a commutative algebra and a Lie algebra with additional structure which generalizes the mutual structure of interaction between the algebra of smooth functions and the Lie algebra of smooth vector fields on a smooth manifold. More precisely:

Definition 5.1. A *Lie-Rinehart* algebra consists of a commutative algebra A and a Lie-algebra L such that L acts on A by derivations and that L has an A-module structure, and these are required to satisfy

$$\begin{aligned}[][\alpha, a\beta] &= \alpha(a)\beta + a[\alpha, \beta],\\ (a\alpha)(b) &= a(\alpha(b)),\end{aligned}$$

where $a, b \in A$ and $\alpha, \beta \in L$.

Definition 5.2. An A-module M which is also a left L-module is called a *left* (A, L)-module provided

$$\alpha(ax) = \alpha(a)x + a\alpha(x) \tag{5.1}$$

$$(a\alpha)(x) = a(\alpha(x)) \tag{5.2}$$

where $a \in A,\ x \in M,\ \alpha \in L$.

We will now explain briefly the Lie-Rinehart algebra associated with a Poisson algebra; more details may be found in [14], [15], and [23]. Thus, let $(A, \{\,\cdot\,,\,\cdot\,\})$ be a Poisson algebra. Let D_A the the A-module of formal differentials of A the elements of which we write as du, for $u \in A$. For $u, v \in A$, the association

$$(du, dv) \longrightarrow \pi(du, dv) = \{u, v\}$$

yields an A-valued A-bilinear skew-symmetric 2-form $\pi = \pi_{\{\cdot,\cdot\}}$ on D_A, referred to as the *Poisson* 2-*form* associated with the Poisson structure $\{\,\cdot\,,\,\cdot\,\}$. The adjoint

$$\pi^\sharp \colon D_A \longrightarrow \mathrm{Der}(A) = \mathrm{Hom}_A(D_A, A)$$

of π is a morphism of A-modules, and the formula

$$[adu, bdv] = a\{u,b\}dv + b\{a,v\}du + abd\{u,v\}$$

yields a Lie bracket $[\,\cdot\,,\,\cdot\,]$ on D_A.

Theorem 5.3 ([14])**.** *The A-module structure on D_A, the bracket $[\cdot,\cdot]$, and the morphism $\pi^\sharp$ of A-modules turn the pair (A, D_A) into a Lie-Rinehart algebra.*

We will write the resulting Lie-Rinehart algebra as $(A, D_{\{\cdot,\cdot\}})$. For intelligibility we recall that, given a Lie-Rinehart algebra (A, L), the Lie algebra L together with the additional A-module structure on L and L-module structure on A is referred to as an $(\mathbb{R}, A)$-*Lie algebra*. Thus $D_{\{\cdot,\cdot\}}$ is an $(\mathbb{R}, A)$-Lie algebra.

When the Poisson algebra A is the algebra of smooth functions $C^\infty(M)$ on a symplectic manifold M, the A-dual $\mathrm{Der}(A) = \mathrm{Hom}_A(D_A, A)$ of D_A amounts to the A-module $\mathrm{Vect}(M)$ of smooth vector fields, and

$$(\pi^\sharp, \mathrm{Id})\colon (D_A, A) \longrightarrow (\mathrm{Vect}(M), C^\infty(M)) \tag{5.3}$$

is a morphism of Lie-Rinehart algebras, where $(\mathrm{Vect}(M), C^\infty(M))$ carries its ordinary Lie-Rinehart structure. The A-module morphism $\pi^\sharp$ is plainly surjective, and the kernel consists of those formal differentials which "vanish at each point of" M.

We return to our general Poisson algebra $(A, \{\,\cdot\,,\,\cdot\,\})$. The Poisson 2-form $\pi_{\{\cdot,\cdot\}}$ determines an *extension*

$$0 \longrightarrow A \longrightarrow \overline{L}_{\{\cdot,\cdot\}} \longrightarrow D_{\{\cdot,\cdot\}} \longrightarrow 0 \tag{5.4}$$

of $(\mathbb{R}, A)$-Lie algebras which is central as an extension of ordinary Lie algebras; in particular, on the kernel A, the Lie bracket is trivial. Moreover, as A-modules,

$$\overline{L}_{\{\cdot,\cdot\}} = A \oplus D_{\{\cdot,\cdot\}}, \tag{5.5}$$

and the Lie bracket on $\overline{L}_{\{\cdot,\cdot\}}$ is given by

$$[(a, du), (b, dv)] = (\{u,b\} + \{a,v\} - \{u,v\}, d\{u,v\}), \quad a, b, u, v \in A. \tag{5.6}$$

Here we have written "$\overline{L}$" rather than simply L to indicate that the extension (5.4) represents the *negative* of the class of $\pi_{\{\cdot,\cdot\}}$ in Poisson cohomology $\mathrm{H}^2_{\mathrm{Poisson}}(A, A)$, cf. [14]. When $(A, \{\,\cdot\,,\,\cdot\,\})$ is the smooth symplectic Poisson algebra of an ordinary smooth symplectic manifold, (perhaps) up to sign, the class of $\pi_{\{\cdot,\cdot\}}$ comes essentially down to the cohomology class represented by the symplectic structure.

The following concept was introduced in [15].

Definition 5.4. Given an $(A \otimes \mathbb{C})$-module M, we refer to an $(A, \overline{L}_{\{\cdot,\cdot\}})$-module structure

$$\chi \colon \overline{L}_{\{\cdot,\cdot\}} \longrightarrow \mathrm{End}_{\mathbb{R}}(M) \tag{5.7}$$

on M as a *prequantum module structure for* $(A, \{\,\cdot\,,\,\cdot\,\})$ provided
(i) the values of χ lie in $\mathrm{End}_{\mathbb{C}}(M)$, that is to say, for $a \in A$ and $\alpha \in D_{\{\cdot,\cdot\}}$, the operators $\chi(a, \alpha)$ are complex linear transformations, and
(ii) for every $a \in A$, with reference to the decomposition (5.5), we have

$$\chi(a, 0) = i\, a\, \mathrm{Id}_M. \tag{5.8}$$

A pair (M, χ) consisting of an $(A \otimes \mathbb{C})$-module M and a prequantum module structure will henceforth be referred to as a *prequantum module* (for $(A, \{\,\cdot\,,\,\cdot\,\})$).

Prequantization now proceeds in the following fashion, cf. [14]: The assignment to $a \in A$ of $(a, da) \in \overline{L}_{\{\cdot,\cdot\}}$ yields a morphism ι of real Lie algebras from A to $\overline{L}_{\{\cdot,\cdot\}}$; thus, for any prequantum module (M, χ), the composite of ι with $-i\chi$ is a representation $a \mapsto \widehat{a}$ of the A underlying real Lie algebra having M, viewed as a complex vector space, as its representation space; this is a representation by $\mathbb{C}$-linear operators so that any constant acts by multiplication, that is, for any real number r, viewed as a member of A,

$$\widehat{r} = r\, \mathrm{Id} \tag{5.9}$$

and so that, for $a, b \in A$,

$$\widehat{\{a, b\}} = i\, [\widehat{a}, \widehat{b}] \qquad \text{(the Dirac condition)}. \tag{5.10}$$

More explicitly, these operators are given by the formula

$$\widehat{a}(x) = \frac{1}{i}\chi(0, da)(x) + ax, \quad a \in A,\ x \in M. \tag{5.11}$$

In this fashion, prequantization, that is to say, the first step in the realization of the correspondence principle in one direction, can be made precise in certain singular situations.

When $(A, \{\,\cdot\,,\,\cdot\,\})$ is the Poisson algebra of smooth functions on an ordinary smooth symplectic manifold, this prequantization factors through the morphism (5.3) of Lie-Rinehart algebras in such a way that, on the target, the construction comes down to the ordinary prequantization construction.

Remark. In the physics literature, Lie-Rinehart algebras were explored in a paper by Kastler and Stora under the name *Lie-Cartan pairs* [31].

6 Quantization on stratified Kähler spaces

In the paper [21] we have shown that the *holomorphic* quantization scheme may be extended to stratified Kähler spaces. We recall the main steps:
1) The notion of ordinary Kähler polarization generalizes to that of *stratified Kähler polarization*. This concept is defined in terms of the Lie-Rinehart algebra associated with the stratified symplectic Poisson structure; it specifies *polarizations on the strata* and, moreover, encapsulates the *mutual positions of polarizations on the strata*.
Under the circumstances of Theorem 3.2, *symplectic reduction carries a Kähler polarization preserved by the symmetries into a stratified Kähler polarization*.
2) The notion of prequantum bundle generalizes to that of *stratified prequantum module*. Given a stratified Kähler space, a stratified prequantum module is, roughly speaking, a system of prequantum modules in the sense of Definition 5.4, one for the closure of each stratum, together with appropriate morphisms among them which reflect the stratification.
3) The notion of quantum Hilbert space generalizes to that of *costratified quantum Hilbert space* in such a way that the costratified structure reflects the stratification on the classical level. *Thus the costratified Hilbert space structure is a quantum structure which has the classical singularities as its shadow.*
4) The main result says that $[Q, R] = 0$, that is, quantization commutes with reduction [21]:

Theorem 6.1. *Under the circumstances of* Theorem 3.2*, suppose that the Kähler manifold is quantizable (that is, suppose that the cohomology class of the Kähler form is integral). When a suitable additional condition is satisfied, reduction after quantization coincides with quantization after reduction in the sense that not only the reduced and unreduced quantum phase spaces correspond but the (invariant) unreduced and reduced quantum observables as well.*

What is referred to here as 'suitable additional condition' is a condition on the behaviour of the gradient flow. For example, when the Kähler manifold is compact, the condition will automatically be satisfied.

On the reduced level, the resulting classical phase space involves in general singularities and is a stratified Kähler space; the appropriate quantum phase space is then a costratified Hilbert space.

7 An illustration arising from angular momentum and holomorphic nilpotent orbits

Let s and ℓ be non-zero natural numbers. The unreduced classical momentum phase space of ℓ particles in $\mathbb{R}^s$ is real affine space of real dimension $2s\ell$. For example, for our solar system, $s=3$, and ℓ is the number of celestial bodies we take into account, that is, the sun, the planets, their moons, asteroids, etc., and the true physical phase space is the reduced space subject to the (physically reasonable) constraint that the total angular momentum of the solar system be constant and non-zero. The shifting trick reduces this case to that of total angular momentum zero relative to the planar orthogonal group. The subsequent discussion implies that the reduced phase space relative to the planar orthogonal group is the space of complex symmetric $(\ell \times \ell)$-matrices of rank at most equal to 2. The true reduced phase space we are looking for then fibers over a semisimple orbit in $\mathfrak{sp}(\ell,\mathbb{R})$ with fiber the space of complex symmetric $(\ell \times \ell)$-matrices of rank at most equal to 2. The additional requirement that the total energy be constant then reduces the system by one more degree of freedom.

We return to the general case. Identify real affine space of real dimension $2s\ell$ with the vector space $(\mathbb{R}^{2s})^{\times \ell}$ as usual, endow $\mathbb{R}^s$ with the standard inner product, $\mathbb{R}^{2\ell}$ with the standard symplectic structure, and thereafter $(\mathbb{R}^{2s})^{\times \ell}$ with the obvious induced inner product and symplectic structure. The isometry group of the inner product on $\mathbb{R}^s$ is the orthogonal group $\mathrm{O}(s,\mathbb{R})$, the group of linear transformations preserving the symplectic structure on $\mathbb{R}^{2\ell}$ is the symplectic group $\mathrm{Sp}(\ell,\mathbb{R})$, and the actions extend to linear $\mathrm{O}(s,\mathbb{R})$- and $\mathrm{Sp}(\ell,\mathbb{R})$-actions on $(\mathbb{R}^{2s})^{\times \ell}$ in an obvious manner. As usual, denote the Lie algebras of $\mathrm{O}(s,\mathbb{R})$ and $\mathrm{Sp}(\ell,\mathbb{R})$ by $\mathfrak{so}(s,\mathbb{R})$ and $\mathfrak{sp}(\ell,\mathbb{R})$, respectively.

The $\mathrm{O}(s,\mathbb{R})$- and $\mathrm{Sp}(\ell,\mathbb{R})$-actions on $(\mathbb{R}^{2s})^{\times \ell}$ are hamiltonian. To spell out the $\mathrm{O}(s,\mathbb{R})$-momentum mapping having the value zero at the origin, identify $\mathfrak{so}(s,\mathbb{R})$ with its dual $\mathfrak{so}(s,\mathbb{R})^*$ by interpreting $a \in \mathfrak{so}(s,\mathbb{R})$ as the linear functional on $\mathfrak{so}(s,\mathbb{R})$ which assigns $\mathrm{tr}(a^t x)$ to $x \in \mathfrak{so}(s,\mathbb{R})$; here ${}^t x$ refers to the transpose of the matrix x. We note that, for $s \geq 3$,

$$(s-2)\mathrm{tr}(a^t b) = -\beta(a,b), \ \ a,b \in \mathfrak{so}(s,\mathbb{R}),$$

where β is the Killing form of $\mathfrak{so}(s,\mathbb{R})$. Moreover, for a vector $\mathbf{x} \in \mathbb{R}^s$, realized as a column vector, let ${}^t\mathbf{x}$ be its transpose, so that ${}^t\mathbf{x}$ is a row vector. With these preparations out of the way, the *angular momentum mapping*

$$\mu_{\mathrm{O}} \colon (\mathbb{R}^{2s})^{\times \ell} \longrightarrow \mathfrak{so}(s,\mathbb{R})$$

with reference to the origin is given by

$$\mu_{\mathrm{O}}(\mathbf{q}_1,\mathbf{p}_1,\dots,\mathbf{q}_\ell,\mathbf{p}_\ell) = \mathbf{q}_1{}^t\mathbf{p}_1 - \mathbf{p}_1{}^t\mathbf{q}_1 + \dots + \mathbf{q}_\ell{}^t\mathbf{p}_\ell - \mathbf{p}_\ell{}^t\mathbf{q}_\ell.$$

Likewise, identify $\mathfrak{sp}(\ell,\mathbb{R})$ with its dual $\mathfrak{sp}(\ell,\mathbb{R})^*$ by interpreting $a \in \mathfrak{sp}(\ell,\mathbb{R})$ as the linear functional on $\mathfrak{sp}(\ell,\mathbb{R})$ which assigns $\frac{1}{2}\mathrm{tr}(ax)$ to $x \in \mathfrak{sp}(\ell,\mathbb{R})$; we remind the reader that the *Killing* form β of $\mathfrak{sp}(\ell,\mathbb{R})$ is given by

$$\beta(a,b) = 2(\ell+1)\mathrm{tr}(ab)$$

where $a, b \in \mathfrak{sp}(\ell, \mathbb{R})$. The $\mathrm{Sp}(\ell, \mathbb{R})$-momentum mapping

$$\mu_{\mathrm{Sp}} \colon (\mathbb{R}^{2s})^{\times \ell} \longrightarrow \mathfrak{sp}(\ell, \mathbb{R})$$

having the value zero at the origin is given by the assignment to

$$[\mathbf{q}_1, \mathbf{p}_1, \ldots, \mathbf{q}_\ell, \mathbf{p}_\ell] \in (\mathbb{R}^s \times \mathbb{R}^s)^{\times \ell}$$

of

$$\left[\begin{array}{cc} [\mathbf{q}_j \mathbf{p}_k] & -[\mathbf{q}_j \mathbf{q}_k] \\ [\mathbf{p}_j \mathbf{p}_k] & -[\mathbf{p}_j \mathbf{q}_k] \end{array}\right] \in \mathfrak{sp}(\ell, \mathbb{R}),$$

where $[\mathbf{q}_j \mathbf{p}_k]$ etc. denotes the $(\ell \times \ell)$-matrix having the inner products $\mathbf{q}_j \mathbf{p}_k$ etc. as entries.

Consider the $\mathrm{O}(s, \mathbb{R})$-reduced space

$$N_0 = \mu_{\mathrm{O}}^{-1}(0) \big/ \mathrm{O}(s, \mathbb{R}).$$

The $\mathrm{Sp}(\ell, \mathbb{R})$-momentum mapping induces an embedding of the reduced space N_0 into $\mathfrak{sp}(\ell, \mathbb{R})$. We now explain briefly how the image of N_0 in $\mathfrak{sp}(\ell, \mathbb{R})$ may be described. More details may be found in [20], see also [22].

Choose a positive complex structure J on $\mathbb{R}^{2\ell}$ which is compatible with ω in the sense that $\omega(J\mathbf{u}, J\mathbf{v}) = \omega(\mathbf{u}, \mathbf{v})$ for every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{2\ell}$; here 'positive' means that the associated real inner product $\cdot$ on $\mathbb{R}^{2\ell}$ given by $\mathbf{u} \cdot \mathbf{v} = \omega(\mathbf{u}, J\mathbf{v})$ for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{2\ell}$ is positive definite. The subgroup of $\mathrm{Sp}(\ell, \mathbb{R})$ which preserves the complex structure J is a maximal compact subgroup of $\mathrm{Sp}(\ell, \mathbb{R})$; relative to a suitable orthonormal basis, this group comes down to a copy of the ordinary unitary group $\mathrm{U}(\ell)$. Furthermore, the complex structure J induces a Cartan decomposition

$$\mathfrak{sp}(\ell, \mathbb{R}) = \mathfrak{u}(\ell) \oplus \mathfrak{p}; \tag{7.1}$$

here $\mathfrak{u}(\ell)$ is the Lie algebra of $\mathrm{U}(\ell)$, the symmetric constituent $\mathfrak{p}$ decomposes as the direct sum

$$\mathfrak{p} \cong \mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^\ell] \oplus \mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^\ell]$$

of two copies of the real vector space $\mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^\ell]$ of real symmetric $(\ell \times \ell)$-matrices, and the complex structure J induces a complex structure on $\mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^\ell] \oplus \mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^\ell]$ in such a way that the resulting complex vector space is complex linearly isomorphic to the complex vector space $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$ of complex symmetric $(\ell \times \ell)$-matrices in a canonical fashion. We refer to a nilpotent orbit $\mathcal{O}$ in $\mathfrak{sp}(\ell, \mathbb{R})$ as being *holomorphic* if the orthogonal projection from $\mathfrak{sp}(\ell, \mathbb{R})$ to $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$, restricted to $\mathcal{O}$, is a diffeomorphism from $\mathcal{O}$ onto its image in $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$. The diffeomorphism from a holomorphic nilpotent orbit $\mathcal{O}$ onto its image in $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$ extends to a homeomorphism from the closure $\overline{\mathcal{O}}$ onto its image in $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$, and the closures of the holomorphic nilpotent orbits constitute an ascending sequence

$$0 \subseteq \overline{\mathcal{O}}_1 \subseteq \cdots \subseteq \overline{\mathcal{O}}_k \subseteq \cdots \subseteq \overline{\mathcal{O}}_\ell \subseteq \mathfrak{sp}(\ell, \mathbb{R}),\ 1 \leq k \leq \ell, \tag{7.2}$$

such that the orthogonal projection from $\mathfrak{sp}(\ell,\mathbb{R})$ to $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$, restricted to $\overline{\mathcal{O}}_\ell$, is a homeomorphism from $\overline{\mathcal{O}}_\ell$ onto $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$. For $1 \leq k \leq \ell$, this orthogonal projection, restricted to $\overline{\mathcal{O}}_k$, is a homeomorphism from $\overline{\mathcal{O}}_k$ onto the space of complex symmetric $(\ell \times \ell)$-matrices of rank at most equal to k; in particular, each space of the kind $\overline{\mathcal{O}}_k$ is a *stratified* space, the stratification being given by the rank of the corresponding complex symmetric $(\ell \times \ell)$-matrices.

The Lie bracket of the Lie algebra $\mathfrak{sp}(\ell,\mathbb{R})$ induces a Poisson bracket on the algebra $C^\infty(\mathfrak{sp}(\ell,\mathbb{R})^*)$ of smooth functions on the dual $\mathfrak{sp}(\ell,\mathbb{R})^*$ of $\mathfrak{sp}(\ell,\mathbb{R})$ in a canonical fashion. Via the identification of $\mathfrak{sp}(\ell,\mathbb{R})$ with its dual, the Lie bracket on $\mathfrak{sp}(\ell,\mathbb{R})$ induces a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$. Indeed, the assignment to $a \in \mathfrak{sp}(\ell,\mathbb{R})$ of the linear function

$$f_a \colon \mathfrak{sp}(\ell,\mathbb{R}) \longrightarrow \mathbb{R}$$

given by $f_a(x) = \frac{1}{2}\mathrm{tr}(ax)$ induces a linear isomorphism

$$\mathfrak{sp}(\ell,\mathbb{R}) \longrightarrow \mathfrak{sp}(\ell,\mathbb{R})^*; \tag{7.3}$$

let

$$[\,\cdot\,,\,\cdot\,]^* \colon \mathfrak{sp}(\ell,\mathbb{R})^* \otimes \mathfrak{sp}(\ell,\mathbb{R})^* \longrightarrow \mathfrak{sp}(\ell,\mathbb{R})^*$$

be the bracket on $\mathfrak{sp}(\ell,\mathbb{R})^*$ induced by the Lie bracket on $\mathfrak{sp}(\ell,\mathbb{R})$. The Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on the algebra $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$ is given by the formula

$$\{f,h\}(x) = [f'(x),h'(x)]^*(x),\ x \in \mathfrak{sp}(\ell,\mathbb{R}).$$

The isomorphism (7.3) induces an embedding of $\mathfrak{sp}(\ell,\mathbb{R})$ into $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$, and this embedding is plainly a morphism

$$\delta \colon \mathfrak{sp}(\ell,\mathbb{R}) \longrightarrow C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$$

of Lie algebras when $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$ is viewed as a real Lie algebra via the Poisson bracket. In the literature, a morphism of the kind δ is referred to as a *comomentum* mapping.

Let $\mathcal{O}$ be a holomorphic nilpotent orbit. The embedding of $\overline{\mathcal{O}}$ into $\mathfrak{sp}(\ell,\mathbb{R})$ induces a map from the algebra $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$ of ordinary smooth functions on $\mathfrak{sp}(\ell,\mathbb{R})$ to the algebra $C^0(\overline{\mathcal{O}})$ of continuous functions on $\overline{\mathcal{O}}$, and we denote the image of $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$ in $C^0(\overline{\mathcal{O}})$ by $C^\infty(\overline{\mathcal{O}})$. By construction, each function in $C^\infty(\overline{\mathcal{O}})$ is the restriction of an ordinary smooth function on the ambient space $\mathfrak{sp}(\ell,\mathbb{R})$. Since each stratum of $\overline{\mathcal{O}}$ is an ordinary smooth closed submanifold of $\mathfrak{sp}(\ell,\mathbb{R})$, the functions in $C^\infty(\overline{\mathcal{O}})$, restricted to a stratum of $\overline{\mathcal{O}}$, are ordinary smooth functions on that stratum. Hence $C^\infty(\overline{\mathcal{O}})$ is a *smooth structure* on $\overline{\mathcal{O}}$. The algebra $C^\infty(\overline{\mathcal{O}})$ is referred to as the algebra of WHITNEY-smooth functions on $\overline{\mathcal{O}}$, relative to the embedding of $\overline{\mathcal{O}}$ into the affine space $\mathfrak{sp}(\ell,\mathbb{R})$. Under the identification (7.3), the orbit $\mathcal{O}$ passes to a *coadjoint* orbit. Consequently, under the surjection

$C^\infty(\mathfrak{sp}(\ell,\mathbb{R})) \to C^\infty(\overline{\mathcal{O}})$, the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on the algebra $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$ descends to a Poisson bracket on $C^\infty(\overline{\mathcal{O}})$, which we still denote by $\{\,\cdot\,,\,\cdot\,\}$, with a slight abuse of notation. This Poisson algebra turns $\overline{\mathcal{O}}$ into a stratified symplectic space. Combined with the complex analytic structure coming from the projection from $\overline{\mathcal{O}}$ onto the corresponding space of complex symmetric $(\ell\times\ell)$-matrices, in this fashion, the space $\overline{\mathcal{O}}$ acquires a *stratified Kähler space* structure. The composite of the above comomentum mapping δ with the projection from $C^\infty(\mathfrak{sp}(\ell,\mathbb{R}))$ to $C^\infty(\overline{\mathcal{O}})$ yields an embedding

$$\delta_{\mathcal{O}}\colon \mathfrak{sp}(\ell,\mathbb{R}) \longrightarrow C^\infty(\overline{\mathcal{O}}) \tag{7.4}$$

which is still a morphism of Lie algebras and therefore a comomentum mapping in the appropriate sense.

The $\mathrm{Sp}(\ell,\mathbb{R})$-momentum mapping induces an embedding of the reduced space N_0 into $\mathfrak{sp}(\ell,\mathbb{R})$ which identifies N_0 with the closure $\overline{\mathcal{O}}_{\min(s,\ell)}$ of the holomorphic nilpotent orbit $\mathcal{O}_{\min(s,\ell)}$ in $\mathfrak{sp}(\ell,\mathbb{R})$. In this fashion, the reduced space N_0 inherits a stratified Kähler structure. Since the $\mathrm{Sp}(\ell,\mathbb{R})$-momentum mapping induces an identification of N_0 with $\overline{\mathcal{O}}_s$ for every $s \leq \ell$ in a compatible manner, the ascending sequence (7.2), and in particular the notion of holomorphic nilpotent orbit, is actually independent of the choice of complex structure J on $\mathrm{R}^{2\ell}$. For a single particle, i. e. $\ell = 1$, the description of the reduced space N_0 comes down to that of the semicone given in Section 2.1 above.

Thus, when the number ℓ of particles is at most equal to the (real) dimension s of the space $\mathbb{A}^s$ in which these particles move, as a space, the reduced space N amounts to a copy of complex affine space of dimension $\frac{\ell(\ell+1)}{2}$ and hence to a copy of real affine space of dimension $\ell(\ell+1)$. When the number ℓ of particles exceeds the (real) dimension s of the space in which the particles move, as a space, the reduced space N amounts to a copy of the complex affine variety of complex symmetric matrices of rank at most equal to s.

8 Quantization in the situation of the previous class of examples

In the situation of the previous section, we will now explain briefly the quantization procedure developed in [21]. Suppose that $s \leq \ell$ (for simplicity), let $m = s\ell$, and endow the affine coordinate ring of $\mathbb{C}^m$, that is, the polynomial algebra $\mathbb{C}[z_1,\dots,z_m]$, with the inner product $\cdot$ given by the standard formula

$$\psi\cdot\psi' = \int \psi\overline{\psi'}\mathrm{e}^{-\frac{\mathbf{z}\overline{\mathbf{z}}}{2}}\varepsilon_m, \quad \varepsilon_m = \frac{\omega^m}{(2\pi)^m m!}, \tag{8.1}$$

where ω refers to the symplectic form on $\mathbb{C}^m$. Furthermore, endow the polynomial algebra $\mathbb{C}[z_1,\dots,z_m]$ with the induced $\mathrm{O}(s,\mathbb{R})$-action. By construction, the affine complex coordinate ring $\mathbb{C}[\overline{\mathcal{O}}_s]$ of $\overline{\mathcal{O}}_s$ is canonically isomorphic to the algebra

$$\mathbb{C}[z_1, \ldots, z_m]^{\mathrm{O}(s,\mathbb{R})}$$

of $\mathrm{O}(s, \mathbb{R})$-invariants in $\mathbb{C}[z_1, \ldots, z_m]$. The restriction of the inner product $\cdot$ to $\mathbb{C}[\overline{\mathcal{O}}_s]$ turns $\mathbb{C}[\overline{\mathcal{O}}_s]$ into a pre-Hilbert space, and HILBERT space completion yields a HILBERT space which we write as $\widehat{\mathbb{C}}[\overline{\mathcal{O}}_s]$. This is the Hilbert space which arises by *holomorphic quantization* on the stratified Kähler space $\overline{\mathcal{O}}_s$; see [21] for details. On this Hilbert space, the elements of the Lie algebra $\mathfrak{u}(\ell)$ of the unitary group $\mathrm{U}(\ell)$ act in an obvious fashion; indeed, the elements of $\mathfrak{u}(\ell)$, viewed as functions in $C^\infty(\overline{\mathcal{O}}_s)$, are classical observables which are directly quantizable, and quantization yields the obvious $\mathfrak{u}(\ell)$-representation on $\mathbb{C}[\overline{\mathcal{O}}_s]$. This construction may be carried out for any $s \leq \ell$ and, for each $s \leq \ell$, the resulting quantizations yields a *costratified Hilbert space* of the kind

$$\mathbb{C} \longleftarrow \widehat{\mathbb{C}}[\overline{\mathcal{O}_1}] \longleftarrow \ldots \longleftarrow \widehat{\mathbb{C}}[\overline{\mathcal{O}_s}].$$

Here each arrow is just a restriction mapping and is actually a morphism of representations for the corresponding quantizable observables, in particular, a morphism of $\mathfrak{u}(\ell)$-representations; each arrow amounts essentially to an orthogonal projection. Plainly, the costratified structure integrates to a costratified $\mathrm{U}(\ell)$-representation, i. e. to a corresponding system of $\mathrm{U}(\ell)$-representations. The resulting costratified quantum phase space for $\overline{\mathcal{O}_s}$ is a kind of *singular* Fock space. This quantum phase space is entirely given in terms of *data on the reduced level*.

Consider the unreduced classical harmonic oscillator energy E which is given by $E = z_1\overline{z}_1 + \cdots + z_m\overline{z}_m$; it quantizes to the Euler operator (quantized harmonic oscillator hamiltonian). For $s \leq \ell$, the reduced classical phase space Q_s of ℓ harmonic oscillators in $\mathbb{R}^s$ with total angular momentum zero and fixed energy value which is here encoded in the even number $2k$ fits into an ascending sequence

$$Q_1 \subseteq \cdots \subseteq Q_s \subseteq \cdots \subseteq Q_\ell \cong \mathbb{C}\mathrm{P}^d \tag{8.2}$$

of stratified Kähler spaces where

$$\mathbb{C}\mathrm{P}^d = \mathrm{P}(\mathrm{S}^2[\mathbb{C}^\ell]), \quad d = \frac{\ell(\ell+1)}{2} - 1.$$

The sequence (8.2) arises from the sequence (7.2) by *projectivization*. The parameter k (energy value $2k$) is encoded in the Poisson structure. Let $\mathcal{O}(k)$ be the k'th power of the hyperplane bundle on $\mathbb{C}\mathrm{P}^d$, let

$$\iota_{Q_s}\colon Q_s \longrightarrow Q_\ell \cong \mathbb{C}\mathrm{P}^d$$

be the inclusion, and let $\mathcal{O}_{Q_s}(k) = \iota_{Q_s}^*\mathcal{O}(k)$. The quantum *Hilbert* space amounts now to the space of holomorphic sections of $\iota_{Q_s}^*\mathcal{O}(k)$, and the resulting *costratified quantum Hilbert space* has the form

$$\Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_1}(k)) \longleftarrow \ldots \longleftarrow \Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_s}(k)).$$

Each vector space $\Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_{s'}}(k))$ $(1 \leq s' \leq s)$ is a finite-dimensional representation space for the quantizable observables in $C^\infty(Q_s)$, in particular, a $\mathfrak{u}(\ell)$-representation, and this representation integrates to a $\mathrm{U}(\ell)$-representation, and each arrow is a morphism of representations; similarly as before, these arrows are just restriction maps.

We will now give a description of the decomposition of the space

$$\Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_\ell}(k)) = S^k_{\mathbb{C}}[\mathfrak{p}^*]$$

of homogeneous degree k polynomial functions on $\mathfrak{p} = S^2_{\mathbb{C}}[\mathbb{C}^\ell]$ into its *irreducible* $\mathrm{U}(\ell)$-representations in terms of highest weight vectors. To this end we note that coordinates $x_1, \ldots, x_\ell$ on $\mathbb{C}^\ell$ give rise to coordinates of the kind $\{x_{i,j} = x_{j,i};\, 1 \leq i, j \leq \ell\}$ on $S^2_{\mathbb{C}}[\mathbb{C}^\ell]$, and the determinants

$$\delta_1 = x_{1,1},\ \delta_2 = \begin{vmatrix} x_{1,1} & x_{1,2} \\ x_{1,2} & x_{2,2} \end{vmatrix},\ \delta_3 = \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{1,2} & x_{2,2} & x_{2,3} \\ x_{1,3} & x_{2,3} & x_{3,3} \end{vmatrix},\ \text{etc.}$$

are highest weight vectors for certain $\mathrm{U}(\ell)$-representations. For $1 \leq s \leq r$ and $k \geq 1$, the $\mathrm{U}(\ell)$-representation $\Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_s}(k))$ is the sum of the irreducible representations having as highest weight vectors the monomials

$$\delta_1^\alpha \delta_2^\beta \ldots \delta_s^\gamma, \quad \alpha + 2\beta + \cdots + s\gamma = k,$$

and the restriction morphism

$$\Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_s}(k)) \longrightarrow \Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_{s-1}}(k))$$

has the span of the representations involving δ_s explicitly as its kernel and, restricted to the span of those irreducible representations which do *not* involve δ_s, this morphism is an isomorphism.

This situation may be interpreted in the following fashion: The composite

$$\mu_{2k} \colon \overline{\mathcal{O}}_s \subseteq \mathfrak{sp}(\ell, \mathbb{R}) \cong \mathfrak{sp}(\ell, \mathbb{R})^* \longrightarrow \mathfrak{u}(\ell)^*$$

is a singular momentum mapping for the $\mathrm{U}(\ell)$-action on $\overline{\mathcal{O}}_s$; actually, the adjoint $\mathfrak{u}(\ell) \to C^\infty(\overline{\mathcal{O}}_s)$ of μ^{2k} amounts to the composite of (7.4) with the inclusion of $\mathfrak{u}(\ell)$ into $\mathfrak{sp}(\ell, \mathbb{R})$. The *irreducible* $\mathrm{U}(\ell)$*-representations which correspond to the coadjoint orbits in the image*

$$\mu_{2k}(O_{s'} \setminus O_{s'-1}) \subseteq \mathfrak{u}(\ell)^*$$

of the stratum $O_{s'} \setminus O_{s'-1}$ $(1 \leq s' \leq s)$ *are precisely the irreducible representations having as highest weight vectors the monomials*

$$\delta_1^\alpha \delta_2^\beta \ldots \delta_{s'}^\gamma \quad (\alpha + 2\beta + \cdots + s'\gamma = k)$$

involving $\delta_{s'}$ *explicitly, i. e. with* $\gamma \geq 1$.

9 Holomorphic half-form quantization on the complexification of a compact Lie group

Recall that, given a general compact Lie group K, via the diffeomorphism (2.1), the complex structure on $K^{\mathbb{C}}$ and the cotangent bundle symplectic structure on T^*K combine to K-bi-invariant Kähler structure. A global Kähler potential is given by the function κ defined by by

$$\kappa(x\,\mathrm{e}^{iY}) = |Y|^2,\ x \in K,\ Y \in \mathfrak{k}.$$

The function κ being a Kähler potential signifies that the symplectic structure on $\mathrm{T}^*K \cong K^{\mathbb{C}}$ is given by $i\partial\overline{\partial}\kappa$. Let ε denote the symplectic (or Liouville) volume form on $\mathrm{T}^*K \cong K^{\mathbb{C}}$, and let η be the real K-bi-invariant (analytic) function on $K^{\mathbb{C}}$ given by

$$\eta(x\,\mathrm{e}^{iY}) = \sqrt{\left|\frac{\sin(\mathrm{ad}(Y))}{\mathrm{ad}(Y)}\right|},\ x \in K,\ Y \in \mathfrak{k},$$

cf. [12] (2.10). Thus η^2 is the density of Haar measure on $K^{\mathbb{C}}$ relative to Liouville measure ε.

Half-form Kähler quantization on $K^{\mathbb{C}}$ leads to the Hilbert space

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)$$

of holomorphic functions on $K^{\mathbb{C}}$ that are square-integrable relative to $\mathrm{e}^{-\kappa/\hbar}\eta\varepsilon$ [12]. Thus the scalar product in this Hilbert space is given by

$$\langle\psi_1, \psi_2\rangle = \frac{1}{\mathrm{vol}(K)} \int_{K^{\mathbb{C}}} \overline{\psi_1}\psi_2 \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon.$$

Relative to left and right translation, $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)$ is a unitary $(K \times K)$-representation, and the Hilbert space associated with $\mathcal{P}$ by reduction after quantization is the subspace

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K$$

of K-invariants relative to conjugation.

Let ε_T denote the Liouville volume form of $\mathrm{T}^*T \cong T^{\mathbb{C}}$. There is a function γ on this space, made explicit in [28], such that the restriction mapping induces an isomorphism

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K \longrightarrow \mathcal{H}L^2(T^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\gamma\varepsilon_T)^W \tag{9.1}$$

of Hilbert spaces where the scalar product in $\mathcal{H}L^2(T^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\gamma\varepsilon_T)^W$ is given by

$$\frac{1}{\mathrm{vol}(K)} \int_{T^{\mathbb{C}}} \overline{\psi_1}\psi_2 e^{-\kappa/\hbar}\gamma\varepsilon_T\,. \tag{9.2}$$

10 Singular quantum structure: costratified Hilbert space

Let N be a stratified space. Thus N is a disjoint union $N = \cup N_\lambda$ of locally closed subspaces N_λ, called *strata*, each stratum being an ordinary smooth manifold, and the mutual positions of the strata are made precise in a way not spelled out here. Let $\mathcal{C}_N$ be the category whose objects are the strata of N and whose morphisms are the inclusions $Y' \subseteq \overline{Y}$ where Y and Y' range over strata. We define a *costratified Hilbert space* relative to N or associated with the stratification of N to be a system which assigns a Hilbert space $\mathcal{C}_Y$ to each stratum Y, together with a bounded linear map $\mathcal{C}_{Y_2} \to \mathcal{C}_{Y_1}$ for each inclusion $Y_1 \subseteq \overline{Y_2}$ such that, whenever $Y_1 \subseteq \overline{Y_2}$ and $Y_2 \subseteq \overline{Y_3}$, the composite of $\mathcal{C}_{Y_3} \to \mathcal{C}_{Y_2}$ with $\mathcal{C}_{Y_2} \to \mathcal{C}_{Y_1}$ coincides with the bounded linear map $\mathcal{C}_{Y_3} \to \mathcal{C}_{Y_1}$ associated with the inclusion $Y_1 \subseteq \overline{Y_3}$.

We now explain the construction of the costratified Hilbert space associated with the reduced phase space $\mathcal{P}$. This costratified structure is a *quantum analogue* of the *orbit type stratification*.

In the Hilbert space

$$\mathcal{H} = \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K \cong \mathcal{H}L^2(T^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\gamma\varepsilon_T)^W,$$

we single out subspaces associated with the strata in an obvious manner. For the special case

$$K = \mathrm{SU}(2),\ \mathcal{P} = \mathrm{T}^*K//K \cong \mathbb{C},$$

this comes down to the following procedure:

The elements of $\mathcal{H}$ are ordinary holomorphic functions on $K^{\mathbb{C}}$. Being K-invariant, they are determined by their restrictions to $\mathrm{T}^{\mathbb{C}}$; these are W-invariant holomorphic functions on $\mathrm{T}^{\mathbb{C}}$, and these W-invariant holomorphic functions, in turn, are determined by the holomorphic functions on

$$\mathcal{P} = K^{\mathbb{C}}//K^{\mathbb{C}} \cong T^{\mathbb{C}}/W \cong \mathbb{C}$$

which they induce on that space. In terms of the realization of $\mathcal{P}$ as the complex line $\mathbb{C}$, the stratification of $\mathcal{P}$ reproduced in Subsection 2.3 above is given by the decomposition $\mathbb{C} = \mathcal{P}_+ \cup \mathcal{P}_- \cup \mathcal{P}_1$ of $\mathbb{C}$ into

$$\mathcal{P}_+ = \{2\} \subseteq \mathbb{C},\ \mathcal{P}_- = \{-2\} \subseteq \mathbb{C},\ \mathcal{P}_1 = \mathbb{C} \setminus \mathcal{P}_0 = \mathbb{C} \setminus \{2, -2\}.$$

The closed subspaces

$$\mathcal{V}_+ = \{f \in \mathcal{H}; f\big|_{\mathcal{P}_+} = 0\} \subseteq \mathcal{H}$$
$$\mathcal{V}_- = \{f \in \mathcal{H}; f\big|_{\mathcal{P}_-} = 0\} \subseteq \mathcal{H}$$

are Hilbert spaces, and we *define* the Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ to be the orthogonal complements in $\mathcal{H}$ so that

$$\mathcal{H} = \mathcal{V}_+ \oplus \mathcal{H}_+ = \mathcal{V}_- \oplus \mathcal{H}_-;$$

moreover, we take $\mathcal{H}_1$ to be the entire space $\mathcal{H}$. The resulting system

$$\{\mathcal{H};\ \mathcal{H}_1,\ \mathcal{H}_+,\ \mathcal{H}_-\},$$

together with the corresponding orthogonal projections, is the *costratified Hilbert space* associated with the stratification of $\mathcal{P}$. By construction, this costratified Hilbert space structure is a *quantum analogue* of the *orbit type stratification* of $\mathcal{P}$.

11 The holomorphic Peter-Weyl theorem

Choose a dominant Weyl chamber in the maximal torus $\mathfrak{t}$. Given the highest weight λ (relative to the chosen dominant Weyl chamber), we will denote by $\chi_\lambda^{\mathbb{C}}$ the irreducible character of $K^{\mathbb{C}}$ associated with λ.

Theorem 11.1 (Holomorphic Peter-Weyl theorem). *The Hilbert space*

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)$$

contains the vector space $\mathbb{C}[K^{\mathbb{C}}]$ of representative functions on $K^{\mathbb{C}}$ as a dense subspace and, as a unitary $(K \times K)$-representation, this Hilbert space decomposes as the direct sum

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon) \cong \widehat{\oplus}_{\lambda\in\widehat{K^{\mathbb{C}}}} V_\lambda^* \otimes V_\lambda$$

of $(K\times K)$-isotypical summands, each such summand being written here as $V_\lambda^*\otimes V_\lambda$ where V_λ refers to the irreducible K-representation associated with the highest weight λ.

A proof of this theorem and relevant references can be found in [26]. The holomorphic Peter-Weyl theorem entails that the irreducible characters $\chi_\lambda^{\mathbb{C}}$ of $K^{\mathbb{C}}$ constitute a Hilbert space basis of

$$\mathcal{H} = \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K.$$

Given the highest weight λ, we will denote by χ_λ the corresponding irreducible character of K; plainly, χ_λ is the restriction to K of the character $\chi_\lambda^{\mathbb{C}}$. As usual, let $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$, the half sum of the positive roots and, for a highest weight λ, let

$$C_\lambda := (\hbar\pi)^{\dim(K)/2}\mathrm{e}^{\hbar|\lambda+\rho|^2}, \tag{11.1}$$

where $|\lambda+\rho|$ refers to the norm of $\lambda+\rho$ relative to the inner product on $\mathfrak{k}$. In view of the ordinary Peter-Weyl theorem, the $\{\chi_\lambda\}$'s constitute an *orthonormal* basis of the Hilbert space $L^2(K, \mathrm{d}x)^K$.

Theorem 11.2. *The assignment to χ_λ of $C_\lambda^{-1/2}\chi_\lambda^{\mathbb{C}}$, as λ ranges over the highest weights, yields a unitary isomorphism*

$$L^2(K,\mathrm{d}x)^K \longrightarrow \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K \tag{11.2}$$

of Hilbert spaces.

By means of this isomorphism, the costratified Hilbert space structure arising from *stratified Kähler* quantization as explained earlier carries over to the Schrödinger quantization.

12 Quantum Hamiltonian and Peter-Weyl decomposition

In the Kähler quantization, only the constants are quantizable while in the Schrödinger quantization, functions that are at most quadratic in generalized momenta are quantizable. In particular, the classical Hamiltonian (2.2) of our model is quantizable in the Schrödinger quantization, having as associated quantum Hamiltonian the operator

$$H = -\frac{\hbar^2}{2}\Delta_K + \frac{\nu}{2}(3-\chi_1) \tag{12.1}$$

on $L^2(K,dx)^K$. The operator Δ_K, in turn, arises from the non-positive Laplace-Beltrami operator $\tilde{\Delta}_K$ associated with the bi-invariant Riemannian metric on K as follows: The operator $\tilde{\Delta}_K$ is essentially self-adjoint on $C^\infty(K)$ and has a unique extension Δ_K to an (unbounded) self-adjoint operator on $L^2(K,dx)$. The spectrum being discrete, the domain of this extensions is the space of functions of the form $f=\sum_n \alpha_n\varphi_n$ such that $\sum_n|\alpha_n|^2\lambda_n^2<\infty$ where the φ_n's range over the eigenfunctions and the λ_n's over the eigenvalues of $\tilde{\Delta}_K$.

Since the metric is bi-invariant, so is Δ_K, whence Δ_K restricts to a self-adjoint operator on $L^2(K,\mathrm{d}x)^K$, which we still write as Δ_K. By means of the isomorphism (11.2), we then transfer the Hamiltonian, in particular, the operator Δ_K, to a self-adjoint operators on $\mathcal{H}$. Schur's lemma then tells us the following:
(1) Each isotypical $(K\times K)$-summand $L^2(K,dx)_\lambda$ of $L^2(K,dx)$ in the Peter-Weyl decomposition is an eigenspace for Δ_K;
(2) the representative functions are eigenfunctions for Δ_K;
(3) the eigenvalue $-\varepsilon_\lambda$ of Δ_K corresponding to the highest weight λ is given by

$$\varepsilon_\lambda = (|\lambda+\rho|^2-|\rho|^2).$$

Thus, in the holomorphic quantization on $\mathrm{T}^*K\cong K^{\mathbb{C}}$, the free energy operator (i. e. without potential energy term) arises as the unique extension of the operator $-\frac{1}{2}\Delta_K$ on $\mathcal{H}$ to an unbounded self-adjoint operator, and the spectral decomposition thereof refines to the holomorphic Peter-Weyl decomposition of $\mathcal{H}$.

13 The lattice gauge theory model arising from SU(2)

In the rest of the paper we will discuss somewhat informally, for the special case where the underlying compact group is $K = \mathrm{SU}(2)$, some of the implications for the physical interpretation ; see [29] for a leisurely somewhat more complete introduction and [28] for a systematic description.

To begin with, we write out the requisite data for the special case under consideration. We denote the roots of $K = \mathrm{SU}(2)$ relative to the dominant Weyl chamber chosen earlier by α and $-\alpha$, so that $\varrho = \frac{1}{2}\alpha$. The invariant inner product on the Lie algebra $\mathfrak{k}$ of K is of the form

$$-\frac{1}{2\beta^2}\mathrm{tr}(Y_1Y_2)\,, \qquad Y_1, Y_2 \in \mathfrak{k}\,, \tag{13.1}$$

with a scaling factor $\beta > 0$ which we will leave unspecified (e.g., $\beta = \frac{1}{\sqrt{8}}$ for the Killing form). Then

$$|\alpha|^2 = 4\beta^2\,, \qquad |\varrho|^2 = \beta^2\,.$$

The highest weights are $\lambda_n = \frac{n}{2}\alpha$, where $n = 0, 1, 2, \dots$ (twice the spin). Then

$$\varepsilon_n \equiv \varepsilon_{\lambda_n} = \beta^2 n(n+2)\,, \qquad C_n \equiv C_{\lambda_n} = (\hbar\pi)^{3/2}\mathrm{e}^{\hbar\beta^2(n+1)^2}\,, \tag{13.2}$$

cf. (11.1) for the significance of the notation C_{λ_n}. We will now write the complex characters $\chi^{\mathbb{C}}_{\lambda_n}$ as $\chi^{\mathbb{C}}_n$ $(n \geq 0)$. On $T^{\mathbb{C}}$, these complex characters are given by

$$\chi^{\mathbb{C}}_n\big(\mathrm{diag}(z, z^{-1})\big) = z^n + z^{n-2} + \cdots + z^{-n}\,, \qquad z \in \mathbb{C} \setminus \{0\}\,, \tag{13.3}$$

whereas, on T, the corresponding real characters take the form

$$\chi_n\big(\mathrm{diag}(\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x})\big) = \frac{\sin\big((n+1)x\big)}{\sin(x)}\,, \qquad x \in \mathbb{R}\,, \quad n \geq 0\,. \tag{13.4}$$

The Weyl group W permutes the two entries of the elements in T. Hence, the reduced configuration space $\mathcal{X} = T/W$ can be parametrized by $x \in [0, \pi]$ through $x \mapsto \mathrm{diag}(\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x})$. In this parametrization, the measure v on T is given by

$$v\,\mathrm{d}t = \frac{\mathrm{vol}(K)}{\pi}\sin^2(x)\,\mathrm{d}x\,.$$

It follows that the assignment to $\psi \in C^\infty(T)^W$ of the function

$$x \mapsto \sqrt{2}\,\sin x\;\psi\big(\mathrm{diag}(\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x})\big)\,, \qquad x \in [0, \pi]\,,$$

defines a Hilbert space isomorphism from $L^2(K, \mathrm{d}x)^K$, realized as a Hilbert space of W-invariant L^2-functions on T, onto the ordinary $L^2[0, \pi]$, where the inner product in $L^2[0,\pi]$ is normalized so that the constant function with value 1 has norm 1. In particular, given $n \geq 0$, the character χ_n is mapped to the function given by the expression

$$\chi_n(x) = \sqrt{2}\,\sin((n+1)x)\,. \tag{13.5}$$

In view of the isomorphism between $L^2(K,\mathrm{d}x)^K$ and $L^2[0,\pi]$ and the isomorphism (11.2), we can work in an abstract Hilbert space $\mathcal{H}$ with a distinguished orthonormal basis $\{|n\rangle : n = 0,1,2,\dots\}$. We achieve the passage to the holomorphic realization $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K$, to the Schrödinger realization $L^2(K,\mathrm{d}x)^K$, and to the ordinary L^2-realization $L^2[0,\pi]$ by substitution of, respectively, $C_n^{-1/2}\chi_n^{\mathbb{C}}$, χ_n, and $\sqrt{2}\sin(n+1)x$, for $|n\rangle$. We remark that plotting wave functions in the realization of $\mathcal{H}$ by $L^2[0,\pi]$ has the advantage that, directly from the graph, one can read off the corresponding probability densities with respect to Lebesgue measure on the parameter space $[0,\pi]$.

We determine the subspaces $\mathcal{H}_\tau$ for the special case $K = \mathrm{SU}(2)$. The orbit type strata are $\mathcal{P}_+$, $\mathcal{P}_-$ and $\mathcal{P}_1$, where $\mathcal{P}_\pm$ consists of the class of $\pm\mathbf{1}$ and $\mathcal{P}_1 = \mathcal{P} \setminus (\mathcal{P}_+ \cup \mathcal{P}_-)$. (Recall that via the complex analytic isomorphism (2.3), $\mathcal{P}_\pm$ is identified with the subset $\{\pm 2\}$ of $\mathbb{C}$.) Since $\mathcal{P}_1$ is dense in $\mathcal{P}$, the space $\mathcal{V}_1$ reduces to zero and so $\mathcal{H}_1 = \mathcal{H}$. By definition, the subspaces $\mathcal{V}_+$ and $\mathcal{V}_-$ consist of the functions $\psi \in \mathcal{H}$ that satisfy the constraints

$$\psi(\mathbf{1}) = 0\,, \qquad \psi(-\mathbf{1}) = 0\,, \tag{13.6}$$

respectively. One can check that the system $\{\chi_n^{\mathbb{C}} - (n+1)\chi_0^{\mathbb{C}} \ :\ n = 1,2,3,\dots\}$ forms a basis in $\mathcal{V}_+$ and that the system $\{\chi_n^{\mathbb{C}} + (-1)^n \frac{n+1}{2}\chi_1^{\mathbb{C}} \ :\ n = 0,2,3,\dots\}$ forms a basis in $\mathcal{V}_-$. Taking the orthogonal complements, we arrive at the following.

Theorem 13.1. *The subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ have dimension* 1*. They are spanned by the normalized vectors*

$$\psi_+ \ := \ \frac{1}{N}\sum\nolimits_{n=0}^{\infty}(n+1)\,\mathrm{e}^{-\hbar\beta^2\,(n+1)^2/2}\,|n\rangle\,, \tag{13.7}$$

$$\psi_- \ := \ \frac{1}{N}\sum\nolimits_{n=0}^{\infty}(-1)^n\,(n+1)\,\mathrm{e}^{-\hbar\beta^2\,(n+1)^2/2}\,|n\rangle\,, \tag{13.8}$$

respectively. The normalization factor N is determined by the identity

$$N^2 = \sum_{n=1}^{\infty} n^2\,\mathrm{e}^{-\hbar\beta^2\,n^2}\,.$$

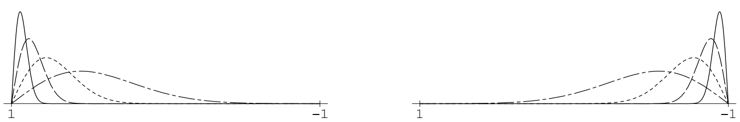
Hence, in Dirac notation, the orthogonal projections $\Pi_\pm \colon \mathcal{H} \to \mathcal{H}_\pm$ are given by the expressions

$$\Pi_\pm = |\psi_\pm\rangle\,\langle\psi_\pm|\,. \tag{13.9}$$

In terms of the θ-constant $\theta_3(Q) = \sum_{k=-\infty}^{\infty} Q^{k^2}$, the normalization factor N is determined by the identity

$$N^2 = \frac{1}{2} \mathrm{e}^{-\hbar\beta^2} \theta_3'(\mathrm{e}^{-\hbar\beta^2})\,. \tag{13.10}$$

The following figure shows plots of $\psi_\pm$ in the realization of $\mathcal{H}$ via $L^2[0,\pi]$ for $\hbar\beta^2 = 1/128$ (continuous line), 1/32 (long dash), 1/8 (short dash), 1/2 (alternating short-long dash).



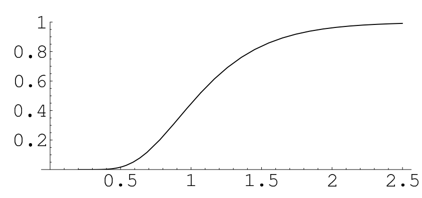
ψ_+ ψ_-

14 Tunneling between strata

Computing the inner product of ψ_+ and ψ_-,

$$\langle \psi_+, \psi_- \rangle = \frac{1}{N^2} \sum_{n=1}^{\infty} (-1)^{n+1}\, n^2\, \mathrm{e}^{-\hbar\beta^2 n^2} = -\frac{\theta_3'\big(-\mathrm{e}^{-\hbar\beta^2}\big)}{\theta_3'\big(\mathrm{e}^{-\hbar\beta^2}\big)}\,,$$

we observe that the subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ are not orthogonal. They share a certain overlap which depends on the combined parameter $\hbar\beta^2$. The absolute square $|\langle \psi_+, \psi_- \rangle|^2$ yields the tunneling probability between the strata $\mathcal{P}_+$ and $\mathcal{P}_-$, i. e., the probability for a state prepared at $\mathcal{P}_+$ to be measured at $\mathcal{P}_-$ and vice versa. The following figure shows a plot of the tunneling probability against $\hbar\beta^2$. For large values, this probability tends to 1 whereas for $\hbar\beta^2 \to 0$, i.e., in the semiclassical limit, it vanishes.



15 Energy eigenvalues and eigenstates

Passing to the realization of $\mathcal{H}$ via $L^2[0,\pi]$ and applying the general formula for the radial part of the Laplacian on a compact group, see [13, §II.3.4], from the description (12.1) of the quantum Hamiltonian, viz.

$$H = -\frac{\hbar^2}{2}\Delta_K + \frac{\nu}{2}(3 - \chi_1),$$

we obtain the formal expression

$$-\frac{\hbar^2\beta^2}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+1\right)+\frac{\nu}{2}(3-\chi_1)$$

for H on $L^2[0,\pi]$. Hence the stationary Schrödinger equation can be written as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+2\tilde{\nu}\cos(x)+\left(\frac{2E}{\hbar^2\beta^2}+1-3\tilde{\nu}\right)\right)\psi(x)=0\,, \tag{15.1}$$

where $\tilde{\nu}=\frac{\nu}{\hbar^2\beta^2}\equiv\frac{1}{\hbar^2\beta^2 g^2}$, and E refers to the eigenvalue. The change of variable $y=(x-\pi)/2$ leads to the Mathieu equation

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}f(y)+(a-2q\cos(2y))f(y)=0\,, \tag{15.2}$$

where

$$a=\frac{8E}{\hbar^2\beta^2}+4-12\tilde{\nu}\,,\qquad q=4\tilde{\nu}\,; \tag{15.3}$$

here f refers to a Whitney smooth function on the interval $[-\pi/2,0]$ satisfying the boundary conditions

$$f(-\pi/2)=f(0)=0\,. \tag{15.4}$$

For the theory of the Mathieu equation and its solutions, called *Mathieu functions*, see [1]. For certain characteristic values of the parameter a depending analytically on q and usually denoted by $b_{2n+2}(q)$, $n=0,1,2,\dots$, solutions satisfying (15.4) exist. Given $a=b_{2n+2}(q)$, the corresponding solution is unique up to a complex factor and can be chosen to be real-valued. It is usually denoted by $\mathrm{se}_{2n+2}(y;q)$, where 'se' stands for *sine elliptic*.

Thus, in the realization of $\mathcal{H}$ via $L^2[0,\pi]$, the stationary states are given by

$$\xi_n(x)=(-1)^{n+1}\sqrt{2}\left(\mathrm{se}_{2n+2}\left(\frac{x-\pi}{2};4\tilde{\nu}\right)\right)\,,\qquad n=0,1,2,\dots\,, \tag{15.5}$$

and the corresponding eigenvalues by

$$E_n=\frac{\hbar^2\beta^2}{2}\left(\frac{b_{2n+2}(4\tilde{\nu})}{4}+3\tilde{\nu}-1\right)\,.$$

The factor $(-1)^{n+1}$ ensures that, for $\tilde{\nu}=0$, we get $\xi_n=\chi_n$. According to [1, §20.5], for any value of the parameter q, the functions

$$\sqrt{2}\,\mathrm{se}_{2n+2}(y;q)\,,\qquad n=0,1,2,,\dots\,,$$

form an orthonormal basis in $L^2[-\pi/2,0]$ and the characteristic values satisfy $b_2(q)<b_4(q)<b_6(q)<\cdots$. Hence, the ξ_n's form an orthonormal basis in $\mathcal{H}$ and the eigenvalues E_n are nondegenerate.

Figure 2 shows the energy eigenvalues E_n and the level separation $E_{n+1}-E_n$ for $n=0,\dots,8$ as functions of $\tilde{\nu}$. Figure 3 displays the eigenfunctions ξ_n, $n=0,\dots,3$, for $\tilde{\nu}=0,3,6,12,24$. The plots have been generated by means of the built-in Mathematica functions `MathieuS` and `MathieuCharacteristicB`.

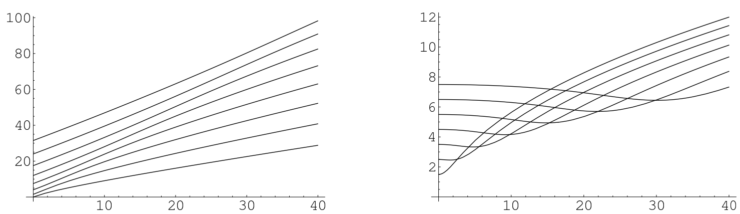


Figure 2: Energy eigenvalues E_n (left) and transition energy values $E_{n+1} - E_n$ (right) for $n = 0, \ldots, 7$ in units of $\hbar^2\beta^2$ as functions of $\tilde{\nu}$.

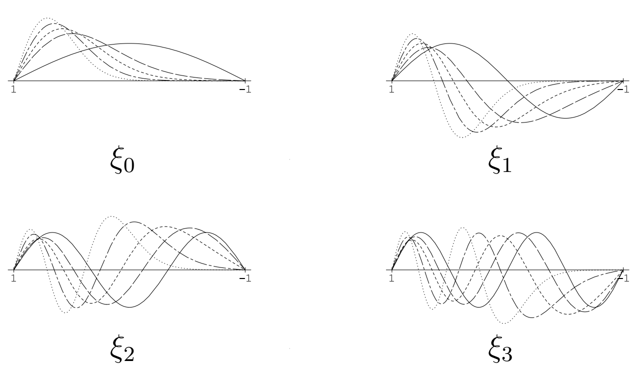


Figure 3: Energy eigenfunctions $\xi_0, \ldots, \xi_3$ for $\tilde{\nu} = 0$ (continuous line), 3 (long dash), 6 (short dash), 12 (alternating short-long dash), 24 (dotted line).

16 Expectation values of the costratification orthoprojectors

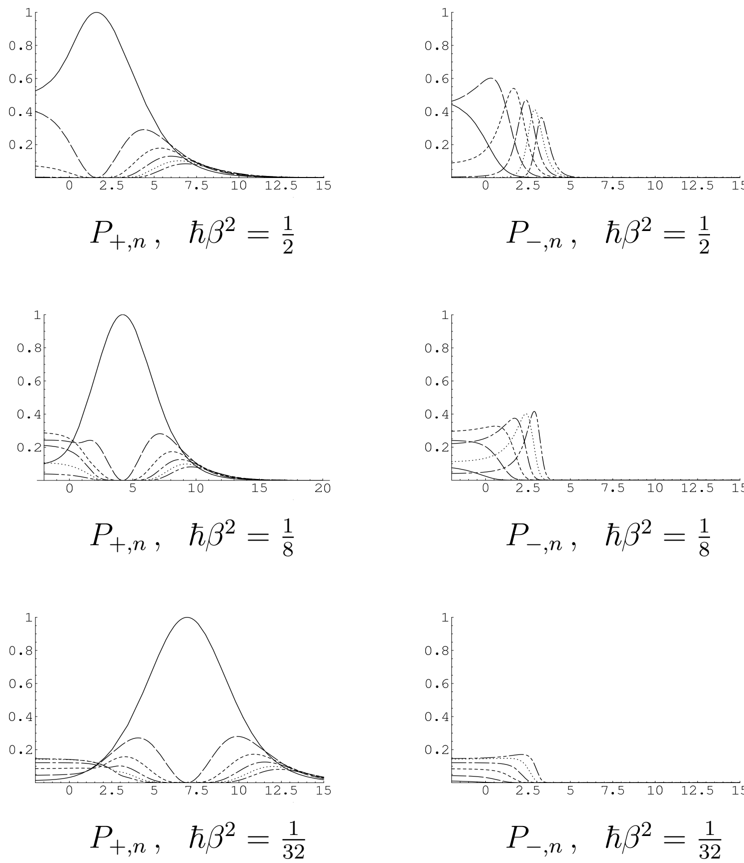


Figure 4: Expectation values $P_{+,n}$ and $P_{-,n}$ for $n = 0$ (continuous line), $n = 1$ (long dash), $n = 2$ (short dash), $n = 3$ (long-short dash), $n = 4$ (dotted line) and $n = 5$ (long-short-short dash), plotted over $\log \tilde{\nu}$ for $\hbar\beta^2 = \frac{1}{2}, \frac{1}{8}, \frac{1}{32}$.

On the level of the observables, the costratification is given by the orthoprojectors $\Pi_\pm$ onto the subspaces $\mathcal{H}_\pm$. We discuss their expectation values in the energy eigenstates,

$$P_{\pm,n} := \langle \xi_n | \Pi_\pm \xi_n \rangle\,,$$

i.e., the probability that the system prepared in the stationary state ξ_n is measured in the subspace $\mathcal{H}_\pm$. According to (13.9),

$$P_{\pm,n} = |\langle \xi_n | \psi_\pm \rangle|^2\,. \tag{16.1}$$

As se_{2n+2} is odd and π-periodic, it can be expanded as

$$\mathrm{se}_{2n+2}(y;q) = \sum\nolimits_{k=0}^{\infty} B^{2n+2}_{2k+2}(q) \sin((2k+2)y)\,,$$

with Fourier coefficients $B^{2n+2}_{2k+2}(q)$ satisfying certain recurrence relations [1, §20.2]. Due to (13.5) ,

$$\langle \xi_n | k \rangle = (-1)^{n+k} B^{2n+2}_{2k+2}(4\tilde{\nu}) \,, \tag{16.2}$$

whence (13.7) and (13.8) yield the expressions

$$\langle \xi_n | \psi_+ \rangle = \frac{(-1)^n}{N} \sum\nolimits_{k=0}^{\infty} (-1)^k \, (k+1) \, \mathrm{e}^{-\hbar\beta^2(k+1)^2/2} \, B^{2n+2}_{2k+2}(4\tilde{\nu}), \tag{16.3}$$

$$\langle \xi_n | \psi_- \rangle = \frac{(-1)^n}{N} \sum\nolimits_{k=0}^{\infty} (k+1) \, \mathrm{e}^{-\hbar\beta^2(k+1)^2/2} \, B^{2n+2}_{2k+2}(4\tilde{\nu}). \tag{16.4}$$

Together with (16.1), this procedure leads to formulas for $P_{\pm,n}$. The functions $P_{\pm,n}$ depend on the parameters $\hbar$, β^2 and ν only via the combinations $\hbar\beta^2$ and $\tilde{\nu} = \nu/(\hbar^2\beta^2)$. Figure 4 displays $P_{\pm,n}$ for $n = 0, \ldots, 5$ as functions of $\tilde{\nu}$ for three specific values of $\hbar\beta^2$, thus treating $\tilde{\nu}$ and $\hbar\beta^2$ as independent parameters. This is appropriate for the discussion of the dependence of the functions $P_{\pm,n}$ on the coupling parameter g for fixed values of $\hbar$ and β^2. The plots have been generated by Mathematica through numerical integration.

For $n = 0$, the function $P_{+,n}$ has a dominant peak which is enclosed by less prominent maxima of the other $P_{+,n}$'s and moves to higher $\tilde{\nu}$ when $\hbar\beta^2$ decreases. That is to say, for a certain value of the coupling constant, the state ψ_+ which spans $\mathcal{H}_+$ seems to coincide almost perfectly with the ground state. If the two states coincided exactly then (16.2) would imply that, for a certain value of q, the coefficients $B^{2n+2}_{2k+2}(q)$ would be given by $(-1)^{n+k}\frac{1}{N}(k+1)\mathrm{e}^{-\hbar\beta^2(k+1)^2/2}$. However, this is not true; the latter expressions do not satisfy the recurrence relations valid for the coefficients $B^{2n+2}_{2k+2}(q)$ for any value of q.

17 Outlook

For $K = \mathrm{SU}(2)$ it remains to discuss the dynamics relative to the costratified structure and to explore the probability flow into and out of the subspaces $\mathcal{H}_\pm$. More generally, it would be worthwhile carrying out this program for $K = \mathrm{SU}(n)$, $n \geq 3$. For $K = \mathrm{SU}(3)$, the orbit type stratification of the reduced phase space consists of a 4-dimensional stratum, a 2-dimensional stratum, and three isolated points. Thereafter the approach should be extended to arbitrary lattices.

The notion of costratified Hilbert space implements the stratification of the reduced classical phase space on the level of states. The significance of the stratification for the quantum observables remains to be clarified. Then the physical role of this stratification can be studied in more realistic models like the lattice QCD of [30, 32, 33].

A number of applications of the theory of stratified Kähler spaces have already been mentioned. Using the approach to lattice gauge theory in [19], we intend to develop elsewhere a rigorous approach to the quantization of certain lattice gauge theories by means of the Kähler quantization scheme for stratified Kähler spaces explained in the present paper. We plan to apply this scheme in particular to situations of the kind explored in [34]–[36] and to compare it with the approach to quantization in these papers. Constrained quantum systems occur in molecular mechanics as well, see e. g. [45] and the references there. Perhaps the Kähler quantization scheme for stratified Kähler spaces will shed new light on these quantum systems.

References

[1] M. Abramowitz, I.A. Stegun: Handbook of Mathematical Functions (abridged edition), Verlag Harri Deutsch (1984)

[2] J. M. Arms, J. E. Marsden, and V. Moncrief: Symmetry and bifurcation of moment mappings. *Comm. Math. Phys.* **78** (1981), 455–478

[3] J. M. Arms, J. E, Marsden, and V. Moncrief: The structure of the space of solutions of Einstein's equations. II. Several Killing fields and the Einstein-Yang-Mills equations. *Ann. Physics.*, **144** (1) (1982), 81–106

[4] M. Asorey, F. Falceto, J.L. Lopez and G. Luzón: Nodes, monopoles and confinement, `hep-th/9502024`

[5] M. F. Atiyah: The geometry and physics of knots. Cambridge University Press, Cambridge, U. K. (1990)

[6] M. F. Atiyah and R. Bott: The Yang-Mills equations over Riemann surfaces. *Phil. Trans. R. Soc. London A* **308** (1982), 523–615

[7] S. Charzyński, J. Kijowski, G. Rudolph, and M. Schmidt: On the stratified classical configuration space of lattice QCD. *J. Geom. and Physics* **55** (2005), 137–178

[8] R.H. Cushman and L.M. Bates: Global Aspects of Classical Integrable Systems. Birkhäuser-Verlag (1997)

[9] C. Emmrich and H. Römer: Orbifolds as configuration spaces of systems with gauge symmetries. *Commun. Math. Phys.* **129** (1990), 69–94

[10] M. Goresky and R. MacPherson: Intersection homology theory. *Topology*, **19** (1980), 135–162

[11] R. C. Gunning and H. Rossi: Analytic functions of several complex variables. Prentice-Hall Inc., Englewood Cliffs, N.J. (1965)

[12] B.C. Hall: Geometric quantization and the generalized Segal-Bargmann transform for Lie groups of compact type. *Commun. Math. Phys.* **226** (2002), 233–268, `quant-ph/0012015`

[13] S. Helgason: Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions, Academic Press (1984)

[14] J. Huebschmann: Poisson cohomology and quantization. *J. reine angew. Math.* **408** (1990), 57–113

[15] J. Huebschmann: On the quantization of Poisson algebras. In: Symplectic Geometry and Mathematical Physics, Actes du colloque en l'honneur de Jean-Marie Souriau, P. Donato, C. Duval, J. Elhadad, G. M. Tuynman, eds.; Progress in Mathematics, Vol. 99, Birkhäuser-Verlag, Boston · Basel · Berlin, 204–233 (1991)

[16] J. Huebschmann: Poisson geometry of flat connections for SU(2)-bundles on surfaces. *Math. Z.*, **221** (1996), 243–259, `hep-th/9312113`

[17] J. Huebschmann: Symplectic and Poisson structures of certain moduli spaces. *Duke Math. J.* **80** (1995), 737–756, `hep-th/9312112`

[18] J. Huebschmann: Singularities and Poisson geometry of certain representation spaces. In: Quantization of Singular Symplectic Quotients, N. P. Landsman, M. Pflaum, M. Schlichenmaier, eds., Workshop, Oberwolfach, August 1999, Progress in Mathematics, Vol. 198, Birkhäuser Verlag, Boston · Basel · Berlin, 119–135 (2001), `math.dg/0012184`

[19] J. Huebschmann: Extended moduli spaces, the Kan construction, and lattice gauge theory. *Topology* **38** (1999), 555–596

[20] J. Huebschmann: Kähler spaces, nilpotent orbits, and singular reduction. *Memoirs of the AMS*, **172 (814)** (2004), Amer. Math. Soc., Providence R.I., `math.DG/0104213`

[21] J. Huebschmann: Kähler reduction and quantization. *J. reine angew. Math.* **591** (2006), 75–109, `math.SG/0207166`

[22] J. Huebschmann: Singular Poisson-Kähler geometry of Scorza varieties and their ambient spaces. *Diff. Geometry and its Applications*, **23** (2005), 79–93, `math.DG/0405165`

[23] J. Huebschmann: Lie-Rinehart algebras, descent, and quantization. *Fields Institute Communications*, **43** (2004), 295–316 `math.SG/0303016`

[24] J. Huebschmann: Stratified Kähler structures on adjoint quotients. *J. Geom. Phys.* **58** (2008), 833–848, `math.DG/0404141`

[25] J. Huebschmann: Singular Poisson-Kähler geometry of certain adjoint quotients. In: *The mathematical legacy of C. Ehresmann*, Proceedings, Bedlewo, 2005, *Banach Center Publ.* **76** (2007), 325–347, `math.SG/0610614`

[26] J. Huebschmann: Kirillov's character formula, the holomorphic Peter-Weyl theorem and the Blattner-Kostant-Sternberg pairing. *J. Geom. Phys.* **58** (2008), 833-848; `math.DG/0610613`

[27] J. Huebschmann: Classical phase space singularities and quantization. In: *Quantum Theory and Symmetries IV*, V.K. Dobrev, ed., Heron Press, Sofia 2006, pp. 51–68, `math-ph/0610047`.

[28] J. Huebschmann, G. Rudolph and M. Schmidt: A gauge model for quantum mechanics on a stratified space. *Commun. Math. Physics* **286** (2009), 459–494, `hep-th/0702017`

[29] J. Huebschmann, G. Rudolph and M. Schmidt: Quantum mechanical tunneling on a space with singularities. In: *Lie theorey and its application in Physics. VII*, ed. V.K. Dobrev et al., Heron Press, Sofia 2008, pp. 190–210.

[30] P.D. Jarvis, J. Kijowski, G. Rudolph: On the structure of the observable algebra of QCD on the lattice. *J. Phys.* **A 38** (2005), 5359–5377

[31] D. Kastler and R. Stora: A differential geometric setting for BRS transformations and anomalies. I. II. *J. Geom. Phys.* **3** (1986), 437–482; 483–506

[32] J. Kijowski, G. Rudolph: On the structure of the observable algebra of QCD on the lattice. *J. Math. Phys.* **43** (2002), 1796–1808

[33] J. Kijowski, G. Rudolph: On the structure of the observable algebra of QCD on the lattice. *J. Math. Phys.* **46** (2005), 032303

[34] J. Kijowski, G. Rudolph, and C. Śilwa: On the structure of the observable algebra for QED on the lattice. *Lett. Math. Phys.* **43** (1998), 299–308

[35] J. Kijowski and G. Rudolph: On the Gauss law and global charge for quantum chromodynamics. *J. Math. Phys.* **43** (2002), 1796–1808

[36] J. Kijowski, G. Rudolph, and C. Śilwa: Charge superselection sectors for scalar QED on the lattice. *Ann. Inst. H. Poincaré* **4** (2003), 1–32

[37] M. S. Narasimhan and C. S. Seshadri: Stable and unitary vector bundles on a compact Riemann surface. *Ann. of Math.* **82** (1965), 540–567

[38] G. Rinehart: Differential forms for general commutative algebras. *Trans. Amer. Math. Soc.* **108** (1963), 195–222

[39] G. Rudolph, M. Schmidt and I. P. Volobuev: Classification of gauge orbit types for SU(n)-gauge theories. *Math. Phys. Anal. Geom.* **5** (2002), 201–241

[40] G. Rudolph, M. Schmidt and I. P. Volobuev: Partial ordering of gauge orbit types for SU(n)-gauge theories. *J. Geom. Phys.* **42** (2002), 106–138

[41] M. Schmidt: How to study the physical relevance of gauge orbit singularities? *Rep. Math. Phys.* **51** (2003), 325–333

[42] C. S. Seshadri: Spaces of unitary vector bundles on a compact Riemann surface. *Ann. of Math.* **85** (1967), 303–336

[43] C. S. Seshadri: Fibrés vectoriels sur les courbes algébriques. Astérisque Vol. 96, Soc. Math. de France (1982)

[44] R. Steinberg: Regular elements of semisimple algebraic groups. *Pub. Math. I. H. E. S.* **25** (1965), 49–80

[45] S. Tanimura and T. Iwai: Reduction of quantum system on Riemannian manifolds with symmetry and application to molecular mechanics. *J. of Math. Phys.* **41** (2000), 1814–1842, `math.PH/9907005`

Johannes Huebschmann
USTL, UFR de Mathématiques
CNRS-UMR 8524
59655 Villeneuve d'Ascq Cedex, France
Johannes.Huebschmann@math.univ-lille1.fr

Travaux mathématiques, Volume 19 (2011), 65–96,

Problems on asymptotic analysis over convex polytopes

by Tatsuya Tate[1]

Abstract

In this paper, a survey of results in two topics on asymptotic analysis over convex polytopes, obtained in the papers [22, 25], one of which is related to representation theory of compact Lie groups and another is asymptotic formulas of sections of a line bundle over a toric Kähler manifold, is given.

1 Introduction

Convex polytopes often appear in many areas of mathematics. In particular, they play essential roles in representation theory of compact Lie groups and the theory of toric varieties. Combinatorial aspects of polytopes describe some algebraic structures in representation theory and geometrical structures in the theory of toric varieties. In representation theory, multiplicities of weights or irreducible summands are important quantities. But, many of the well-known formulas on multiplicities are given as alternating sums, and it would not be so easy to find effective estimates for these quantities. Then, as in [12], it would be reasonable to find asymptotic formulas for these quantities. Problems on asymptotic behavior of sections of line bundles over compact Kähler manifolds are intensively investigated. They are interesting problems in themselves, and also they often provide important information for complex geometrical problems. There is an enormous literature in this direction. We just refer to [6] for this direction.

In this paper, we give a survey of results on asymptotic analysis in these two topics, obtained in the papers [22, 25]. Let us give a brief account on the materials discussed here. First, we give an asymptotic formula of a quantity called a *lattice path counting function*. This quantity is defined as the number of lattice paths on a lattice in a vector space starting from the origin each step of which is in a fixed finite subset of the lattice. This quantity is a natural generalization of the binomial coefficient, and it goes well with the probability theory. Main result for this quantity is regarded as a result on large deviation, but we also give other asymptotic formulas, for example, corresponding to the local central limit theorem.

[1]Research partially supported by JSPS Grant-in-Aid for Scientific Research (No. 21740117)

Second topic is on an asymptotic behavior of distribution functions of sections of line bundles over a compact toric Kähler manifold, which we call *toric monomials*. In general, problems on asymptotic behavior of eigenfunctions of elliptic operators with discrete spectrum is very difficult. Indeed, one of simplest problems in this direction would be to find weak limits of modulus square of eigenfunctions. But, this problem is already hard. Indeed, it is known as quantum ergodicity problems when the classical counterpart is chaotic and there are many open problems. Even if the classical dynamical system is completely integrable, this problem is still difficult to resolve completely. (To our knowledge, one can find complete answer to this problem only in the case of the standard sphere. See [15].) So then, it would be useful to find reasonable and simple 'toy model' where one can settle almost all problems in this topic, such as weak limits, estimation of supremum norm, asymptotics of L^p-norm, pointwise asymptotics and asymptotic behavior of distribution functions. The toric varieties often provide a simple model for difficult problems, and this is the case with us. Namely, the projective toric Kähler manifolds are regarded as compactified phase spaces with completely integrable systems (torus actions on toric manifolds) whose joint eigenfunctions are toric monomials. So, our toric monomials are regarded as a model of (micro-local lifts of) joint eigenfunctions for completely integrable system.

Throughout this paper, the parameter which is made to tend to infinity is denoted by N. We note here that in each topic we are going to address the parameter N has a physical meaning. For the lattice path counting functions and the multiplicities of weights, the parameter N can be regarded as 'number of particles', because it is the parameter for tensor powers of a fixed representation and the 'classical phase space' of it would be N-fold product of a coadjoint orbit. Hence the limit $N \to \infty$ would be regarded as a thermodynamic limit. For the asymptotics of distribution functions of toric monomials, the limit $N \to \infty$ represents a semiclassical limit, because it is the parameter for the tensor power of a fixed line bundle over a toric Kähler manifold.

The organization of this paper is as follows. In Section 2, we define the lattice path counting functions and investigate its properties. In particular, we give an asymptotic formula (2.14) for the lattice path counting function. The formula (2.14) is a general formula, and we then use the formula (2.14) to give various asymptotic properties of the lattice path counting functions. These asymptotic formulas are used, in Section 3, to find asymptotic formulas for multiplicities of weights and irreducibles in the high tensor powers of a fixed irreducible representation of a compact Lie group. Section 4 is devoted to the study of toric monomials of a projective smooth toric variety. In particular, we give a sketch of proof of an asymptotic formula for the rescaled distribution functions of toric monomials.

Acknowledgments This paper was written based on the lectures given by the author in the International School on Geometry and Quantization held at the University of Luxembourg (August 31 – September 5, 2009). The International

School and the subsequent International Conference was very impressive meetings. The author would like to express his special thanks to the organizers, in particular, professor Schlichenmaier for his great endeavor for the successful meetings. He would also like to thank to the administrators in the University of Luxembourg for their hospitality.

2 Asymptotic behavior of lattice path counting functions

In this section, we consider a problem on asymptotic behavior of lattice paths. In particular, we give various asymptotic results for the lattice path counting functions, which are explained their naturality along with their probabilistic background.

2.1 Lattice path counting functions

To begin with, let us prepare notation. Let X be a real vector space of dimension m, and let I be a lattice in X, that is, I is a co-compact discrete subgroup of X. Let X^* be the dual space of X, and let I^* be the dual lattice of I, that is, I^* is the lattice in X^* defined by $I^* = \{\gamma \in X^* ;\, \langle \gamma, x \rangle \in \mathbb{Z},\, x \in I\}$. Let $S \subset I^*$ be a finite subset which is assumed to satisfy the following non-degeneracy condition;

$$\mathrm{span}_{\mathbb{R}}\{\alpha - \beta ;\, \alpha, \beta \in S\} = X^*. \tag{2.1}$$

For each positive integer N, we define the set $S(N)$ of lattice paths of length N with steps in S by

$$S(N) = \{\gamma \in I^* ;\, \gamma = \beta_1 + \cdots + \beta_N \ \text{ for some } \ \beta_1, \ldots, \beta_N \in S\}. \tag{2.2}$$

Fix a positive function $c : S \to \mathbb{R}_{>0}$ on S which we call a weight function. Then, the main object in this section is the *weighted lattice path counting function* $\mathcal{P}_N^c : I^* \to \mathbb{R}$ with weight c defined by

$$\mathcal{P}_N^c(\gamma) = \begin{cases} \displaystyle\sum_{\substack{\beta_1, \ldots, \beta_N \in S \\ \gamma = \beta_1 + \cdots + \beta_N}} c(\beta_1) \cdots c(\beta_N) & \text{if } \gamma \in S(N), \\ 0 & \text{if } \gamma \notin S(N). \end{cases} \tag{2.3}$$

The function $\mathcal{P}_N^c$ often appears especially in probability theory and representation theory. We will explain a representation theoretical aspect of the function $\mathcal{P}_N^c$ in the next section. In this section, we discuss its probabilistic aspect and derive various asymptotic formulas for $\mathcal{P}_N^c$ as $N \to \infty$. Here is a typical example.

Example 2.1. Set $X = \mathbb{R}^m$ and use the standard Euclidean inner product to identify X^* with $\mathbb{R}^m$. We take the standard lattice $\mathbb{Z}^m$ for $I = I^*$. Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{Z}^m$ and let $\Sigma = \mathrm{ch}\,(0, e_1, \ldots, e_m)$. Here, for a subset $A \subset X^*$, $\mathrm{ch}\,(A)$ denotes the convex hull of A. Fix a positive integer p and set $S = (p\Sigma) \cap \mathbb{Z}^m$, the lattice points in the dilated polytope $p\Sigma$. Define the weight function $c : S \to \mathbb{R}_{>0}$ by

$$c(\beta) = \binom{p}{\beta} = \frac{p!}{\beta_1! \cdots \beta_m!(p - |\beta|)!},$$

where $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$ and $|\beta| = \sum_{j=1}^m \beta_j$. Then, it is easy to show that $S(N) = (Np\Sigma) \cap \mathbb{Z}^m$ and

$$\mathcal{P}_N^c(\gamma) = \binom{Np}{\gamma}, \quad \gamma \in S(N).$$

Remark 2.2. For our finite set S in X^*, set $P = \mathrm{ch}\,(S)$. By definition, P is a convex polytope in X^*. Clearly the set $S(N)$ of lattice paths of length N with each step in S is contained in $NP \cap I^*$. However, in general, it is not necessary to have $S(N) = (NP) \cap I^*$. Indeed, let $X = X^* = \mathbb{R}^3$, $I = I^* = \mathbb{Z}^3$ and $S = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 2)\}$. The set $S \setminus \{(0, 0, 0)\}$ forms a basis of $\mathbb{R}^3$ (but not of $\mathbb{Z}^3$), and hence P is a simplex. Then, the point $(1, 1, 1)$ is in $(2P) \cap \mathbb{Z}^3$ but not in $S(2)$. It is a bit subtle problem whether or not we have $(NP) \cap \mathbb{Z}^m = S(N)$ for general S. It is related to the (projective) normality of the toric variety defined by the finite set S. See Section 4.

2.2 Asymptotic behavior of the binomial coefficients

In Example 2.1 we see that the lattice path counting functions $\mathcal{P}_N^c$ are regarded as a generalization of the binomial (multinomial) coefficients. In elementary probability theory, asymptotic properties of the binomial coefficients $\binom{N}{k_N}$ as $N \to \infty$ are related to the (local) central limit theorem or de Moivre-Laplace theorem. (In probability theory, the parameter N is the number of Bernoulli trials.) We just remind to the readers the following asymptotic properties of the binomial coefficients. In the following, we set $d_N(k) = k - N/2$. (For the proof, one just use Stirling's formula.)

$$\binom{N}{k} \sim \begin{cases} \text{(CL)} \quad 2^N \sqrt{\frac{2}{\pi N}} e^{-\frac{2d_N(k)^2}{N}} & \text{if } d_N(k) = o(N^{2/3}), \\ \text{(MD)} \quad 2^N \sqrt{\frac{2}{\pi N}} e^{-\frac{2d_N(k)^2}{N} - \frac{Nf(2d_N(k)/N)}{2}} & \text{if } d_N(k) = o(N), \\ \text{(SD)} \quad \frac{e^{-N(a \log a + (1-a)\log(1-a))}}{\sqrt{2\pi N a(1-a)}} & \text{if } k \sim aN,\ 0 < a < 1, \\ \text{(RE)} \quad \frac{N^{k_o}}{k_o!} & \text{if } k = k_o,\ N - k_o. \end{cases} \tag{2.4}$$

Here, in the second line, the function $f(x)$ is given by

$$f(x) + x^2 = g(x) := (1+x)\log(1+x) + (1-x)\log(1-x). \tag{2.5}$$

Let us consider the above asymptotic behavior of the binomial coefficients. In the case of (CL) and (MD), the exponent in the exponential is given by

$$-\frac{N}{2}g(2d_N(k)/2) = -\frac{2d_N(k)^2}{N} - \frac{N}{2}f(2d_N(k)/N).$$

But in the case of (CL), the first term x^2 of the function $g(x)$ dominates the decay rate because $Nf(2d_N(k)/N) = o(N^{-1/3})$. We call the case (CL) central limit region for k since the asymptotic form of the binomial coefficients in this region is Gaussian. (Note that, in the central limit region, if $d_N(k) = o(N^{1/2})$, the exponent $d_N(k)^2/N$ is bounded.) In the next case (MD), called moderate deviations, the second term is of order $o(N)$ which is the same as that of the first term. Thus, one can not ignore the second term in this region. Both cases of (CL) and (MD), the growth is governed by the exponent $\log 2$. But, in the case of strong deviations (SD), the growth is governed by the positive number $a\log(1/a) + (1-a)\log(1/(1-a))$ which is strictly less than $\log 2$ if $a \neq 1/2$. Finally, in the case of (RE), which we call the region of rare events, the binomial coefficients have polynomial growth rate rather than the exponential one.

2.3 Probabilistic aspects

In the previous subsection, we described the asymptotic behavior of the binomial coefficients, which is related to the central limit theorem and other limit theorems in probability theory. In this subsection, we give an account on probabilistic aspects of the lattice path counting function $\mathcal{P}_N^c(\gamma)$. Consider the function k_S^c on X defined by

$$k_S^c(\tau) = \sum_{\alpha \in S} c(\alpha) e^{\langle \alpha, \tau \rangle}, \qquad \tau \in X. \tag{2.6}$$

Then, it is easy to show that

$$k_S^c(\tau)^N = \sum_{\gamma \in I^*} \mathcal{P}_N^c(\gamma) e^{\langle \gamma, \tau \rangle}. \tag{2.7}$$

Thus, if we set $V(S,c) = \sum_{\alpha \in S} c(\alpha)$, then $V(S,c) = k_S^c(0)$ and $V(S,c)^N = \sum_{\gamma \in I^*} \mathcal{P}_N^c(\gamma)$. Therefore, for any positive integer N, the measures

$$\begin{aligned} dm_N &= \frac{1}{V(S,c)^N} \sum_{\gamma \in I^*} \mathcal{P}_N^c(\gamma) \delta_{\gamma/N}, \\ d\mu_N &= \frac{1}{V(S,c)^N} \sum_{\gamma \in I^*} \mathcal{P}_N^c(\gamma) \delta_{\frac{1}{\sqrt{N}}(\gamma - N m_{S,c}^*)}, \end{aligned} \tag{2.8}$$

are probability measures on X^*. Here $m^*_{S,c} \in \mathrm{Int}\,(P)$ denotes the center of mass given by

$$m^*_{S,c} = \frac{1}{V(S,c)} \sum_{\alpha \in S} c(\alpha)\alpha, \tag{2.9}$$

and δ_x denotes the Dirac delta measure at x. The measure dm_N is supported on the polytope $P = \mathrm{ch}\,(S)$ while the support of the measure $d\mu_N$ is larger than P; it is supported on $\sqrt{N}P$ if the center of mass $m^*_{S,c}$ is the origin. The limit theorem we would like to mention first is the following.

Theorem 2.3. *The measures dm_N tend weakly to $\delta_{m^*_{S,c}}$ as $N \to \infty$.*

The above theorem, which is known as the *law of large numbers*, suggests that the normalized lattice path counting function $V(S,c)^{-N}\mathcal{P}^c_N(\gamma)$ decreases when γ is far from $Nm^*_{S,c}$. The measure $d\mu_N$ measures its decay when $\gamma - Nm^*_{S,c} = O(N^{1/2})$. The precise statement is given as a *central limit theorem*.

Theorem 2.4. *The measures $d\mu_N$ tend weakly to the Gaussian measure*

$$\frac{e^{-\langle A^{-1}x,x\rangle/2}}{(2\pi)^{m/2}\sqrt{\det A}}\,dx$$

as $N \to \infty$, where the positive definite symmetric matrix A is given by

$$A = \frac{1}{V(S,c)} \sum_{\alpha \in S} \alpha \otimes \alpha - m^*_{S,c} \otimes m^*_{S,c}. \tag{2.10}$$

For simplicity, we explain in the case where $m^*_{S,c}$ is the origin. If F is a subset in X^*, according to the central limit theorem, $\mu_N(F) = m_N(N^{-1/2}F)$ tends to $C\int_F e^{-\langle A^{-1}x,x\rangle/2}\,dx = CN^{-m/2}\int_{N^{1/2}F} e^{-\langle A^{-1}y,y\rangle/2N}\,dy$. Thus, when $\gamma \in N^{1/2}F$, that is $\gamma = O(N^{1/2})$, the central limit theorem suggests that, on average, the behavior of the quantity $V(S,c)^{-N}\mathcal{P}^c_N(\gamma)$ would be expressed as $CN^{-m/2}e^{-\langle A^{-1}\gamma,\gamma\rangle/2N}$. When, $\gamma - Nm^*_S = O(N)$, which means that γ is in the region of strong deviations as explained for the case of binomial coefficients, the averaged behavior of $V(S,c)^{-N}\mathcal{P}^c_N(\gamma)$ is described by the following theorem, which is known as the *large deviation principle*.

Theorem 2.5. *Set $I^c_S(x) = \sup_{\tau \in X}\{\langle x,\tau\rangle - \log(k^c_S(\tau)/V(S,c))\}$, $x \in X^*$. Then the function I_S is lower semi-continuous, and, for any closed set F and open set U of the polytope P, the measures dm_N satisfies the following.*

$$\limsup_{N\to\infty} \frac{1}{N}\log m_N(F) \le -\inf_{x\in F} I^c_S(x), \quad \liminf_{N\to\infty} \frac{1}{N}\log m_N(U) \ge -\inf_{x\in U} I^c_S(x).$$

The function I_S^c in Theorem 2.5 is called the *rate function* in the theory of large deviations. Theorem 2.5 says that when $\gamma - Nm_S^* = O(N)$ (that is, $\gamma/N \in F$ for fixed subset F), $V(S,c)^{-N}\mathcal{P}_N^c(\gamma)$ behaves, on average, like $e^{-NI_S^c(\gamma)}$. Theorems 2.3, 2.4 and 2.5 are proved by a standard method, although complete proofs can be found in [25]. In the next subsection, we derive much more precise asymptotic formulas for $V(S,c)^{-N}\mathcal{P}_N^c(\gamma)$, which support the above discussion.

2.4 Asymptotic behavior of lattice paths counting functions

The various aspects of asymptotic behavior of the binomial coefficients as explained above suggest that our weighted lattice path counting function $\mathcal{P}_N^c(\gamma)$ would also has similar asymptotic behavior. Indeed this is true. In the rest of this section, we give such results. To introduce such results, let us prepare some more notation. In the following, the setting up described in Subsection 2.1 is used. Since the differences $\alpha - \beta$ $(\alpha, \beta \in S)$ spans the whole space X^* as in the assumption (2.1), these spans over $\mathbb{Z}$ a sublattice of I^*, which is denoted by $L(S)^*$. Then, its dual lattice $L(S)$ in X contains the original lattice I. We set $Z(S) = I^*/L(S)^*$, which is a finite abelian group. It is easy to see that the Hessian of the function $\log k_S^c$,

$$A_S^c(\tau) := \nabla^2 \log k_S^c(\tau), \qquad \tau \in X, \tag{2.11}$$

is positive definite, and hence $\log k_S^c(\tau)$ is a convex function on X. By using this fact, one can show that the gradient,

$$\mu_S^c(\tau) := \nabla \log k_S^c(\tau), \qquad \tau \in X, \tag{2.12}$$

defines a *diffeomorphism* $\mu_S^c : X \to \mathrm{Int}\,(P)$, where $\mathrm{Int}\,(P)$ is the interior of the polytope $P = \mathrm{ch}\,(S)$. (Note that by the assumption (2.1), the polytope P is of dimension m.) See [7] or [8] for the proof of this fact. Denote the inverse map of $\mu_S^c : X \to \mathrm{Int}\,(P)$ by $\tau_S^c : \mathrm{Int}\,(P) \to X$. Define the smooth function δ_S^c on $\mathrm{Int}\,(P)$ by

$$\delta_S(x) = \log k_S^c(\tau_S^c(x)) - \langle\, x, \tau_S^c(x)\,\rangle, \qquad x \in \mathrm{Int}\,(P). \tag{2.13}$$

Theorem 2.6. *Take $x_o \in \mathrm{Int}\,(P)$ and $\gamma_N \in (NP) \cap I^*$ such that $\gamma_N = Nx_o + o(N)$. Then, we have*

$$\mathcal{P}_N^c(\gamma_N) = (2\pi N)^{-m/2} \frac{|Z(S)| e^{N\delta_S^c(\gamma_N/N)}}{\sqrt{\det A_S^c(\tau_S^c(\gamma_N/N))}} (1 + O(N^{-1})). \tag{2.14}$$

In particular, we have

$$\lim_{N\to\infty} \frac{1}{N} \log \mathcal{P}_N^c(\gamma_N) = \delta_S^c(x_o). \tag{2.15}$$

Remark 2.7. In [25], the assumption that, the lattice points γ_N is in $S(N)$ for every sufficiently large N, is imposed. However, in the proof of Theorem 2.6, we only use the integral formula (2.20) below, and this comes from the fact that $\mathcal{P}_N^c(\gamma)$ is the coefficients of $e^{i\langle \gamma, \varphi \rangle}$ in the Fourier series $k_S^c(i\varphi)^N$. Thus, we do not need to impose such an assumption.

Remark 2.8. It is easy to show that the rate function I_S^c in the large deviation principle (Theorem 2.5) is given by

$$I_S^c(x) = -\delta_S^c(x) + \log V(S,c), \quad x \in \mathrm{Int}\,(P).$$

Example 2.9. Let us examine the formula (2.15) for the binomial coefficients. As in Example 2.1 with $m = 1$, let p be a positive integer, and let $S = \Sigma \cap \mathbb{Z} = \{0, 1, \dots, p\}$ with $\Sigma = \mathrm{ch}\,(0,p) = [0,p]$. Define $c : S \to \mathbb{R}_{>0}$ by $c(\beta) = \binom{p}{\beta}$. Then, as in Example 2.1, we have $\mathcal{P}_N^c(\gamma) = \binom{Np}{\gamma}$. In this case, the finite abelian group $Z(S)$ is trivial. The function k_S^c on $X^* = \mathbb{R}$ is given by $k_S^c(\tau) = (1+e^\tau)^p$, and hence

$$\mu_S^c(\tau) = \frac{pe^\tau}{1+e^\tau}, \quad \tau_S^c(x) = \log\left(\frac{x}{p-x}\right), \quad x \in \mathrm{Int}\,(P),\ \tau \in X = \mathbb{R}.$$

This shows that the function δ_S^c in this case is given by

$$\delta_S^c(x) = \log\left(\frac{p^p}{x^x(p-x)^{p-x}}\right), \qquad x \in \mathrm{Int}\,(P) = (0,p).$$

Then, the formula (2.15) can be deduced easily from Stirling's formula.

In the previous sections, we saw that $\mathcal{P}_N^c(\gamma_N)$ behaves differently when γ_N have different behavior as $N \to \infty$. In turn, Theorem 2.6 contains only one asymptotic formula (2.14). However, the asymptotic formula (2.14) in Theorem 2.6 is rather general. Indeed one can prove various asymptotic results from this formula similar to what was described for binomial coefficients. Let us explain how one can deduce them from just one formula (2.14). First, we note that since the weight function c is positive everywhere on the finite set S, the center of mass $m_{S,c}^*$ defined in (2.9) is in $\mathrm{Int}\,(P)$. Hence we can take $\gamma_N = Nm_S^* + d_N$ with $d_N = o(N^s)$, $0 \le s \le 2/3$ for the sequence γ_N in Theorem 2.6. A direct computation using the Taylor expansions around $x = m_{S,c}^*$ of the functions $\sqrt{\det A_S^c(\tau_S^c(x))}$ and $\delta_S^c(x)$ show that

$$\sqrt{\det A_S^c(\tau_S^c(\gamma_N/N))} = \sqrt{\det A}(1 + O(N^{-(1-s)})),$$
$$N\delta_S^c(\gamma_N/N) = N\log V(S,c) - \langle A^{-1}d_N, d_N \rangle/(2N) + o(N^{3s-2}),$$

where the symmetric matrix A is defined in (2.10). From these combined with the formula (2.14), we obtain the following local central limit theorem.

Theorem 2.10. *Let* $0 \leq s \leq 2/3$ *and* $\gamma_N = N m^*_{S,c} + d_N$ *with* $d_N = o(N^s)$. *Then, we have*

$$\mathcal{P}^c_N(\gamma_N) = (2\pi N)^{-m/2} \frac{|Z(S)| V(S,c)^N e^{-\langle A^{-1} d_N, d_N \rangle/(2N)}}{\sqrt{\det A}} (1 + o(N^{3s-2})).$$

Next, we take $\gamma_N = N\alpha + f$ for some $f \in L(S)^*$ and $\alpha \in S \cap \mathrm{Int}\,(P)$. Then, one can apply directly Theorem 2.6. But in this case, one can also take $\gamma_N = N\alpha$ in the sketch of proof of Theorem 2.6 explained in the next subsection, and one has the following.

Theorem 2.11. *Let* $f \in L(S)^*$ *and let* $\alpha \in S \cap \mathrm{Int}\,(P)$. *Then we have*

$$\mathcal{P}^c_N(N\alpha + f) = (2\pi N)^{-m/2} \frac{|Z(S)| e^{-\langle f, \tau^c_S(\alpha) \rangle + N\delta^c_S(\alpha)}}{\sqrt{\det A^c_S(\tau^c_S(\alpha))}} (1 + O(N^{-1})).$$

Remark 2.12. Theorem 2.11 is a result on large deviations. For results on large deviations in a more general setting and from a geometrical point of view, see [16].

2.5 Method of stationary phase and sketch of proof

In this subsection, we give a sketch of proof of Theorem 2.6. To prove Theorem 2.6, we use a theorem on the method of stationary phase. First of all, let us give some account on this method.

Let $U \subset \mathbb{R}^m$ be an open set. Let $u \in C^\infty_0(U)$ and $\Phi \in C^\infty(U)$ with $\mathrm{Re}\,\Phi \geq 0$. Consider the following integral

$$I_N(u) = \int_U e^{-N\Phi(x)} u(x)\, dx. \tag{2.16}$$

We call the function Φ the phase function for the integral $I_N(u)$. The method of stationary phase is a method for studying asymptotic behavior of the integral of the form $I_N(u)$ as $N \to \infty$. To explain this method, suppose first that $\nabla\Phi \neq 0$ near $\mathrm{supp}\,(u)$. In this case, the first order differential operator,

$$L = -\frac{1}{|\nabla\Phi(x)|^2} \sum_{j=1}^m \overline{\frac{\partial\Phi}{\partial x_j}} \frac{\partial}{\partial x_j},$$

is well-defined near $\mathrm{supp}\,(u)$, where $|\nabla\Phi(x)|^2 = \sum_{j=1}^m \left|\frac{\partial\Phi}{\partial x_j}\right|^2$. Then it is straightforward to see that $L(e^{-N\Phi}) = Ne^{-N\Phi}$. Substituting this into the definition (2.16) of the integral $I_N(u)$ and integrating by parts show

$$I_N(u) = \frac{1}{N} \int_U e^{-N\Phi} ({}^t L u)\, dx,$$

where ${}^tL$ is the adjoint operator of L given by ${}^tLu = \sum_{j=1}^m \frac{\partial}{\partial x_j}\left(\frac{1}{|\nabla\Phi|^2}\overline{\frac{\partial\Phi}{\partial x_j}}u\right)$. Repeating this procedure, one has $I_N(u) = O(N^{-\infty})$, namely, for any positive integer k, one has $|I_N(u)| \leq C_k N^{-k}$ with a positive constant C_k.

Next, suppose that the phase function Φ satisfies $\mathrm{Re}\,(\Phi) > 0$ near $\mathrm{supp}\,(u)$. Then, since $\mathrm{supp}\,(u)$ is assumed to be compact, one can find a positive constant α such that $\mathrm{Re}\,(\Phi) \geq \alpha$ near $\mathrm{supp}\,(u)$. This shows that $I_N(u) = O(e^{-N\alpha})$. Therefore, one find that the contribution to $I_N(u)$ as $N \to \infty$ comes from neighborhoods of points $x \in U$ where $\mathrm{Re}\,(\Phi)(x) = 0$ and $\nabla\Phi(x) = 0$. Traditional method of stationary phase considers the case where the phase function Φ is pure imaginary, namely $\mathrm{Re}\,(\Phi) \equiv 0$. In this case, we set $\Phi = i\phi$ with a real-valued function ϕ. Suppose also that there exists a point $x_o \in U$ such that $\nabla\phi(x_o) = 0$, the Hessian $\nabla^2\phi(x_o)$ is non-degenerate and $\nabla\phi(x) \neq 0$ for points x different from x_o. Then, by the Morse lemma, there exists a neighborhood V of x_o and a diffeomorphism $\kappa : V \to \kappa(V) \subset \mathbb{R}^m$ such that $\kappa(x_o) = 0$, $\nabla\kappa(x_o) = \mathrm{Id}$ and

$$\phi\circ\kappa^{-1}(y) = \phi(x_o) + \langle\, Ay, y\,\rangle/2, \quad A := \nabla^2\phi(x_o).$$

Changing a variable $x = \kappa(y)$ will show that $I_N(u) = \tilde{I}_N(\tilde{u}) + O(N^{-\infty})$, where $\tilde{u}$ equals $|\det\nabla\kappa^{-1}(y)|u(\kappa^{-1}y)$ times a cut-off function near the origin and

$$\tilde{I}_N(\tilde{u}) = e^{-iN\phi(x_o)}\int_{\kappa(V)} e^{-iN\langle\, Ay,y\,\rangle/2}\tilde{u}(y)\,dy.$$

Using Plancherel formula and the well-known formula,

$$\mathcal{F}^{-1}[e^{-iN\langle\, Ay,y\,\rangle/2}](\xi) = \frac{e^{-i\pi\,\mathrm{sgn}(A)/4}}{(2\pi N)^{m/2}|\det A|^{1/2}}e^{i\langle\, A^{-1}\xi,\xi\,\rangle/(2N)},$$

where $\mathcal{F}^{-1}$ is the inverse of the Fourier transform $\mathcal{F}$, shows that the integral $I_N(u)$ equals

$$\frac{e^{-iN\phi(x_o)-i\pi\,\mathrm{sgn}(A)/4}}{(2\pi N)^{m/2}|\det A|^{1/2}}\int_{\mathbb{R}^m} e^{i\langle\, A^{-1}\xi,\xi\,\rangle/(2N)}\widehat{\tilde{u}}(\xi)\,d\xi$$

modulo terms of order $O(N^{-\infty})$. Then, a Taylor expansion of the exponential function shows that the integral $I_N(u)$ has the following asymptotic expansion

$$I_N(u) \sim \left(\frac{2\pi}{N}\right)^{m/2}\frac{e^{-iN\phi(x_o)-i\pi\,\mathrm{sgn}(\nabla^2\phi(x_o))/4}}{|\det\nabla^2\phi(x_o)|^{1/2}}\sum_{k\geq 0}(A_k u)(x_o)N^{-k}, \tag{2.17}$$

where A_k is a differential operator of order $2k$ with $A_0 = I$. This is a usual method of stationary phase. However, in the above lines, we used the Morse lemma, and the differential operators A_k contains derivatives of the diffeomorphism κ. Hence it is not suitable to compute lower order terms explicitly. Furthermore, as is seen below, in our case, the phase function Φ itself depends on the parameter N. So, one can not apply, at least directly, the above method. Fortunately, there is a version of the method of stationary phase which is quite useful.

Theorem 2.13. *Let U be an open set in $\mathbb{R}^m$ and let K be a compact set in U. Let $u \in C_0^\infty(K)$ and let $\Phi \in C^\infty(U)$ such that $\mathrm{Re}\,(\Phi) \geq 0$. Suppose that there exists a point $x_o \in K$ such that $\mathrm{Re}\,(\Phi(x_o)) = 0$, $\nabla\Phi(x_o) = 0$, $\det \nabla^2\Phi(x_o) \neq 0$ and that $\nabla\Phi(x) \neq 0$ for $x \in K$ different from x_o. Then, for each positive integer k,*

$$I_N(u) = \left(\frac{2\pi}{N}\right)^{m/2} \frac{e^{-N\Phi(x_o)}}{\sqrt{\det \nabla^2\Phi(x_o)}} \sum_{j=0}^{k-1} (L_j u)(x_o) N^{-j} + R_k(N), \tag{2.18}$$

with the error estimate

$$|R_k(N)| \leq C_k(\Phi) \|u\|_{C^{2k}(U)} N^{-k}, \tag{2.19}$$

where $C_k(\Phi)$ is a positive constant. The differential operator L_j at x_o is given by

$$(L_j u)(x_o) = (-1)^j \sum_{\substack{\nu,\mu\geq 0 \\ \nu-\mu=j, 2\nu\geq 3\mu}} \frac{1}{2^\nu \mu! \nu!} [\langle\, \nabla^2\Phi(x_o)^{-1} D, D\,\rangle^\nu (g_\Phi^\mu u)](x_o),$$

$$g_\Phi(x) = \Phi(x) - \Phi(x_o) - \frac{1}{2}\langle\, \nabla^2\Phi(x_o)(x - x_o), x - x_o\,\rangle.$$

Furthermore, suppose that B is a subset of $C^\infty(U)$ such that;

- *every $\Phi \in B$ satisfies $\mathrm{Re}\,(\Phi) \geq 0$, $\mathrm{Re}\,(\Phi(x_o)) = 0$, $\nabla\Phi(x_o) = 0$, $\det \nabla^2\Phi(x_o) \neq 0$ and that $\nabla\Phi(x) \neq 0$ for $x \in K$ different from x_o, where x_o is fixed;*
- *$\|\Phi\|_{C^{3k+1}(U)}$ is bounded from above uniformly in $\Phi \in B$;*
- *$|x - x_o|/|\nabla\Phi(x)|$ is bounded from above uniformly in $x \in U$ and $\Phi \in B$.*

Then, the constant $C_k(\Phi)$ in (2.19) *can be taken to be independent of $\Phi \in B$.*

See [13, Section 7] for the proof of the above theorem.

Remark 2.14. It is easy to see that the third condition for $B \subset C^\infty(U)$ in Theorem 2.13 can be replaced by $\sup_{\Phi\in B} \|\nabla^2\Phi(x_o)^{-1}\| < \infty$. More precisely, if $\|\Phi\|_{C^3} \leq \alpha$ and $\|\nabla^2\Phi(x_o)^{-1}\| \leq \beta$ for each $\Phi \in B$, then, one can show that, when $|x - x_o| \leq 1/2\alpha\beta$, we have $\dfrac{|x - x_o|}{|\nabla\Phi(x)|} \leq 2\beta$. In particular, in the case where we can shrink the domain of integration suitably, the assumption that $\nabla\Phi(x) \neq 0$ for x different from x_o is satisfied if the Hessian at x_o is non-degenerate and its inverse is bounded from above.

We now give a sketch of proof of Theorem 2.6. We extend elements in X^* to the complex linear form on $X \otimes \mathbb{C}$. We write elements in $X \otimes \mathbb{C}$ as $w = \tau + i\varphi$, $\tau, \varphi \in X$. Then, the function k_S^c on X is naturally extended to $X \otimes \mathbb{C}$, and by (2.7), the lattice path counting function $\mathcal{P}_N^c(\gamma)$ has the following integral representation,

$$\mathcal{P}_N^c(\gamma) = \frac{1}{(2\pi)^m} \int_{T^m} e^{-i\langle\, \gamma,\varphi\,\rangle} k(i\varphi)^N \, d\varphi, \tag{2.20}$$

where $T^m = X/2\pi I$ is an m-dimensional torus, and the Lebesgue measure $d\varphi$ is normalized so that the volume of T^m is $(2\pi)^m$. Since the function $k(z) = k_S^c(\tau + i\varphi)$ ($z = e^{\tau+i\varphi}$, $\tau, \varphi \in X$) is holomorphic on the complex torus $\exp(X \otimes \mathbb{C}) \cong (\mathbb{C}^*)^m$, we can change the contour in the integral to obtain

(2.21)

$$\mathcal{P}_N^c(\gamma_N) = \frac{1}{(2\pi)^m}[k_S^c(\tau)e^{-\langle \gamma_N/N, \tau \rangle}]^N \int_{T^m} e^{-iN\langle \gamma_N/N, \varphi \rangle} \left(\frac{k_S^c(\tau + i\varphi)}{k_S^c(\tau)} \right)^N d\varphi,$$

which is valid for arbitrary $\tau \in X$. It is easy to show that $|k_S^c(\tau + i\varphi)| \leq k_S^c(\tau)$, and the equality holds if and only if $\varphi \in 2\pi L(S)$. Since $I \subset L(S)$, there is a natural surjective homomorphism $\pi_S : T^m \to T(S) := X/2\pi L(S)$. Then, the above equality condition is equivalent to say that $|k_S^c(\tau + i\varphi)| = k_S^c(\tau)$ if and only if $\varphi \,(\bmod\, 2\pi I)$ is in the kernel $\ker(\pi_S)$ of π_S, and which is naturally isomorphic to the finite abelian group $Z(S)$. Since (2.21) holds for any $\tau \in X$, we choose τ as $\tau_N = \tau_S^c(\gamma_N/N)$. Then, we have

$$e^{\delta_S^c(\gamma_N/N)} = k_S^c(\tau_N)e^{-\langle \gamma_N/N, \tau_N \rangle}.$$

We take a neighborhood U of the identity $0 \in \ker(\pi_S) \cong Z(S)$ so that $U \cap \ker(\pi_S) = \{0\}$ and take a cut-off function $\rho \in C_0^\infty(U)$ which is 1 near $\varphi = 0$. For any $g \in \ker(\pi_S)$, we set $U_g = U + g$ and $\rho_g(\varphi) = \rho(\varphi - g)$. Then, if we take U so small, there exists a constant $a > 0$ such that

$$\mathcal{P}_N^c(\gamma_N) = \frac{e^{N\delta_S^c(\gamma_N/N)}}{(2\pi)^m} \sum_{g \in \ker(\pi_S)} I_{N,g} \tag{2.22}$$

modulo terms of order $O(e^{-aN})$, where $I_{N,g}$ is given by

(2.23)

$$I_{N,g} = \int_{U_g} e^{-N\Phi_{N,g}(\varphi)} \rho_g(\varphi)\, d\varphi, \quad \Phi_{N,g} = i\langle \gamma_N/N, \varphi \rangle - \log\left(\frac{k_S^c(\tau_N + i\varphi)}{k_S^c(\tau_N)} \right).$$

Note that, if we introduce the function

$$\Phi(\tau, \varphi) := i\langle \mu_S^c(\tau), \varphi \rangle - \log\left(\frac{k_S^c(\tau + i\varphi)}{k_S^c(\tau)} \right)$$

on $B \times U_g$, where B is a closed ball with center $\tau_S^c(x_o)$, then we have $\Phi_{N,g}(\varphi) = \Phi(\tau_N, \varphi)$. From this expression, one can show that $\|\Phi_{N,g}\|_{C^k(U_g)}$ is bounded from above independently of N, where k is any integer greater than $[m/2]+1$. We take a representative $\varphi_g \in X$ of $g \in \ker(\pi_S)$ and identify U_g with a neighborhood of φ_g. Note that $\operatorname{Re}\Phi(\tau, \varphi) \geq 0$ and the equality holds for $(\tau, \varphi) \in B \times \overline{U_g}$ if and only if $\varphi = \varphi_g$. Then, we see that $\nabla\Phi_{N,g}(\varphi_g) = 0$. Furthermore, $\operatorname{Re}\Phi_{N,g}(\varphi) = 0$ on U_g if and only if $\varphi = \varphi_g$. A direct computation shows $e^{N\Phi_{N,g}(\varphi_g)} = 1$ and $\nabla^2\Phi_{N,g}(\varphi_g) = A_S^c(\tau_N)$. Since τ_N tends to $\tau_S^c(x_o)$ and since $x_o \in \operatorname{Int}(P)$, $A_S^c(\tau_N)$ has inverse whose norm is bounded uniformly in N. Therefore, Theorem 2.13 is applied and a direct computation shows Theorem 2.6.

3 Asymptotics of multiplicities in high tensor powers

In Section 2, we derived asymptotic formula for the lattice path counting function $\mathcal{P}_N^c(\gamma)$ for general weight function c on the set S of steps. In this section, we give an application of the formula in Theorem 2.11 to representation theory of compact connected Lie groups.

3.1 Quick review of representation theory of compact Lie groups

The representations we are going to consider is them for compact connected Lie groups. For structure theory and representation theory of compact Lie groups, we refer the readers to [3]. In the following we prepare and review some terminology for representation theory of compact Lie groups. Let G be a compact connected Lie group. For simplicity, we assume that G is semi-simple, that is, assume that the center $Z(G)$ of G is finite. Let T be a maximal torus in G and let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebra of G and T, respectively. Let $\mathfrak{g}^*$ and $\mathfrak{t}^*$ be the dual space of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. Since T is abelian and compact, the exponential map $\exp : \mathfrak{t} \to T$ is a surjective homomorphism, and its kernel $I = \ker(\exp)$ is a lattice in $\mathfrak{t}$ so that $T = \mathfrak{t}/I$. The dual lattice I^* of I is called the weight lattice or the lattice of integral forms. The Weyl group W is the quotient group $N(T)/T$ of the normalizer $N(T)$ of T by T, which is known to be a finite group. The maximal torus T acts on the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ by the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}^{\mathbb{C}})$, and then $\mathfrak{g}^{\mathbb{C}}$ is decomposed into irreducible components of T-action as

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where, for each $0 \neq \alpha \in I^*$, we set

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}^{\mathbb{C}}\,;\, \mathrm{Ad}\,(\exp(\varphi))x = e^{2\pi i \langle \alpha, \varphi \rangle} x, \varphi \in \mathfrak{t}\}.$$

The set R, called the root system, is defined by $R = \{\alpha \in \mathfrak{t}^* \setminus \{0\}\,;\, \mathfrak{g}_\alpha \neq 0\}$. The elements in R are called the roots. Since we have assumed that the group G is semi-simple, we have $\mathrm{span}_{\mathbb{R}}(R) = \mathfrak{t}^*$. Furthermore, there exists a basis $\{\alpha_1, \dots, \alpha_m\} \subset R$ of $\mathfrak{t}^*$ ($m = \dim T$) such that each $\alpha \in R$ can be represented as a $\mathbb{Z}$-linear combination of $\{\alpha_1, \dots, \alpha_m\}$,

$$\alpha = \sum_{j=1}^m n_j(\alpha)\alpha_j,$$

where $n_j(\alpha) \geq 0$ for all j or $n_j(\alpha) \leq 0$ for all j. Such a basis $\{\alpha_1, \dots, \alpha_m\} \subset R$ is called a system of simple roots. Let $R_+ = \{\alpha \in R\,;\, n_j(\alpha) \geq 0,\ j = 1, \dots, m\}$ and set $R_- = -R_+$. Then, it is well-known that $R_- \subset R$ and $R = R_+ \cup R_-$ (disjoint union). The elements in R_+ are called the positive roots. The Weyl group W is finite and acts on $\mathfrak{t}$ and $\mathfrak{t}^*$. We choose a W-invariant inner product on $\mathfrak{t}$ which is denoted $\langle \cdot, \cdot \rangle$, and this inner product naturally induces a W-invariant inner product on $\mathfrak{t}^*$ which we continue to write as $\langle \cdot, \cdot \rangle$. A positive Weyl chamber, denoted by C, is a cone in $\mathfrak{t}^*$ defined by

$$C = \{\varphi \in \mathfrak{t}^* \,;\, \langle \varphi, \alpha \rangle > 0,\ \alpha \in R_+\}.$$

Then, the famous Weyl character formula states that there exists a bijection between $\overline{C} \cap I^*$ and the set of characters (restricted to the maximal torus T) of irreducible representations of G. Furthermore, for each $\lambda \in \overline{C} \cap I^*$, the corresponding irreducible representation, denoted by V_λ, has the character χ_λ on T given by

$$\chi_\lambda(t) = \frac{A(\lambda+\rho)(\varphi)}{\Delta(\varphi)}, \quad t = \exp(\varphi) \in T,\ \varphi \in \mathfrak{t}, \tag{3.1}$$

where ρ is half the sum of the positive roots, $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$, and for $\alpha \in \mathfrak{t}^*$, the alternating sum $A(\alpha)$ is defined by

$$A(\alpha)(\varphi) = \sum_{w \in W} \operatorname{sgn}(w) e^{2\pi i \langle w\alpha, \varphi \rangle}, \quad \varphi \in \mathfrak{t}, \tag{3.2}$$

where $\operatorname{sgn}(w)$ is the determinant of the transformation $w : \mathfrak{t} \to \mathfrak{t}$. The function $\Delta(\varphi)$ is defined by $\Delta(\varphi) = A(\rho)(\varphi)$, which is called the Weyl denominator. Note that, since W preserves the inner product on $\mathfrak{t}$, $\operatorname{sgn}(w) = \pm 1$. The integral form $\lambda \in \overline{C} \cap I^*$ is called the dominant weight of the irreducible representation V_λ.

3.2 Multiplicities in high tensor powers

Let V_λ be the irreducible representation of G with the dominant weight $\lambda \in \overline{C} \cap I^*$, and let N be a positive integer. Then, the tensor product $V_\lambda^{\otimes N}$ is a representation space of T, and hence one has a weight space decomposition

$$V_\lambda^{\otimes N} = \bigoplus_{\mu \in I^*} V_{\lambda,N}(\mu), \quad V_{\lambda,N}(\mu) = \{v \in V_\lambda^{\otimes N} \,;\, \exp(\varphi) v = e^{2\pi i \langle \mu, \varphi \rangle} v,\ \varphi \in \mathfrak{t}\}. \tag{3.3}$$

We set

$$m_N(\lambda; \mu) = \dim_{\mathbb{C}} V_{\lambda,N}(\mu), \quad \mu \in I^*,$$

and call $m_N(\lambda; \mu)$ the multiplicity of the weight μ in $V_\lambda^{\otimes N}$. The space $V_\lambda^{\otimes N}$ is also a representation space of the compact Lie group G, and hence it can be decomposed into irreducible summands,

$$V_\lambda^{\otimes N} = \bigoplus_{\mu \in \overline{C} \cap I^*} a_N(\lambda; \mu) V_\mu,$$

where $a_N(\lambda;\mu) \in \mathbb{Z}_+$ is the number of times V_μ appears in $V_\lambda^{\otimes N}$. We call $a_N(\lambda;\mu)$ the multiplicity of the irreducible representation V_μ in $V_\lambda^{\otimes N}$. A natural problem in this setting is whether or not one can find an effective formula for the multiplicities $m_N(\lambda;\mu)$ or $a_N(\lambda;\mu)$ in terms of N, λ and μ. For example, when $N = 2$, Steinberg's formula states that $a_2(\lambda;\mu)$ can be written as

$$a_2(\lambda;\mu) = \sum_{v,w\in W} \operatorname{sgn}(vw)\mathfrak{p}(v(\lambda+\rho) + w(\lambda+\rho) - (\mu+2\rho)), \tag{3.4}$$

where $\mathfrak{p}$ is Kostant's partition function,

$$\mathfrak{p}(\lambda) = \sharp\left\{ (n_\alpha | \alpha \in R_+)\,;\, n_\alpha \in \mathbb{Z}_{\geq 0},\ \lambda = \sum_{\alpha\in R_+} n_\alpha \alpha \right\}. \tag{3.5}$$

One can also use Steinberg's formula repeatedly to represent $a_N(\lambda;\mu)$ in terms of Kostant's partition function. However, this formula is an alternating sum and it is easy to imagine that the result becomes quite complicated as N becomes large. Hence it would not be so easy to estimate how large the multiplicity $a_N(\lambda;\mu)$ is from this formula for large N.

In the rest of this section, we give results on the asymptotics of the multiplicities $m_N(\lambda;\mu)$ and $a_N(\lambda;\mu)$ which is an application of Theorem 2.11. For any $\lambda \in \overline{C} \cap I^*$, define $S_\lambda = \{\mu \in I^*\,;\, m_1(\lambda;\mu) \neq 0\}$. Namely, S_λ is the set of weights occurring in the irreducible representation V_λ. Let $P(\lambda)$ denote the convex hull of the orbit $W\cdot\lambda$ of the Weyl group through λ. Let Λ^* be the lattice in $\mathfrak{t}^*$ spanned by the root system R over $\mathbb{Z}$, which is often called the root lattice. Define the map $\mu_\lambda : \mathfrak{t} \to \mathfrak{t}^*$ by

$$\mu_\lambda(x) = \frac{1}{\sum_{\nu\in S_\lambda} m_1(\lambda;\nu)e^{\langle \nu,x\rangle}} \sum_{\mu\in S_\lambda} m_1(\lambda;\mu)e^{\langle \mu,x\rangle}\mu. \tag{3.6}$$

It is well-known that $W\cdot\lambda \subset S_\lambda \subset P(\lambda)$. Since each coefficient of $\mu \in S_\lambda$ in the definition the map μ_λ is positive, the image of the map μ_λ is contained in the (relative) interior $\operatorname{Int}(P(\lambda))$ of the polytope $P(\lambda)$. Furthermore, as for the case of the map μ_S^c defined by (2.12), it turns out that the map μ_λ is a diffeomorphism from $\mathfrak{t}$ onto $\operatorname{Int}(P(\lambda))$ if the dominant weight λ is in the interior C of the closed positive Weyl chamber $\overline{C}$. (See below for this point.) Denote by $\tau_\lambda : \operatorname{Int}(P(\lambda)) \to \mathfrak{t}$ the inverse of the map μ_λ.

Theorem 3.1. *Let $\lambda \in C \cap I^*$ and $\nu_o \in S_\lambda$. Suppose that ν_o is in the interior of the polytope $P(\lambda)$. Take $f \in \Lambda^*$. Then, we have the following formula.*

$$m_N(\lambda; N\nu_o + f) = (2\pi N)^{-m/2}\frac{|Z(G)|e^{N\delta_\lambda(\nu_o) - \langle f,\tau_\lambda(\nu_o)\rangle}}{\sqrt{\det A_\lambda(\nu_o)}}(1+O(N^{-1})), \tag{3.7}$$

where $Z(G)$ *is the center of* G*, and the function* δ_λ *on* $\mathrm{Int}\,(P(\lambda))$ *and the positive definite matrix* $A_\lambda(\nu_o)$ *is given by*

$$\delta_\lambda(x) = \log\left(\sum_{\mu\in S_\lambda} m_1(\lambda;\mu)e^{\langle\,\mu,\tau_\lambda(x)\,\rangle}\right) - \langle\, x,\tau_\lambda(x)\,\rangle, \quad x\in \mathrm{Int}\,(P(\lambda)),$$

$$A_\lambda(\nu_o) = \sum_{\mu\in S_\lambda}\frac{m_1(\lambda;\mu)e^{\langle\,\mu,\tau_\lambda(\nu_o)\,\rangle}}{\sum_{\nu\in S_\lambda} m_1(\lambda;\nu)e^{\langle\,\nu,\tau_\lambda(\nu_o)\,\rangle}}\mu\otimes\mu - \nu_o\otimes\nu_o.$$

Note that we have some other asymptotic results for the multiplicities $m_N(\lambda;\mu)$ of weights in tensor power $V_\lambda^{\otimes N}$. See [25]. By using Theorem 3.1, one can find the following asymptotic result for the multiplicities of irreducible representations in $V_\lambda^{\otimes N}$.

Theorem 3.2. *Let* $\lambda\in C\cap I^*$ *and let* $\nu_o\in\overline{C}\cap S_\lambda\cap\mathrm{Int}\,(P(\lambda))$*. Then, we have the following formula.*

(3.8)

$$a_N(\lambda;N\nu_o) = (2\pi N)^{-m/2}e^{N\delta_\lambda(\nu_o)}\left(\frac{|Z(G)|\Delta(\tau_\lambda(\nu_o)/(2\pi i))e^{-\langle\,\rho,\tau_\lambda(\nu_o)\,\rangle}}{\sqrt{\det A_\lambda(\nu_o)}} + O(N^{-1})\right),$$

where the Weyl denominator Δ *is extended to the complexification* $\mathfrak{t}\otimes\mathbb{C}$*.*

Remark 3.3. By using the Weyl denominator formula, we have

$$\Delta(\tau_\lambda(\nu_o)/(2\pi i)) = \prod_{\alpha\in R_+}\left(e^{\langle\,\alpha,\tau_\lambda(\nu_o)\,\rangle/2} - e^{-\langle\,\alpha,\tau_\lambda(\nu_o)\,\rangle/2}\right).$$

So, for example, when $\tau_\lambda(\nu_o)$ is in a wall of a Weyl chamber, that is, there is a $\alpha\in R_+$ such that $\langle\,\alpha,\tau_\lambda(\nu_o)\,\rangle = 0$, we have $\Delta(\tau_\lambda(\nu_o)/(2\pi i)) = 0$, and hence the leading term in the asymptotic formula (3.8) vanishes. In this case, the formula (3.8) is not relevant to estimate the multiplicity $a_N(\lambda;N\nu_o)$.

Remark 3.4. Theorems 3.1 and 3.2 are formulas in large deviation. The local central limit theorems for the multiplicities $a_N(\lambda;\mu)$, $m_N(\lambda;\mu)$ also hold true. See [2], [25] for these formulas.

3.3 Multiplicities versus lattice path counting functions

In this subsection, we give a sketch of proof of Theorems 3.1 and 3.2. Indeed, these are proved by using the asymptotic formula in Theorem 2.11 of lattice path counting function $\mathcal{P}_N^c(\gamma)$. Let us explain how the lattice path counting function comes into the discussion. To use the lattice path model, we need to specify the vector space X, the lattice I, the finite set S in the dual lattice I^* satisfying the non-degeneracy condition (2.1) and the weight function $c: S\to\mathbb{R}_{>0}$.

In the representation theoretical setting, we take $\mathfrak{t}$ for the vector space X and the integer lattice ker(exp) for the lattice I. The finite set S is the set S_λ of weights occurring in the fixed irreducible representation V_λ. We define the weight function c_λ on S_λ by setting $c_\lambda(\mu) = m_1(\lambda;\mu)$. Then, we can consider the lattice path counting function $\mathcal{P}_N^\lambda = \mathcal{P}_N^{c_\lambda}$ on I^*. To be precise, we need to check that S_λ satisfies the condition (2.1) and to specify the lattice $L(S_\lambda)^*$. Let $\{\alpha_1, \dots, \alpha_m\}$ be the system of simple roots which defines the fixed positive Weyl chamber C. We identify $\mathfrak{t}$ and $\mathfrak{t}^*$ by using the fixed W-invariant inner product. For any root $\alpha \in R$, define $\alpha^\vee = \frac{2}{\langle \alpha,\alpha \rangle}\alpha \in \mathfrak{t}$. The vector $\alpha^\vee$ is called an inverse root (or co-root). It is well-known that the set $R^\vee$ of all $\alpha^\vee$, $\alpha \in R$, is again a root system, but what we need is the fact that $R^\vee$ is contained in the integer lattice I. From this and the assumption that $\lambda \in C \cap I^*$, that is λ is not in the wall of the chamber C, the pairing $\langle \lambda, \alpha^\vee \rangle$ is a positive integer for each $\alpha \in R_+$. The reflection $s_\alpha : \mathfrak{t}^* \to \mathfrak{t}^*$ with respect to α is given by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \quad x \in \mathfrak{t}^*.$$

The reflection with respect to the simple roots α_j $(j = 1, \dots, m)$ is denoted by s_j. Then, it is well-known that $s_j \in W$ (actually, s_j's generate W) and the set of weights S_λ in V_λ is invariant under the action of the Weyl group W. Since, $\lambda \in S_\lambda$ with $m_1(\lambda;\lambda) = 1$, we see

$$\alpha_j = \frac{1}{\langle \lambda, \alpha_j^\vee \rangle}(\lambda - s_j(\lambda)) \in \operatorname{span}_{\mathbb{R}}\{\mu - \nu\,;\, \mu, \nu \in S_\lambda\}.$$

This shows that S_λ satisfies the condition (2.1). Indeed, one can say more. It is well-known that $\lambda - j\alpha$ is contained in S_λ for any positive root α and any integer j with $0 \le j \le \langle \lambda, \alpha^\vee \rangle$ (see [14]). This shows that the lattice $L(S_\lambda)^*$ coincides with the root lattice Λ^*. Then, the finite group $Z(S_\lambda)$, which is defined as the quotient $I^*/L(S_\lambda)^* \cong L(S_\lambda)/I$, is isomorphic to the quotient Λ/I, where Λ is the lattice in $\mathfrak{t}$ dual to Λ^*. The latter group Λ/I is known to be isomorphic to the center $Z(G)$. Thus, the lattice path counting function $\mathcal{P}_N^\lambda(\gamma)$ satisfies the assumptions made for Theorem 2.11, and hence we can apply it. But then the crucial fact is that we have

$$\mathcal{P}_N^\lambda(\mu) = m_N(\lambda;\mu), \quad \mu \in I^*. \tag{3.9}$$

Indeed, in this case the function $k_\lambda := k_{S_\lambda}^{c_\lambda}$ defined in (2.6) is given by

$$k_\lambda(\tau) = \sum_{\mu \in S_\lambda} m_1(\lambda;\mu) e^{\langle \mu, \tau \rangle}, \quad \tau \in \mathfrak{t},$$

which coincides with the character $\chi_\lambda(\tau/(2\pi i))$. Since χ_λ^N is the character of the representation $V_\lambda^{\otimes N}$ of G, (3.9) follows from (2.7) and (3.3). Hence, Theorem 3.1

follows directly from Theorem 2.11. To prove Theorem 3.2, we just need to use Theorem 3.1 with $f = \rho - w\rho$, $w \in W$ (these are indeed elements in $L(S)^* = \Lambda^*$), and the following identity;

$$a_N(\lambda; \mu) = \sum_{w \in W} \operatorname{sgn}(w) m_N(\lambda; \mu + \rho - w\rho). \tag{3.10}$$

This formula is obtained in [10]. To prove this, we observe that the character χ_λ^N of $V_\lambda^{\otimes N}$ can be written as

$$\chi_\lambda^N = \sum_{\mu \in \overline{C} \cap I^*} a_N(\lambda; \mu) \chi_\mu. \tag{3.11}$$

Then, multiplying this identity by the Weyl denominator Δ and using the Weyl character formula (3.1), we see

$$\Delta \chi_\lambda^N = \sum_{\mu \in \overline{C} \cap I^*,\, w \in W} \operatorname{sgn}(w) a_N(\lambda; \mu) e^{2\pi i w(\mu + \rho)}. \tag{3.12}$$

But, the weight decomposition (3.3) tells us that

$$\Delta \chi_\lambda^N = \sum_{\gamma \in I^*,\, w \in W} \operatorname{sgn}(w) m_N(\lambda; \gamma) e^{2\pi i(\gamma + w\rho)}. \tag{3.13}$$

In (3.12), note that, when $\mu \in \overline{C}$ we have $\mu + \rho \in C$, and hence $w(\mu + \rho) = \mu + \rho$ if and only if $w = 1$. Thus, the coefficient of $e^{2\pi i(\mu+\rho)}$ in (3.12) is $a_N(\lambda; \mu)$ while that in (3.13) is the right hand side of (3.10). From this, we conclude (3.10) and hence Theorem 3.2.

4 Distribution laws for toric monomials

In the previous sections, we consider the lattice path counting function or multiplicities of group representations. In these topics, the limit $N \to \infty$ can be regarded as a kind of thermodynamic limit because N can be regarded as a 'number of particles'. In turn, the problem we are going to address in this section is in the semiclassical limit. Namely, we consider asymptotic behavior of sections of a line bundle over a projective toric varieties.

4.1 Toric varieties from monomial embeddings

In this section, for simplicity, we set $X = X^* = \mathbb{R}^m$ and $I = I^* = \mathbb{Z}^m$. Let $S \subset \mathbb{Z}^m$ be a finite set and put $s = \sharp S$. As in the previous sections, we fix a positive function c on S. Assume that the set S satisfies the following stronger assumption than (2.1):

$$\operatorname{span}_{\mathbb{Z}}\{\alpha - \beta\,;\, \alpha, \beta \in S\} = \mathbb{Z}^m. \tag{4.1}$$

We denote the standard coordinates on $\mathbb{C}^s$ by $\zeta = (\zeta_\alpha)_{\alpha \in S}$ and the homogeneous coordinates of points in the complex projective space $\mathbb{C}P^{s-1}$ of dimension $s-1$ by $[\zeta] = [\zeta_\alpha]_{\alpha \in S}$, $\zeta \in \mathbb{C}^s \setminus \{0\}$. Denote by $T^m_{\mathbb{C}} = (\mathbb{C}^*)^m$ a complex torus of dimension m and consider the map

$$\Phi_S : T^m_{\mathbb{C}} \to \mathbb{C}P^{s-1}, \quad \Phi_S(t) = [c(\alpha)^{1/2} t^\alpha]_{\alpha \in S}, \tag{4.2}$$

where, for $t = (t_1, \dots, t_m) \in T^m_{\mathbb{C}}$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m$, we set $t^\alpha = t_1^{\alpha_1} \cdots t_m^{\alpha_m}$. The condition (4.1) assures that the map Φ_S is injective and is an embedding, which we call a *monomial embedding*. Define

$$\mathcal{O}_S = \Phi_S(T^m_{\mathbb{C}}), \quad M_S = \overline{\mathcal{O}_S}^{\text{Zariski}}, \tag{4.3}$$

where $\overline{\mathcal{O}_S}^{\text{Zariski}}$ denotes the Zariski closure of $\mathcal{O}_S$, which means that M_S is the smallest algebraic variety containing $\mathcal{O}_S$. We call the projective variety M_S a *toric variety*. Usually, toric varieties are, by definition, algebraic varieties which is irreducible, normal, and on which $T^m_{\mathbb{C}}$ acts algebraically with an open dense orbit. Our varieties of the form M_S admit these properties except the *normality*. The structures and properties of the varieties of the form M_S are described in [9] and in the article by A. Cannas da Silva in [1]. In this section, we give a brief account on these varieties. First, we give just one example. For other examples, see [1] where one can find many examples and exercises.

Example 4.1. Let $m = 1$. Take a positive integer p. Set $S = \{0, 1, \dots, p\}$. Take $c \equiv 1$. Then, the monomial embedding $\Phi_S : \mathbb{C}^* \to \mathbb{C}P^p$ is given by $\Phi_S(t) = [1 : t : t^2 : \cdots : t^p]$. Hence the variety M_S coincides with the image of the Veronese embedding $V : \mathbb{C}P^1 \to \mathbb{C}P^p$ given by

$$V([z_1 : z_2]) = [z_2^p : z_1 z_2^{p-1} : \cdots : z_1^{p-1} z_2 : z_1^p].$$

This shows that M_S is isomorphic to $\mathbb{C}P^1$.

We introduced the variety M_S by using the positive function c on S. But, the structure of M_S does not depend on the choice of the function c. Indeed, let X be the variety of the form M_S obtained by letting $c \equiv 1$. Let $C \in \mathrm{GL}(s, \mathbb{C})$ be the diagonal matrix whose components are given by $c(\alpha)^{1/2}$, $\alpha \in S$. Then, we have $M_S = CX$. However, when the variety M_S is smooth, the Kähler structure on M_S induced by the Fubini-Study form on $\mathbb{C}P^{s-1}$ depends on the choice of the weight function c.

Thus, for simplicity, we set $c \equiv 1$ in the rest of this subsection. Let $\mathbb{Z}^s_+$ denotes the set of lattice points in $\mathbb{R}^s$ whose components are all non-negative integers. Then it is not hard to show that, the homogeneous ideal $I_S \subset \mathbb{C}[\mathbb{Z}^s_+]$ defining the variety M_S, where $\mathbb{C}[\mathbb{Z}^s_+]$ denotes the algebra of polynomials in s-variables over $\mathbb{C}$, is generated by

$$\left\{ \zeta^\nu - \zeta^{\nu'} \,\middle|\, \nu, \nu' \in \mathbb{Z}^s_+,\ \sum_{\alpha \in S} \nu_\alpha \alpha = \sum_{\alpha \in S} \nu'_\alpha \alpha,\ \sum_\alpha \nu_\alpha = \sum_\alpha \nu'_\alpha \right\}.$$

To look closer at the ideal I_S, we set $A(S) := \{(\alpha, 1) \in \mathbb{Z}^{m+1}\,;\, \alpha \in S\}$. Let $S_{A(S)}$ denote the additive semigroup generated by $A(S)$ and let $\mathbb{C}[S_{A(S)}]$ the semigroup algebra. As an algebra, $\mathbb{C}[S_{A(S)}]$ has generators $(z, w)^{(\alpha,1)} = z^\alpha w$ ($\alpha \in S$), where z and w are a complex m-variables and a complex variable, respectively. Let $\pi : \mathbb{R}^s \to \mathbb{R}^{m+1}$ be the linear map defined by $\pi(x) = \sum_\alpha x_\alpha(\alpha, 1)$, $x = (x_\alpha)_{\alpha \in S}$, and let $\hat{\pi} : \mathbb{C}[\mathbb{Z}^s_+] \to \mathbb{C}[S_{A(S)}]$ be a surjective homomorphism defined by

$$\hat{\pi}(\zeta^\nu) = z^{\sum_\alpha \nu_\alpha \alpha} w^{\sum_\alpha \nu_\alpha} = (z, w)^{\pi(\nu)}.$$

The following lemma is easy to prove and hence we omit the proof.

Lemma 4.2. *We have* $\ker(\hat{\pi}) = I_S$. *In particular,* I_S *is a prime homogeneous ideal and* M_S *is irreducible. The homogeneous coordinate ring of* M_S *is isomorphic to the semigroup ring* $\mathbb{C}[S_{A(S)}]$.

Let $\phi : T^m_{\mathbb{C}} \to T^s_{\mathbb{C}}$ be an injective homomorphism defined by $\phi(t) = (t^\alpha)_{\alpha \in S}$. Then, $T^m_{\mathbb{C}}$ acts on $\mathbb{C}P^{s-1}$ through the homomorphism ϕ and the monomial embedding $\Phi_S : T^m_{\mathbb{C}} \to \mathbb{C}P^{s-1}$ is equivariant. Clearly, the image $\mathcal{O}_S$ of Φ_S is an orbit of $T^m_{\mathbb{C}}$-action on $\mathbb{C}P^{s-1}$. Furthermore, it is not hard to show that $\mathcal{O}_S = \{[\zeta] \in M_S\,;\, \zeta_\alpha \neq 0,\ \alpha \in S\}$, and hence $\mathcal{O}_S$ is open in M_S. Thus, up to the normality, the variety M_S is toric.

In general, a projective variety X in $\mathbb{C}P^{s-1}$ is said to be *normal* if the local ring o_p is integrally closed the function field of X for each $p \in X$ (see [11]). For each $\alpha \in S$, let $U_\alpha \subset \mathbb{C}P^{s-1}$ be the open set given by $\{\zeta_\alpha \neq 0\}$. We know that $U_\alpha \cong \mathbb{C}^{s-1}$ and $\{U_\alpha\}_{\alpha \in S}$ covers $\mathbb{C}P^{s-1}$. When $X \subset \mathbb{C}P^{s-1}$ is a projective variety, each $U_\alpha \cap X$ is an affine variety. Then, the normality of X is equivalent to the condition that the affine coordinate ring of $X \cap U_\alpha$ is integrally closed for each $\alpha \in S$. To describe the conditions for the normality of our variety M_S, let us prepare some more notation. We set $P = \mathrm{ch}\,(S)$, the convex hull of S. Then, the set $\mathcal{V}(P)$ of vertices of P is in S. For any $p \in \mathcal{V}(P)$, let $S_p \subset \mathbb{Z}^m$ be the semigroup generated by $\{\alpha - p\,;\, \alpha \in S\}$. Then, we have the following theorem.

Theorem 4.3. *Suppose that the finite set* S *in* $\mathbb{Z}^m$ *satisfies the condition* (4.1). *Then, the following conditions are equivalent.*

1. *The projective variety* M_S *defined by* (4.3) *is normal.*
2. *For all* $p \in \mathcal{V}(P)$, *we have* $S_p = K(S_p) \cap \mathbb{Z}^m$, *where* $K(S_p)$ *denotes the cone generated by* S_p.
3. *There exists a positive integer* N_o *such that for any integer* $N \geq N_o$, *we have* $S(N) = (NP) \cap \mathbb{Z}^m$, *where* $S(N)$ *is defined in* (2.2)

Note that the second condition in Theorem 4.3 comes from the fact that the family of open sets $\{U_p \cap M_S\}_{p \in \mathcal{V}(P)}$ is an open covering. See [21, Lemma 13.10, Theorem 13.11] for the proofs of this fact and Theorem 4.3.

Remark 4.4. A projective variety X is said to be projectively normal if its homogeneous coordinate ring is integrally closed. For the toric variety M_S constructed above (with S satisfying (4.1)), the projective normality is equivalent to that we have $S_{A(S)} = K(S_{A(S)}) \cap \mathbb{Z}^{m+1}$. See [9], [21]. Furthermore, under the assumption (4.1), one can show that this condition holds if and only if $S(N) = (NP) \cap \mathbb{Z}^m$ for *any* positive integer N. As we will see in the next section, if $S = P \cap \mathbb{Z}^m$ with the Delzant lattice polytope P, the corresponding toric variety M_S is smooth and normal. However, even in this case, it is not clear whether M_S is projectively normal or not. See [4], [19] for this issue.

4.2 Smooth projective toric varieties

We have constructed a toric variety M_S from a finite set $S \subset \mathbb{Z}^m$ satisfying the condition (4.1) through the monomial embedding $\Phi_S : T^m_{\mathbb{C}} \to \mathbb{C}P^{s-1}$, $s = \sharp S$. Our interest is in asymptotic analysis, and it would be reasonable to use smooth toric variety. In this section, we consider such a variety. From now on, we assume that our finite set S is of the form $S = P \cap \mathbb{Z}^m$ where P is a lattice polytope, which means that each vertex of the polytope P lies in the lattice $\mathbb{Z}^m$. In this case, we write Φ_P, $\mathcal{O}_P$, M_P instead of Φ_S, $\mathcal{O}_S$, M_S, respectively. Furthermore, we assume that the polytope P is Delzant. Recall that a polytope P in $\mathbb{R}^m$ is said to be *Delzant* if, for each vertex p of P, there exist exactly m edges emanating from p and there exists a lattice basis $\{w_1, \ldots, w_m\}$ of $\mathbb{Z}^m$ such that each edge emanating from p lies on the half line $\{p + tw_j \,;\, t \geq 0\}$ for some j. Then, the following fact is well-known.

Proposition 4.5. *The toric variety M_S constructed above is smooth if S is of the form $S = P \cap \mathbb{Z}^m$ with a Delzant lattice polytope P.*

In [9], the corresponding fact is in Corollary 3.2, Chapter 5. There, the conditions for M_S to be smooth is described in a different fashion. However, one can check that the Delzant condition implies these conditions. In the following we give a sketch of proof of Proposition 4.5, which is similar to the proof of the fact that the complex projective space is smooth.

Proof. Define the map $\mu_s : \mathbb{C}P^s \to \mathbb{R}^s$ by

$$\mu_s([\zeta]) = \sum_{\alpha \in S} \frac{|\zeta_\alpha|^2}{\sum_{\beta \in S} |\zeta_\beta|^2} e_\alpha,$$

where e_α ($\alpha \in S$) is the standard basis of $\mathbb{R}^s$. We then define the map $\mu^c_P : M_P \to \mathbb{R}^m$ by the composition

$$\mu^c_P : M_P \overset{\iota_P}{\hookrightarrow} \mathbb{C}P^s \overset{\mu_s}{\to} \mathbb{R}^s \overset{p}{\to} \mathbb{R}^m, \tag{4.4}$$

where the linear map $p : \mathbb{R}^s \to \mathbb{R}^m$ is defined as $p(e_\alpha) = \alpha$, $\alpha \in S = P \cap \mathbb{Z}^m$. Note that the map μ_P^c depends on the choice of the weight function c on $S = P \cap \mathbb{Z}^m$. The map μ_P^c is continuous in the usual topology on M_S and its image coincides with the Delzant polytope P. It is not so hard to show that $\mu_P^c(M_P \setminus \mathcal{O}_P) = \partial P$, where $\mathcal{O}_P = \Phi_S(T_{\mathbb{C}}^m)$ is the image of the monomial embedding $\Phi_S : T_{\mathbb{C}}^m \to \mathbb{C}P^{s-1}$. (To prove this, one will need to use the fact that $\mathcal{O}_P$ is dense in M_P in the usual topology. See [17] for this fact.) Furthermore, one can show that the following holds.

1. For each face f of P, we have $(\mu_P^c)^{-1}(f) = \{[\zeta] \in M_P\,;\, \zeta_\alpha = 0,\ \alpha \in S \setminus f\}$.
2. For each face f, we have $(\mu_P^c)^{-1}(\mathrm{ri} f) = \{[\zeta] \in (\mu_P^c)^{-1}(f)\,;\, \zeta_\alpha \neq 0,\ \alpha \in S \cap f\}$, where $\mathrm{ri} f$ is the relative interior of f in the affine hull of f.
3. For each face f, $(\mu_P^c)^{-1}(\mathrm{ri} f)$ is a $T_{\mathbb{C}}^m$-orbit.
4. For each vertex $p \in \mathcal{V}(P)$, the open set $U_p \cap M_P = \{[\zeta] \in M_P\,;\, \zeta_p \neq 0\}$ is given by

$$U_p \cap M_P = \bigcup_{f:\,\text{face of } P,\ p \in f} (\mu_P^c)^{-1}(\mathrm{ri} f). \tag{4.5}$$

(The correspondence between the open faces of P and the $T_{\mathbb{C}}^m$-orbit in M_P is proved in [9]. But, one can show the above facts in an elementary method similar to the proof of Lemma 3.10 in [24]. Note that the above facts hold for general finite set S satisfying (4.1).) From these facts, the decomposition $M_P = \bigcup_{f:\text{face of} P} (\mu_P^c)^{-1}(\mathrm{ri} f)$ gives the orbit decomposition of the $T_{\mathbb{C}}^m$-action on M_P.

Now, fix a vertex p of P. Since P is Delzant, there exists a lattice basis $\{v_j\}_{j=1}^m$ of $\mathbb{Z}^m$ such that each edge emanating from p lies in a half line $\{p + tv_j\,;\, t \geq 0\}$. Define a matrix Γ_p with integer components by the formula $\Gamma_p v_j = e_j$ ($j = 1, \ldots, m$) where $\{e_j\}_{j=1}^m$ is the standard basis of $\mathbb{Z}^m$. Since $\{v_j\}$ is a lattice basis, the determinant of Γ_p is ± 1. Then, we define a map

$$\phi_p : \mathbb{C}^m \to U_p \cap M_P, \quad \phi_p(w) = [w^{\Gamma_p(\alpha - p)}]_{\alpha \in S}. \tag{4.6}$$

Note that ϕ_p is well-defined because $\Gamma_p(\alpha - p) \in \mathbb{Z}_+^m$. (This follows from the fact that each $\alpha - p$ can be written as a linear combination of v_j with coefficients in $\mathbb{Z}_+$.) Define

$$\psi_p : U_p \cap M_P \to \mathbb{C}^m, \quad \psi_p([\zeta_\alpha]_{\alpha \in S}) = \left(\frac{\zeta_{\alpha_1}}{\zeta_p}, \ldots, \frac{\zeta_{\alpha_m}}{\zeta_p} \right),$$

where $\alpha_j \in S$ is characterized by $v_j = \alpha_j - p$. Let us show that $\phi_p^{-1} = \psi_p$. It is easy to show that $\psi_p \circ \phi_p(w) = w$ for $w \in \mathbb{C}^m$. To prove $\phi_p \circ \psi_p([\zeta]) = [\zeta]$ for

$[\zeta] \in U_p \cap M_P$, we need to use the structure of the orbit decomposition described above. We set $\alpha - p = \sum_{j=1}^m c_j(\alpha) v_j$ with $c_j(\alpha) \in \mathbb{Z}_+$. We fix $[\zeta_\alpha]_{\alpha \in S} \in U_p \cap M_P$ and put

$$\lambda_\alpha = \left(\frac{\zeta_{\alpha_1}}{\zeta_p}\right)^{c_1(\alpha)} \cdots \left(\frac{\zeta_{\alpha_m}}{\zeta_p}\right)^{c_m(\alpha)}, \quad \alpha \in S = P \cap \mathbb{Z}^m.$$

Then, we must show that $[\zeta_\alpha]_{\alpha \in S} = [\lambda_\alpha]_{\alpha \in S}$. There are m facets (faces of codimension 1) of P containing the vertex p, which we denote by $F_1, \dots, F_m$, where F_j is characterized by $\alpha_j \notin F_j$ $(j = 1, \dots, m)$. For any $I \subset \{1, \dots, m\}$, we set

$$f_I = \bigcap_{j \in I} F_j,$$

which is a face of P containing p. All the faces containing p are of the form f_I for some $I \subset \{1, \dots, m\}$. By (4.5), there is a unique $I \subset \{1, \dots, m\}$ such that $[\zeta_\alpha]_{\alpha \in S} \in (\mu_P^c)^{-1}(\mathrm{ri} f_I)$. By the fact that $(\mu_P^c)^{-1}(\mathrm{ri} f_I)$ is a $T_{\mathbb{C}}^m$-orbit, there exist $c \in \mathbb{C}^*$ and $z \in T_{\mathbb{C}}^m$ such that $\zeta_\alpha = cz^\alpha$ $(\alpha \in S \cap f_I)$, $\zeta_\alpha = 0$ $(\alpha \in S \setminus f_I)$. Note that $\alpha_j \notin f_I$ if and only if $j \in I$. From this one can show that $\alpha \in S \setminus f_I$ if and only if $c_j(\alpha) \neq 0$ for some $j \in I$. Thus, we have $\lambda_\alpha = 0$ for $\alpha \in S \setminus f_I$. Since $\zeta_{\alpha_j} = cz^{\alpha_j}$ for $j \notin I$, we have, for $\alpha \in S \cap f_I$,

$$\lambda_\alpha = \prod_{j \notin I} \left(\frac{\zeta_{\alpha_j}}{\zeta_p}\right)^{c_j(\alpha)} = \prod_{j \notin I} (z^{\alpha_j - p})^{c_j(\alpha)} = z^\alpha = c^{-1}\zeta_\alpha,$$

which shows $[\lambda_\alpha]_\alpha = [\zeta_\alpha]_{\alpha \in S}$. Therefore, we have $\psi_p = \phi_p^{-1}$, and hence ϕ_p is a homeomorphism. Now, it is not so hard to show, by a direct computation with the orbit decomposition described above, that the coordinate change $\phi_q^{-1} \circ \phi_p$ $(p, q \in \mathcal{V}(P))$ is holomorphic. □

4.3 Toric monomials

In the rest of this section, let P be a Delzant lattice polytope and let $S = P \cap \mathbb{Z}^m$. Then, we have a compact complex submanifold $M_P := M_S$ in $\mathbb{C}P^{s-1}$. Denote the inclusion of M_P into $\mathbb{C}P^{s-1}$ by $\iota_P : M_P \hookrightarrow \mathbb{C}P^{s-1}$. Let ω_{FS} be the Fubini-Study Kähler form on $\mathbb{C}P^{s-1}$. Then, the 2-form $\omega_P^c = \iota_P^* \omega_{\mathrm{FS}}$ is a Kähler form on M_P. The 2-form ω_P^c is integral in the sense that there exists a line bundle L_P^c over M_P such that $c_1(L_P^c) = [\omega_P^c]$ in (the image of) $H^2(M_P, \mathbb{Z})$. Indeed, let $\mathcal{O}(1) \to \mathbb{C}P^{s-1}$ denote the hyperplane section bundle. The bundle $\mathcal{O}(1)$ is the dual to the tautological line bundle over $\mathbb{C}P^{s-1}$. Then, the pull-back $L_P^c = \iota_P^* \mathcal{O}(1) \to M_P$ has this property.

Our toric variety M_P is smooth and hence is normal as a projective variety. (Normality is checked by the second condition in Theorem 4.3 and the Delzant condition.) Thus, it is equivariantly equivalent to a toric variety constructed from a fan. The fan corresponding to M_P is the 'normal fan' of the Delzant polytope

P ([9]). We do not need to use the fan in this paper, and hence we omit the description of M_P in terms of the fan. However, we mention that we can use the theory of toric variety constructed from the fan, as described in [7], [20]. For example, the space of global holomorphic sections $H^0(M_P, (L_P^c)^{\otimes N})$ of the N-th tensor power of the line bundle L_P^c is decomposed into weight spaces for the $T_{\mathbb{C}}^m$-action as

$$H^0(M_P, (L_P^c)^{\otimes N}) = \bigoplus_{\alpha \in (NP) \cap \mathbb{Z}^m} \mathbb{C}\chi_\alpha^N \quad (N \geq 1),$$

where χ_α^N is a weight vector with weight α. The sections χ_α^N are just monomials on the open orbit $\mathcal{O}_P \cong T_{\mathbb{C}}^m$. We call these sections *toric monomials*. Our purpose is to investigate various asymptotic formulas for sections in $H^0(M_P, (L_P^c)^{\otimes N})$ as $N \to \infty$. So, it is useful to describe concretely the sections χ_α^N $(\alpha \in (NP) \cap \mathbb{Z}^m)$ for every sufficiently large N. Since our variety M_P is normal, there exists a positive integer N_o such that we have

$$H^0(M_P, (L_P^c)^{\otimes N}) = \iota_P^* H^0(\mathbb{C}P^{s-1}, \mathcal{O}(N)) \tag{4.7}$$

for every $N \geq N_o$. (One can also use the third condition in Theorem 4.3 to prove (4.7).) Recall that the holomorphic sections of $\mathcal{O}(N) \to \mathbb{C}P^{s-1}$ are regarded as homogeneous polynomials in $\mathbb{C}^s$ of degree N. In particular, for $N = 1$, define $\lambda_\alpha \in (\mathbb{C}^s)^*$ $(\alpha \in S = P \cap \mathbb{Z}^m)$ as the coordinate functions on $\mathbb{C}^s$. Then, the set $\{\lambda_\alpha\}_{\alpha \in S}$ gives a basis of $H^0(\mathbb{C}P^{s-1}, \mathcal{O}(1))$. Hence, the sections

$$\chi_\alpha = c(\alpha)^{-1/2} \iota_P^* \lambda_\alpha \in H^0(M_P, L_P^c), \quad \alpha \in S,$$

form a basis of $H^0(M_P, L_P^c)$. For $N \geq N_o$, we set

$$\chi_\alpha^N = \chi_{\beta_1} \otimes \cdots \otimes \chi_{\beta_N}, \quad \alpha = \beta_1 + \cdots + \beta_N \in (NP) \cap \mathbb{Z}^m,\ \beta_j \in S.$$

This does not depend on the choice of $\beta_1, \ldots, \beta_N$ for fixed $\alpha \in (NP) \cap \mathbb{Z}^m$. Then the toric monomials χ_α^N, $\alpha \in (NP) \cap \mathbb{Z}^m$ form a basis of $H^0(M_P, (L_P^c)^{\otimes N})$. It is proved by a direct computation that the basis $\{\chi_\alpha^N\,;\, \alpha \in (NP) \cap \mathbb{Z}^m\}$ forms an orthogonal basis with respect to the inner product

$$\langle\, s, t\,\rangle = \int_{M_P} h_P^N(s(z), t(z))\, (\omega_P^c)^m/m!,$$

where h_P^N is the Hermitian metric on $(L_P^c)^{\otimes N}$ induced from the Fubini-Study Hermitian metric on $\mathcal{O}(1)$. Thus, we normalize each χ_α^N as

$$\varphi_\alpha^N = \frac{1}{\|\chi_\alpha^N\|} \chi_\alpha^N, \quad \alpha \in (NP) \cap \mathbb{Z}^m$$

to form an orthonormal basis $\{\varphi_\alpha^N\,;\, \alpha \in (NP) \cap \mathbb{Z}^m\}$ of $H^0(M_P, (L_P^c)^{\otimes N})$. Then, our problem is to investigate asymptotic behavior of $|\varphi_\alpha^N(z)|_P^2 = h_P^N(\varphi_\alpha^N(z), \varphi_\alpha^N(z))$.

Remark that the map $\mu_P^c : M_P \to P \subset \mathbb{R}^m$ defined by (4.4) is the moment map for symplectic action of the real torus T^m on the symplectic manifold (M_P, ω_P^c). The moment map μ_P^c is (by definition) invariant under T^m-action and, when it is restricted to the open orbit $\mathcal{O}_P := \mathcal{O}_S$ $(S = P \cap \mathbb{Z}^m)$, the map $\mu_P^c : \mathcal{O}_P \to \mathrm{Int}\,(P)$ defines, by using the coordinate $z = e^{\tau/2+i\varphi}$ $(\tau, \varphi \in \mathbb{R}^m)$ on $\mathcal{O}_P$, the map $\mathbb{R}^m \to \mathrm{Int}\,(P)$ denoted also by μ_P^c. The map μ_P^c is given explicitly by

$$\mu_P^c(\tau) = \sum_{\alpha \in S} \frac{c(\alpha) e^{\langle \alpha, \tau \rangle}}{\sum_{\beta \in S} c(\beta) e^{\langle \beta, \tau \rangle}} \alpha, \quad \tau \in \mathbb{R}^m. \tag{4.8}$$

The map (4.8) is the same as that defined in (2.12). Then, we have its potential function k_P^c on $\mathbb{R}^m$ defined in (2.6). The function k_P^c defines a function on the open orbit $\mathcal{O}_P$, which is also denoted by k_P^c, by $k_P^c(\Phi_S(z)) = k_S^c(\tau)$, $z = e^{\tau/2+i\varphi} \in T_{\mathbb{C}}^m$, $\tau, \varphi \in \mathbb{R}^m$. An important fact is that the function k_P^c so defined is a Kähler potential of ω_P^c on $\mathcal{O}_P$. Indeed one can check directly that

$$\omega_P^c = \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log k_P^c.$$

From this, the volume form $d\mathrm{vol} = (\omega_P^c)^m/m!$ is given, on the open orbit $\mathcal{O}_P$, by

$$d\mathrm{vol} = \frac{1}{(2\pi)^m} \det A_P^c(\tau)\, d\tau d\varphi, \tag{4.9}$$

where $A_P^c(\tau) = \nabla^2 \log k_P^c(\tau)$ is a positive definite symmetric matrix.

4.4 Asymptotic behavior of toric monomials

There are various aspects of asymptotic behavior of $|\varphi_\alpha^N(z)|_P^2$. In this subsection, we give some of asymptotic results for this functions. The results in this subsection can be found in [22]. Among results in [22], the most typical result is the following. To state the theorem, let us prepare some notation. As mentioned in Section 2, the map $\mu_P^c : \mathbb{R}^m \to \mathrm{Int}\,(P)$ defined in (4.4), (4.8) is a diffeomorphism. We denote its inverse map by $\tau_P^c : \mathrm{Int}\,(P) \to \mathbb{R}^m$. Denote by δ_P^c the function on $\mathrm{Int}\,(P)$ defined by the formula (2.13). Then, we define the function b_P^c on $\mathrm{Int}\,(P) \times \mathrm{Int}\,(P)$ by

$$b_P^c(x, y) = \delta_P^c(y) - \delta_P^c(x) + \langle\, y - x, \tau_P^c(y)\,\rangle. \tag{4.10}$$

For simplicity of notation, we set

$$c(P, x) := \frac{1}{\sqrt{\det A_P^c(\tau_P^c(x))}}, \quad x \in \mathrm{Int}\,(P). \tag{4.11}$$

Theorem 4.6. *Let $D_\alpha(t)$ denote the distribution function of the the push-forward measure $(|\varphi_\alpha^N(z)|_P^2)_* d\mathrm{vol}$ on the real line, which is defined explicitly by*

$$D_\alpha(t) := \mathrm{vol}\left(z \in M_P\,;\, |\varphi_\alpha^N(z)|_P^2 > t\right). \tag{4.12}$$

Suppose that the sequence of lattice points $\alpha_N \in (NP) \cap \mathbb{Z}^m$ satisfies $\alpha_N = Nx_o + O(1)$ with a point x_o in $\mathrm{Int}\,(P)$.

1. *For* $t > 0$*, we have*

$$D_{\alpha_N}(t) \sim \frac{(\pi m)^{m/2}}{c(P, x_o)\Gamma(m/2+1)} \left(\frac{\log N}{N}\right)^{m/2}. \tag{4.13}$$

2. *For* $0 < t < c(P, x_o)$*, we have*

$$\begin{aligned} &\lim_{N\to\infty} \left(\frac{N}{2\pi}\right)^{m/2} D_{\alpha_N}\left(\left(\frac{N}{2\pi}\right)^{m/2} t\right) \\ &\qquad = \frac{1}{c(P, x_o)\Gamma(m/2+1)} \left(\log\left(\frac{c(P, x_o)}{t}\right)\right)^{m/2}. \end{aligned} \tag{4.14}$$

3. *For* $t > 0$*, we have*

$$\lim_{N\to\infty} D_{\alpha_N}(e^{-Nt}) = \mathrm{vol}(x \in \mathrm{Int}\,(P)\,;\, b_P^c(x_o, x) < t), \tag{4.15}$$

where vol *denotes the Euclidean volume.*

Remark 4.7. The formula (4.14) is also proved in [5]. The Kähler structure used in [5] is the one naturally induced from the standard Kähler form on $\mathbb{C}^d$ (d is the number of facets of P) through the GIT description of the toric manifold M_P.

Before giving a sketch of proof, we give an explanation on Theorem 4.6. For simplicity, consider the case where $\alpha_N = N\alpha$ with $\alpha \in \mathrm{Int}\,(P) \cap \mathbb{Z}^m$. The sections $\varphi_{N\alpha}^N$ are expected to concentrate on the fiber $(\mu_P^c)^{-1}(\alpha)$ of the moment map $\mu_P^c : M_P \to P$. (Indeed, one can show that the measure $|\varphi_{N\alpha}^N|_P^2\, d\mathrm{vol}$ tends weakly to the uniform measure on the fiber $(\mu_P^c)^{-1}(\alpha) \cong T^m$.) Since the function $|\varphi_{N\alpha}^N|_P^2$ is invariant under the action of the real torus T^m, and since M_P/T^m is homeomorphic to P, this function induces a function on the polytope P. So, suppose that the function $|\varphi_{N\alpha}^N|_P^2$ is like a Gaussian bump around α. Consider its sub-level sets, $L_N(t) = \{|\varphi_{N\alpha}^N|_P^2 \geq t\}$ in $\mathrm{Int}\,(P)$. When t is a fixed positive constant, which is the case of the formula (4.13), since the Gaussian bump becomes quite sharp around α as N tends to infinity, the volume of the sub-level set $L_N(t)$ becomes small as $N \to \infty$, and how small it becomes does not depend on the constant t because the corresponding measure finally converges to the Dirac delta at α. This is the formula (4.13). Thus, to find a correct limit of $D_{N\alpha}(t)$, we need to rescale the constant t so that $t = t_N$ depends on N. The formula (4.14) gives the correct rescaling when $t = t_N$ becomes large as $N \to \infty$. In this case, since the level sets become upper and upper as N tends to infinity, the rescaling in (4.14) gives the information around the center of the concentration. The formula (4.14) shows that the way of concentration is rather universal, because it does not depend much on the geometry of M_P. In turn, in the formula (4.15), the rescaling $t = t_N$ is made by e^{-Nt} which decays as N tends to infinity. This means that the level sets become lower and lower as N tends to the infinity. In this rescale, the distribution function can grasp the information about the tale of the bump, and the formula (4.15) shows that the tale of the bump contains much geometric information, and such information is contained in the function $b_P^c(\alpha, \cdot)$.

Remark 4.8. We have explained Theorem 4.6 by supposing that the function $|\varphi_{N\alpha}^N|_P^2$ looks like Gaussian. Indeed this is seen by the pointwise asymptotics of this function (see Theorem 4.9 below). However, one could accept this exposition by the following fact. Suppose we are given a Gaussian function

$$g(u) = \frac{e^{-\langle Au, u \rangle/2}}{\sqrt{\det A}}, \quad u \in \mathbb{R}^m$$

on $\mathbb{R}^m$ with the measure

$$d\nu_A = \frac{\det A}{(2\pi)^{m/2}}\, du$$

so that $\displaystyle\int_{\mathbb{R}^m} g(u)\, d\nu_A(u) = 1$. Then, a direct computation tells us that the distribution function is given by

$$\nu_A(u \in \mathbb{R}^m\, ;\, g(u) > t) = \frac{1}{c_A \Gamma(m/2+1)} \left(\log\left(\frac{c_A}{t}\right)\right)^{m/2}$$

for $0 < t \le c_A$ with the constant $c_A = 1/\sqrt{\det A}$. Therefore, one can say that the rescaled distributions in (4.14) for the toric monomials have a universal Gaussian form around the center of the localization.

4.5 Inverting a moment problem

Among various asymptotic formulas in Theorem 4.6, we give a sketch of proof of the formula (4.14). The formula (4.14) is one of consequences of the following theorem about pointwise asymptotic behavior of $|\varphi_{\alpha_N}^N(z)|_P^2$.

Theorem 4.9. *Let $\alpha_N \in (NP) \cap \mathbb{Z}^m$ be a sequence of lattice points in NP such that $\alpha_N = Nx_o + O(1)$ with a point $x_o \in \mathrm{Int}\,(P)$. Then, we have*

$$\begin{aligned} &|\varphi_{\alpha_N}^N(z)|_P^2 = \\ &c(P, x_o)\left(\frac{N}{2\pi}\right)^{m/2} e^{-N[b_P^c(x_o,x)+\langle x_o - \alpha_N/N, \tau_P^c(x) - \tau_P^c(x_o)\rangle]}(1+O(N^{-1})), \end{aligned} \tag{4.16}$$

where we write $z = e^{\tau_P^c(x)/2 + i\varphi}$ with $x \in \mathrm{Int}\,(P)$. This holds uniformly in $z \in T_{\mathbb{C}}^m \cong \mathcal{O}_P$.

Remark 4.10. A special boundary case where $\alpha_N = N\alpha$ with $\alpha \in \partial P \cap \mathbb{Z}^m$ is also handled in [22] by using the local coordinates on $U_p \cap M_P$ ($p \in \mathcal{V}(P)$) described in the proof of Proposition 4.5. In [23], general boundary case is analyzed.

Indeed, taking k-th power of the formula (4.16) in Theorem 4.9, combined with a technical estimate, shows the following asymptotic formula for the L^{2k}-norms.

Theorem 4.11. *Suppose that the sequence $\alpha_N \in (NP) \cap \mathbb{Z}^m$ of lattice points and a point $x_o \in \mathrm{Int}\,(P)$ satisfy the condition in Theorem 4.6. Then, for the L^{2k}-norm $\|\varphi^N_{\alpha_N}\|_{2k}$ of the section $\varphi^N_{\alpha_N}$ has the following asymptotic behavior;*

$$\|\varphi^N_{\alpha_N}\|_{2k}^{2k} = \frac{c(P,x_o)^{k-1}}{k^{m/2}} \left(\frac{N}{2\pi}\right)^{(k-1)m/2} (1 + O_k(N^{-1})), \tag{4.17}$$

where O_k means that the estimate $O(N^{-1})$ depends on k.

Let us explain how one can deduce the formula (4.14) from Theorem 4.11. To take the rescaling in the formula (4.14) into account, let us introduce the measure $d\mathrm{v}_N$ and the function f_N on M_P defined by

$$d\mathrm{v}_N = \left(\frac{N}{2\pi}\right)^{m/2} d\mathrm{vol}_P, \quad f_N(z) = \left(\frac{N}{2\pi}\right)^{-m/4} |\varphi_{\alpha_N}(z)|_P,$$

so that $\|f_N\|_{L^2(d\mathrm{v}_N)} = 1$. According to the formula (4.17), we have

$$\|f_N\|^{2k}_{L^{2k}(d\mathrm{v}_N)} = \frac{c(P,x_o)^{k-1}}{k^{m/2}} (1 + O_k(N^{-1})). \tag{4.18}$$

Consider the push-forward measure $|f_N|^2_* d\mathrm{v}_N$. By using the pointwise asymptotic formula (4.16), one can show that

$$\lim_{N\to\infty} \|f_N\|^2_\infty = c(P,x_o), \tag{4.19}$$

and hence the support of the push-forward measures $|f_N|^2_* d\mathrm{v}_N$ are contained in a bounded set in $[0,+\infty)$ independent of N. The distribution function $F_N(t) := (|f_N|^2_* d\mathrm{v}_N)([t,+\infty))$ of the measure $|f_N|^2_* d\mathrm{v}_N$ is given by the rescaled distribution function, $F_N(t) = (\frac{N}{2\pi})^{m/2} D_{\alpha_N}((\frac{N}{2\pi})^{m/2} t)$, in the formula (4.14). The limit of the k-th moment as $N \to \infty$ of the measure $|f_N|^2_* d\mathrm{v}_N$ is given by

(4.20)

$$\int_{\mathbb{R}} x^k \, d(|f_N|^2_* d\mathrm{v}_N)(x) = \|f_N\|^{2k}_{L^{2k}(d\mathrm{v}_N)} = \frac{c(P,x)^{k-1}}{k^{m/2}} (1 + O_k(N^{-1})) \to \frac{c(P,x)^{k-1}}{k^{m/2}}.$$

Now, we note that the measure

$$d\rho_N(x) = x \, d(|f_N|^2_* d\mathrm{v}_N)(x)$$

on the real line is a probability measure supported in a bounded interval in $[0,+\infty)$ independent of N. Then, if the sequence of probability measures $d\rho_N$ tend weakly to a probability measure, say $d\rho$, the limit measure $d\rho$ would be supported on

$[0, c(P,x)]$ by (4.19), and the distribution function $F_N(t)$ would have a limit because

$$F_N(t) = \int \chi_{(t,+\infty)}(x)\, d(|f_N|_*^2 d\mathrm{v}_N)(x) = \int \frac{1}{x}\chi_{(t,+\infty)}(x)\, d\rho_N(x)$$
$$\to \int \frac{1}{x}\chi_{(t,+\infty)}(x)\, d\rho(x),$$

where $\chi_{(t,+\infty)}$ is the characteristic function of the interval $(t, +\infty)$. Furthermore, by (4.20), we must have

$$\int x^k\, d\rho(x) = \lim_{N\to\infty} \int x^k d\rho_N(x) = \frac{c(P,x)^k}{(k+1)^{m/2}}.$$

So, we arrive at a *moment problem*, that is, to find a probability measure $d\rho$ whose k-th moment is given by $\frac{c(P,x)^k}{(k+1)^{m/2}}$. For this, we have the following lemma.

Lemma 4.12. *Let ρ be a compactly supported probability measure on $\mathbb{R}$. Suppose that there exists a positive integer h and a positive number c such that, for any non-negative integer k,*

$$\int x^k\, d\rho(x) = \frac{c^k}{(k+1)^{h/2}}.$$

Then, we have

$$d\rho(x) = \frac{1}{c\Gamma(h/2)}\chi_{(0,c)}(x)\left(\log\left(\frac{c}{x}\right)\right)^{h/2-1}.$$

See [22, Lemma 4.1] for the proof of Lemma 4.12. From (4.20) and Lemma 4.12, it is not hard to show the formula (4.14). Therefore, what we need is to prove Theorem 4.9.

4.6 Pointwise asymptotic formula

In this subsection, we give a sketch of proof of Theorem 4.9. Since the formula 4.16 is a local estimate on $\mathcal{O}_P \cong T_{\mathbb{C}}^m$, we use the local coordinates $z = e^{\tau/2+i\varphi}$, $\tau, \varphi \in \mathbb{R}^m$, on $T_{\mathbb{C}}^m$. By definition of the Fubini-Study Hermitian metric on $(L_P^c)^{\otimes N} = \iota_P^*\mathcal{O}(N)$, the modulus square of the monomial $\chi_{\alpha_N}^N$ can be written as

$$|\chi_{\alpha_N}^N(z)|_P^2 = e^{-N[\log k_P^c(\tau) - \langle \tau, \alpha_N/N\rangle]}, \quad z = e^{\tau/2+i\varphi}.$$

Hence to consider the pointwise behavior of $\varphi_{\alpha_N}^N = \frac{1}{\|\chi_{\alpha_N}^N\|}\chi_{\alpha_N}^N$, it is enough to consider behavior of the L^2-norm $\|\chi_{\alpha_N}^N\|$. By (4.9), we have

$$\|\chi_{\alpha_N}^N\|^2 = \int_{\mathbb{R}^m} e^{-N[\log k_P^c(\tau) - \langle \tau, \alpha_N/N\rangle]} \det A_P(\tau)\, d\tau. \tag{4.21}$$

The critical point of the function $\log k_P^c(\tau) - \langle \tau, \alpha_N/N \rangle$ is given by $\mu_P^c(\tau) = \alpha_N/N$, that is $\tau = \tau_P^c(\alpha_N/N)$, which depends on the parameter N. So, we discuss as follows. We note that, in (4.21), the function $\det A_P(\tau)$ is a positive integrable function on $\mathbb{R}^m$ by (4.9) and the fact that the open orbit $\mathcal{O}_P$ is dense in M_P. Since $\alpha_N/N = x_o + O(N^{-1})$, we can choose an open ball U in $\mathrm{Int}\,(P)$ around x_o such that $\overline{U} \subset \mathrm{Int}\,(P)$ and $\alpha_N/N \in U$ for every sufficiently large N. As in [22, Lemma 3.3], there exist positive constants R, c such that $\log k_P^c(\tau) - \langle \tau, x \rangle \geq c|\tau|$ for any $(x, \tau) \in \overline{U} \times \mathbb{R}^m$, $|\tau| \geq R$. We may choose $R > 0$ so that $|\tau_P^c(\alpha_N/N)| < R$ for every sufficiently large N. Thus, the integral in (4.21) equals

$$\int_{\mathbb{R}^m} e^{-N[\log k_P^c(\tau) - \langle \tau, \alpha_N/N \rangle]} g(\tau) \det A_P(\tau)\, d\tau \tag{4.22}$$

modulo a term of order $O(e^{-cRN})$, where we inserted a cut-off function $0 \leq g(\tau) \leq 1$ satisfying $g(\tau) = 1$ for $|\tau| \leq 2R$. Changing the integral variable $\tau = \tau_P^c(x)$, the integral in (4.22) is written in the form

$$e^{-N\delta_P^c(x_o)} \int_{\mathrm{Int}\,(P)} e^{-Nb_P^c(x_o,x)} R_N(x_o, x) g(\tau_P^c(x))\, dx, \tag{4.23}$$

where the function $b_P^c(x_o, x)$ is defined in (4.10) and the function $R_N(x_o, x)$ is given by $R_N(x_o, x) = e^{\langle \alpha_N - Nx_o, \tau_P^c(x) \rangle}$. Since $\alpha_N = Nx_o + O(1)$, derivatives of $R_N(x_o, x)$ of any order are bounded by constants on the support of $g(\tau_P^c(x))$. For fixed $x_o \in \mathrm{Int}\,(P)$, it is easy to show that the function $b_P^c(x_o, x)$ has unique critical point at $x = x_o$ with Hessian $A_P^c(\tau_P^c(x_o))$. Since $b_P^c(x_o, x_o) = 0$, a standard argument involving the Morse lemma and the Fourier transform of Gaussian functions as in Section 2 shows

(4.24)

$$\|\chi_{\alpha_N}^N\|^2 = \left(\frac{N}{2\pi}\right)^{-m/2} \sqrt{\det A_P^c(\tau_P^c(x_o))}\, e^{-N[\delta_P^c(x_o) + \langle x_o - \alpha_N/N, \tau_P^c(x_o) \rangle]} (1 + O(N^{-1})).$$

From this and a direct computation, one conclude (4.16).

References

[1] M. Audin, A. Cannas da Silva and E. Lerman, *Symplectic Geometry of Integrable Hamiltonian Systems*, Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser, Basel, 2003.

[2] P. Biane, Estimation asymptotique des multiplicities dans les puissances tensorielles d'un $\mathfrak{g}$-module, *C. R. Acad. Sci. Paris Sér. I Math.* **316** (8) (1993), 849–852.

[3] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, Graduate Texts in Math., 98. Springer-Verlag, New York, 1985.

[4] W. Bruns, J. Gubeladze and Ngô Viêt Trung, Normal polytopes, triangulations, and Koszul algebras, *J. Reine Angew. Math.* **485** (1997), 123–160.

[5] D. Burns, V. Guillemin and Z. Wang, Stability functions, *Geom. Funct. Anal.* **19**, no. 5 (2010), 1258–1295.

[6] S. K. Donaldson, Scalar curvature and projective embeddings, I, *J. Diff. Geom.* **59** (2001), 479–522.

[7] W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies, 131. Princeton University Press, Princeton, 1993.

[8] V. Guillemin, *Moment Maps and Combinatorial Invariants of Hamiltonian T^n-spaces*, Progress in Mathematics, 122. Birkhäuser Boston, Boston, 1994.

[9] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser Boston, Boston, 1994.

[10] D. J. Grabiner and P. Magyar, Random walks in Weyl chambers and the decomposition of tensor powers, *J. Algebraic Combin.* **2** (3) (1993), 239–260.

[11] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math., 52, Springer-Verlag, New York, 1977.

[12] G. J. Heckman, Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups, *Invent. Math.* **67** (1982), 333–356.

[13] L. Hörmander, *The Analysis of Linear Partial Differential Operators, I* (2nd Ed.), Springer-Verlag, Berlin, 1990.

[14] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Math., 9. Springer-Verlag, New York, 1972.

[15] D. Jakobson and S. Zelditch, Classical limits of eigenfunctions for some completely integrable systems, in *Emerging applications of number theory* (Minneapolis, MN, 1996), 329–354, IMA Vol. Math. Appl., 109, Springer-Verlag, New York, 1999.

[16] M. Kotani and T. Sunada, Large deviation and the tangent cone at infinity of a crystal lattice, *Math.Z.* **254** (2006), 837–870.

[17] D. Mumford, *The Red Book of Varieties and Schemes* (2nd Ed.), Springer-Verlag, Berline, 1999.

[18] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, Graduate Texts in Math., 227. Springer-Verlag, New York, 2005.

[19] T. Oda, Problems on Minkowski sums of convex lattice polytopes, submitted at the Oberwolfach Conference "Combinatorial Convexity and Algebraic Geometry", 1997 (arXiv:math.AG/0812.1418).

[20] T. Oda, *Convex Bodies and Algebraic Geometry*, Springer-Verlag, Berline, 1988.

[21] B. Sturmfels, *Gröbner Basis and Convex Polytopes*, University Lecture Series, 8. American Mathematical Society, Providence, RI, 1996.

[22] B. Shiffman, T. Tate and S. Zelditch, Distribution laws for integrable eigenfunctions, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 5, 1497–1546.

[23] J. Song and S. Zelditch, Bergman metrics and geodesics in the space of Kähler metrics on toric varieties, arXiv:math.CV/0707.3082.

[24] T. Tate, Bernstein measures on convex polytopes, *Contemporary Math.* **484** (2009), 295–319.

[25] T. Tate and S. Zelditch, Lattice path combinatorics and asymptotics of multiplicities of weights in tensor powers, *J. Funct. Anal.* **217** (2004), 402–447.

Tatsuya Tate
Graduate School of Mathematics, Nagoya University
Furo-cho, Chikusa-ku, Nagoya, 464-8602, Japan
tate@math.nagoya-u.ac.jp

Travaux mathématiques, Volume 19 (2011), 97–124, © Université du Luxembourg

Berezin-Toeplitz quantization for compact Kähler manifolds. An introduction

by Martin Schlichenmaier

Abstract

The Berezin-Toeplitz operator and Berezin-Toeplitz deformation quantization schemes give quantization methods adapted to a Kähler structure on a manifold to be quantized. Here we present an introduction both to the definitions of its basic objects and to the results.

1 Introduction

For quantizable Kähler manifolds the Berezin - Toeplitz (BT) quantization scheme, both the operator quantization and the deformation quantization, supplies canonically defined quantizations. What makes the Berezin-Toeplitz quantization scheme so attractive is that it does not depend on further choices and that it does not only produce a formal deformation quantization, but one which is deeply related to some operator calculus.

Some time ago, in joint work with Martin Bordemann and Eckhard Meinrenken, the author showed that for compact Kähler manifolds it is a well-defined quantization scheme with correct semi-classical limit [14]. From the point of view of classical mechanics compact Kähler manifolds appear as phase space manifolds of restricted systems or of reduced systems. A typical example of its appearance is given by the spherical pendulum which after reduction has as phase-space the complex projective space.

Very recently, inspired by fruitful applications of the basic techniques of the Berezin-Toeplitz scheme beyond the quantization of classical systems, the interest in it revived considerably. For example these techniques show up in non-commutative geometry. More precisely, they appear in the approach to non-commutative geometry using fuzzy manifolds. The quantum spaces of the Berezin-Toeplitz quantization of level m, defined in Section 3 further down, are finite-dimensional in the compact case and the quantum operator of level m constitute finite-dimensional non-commutative matrix algebras. This is the arena of non-commutative fuzzy manifolds and gauge theories over them. The classical limit, the commutative manifold, is obtained as limit $m \to \infty$.

Another appearance of Berezin-Toeplitz quantization techniques as basic ingredients is in the pioneering work of Jørgen Andersen on the mapping class group (MCG) of surfaces in the context of Topological Quantum Field Theory (TQFT). Andersen gave also a lecture course at the school on his achievements. Beside other results, he was able to proof the asymptotic faithfulness of the mapping class group action on the space of covariantly constant sections of the Verlinde bundle with respect to the Axelrod-Witten-de la Pietra and Witten connection [3, 4], see also [51]. Furthermore, he showed that the MCG does not have Kazhdan's property T. Roughly speaking, a group has *property* T says that the identity representation is isolated in the space of all unitary representations of the group [5]. In these applications the manifolds to be quantized are the moduli spaces of certain flat connections on Riemann surfaces or, equivalently, the moduli space of stable algebraic vector bundles over smooth projective curves. Here further exciting research is going on, in particular, in the realm of TQFT and the construction of modular functors [6], [7, 8].

In general quite often moduli spaces come with a natural quantizable Kähler structure. Hence, it is not surprising that the Berezin-Toeplitz quantization scheme is of importance in moduli space problems. Non-commutative deformations, and a quantization is a non-commutative deformation, yield also informations about the commutative situation. These aspects clearly need further investigations.

It was the goal of the lecture course and it is the goal of this write-up to present a short introduction to the basic definitions and results on Berezin-Toeplitz quantization (both operator and deformation quantization) without proofs and too many details. The language presented was used in other lectures at the school and talks at the conference. The author hopes that it will be equally useful to the reader who aims to get a quick introduction to this exciting field. For a more detailed review, see [53]. There an extended list of references to the original literature and to reviews of other people concentrating on different aspects of the theory can be found, e.g. see [2], [54].

2 The geometric set-up

2.1 Quantizable Kähler manifolds

We will only consider phase-space manifolds which carry the structure of a Kähler manifold (M, ω). Recall that in this case M is a complex manifold (let us say of complex dimension n) and ω, the Kähler form, is a non-degenerate closed positive $(1,1)$-form. This means that the Kähler form ω can be written with respect to local holomorphic coordinates $\{z_i\}_{i=1,\dots,n}$ as

$$\omega = \mathrm{i}\sum_{i,j=1}^{n} g_{ij}(z) dz_i \wedge d\bar{z}_j, \tag{2.1}$$

with local functions $g_{ij}(z)$ such that the matrix $(g_{ij}(z))_{i,j=1,\dots,n}$ is hermitian and positive definite.

Denote by $C^\infty(M)$ the algebra of complex-valued (arbitrary often) differentiable functions with point-wise multiplication as associative product. A symplectic form on a differentiable manifold is a closed non-degenerate 2-form. In particular, we can consider our Kähler form ω as a symplectic form.

For a symplectic manifold M we can introduce on $C^\infty(M)$ a Lie algebra structure, the *Poisson bracket* $\{.,.\}$, in the following way. First we a assign to every $f \in C^\infty(M)$ its *Hamiltonian vector field* X_f, and then to every pair of functions f and g the *Poisson bracket* $\{.,.\}$ via

$$\omega(X_f, \cdot) = df(\cdot), \qquad \{\, f, g\,\} := \omega(X_f, X_g)\ . \tag{2.2}$$

This defines a Lie algebra structure in $C^\infty(M)$. Moreover, we obtain the Leibniz rule

$$\{fg, h\} = f\{g, h\} + \{f, h\}g, \qquad \forall f, g, h \in C^\infty(M).$$

Such a compatible structure is called a *Poisson algebra*.

The next step in the geometric set-up is the choice of a quantum line bundle. A *quantum line bundle* for a given symplectic manifold (M, ω) is a triple (L, h, ∇), where L is a complex line bundle, h a Hermitian metric on L, and ∇ a connection compatible with the metric h such that the (pre)quantum condition

$$\begin{gathered}\mathrm{curv}_{L,\nabla}(X, Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = -\,\mathrm{i}\,\omega(X, Y),\\ \text{resp.}\quad \mathrm{curv}_{L,\nabla} = -\,\mathrm{i}\,\omega\end{gathered} \tag{2.3}$$

is fulfilled. A symplectic manifold is called *quantizable* if there exists a quantum line bundle.

In the situation of Kähler manifolds we require for a quantum line bundle that it is holomorphic and that the connection is compatible both with the metric h and the complex structure of the bundle. In fact, by this requirement ∇ will be uniquely fixed. If we choose local holomorphic coordinates on the manifold and and a local holomorphic frame of the bundle the metric h will be represented by a function $\hat{h}$. In this case the curvature of the bundle can be given by $\overline{\partial}\partial \log \hat{h}$ and the quantum condition reads as

$$\mathrm{i}\,\overline{\partial}\partial \log \hat{h} = \omega\ . \tag{2.4}$$

2.2 Examples

(a) Of course, $\mathbb{C}^n$ is a Kähler manifold with Kähler form

$$\omega = \mathrm{i} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k \ . \tag{2.5}$$

The Poisson bracket writes as

$$\{f,g\} = \mathrm{i} \sum_{k=1}^{n} \left(\frac{\partial f}{\partial \overline{z}_k} \cdot \frac{\partial g}{\partial z_k} - \frac{\partial f}{\partial z_k} \frac{\partial g}{\partial \overline{z}_k} \right) \ . \tag{2.6}$$

The quantum line bundle is the trivial line bundle with hermitian metric fixed by the function $\hat{h}(z) = \exp(-\sum_{-k} \overline{z}_k z_k)$.

(b) The **Riemann sphere** is the complex projective line $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \cong S^2$. With respect to the quasi-global coordinate z the form can be given as

$$\omega = \frac{\mathrm{i}}{(1+z\overline{z})^2} dz \wedge d\overline{z} \ . \tag{2.7}$$

For the Poisson bracket one obtains

$$\{f,g\} = \mathrm{i}\,(1+z\overline{z})^2 \left(\frac{\partial f}{\partial \overline{z}} \cdot \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial \overline{z}} \right) \ . \tag{2.8}$$

Recall that the points in $\mathbb{P}^1(\mathbb{C})$ correspond to lines in $\mathbb{C}^2$ passing through the origin. If we assign to every point in $\mathbb{P}^1(\mathbb{C})$ the line it represents we obtain a holomorphic line bundle, called the tautological line bundle. The hyper plane section bundle is dual to the tautological bundle. It turns out that it is a quantum line bundle. Hence $\mathbb{P}^1(\mathbb{C})$ is quantizable.

(c) The above example generalizes to the n-dimensional **complex projective space** $\mathbb{P}^n(\mathbb{C})$. The Kähler form is given by the Fubini-Study form

$$\omega_{FS} := \mathrm{i} \frac{(1+|w|^2)\sum_{i=1}^{n} dw_i \wedge d\overline{w}_i - \sum_{i,j=1}^{n} \overline{w}_i w_j dw_i \wedge d\overline{w}_j}{(1+|w|^2)^2} \ . \tag{2.9}$$

The coordinates w_j, $j = 1, \ldots, n$ are affine coordinates $w_j = z_j/z_0$ on the affine chart $U_0 := \{(z_0 : z_1 : \cdots : z_n) \mid z_0 \neq 0\}$. Again, $\mathbb{P}^n(\mathbb{C})$ is quantizable with the hyper plane section bundle as quantum line bundle.

(d) The (complex-) **one-dimensional torus** can be given as $M = \mathbb{C}/\Gamma_\tau$ where $\Gamma_\tau := \{n + m\tau \mid n, m \in \mathbb{Z}\}$ is a lattice with $\tau \in \mathbb{C}$, $\operatorname{Im} \tau > 0$. As Kähler form we take

$$\omega = \frac{\mathrm{i}\,\pi}{\operatorname{Im} \tau} dz \wedge d\overline{z} \ , \tag{2.10}$$

with respect to the coordinate z on the covering space $\mathbb{C}$. Clearly, this form is invariant under the lattice Γ_τ and hence well-defined on M. For the Poisson bracket one obtains

$$\{f, g\} = \mathrm{i}\,\frac{\operatorname{Im}\tau}{\pi}\left(\frac{\partial f}{\partial \overline{z}}\cdot\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial \overline{z}}\right) . \tag{2.11}$$

The corresponding quantum line bundle is the theta line bundle of degree 1, i.e. the bundle whose global sections are scalar multiples of the Riemann theta function.

(e) The **unit disc**

$$\mathcal{D} := \{z \in \mathbb{C} \mid |z| < 1\} \tag{2.12}$$

is a (non-compact) Kähler manifold. The Kähler form is given by

$$\omega = \frac{2\,\mathrm{i}}{(1 - z\overline{z})^2} dz \wedge d\overline{z} . \tag{2.13}$$

For every **compact Riemann surface** M of genus $g \geq 2$ the unit disc $\mathcal{D}$ is the universal covering space and M can be given as a quotient of $\mathcal{D}$ by a Fuchsian subgroup of $SU(1,1)$, whose elements act by fractional linear transformations. Recall for $R = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$ with $|a|^2 - |b|^2 = 1$ (as an element of $SU(1,1)$) the fractional linear transformation is given as

$$z \mapsto R(z) := \frac{az + b}{\overline{b}z + \overline{a}} . \tag{2.14}$$

The Kähler form (2.13) is invariant under fractional linear transformations. Hence, it defines a Kähler form on M. The quantum bundle is the canonical bundle, i.e. the bundle whose local sections are the holomorphic differentials. Its global sections can be identified with automorphic forms of weight 2 with respect to the Fuchsian group.

2.3 Conditions for being quantizable

The above examples might create the wrong impression that every Kähler manifold is quantizable. This is not the case. Above we introduced one-dimension tori. Higher dimensional tori can be given as Kähler manifold in a completely analogous manner as quotients $\mathbb{C}^n/L$ were L is a $2n-$dimensional lattice. But only those higher dimensional complex tori are quantizable which are abelian varieties, i.e. which admit enough theta functions. It is well-known that for $n \geq 2$ a generic torus will not be an abelian variety.

In the language of differential geometry a line bundle is called a positive line bundle if its curvature form (up to a factor of 1/i) is a positive form. As the Kähler form is positive the quantum condition (2.3) yields that a quantum line bundle L is a positive line bundle.

Now let M is a quantizable **compact** Kähler manifold with quantum line bundle L. Kodaira's embedding theorem says that L is ample, i.e. that there exists a certain tensor power L^{m_0} of L such that the global holomorphic sections of L^{m_0} can be used to embed the phase space manifold M into a projective space of suitable dimension.

The embedding is defined as follows. Let $\Gamma_{hol}(M, L^{m_0})$ be the vector space of global holomorphic sections of the bundle L^{m_0}. Fix a basis $s_0, s_1, \dots, s_N$. We choose local holomorphic coordinates z for M and a local holomorphic frame $e(z)$ for the bundle L. After these choices the basis elements can be uniquely described by local holomorphic functions $\hat{s}_0, \hat{s}_1, \dots, \hat{s}_N$ defined via $s_j(z) = \hat{s}_j(z)e(z)$. The embedding is given by the map

$$M \hookrightarrow \mathbb{P}^N(\mathbb{C}), \quad z \mapsto \phi(z) = (\hat{s}_0(z) : \hat{s}_1(z) : \dots : \hat{s}_N(z)) \ . \tag{2.15}$$

Note that the point $\phi(z)$ in projective space neither depends on the choice of local coordinates nor on the choice of the local frame for the bundle L. Furthermore, a different choice of basis correspond to a $\mathrm{PGL}(N, \mathbb{C})$ action on the embedding space and hence the embeddings are projectively equivalent. The "map" (2.15) could be given for every line bundle having nontrivial global sections. But it might happen that all sections have common zeros. For those points the map will not be defined. Furthermore, to be an embedding it should separate points and tangent directions. A line bundles whose global holomorphic sections will define an embedding into projective space, is called a *very ample line bundle.*

By this embedding quantizable compact Kähler manifolds are complex submanifolds of projective spaces. By Chow's theorem [52] they can be given as zero sets of homogenous polynomials, i.e. they are smooth projective varieties. The converse is also true. Given a smooth subvariety M of $\mathbb{P}^n(\mathbb{C})$ it will become a Kähler manifold by restricting the Fubini-Study form. The restriction of the hyper plane section bundle will be an associated quantum line bundle.

At this place a warning is necessary. the embedding is only an embedding as complex manifolds, not an isometric embedding as Kähler manifolds. This means that in general $\phi^{-1}(\omega_{FS}) \neq \omega$.

3 Berezin-Toeplitz operators

In this section we will consider an operator quantization. This says that we will assign to each differentiable (differentiable to every order) function f on our Kähler manifold M (i.e. on our "phase space") the Berezin-Toeplitz (BT) quantum operator T_f. More precisely, we will consider a whole family of operators $T_f^{(m)}$. These operators are defined in a canonical way. As we know from the Groenewold-van Howe theorem we cannot expect that the Poisson bracket on M can be represented by the Lie algebra of operators if we require certain desirable conditions, see [1] for further details. The best we can expect is that we obtain it at least "asymptotically". In fact, this is true.

3.1 Definition of the operators

Let (M,ω) be a quantizable Kähler manifold and (L,h,∇) a quantum line bundle. We assume that L is already very ample. We consider all its tensor powers

$$(L^m, h^{(m)}, \nabla^{(m)}). \tag{3.1}$$

Here $L^m := L^{\otimes m}$. If $\hat{h}$ corresponds to the metric h with respect to a local holomorphic frame e of the bundle L then $\hat{h}^m$ corresponds to the metric $h^{(m)}$ with respect to the frame $e^{\otimes m}$ for the bundle L^m. The connection $\nabla^{(m)}$ will be the induced connection.

We introduce a scalar product on the space of sections. We adopt the convention that a hermitian metric (and a scalar product) is anti-linear in the first argument and linear in the second argument. First we take the Liouville form $\Omega = \frac{1}{n!}\omega^{\wedge n}$ as volume form on M and then set for the scalar product and the norm

$$\langle \varphi, \psi \rangle := \int_M h^{(m)}(\varphi,\psi)\,\Omega\ , \qquad ||\varphi|| := \sqrt{\langle \varphi, \varphi \rangle}\ , \tag{3.2}$$

for the space $\Gamma_\infty(M,L^m)$ of global C^∞-sections. Let $\mathrm{L}^2(M,L^m)$ be the L^2-completed space of bounded sections with respect to this norm. Furthermore, let $\Gamma_{hol}^{(b)}(M,L^m)$ be the closed subspace consisting of those global holomorphic sections which are bounded. These spaces are the quantum spaces of the theory. Note that in case that M is compact $\Gamma_{hol}(M,L^m) = \Gamma_{hol}^{(b)}(M,L^m)$ and the spaces are finite-dimensional. Let

$$\Pi^{(m)} : \mathrm{L}^2(M,L^m) \to \Gamma_{hol}^{(b)}(M,L^m) \tag{3.3}$$

be the projection.

Definition 3.1. For $f \in C^\infty(M)$ the *Toeplitz operator* $T_f^{(m)}$ *(of level m)* is defined by

$$T_f^{(m)} := \Pi^{(m)}\,(f\cdot) : \quad \Gamma_{hol}^{(b)}(M,L^m) \to \Gamma_{hol}^{(b)}(M,L^m)\ . \tag{3.4}$$

In words: One takes a holomorphic section s and multiplies it with the differentiable function f. The resulting section $f\cdot s$ will only be differentiable. To obtain a holomorphic section, one has to project it back on the subspace of holomorphic sections.

The linear map

$$T^{(m)} : C^{\infty}(M) \to \mathrm{End}\big(\Gamma^{(b)}_{hol}(M, L^m)\big), \qquad f \to T^{(m)}_f = \Pi^{(m)}(f\cdot)\ , m \in \mathbb{N}_0\ . \tag{3.5}$$

is the *Toeplitz* or *Berezin-Toeplitz quantization map (of level m)*. It will neither be a Lie algebra homomorphism nor an associative algebra homomorphism as in general

$$T^{(m)}_f\, T^{(m)}_g = \Pi^{(m)}\,(f\cdot)\,\Pi^{(m)}\,(g\cdot)\,\Pi^{(m)} \neq \Pi^{(m)}\,(fg\cdot)\,\Pi = T^{(m)}_{fg}.$$

Definition 3.2. The Berezin-Toeplitz (BT) quantization is the map

$$C^{\infty}(M) \to \prod_{m\in\mathbb{N}_0} \mathrm{End}(\Gamma^{(b)}_{hol}(M, L^m)), \qquad f \to (T^{(m)}_f)_{m\in\mathbb{N}_0}. \tag{3.6}$$

In case that M is a compact Kähler manifold on a fixed level m the BT quantization is a map from the infinite-dimensional commutative algebra of functions to a noncommutative finite-dimensional (matrix) algebra. The finite-dimensionality is due to the compactness of M. A lot of classical information will get lost. To recover this information one has to consider not just a single level m but all levels together as done in the above definition. In this way a family of finite-dimensional(matrix) algebras and a family of maps is obtained, which in the classical limit “converges” to the algebra $C^{\infty}(M)$.

3.2 Approximation results for the compact Kähler case

In the following we will only deal with compact quantizable Kähler manifolds. We assume that the quantum line bundle L is already very ample (i.e. its sections give an embedding into projective space). This is not much of a restriction. If L is not very ample we choose $m_0 \in \mathbb{N}$ such that the bundle L^{m_0} is very ample and take this bundle as quantum line bundle with respect to the rescaled Kähler form $m_0\,\omega$ on M. The underlying complex manifold structure will not change.

Recall that in the compact case we have $\Gamma_{hol}(M, L^m) = \Gamma^{(b)}_{hol}(M, L^m)$. Denote for $f \in C^{\infty}(M)$ by $|f|_{\infty}$ the sup-norm of f on M and by

$$||T^{(m)}_f|| := \sup_{\substack{s\in\Gamma_{hol}(M,L^m)\\ s\neq 0}} \frac{||T^{(m)}_f s||}{||s||} \tag{3.7}$$

the operator norm with respect to the norm (3.2) on $\Gamma_{hol}(M, L^m)$. The following theorem was shown in 1994.

Theorem 3.3. *[Bordemann, Meinrenken, Schlichenmaier] [14]*
(a) For every $f \in C^{\infty}(M)$ there exists a $C > 0$ such that

$$|f|_{\infty} - \frac{C}{m} \quad \leq \quad ||T^{(m)}_f|| \quad \leq \quad |f|_{\infty}\ . \tag{3.8}$$

In particular, $\lim_{m\to\infty} ||T_f^{(m)}|| = |f|_\infty$.
(b) For every $f, g \in C^\infty(M)$

$$||m\,\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| \quad = \quad O(\frac{1}{m})\ . \tag{3.9}$$

(c) For every $f, g \in C^\infty(M)$

$$||T_f^{(m)} T_g^{(m)} - T_{f\cdot g}^{(m)}|| \quad = \quad O(\frac{1}{m})\ . \tag{3.10}$$

These results are contained in Theorem 4.1, 4.2, and in Section 5 in [14]. The proofs make reference to the symbol calculus of generalised Toeplitz operators as developed by Boutet-de-Monvel and Guillemin [17]. See [53] for a sketch. Only on the basis of this theorem we are allowed to call our scheme a quantizing scheme. The properties in the theorem might be rephrased as *the BT operator quantization has the correct semiclassical limit*. In other words it is a strict quantization in the sense of Rieffel [44] as formulated in the book by Landsman [36]. This notion is closely related to the notion of continuous field of C^*-algebras.

Let us summarize further properties in the following

Proposition 3.4. *Let* $f, g \in C^\infty(M)$, $n = \dim_{\mathbb{C}} M$ *then*
(a)

$$\lim_{m\to\infty} ||\,[T_f^{(m)}, T_g^{(m)}]\,|| \quad = \quad 0\ . \tag{3.11}$$

(b) The Toeplitz map

$$C^\infty(M) \to \mathrm{End}(\Gamma_{hol}(M, L^m)), \qquad f \to T_f^{(m)},$$

is surjective.
(c)

$$T_f^{(m)^*} = T_{\bar{f}}^{(m)}\ . \tag{3.12}$$

In particular, for real valued functions f *the associated Toeplitz operator* T_f *is selfadjoint.*
(d) Let $A \in \mathrm{End}(\Gamma_{hol}(M, L^m))$ *be a selfadjoint operator then there exists a real valued function* f*, such that* $A = T_f^{(m)}$.
(e) Denote the trace on $\mathrm{End}(\Gamma_{hol}(M, L^m))$ *by* $\mathrm{Tr}^{(m)}$ *then*

$$\mathrm{Tr}^{(m)}\,(T_f^{(m)}) = m^n \left(\frac{1}{\mathrm{vol}(\mathbb{P}^n(\mathbb{C}))}\int_M f\,\Omega + O(m^{-1})\right)\ . \tag{3.13}$$

For the proofs, resp. references to the proofs, I refer to [53]. I like to stress the fact that the Toeplitz map is never injective on a fixed level m. But from $||T^{(m)}_{f-g}|| \to 0$ for $m \to 0$ we can conclude that $f = g$.

There exists another quantum operator in the geometric setting, the operator of geometric quantization introduced by Kostant and Souriau. In a first step the prequantum operator associated to the bundle L^m for the function $f \in C^\infty(M)$ is defined as

$$P^{(m)}_f := \nabla^{(m)}_{X^{(m)}_f} + \mathrm{i}\, f \cdot id. \tag{3.14}$$

Here $\nabla^{(m)}$ is the connection in L^m, and $X^{(m)}_f$ the Hamiltonian vector field of f with respect to the Kähler form $\omega^{(m)} = m \cdot \omega$, i.e. $m\omega(X^{(m)}_f, .) = df(.)$. Next one has to choose a polarization. In general it will not be unique. But in our complex situation there is canonical one by taking the projection to the space of holomorphic sections. This polarization is called *Kähler polarization.* The operator of geometric quantization is then defined by

$$Q^{(m)}_f := \Pi^{(m)} P^{(m)}_f. \tag{3.15}$$

The Toeplitz operator and the operator of geometric quantization (with respect to the Kähler polarization) are related by

Proposition 3.5. *(Tuynman Lemma) Let M be a compact quantizable Kähler manifold then*

$$Q^{(m)}_f = \mathrm{i} \cdot T^{(m)}_{f - \frac{1}{2m}\Delta f}, \tag{3.16}$$

where Δ is the Laplacian with respect to the Kähler metric given by ω.

For the proof see [56], and [13] for a coordinate independent proof.
In particular the operators $Q^{(m)}_f$ and the $T^{(m)}_f$ have the same asymptotic behaviour. It should be noted that for (3.16) the compactness of M is essential.

Remark 3.6. Above we introduced Berezin-Toeplitz operators also for non-compact Kähler manifolds. Unfortunately, in this context the proofs of Theorem 3.3 do not work. One has to study examples or classes of examples case by case and to check whether the corresponding properties are correct. See [53] for list of references in this context.

4 Berezin-Toeplitz deformation quantization

4.1 What is a star product?

There is another approach to quantization. Instead of assigning noncommutative operators to commuting functions one might think about "deforming" the point-wise commutative multiplication of functions into a non-commutative product. It is required to remain associative, the commutator of two elements should relate to the Poisson bracket of the elements, and it should reduce in the "classical limit" to the commutative situation.

It turns out that such a deformation which is valid for all differentiable functions cannot exist. A way out is to enlarge the algebra of functions by considering formal power series over them and to deform the product inside this bigger algebra. A first systematic treatment and applications in physics of this idea were given 1978 by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer [9]. There the notion of *deformation quantization* and *star products* were introduced. Earlier versions of these concepts were around due to Berezin [10], Moyal [39], and Weyl [58]. For a presentation of the history see [54]. We will show that for compact Kähler manifolds M, there is a natural star product.

We start even more general, with a Poisson manifold $(M, \{.,.\})$, i.e. a differentiable manifold with a Poisson bracket for the function such that $(C^\infty(M), \cdot, \{.,.\})$ is a Poisson algebra. Let $\mathcal{A} = C^\infty(M)[[\nu]]$ be the algebra of formal power series in the variable ν over the algebra $C^\infty(M)$.

Definition 4.1. A product $\star$ on $\mathcal{A}$ is called a (formal) star product for M (or for $C^\infty(M)$) if it is an associative $\mathbb{C}[[\nu]]$-linear product which is ν-adically continuous such that

1. $\mathcal{A}/\nu\mathcal{A} \cong C^\infty(M)$, i.e. $\quad f \star g \bmod \nu = f \cdot g$,
2. $\dfrac{1}{\nu}(f \star g - g \star f) \bmod \nu = -\,\mathrm{i}\,\{f, g\}$,

where $f, g \in C^\infty(M)$.

Alternatively we can write

$$f \star g = \sum_{j=0}^{\infty} C_j(f,g)\nu^j \ , \tag{4.1}$$

with $C_j(f,g) \in C^\infty(M)$ such that the C_j are bilinear in the entries f and g. The conditions (1) and (2) can be reformulated as

$$C_0(f,g) = f \cdot g, \qquad \text{and} \qquad C_1(f,g) - C_1(g,f) = -\,\mathrm{i}\,\{f,g\} \ . \tag{4.2}$$

By the ν-adic continuity (4.1) fixes $\star$ on $\mathcal{A}$. A *(formal) deformation quantization* is given by a *(formal) star product*. I will use both terms interchangeable.

There are certain additional conditions for a star product which are sometimes useful.

1. We call it "null on constants", if $1 \star f = f \star 1 = f$, which is equivalent to the fact that the constant function 1 will remain the unit in $\mathcal{A}$. In terms of the coefficients it can be formulated as $C_k(f,1) = C_k(1,f) = 0$ for $k \geq 1$. Here we always assume this to be the case for star products.

2. We call it selfadjoint if $\overline{f \star g} = \overline{g} \star \overline{f}$, where we assume $\bar{\nu} = \nu$.

3. We call it local if

$$\operatorname{supp} C_j(f,g) \subseteq \operatorname{supp} f \cap \operatorname{supp} g, \qquad \forall f,g \in C^\infty(M).$$

From the locality property it follows that the C_j are bidifferential operators and that the global star product defines for every open subset U of M a star product for the Poisson algebra $C^\infty(U)$. Such local star products are also called *differential star products*.

In the usual setting of deformation theory there always exists a trivial deformation. This is not the case here, as the trivial deformation of $C^\infty(M)$ to $\mathcal{A}$, which is nothing else as extending the point-wise product to the power series, is not allowed as it does not fulfill Condition (2) in Definition 4.1 (at least not if the Poisson bracket is non-trivial). In fact the existence problem is highly non-trivial. In the symplectic case different existence proofs, from different perspectives, were given by DeWilde-Lecomte [22], Omori-Maeda-Yoshioka [41], and Fedosov [29]. The general Poisson case was settled by Kontsevich [35].

The next question is the classification of star products.

Definition 4.2. Given a Poisson manifold $(M, \{.,.\})$. Two star products $\star$ and $\star'$ associated to the Poisson structure $\{.,.\}$ are called equivalent if and only if there exists a formal series of linear operators

$$B = \sum_{i=0}^{\infty} B_i \nu^i, \qquad B_i : C^\infty(M) \to C^\infty(M), \tag{4.3}$$

with $B_0 = id$ such that

$$B(f) \star' B(g) = B(f \star g). \tag{4.4}$$

For local star products in the general Poisson setting there are complete classification results. Here I will only consider the symplectic case. To each local star product $\star$ its *Fedosov-Deligne class*

$$cl(\star) \in \frac{1}{\mathrm{i}\,\nu}[\omega] + H^2_{dR}(M)[[\nu]] \tag{4.5}$$

can be assigned. Here $H^2_{dR}(M)$ denotes the 2nd deRham cohomology class of closed 2-forms modulo exact forms and $H^2_{dR}(M)[[\nu]]$ the formal power series with such classes as coefficients. Such formal power series are called *formal deRham classes*. In general we will use $[\alpha]$ for the cohomology class of a form α. This assignment gives a 1:1 correspondence between the formal deRham classes and the equivalence classes of star products.

For contractible manifolds we have $H^2_{dR}(M) = 0$ and hence there is up to equivalence exactly one local star product. This yields that locally all local star products of a manifold are equivalent to a certain fixed one, which is called the Moyal product. For these and related classification results see [23], [31], [12], [40].

For our compact Kähler manifolds we will have many different and even non-equivalent star products. The question is: is there a star product which is given in a natural way? The answer will be yes: the Berezin-Toeplitz star product to be introduced below. First we consider star products respecting the complex structure in a certain sense.

Definition 4.3. (Karabegov [32]) A star product is called *star product with separation of variables* if and only if

$$f \star h = f \cdot h, \quad \text{and} \quad h \star g = h \cdot g, \tag{4.6}$$

for every locally defined holomorphic function g, antiholomorphic function f, and arbitrary function h.

Recall that a local star product $\star$ for M defines a star product for every open subset U of M. We have just to take the bidifferential operators defining $\star$. Hence it makes sense to talk about $\star$-multiplying with local functions.

Proposition 4.4. *A local $\star$ product has the separation of variables property if and only if in the bidifferential operators $C_k(.,.)$ for $k \geq 1$ in the first argument only derivatives in holomorphic and in the second argument only derivatives in antiholomorphic directions appear.*

In Karabegov's original notation the rôles of the holomorphic and antiholomorphic functions is switched. Bordemann and Waldmann [15] called such star products *star products of Wick type*. Both Karabegov and Bordemann-Waldmann proved that there exists for every Kähler manifold star products of separation of variables type. See also Reshetikhin and Takhtajan [43] for yet another construction. But I like to point out that in all these constructions the result is only a formal star product without any relation to an operator calculus, which will be given by the Berezin-Toeplitz star product introduced in the next section.

Another warning is in order. The property of being a star product of separation of variables type will not be kept by equivalence transformations.

4.2 Berezin-Toeplitz star product

Again we restrict the situation to the compact quantizable Kähler case.

Theorem 4.5. *There exists a unique (formal) star product $\star_{BT}$ for M*

$$f \star_{BT} g := \sum_{j=0}^{\infty} \nu^j C_j(f,g), \quad C_j(f,g) \in C^\infty(M), \tag{4.7}$$

in such a way that for $f, g \in C^\infty(M)$ and for every $N \in \mathbb{N}$ we have with suitable constants $K_N(f,g)$ for all m

$$||T_f^{(m)} T_g^{(m)} - \sum_{0 \le j < N} \left(\frac{1}{m}\right)^j T_{C_j(f,g)}^{(m)}|| \le K_N(f,g) \left(\frac{1}{m}\right)^N . \tag{4.8}$$

The star product is null on constants and selfadjoint.

This theorem has been proven immediately after [14] was finished. It has been announced in [46],[47] and the proof was written up in German in [45]. A complete proof published in English can be found in [49].

For simplicity we write

$$T_f^{(m)} \cdot T_g^{(m)} \quad \sim \quad \sum_{j=0}^{\infty} \left(\frac{1}{m}\right)^j T_{C_j(f,g)}^{(m)} \qquad (m \to \infty), \tag{4.9}$$

but we will always assume the strong and precise statement of (4.8).

Next we want to identify this star product. Let K_M be the canonical line bundle of M, i.e. the n^{th} exterior power of the holomorphic 1-differentials. The canonical class δ is the first Chern class of this line bundle, i.e. $\delta := c_1(K_M)$. If we take in K_M the fiber metric coming from the Liouville form Ω then this defines a unique connection and further a unique curvature $(1,1)$-form ω_{can}. In our sign conventions we have $\delta = [\omega_{can}]$.

Together with Karabegov the author showed

Theorem 4.6. *[34] (a) The Berezin-Toeplitz star product is a local star product which is of separation of variable type.*
(b) Its classifying Deligne-Fedosov class is

$$cl(\star_{BT}) = \frac{1}{\mathrm{i}} \left(\frac{1}{\nu}[\omega] - \frac{\delta}{2}\right) \tag{4.10}$$

for the characteristic class of the star product $\star_{BT}$.
(c) The classifying Karabegov form associated to the Berezin-Toeplitz star product is

$$-\frac{1}{\nu}\omega + \omega_{can}. \tag{4.11}$$

Remark 4.7. The Karabegov form

$$\widehat{\omega} = (1/\nu)\omega_{-1} + \omega_0 + \nu\omega_1 + \ldots \tag{4.12}$$

is a formal series, where ω_{-1} is the Kähler form ω of the manifold and the forms $\omega_r,\ r \ge 0$, are closed but not necessarily nondegenerate (1,1)-forms on M. It was shown in [32] that all deformation quantizations with separation of variables on the pseudo-Kähler manifold (M, ω_{-1}) are bijectively parameterized by such formal forms (4.12). They might be considered as formal deformations of $(1/\nu)\omega_{-1}$. The reason that we have $-\frac{1}{\nu}\omega$ in (4.11) is that in Karabegov's terminology the role of the holomorphic and anti-holomorphic variables are switched. For a description of Karabegov's construction, see [53]. There details about the identification of the Berezin-Toeplitz star product in his classification can be found, see also [34].

Remark 4.8. By Tuynman's lemma (3.16) the operators of geometric quantization with Kähler polarization have the same asymptotic behaviour as the BT operators. As the latter defines a star product $\star_{BT}$ it can be used to give also a star product $\star_{GQ}$ associated to geometric quantization. Details can be found in [49]. This star product will be equivalent to the BT star product, but it is not of separation of variables type. The equivalence is given by the $\mathbb{C}[[\nu]]$-linear map induced by

$$B(f) := f - \nu\frac{\Delta}{2}f = (id - \nu\frac{\Delta}{2})f. \tag{4.13}$$

We obtain $B(f) \star_{BT} B(g) = B(f \star_{GQ} g)$.

Remark 4.9. From (3.13) the following complete asymptotic expansion for $m \to \infty$ can be deduced [49], [16]):

$$\operatorname{Tr}^{(m)}(T_f^{(m)}) \quad \sim \quad m^n \left(\sum_{j=0}^{\infty} \left(\frac{1}{m}\right)^j \tau_j(f) \right), \quad \text{with} \quad \tau_j(f) \in \mathbb{C} \ . \tag{4.14}$$

We define the $\mathbb{C}[[\nu]]$-linear map

$$\operatorname{Tr} : C^\infty(M)[[\nu]] \to \nu^{-n}\,\mathbb{C}[[\nu]], \quad \operatorname{Tr} f := \nu^{-n} \sum_{j=0}^{\infty} \nu^j \tau_j(f), \tag{4.15}$$

where the $\tau_j(f)$ are given by the asymptotic expansion (4.14) for $f \in C^\infty(M)$ and for arbitrary elements by $\mathbb{C}[[\nu]]$-linear extension.

Proposition 4.10. *[49] The map* Tr *is a trace, i.e., we have*

$$\operatorname{Tr}(f \star g) = \operatorname{Tr}(g \star f) \ . \tag{4.16}$$

5 Berezin's coherent states, symbols, and transform

5.1 The disc bundle

We will assume that M is a compact quantizable Kähler manifold with very ample quantum line bundle L, i.e. L has enough global holomorphic sections to embed M into projective space. From the bundle[1] (L,h) we pass to its dual $(U,k) := (L^*, h^{-1})$ with dual metric k. Inside of the total space U we consider the circle bundle

$$Q := \{\lambda \in U \mid k(\lambda,\lambda) = 1\}, \tag{5.1}$$

the (open) disc bundle, and (closed) disc bundle respectively

$$D := \{\lambda \in U \mid k(\lambda,\lambda) < 1\}, \qquad \overline{D} := \{\lambda \in U \mid k(\lambda,\lambda) \le 1\}. \tag{5.2}$$

Let $\tau : U \to M$ the projection (maybe restricted to the subbundles).

For the projective space $\mathbb{P}^N(\mathbb{C})$ with the hyperplane section bundle H as quantum line bundle the bundle U is just the tautological bundle. Its fiber over the point $z \in \mathbb{P}^N(\mathbb{C})$ consists of the line in $\mathbb{C}^{N+1}$ which is represented by z. In particular, for the projective space the total space of U with the zero section removed can be identified with $\mathbb{C}^{N+1} \setminus \{0\}$. The same picture remains true for the, via the very ample quantum line bundle in projective space embedded, manifold M. The quantum line bundle will be the pull-back of H (i.e. its restriction to the embedded manifold) and its dual is the pull-back of the tautological bundle.

In the following we use $E \setminus 0$ to denote the total space of a vector bundle E with the image of the zero section removed. Starting from the real valued function $\hat{k}(\lambda) := k(\lambda,\lambda)$ on U we define $\tilde{a} := \frac{1}{2\mathrm{i}}(\partial - \overline{\partial})\log \hat{k}$ on $U \setminus 0$ (the derivation are taken with respect to the complex structure on U) and denote by α its restriction to Q. With the help of the quantization condition (2.3) we obtain $d\alpha = \tau^*\omega$ (with the deRham differential $d = d_Q$) and that in fact $\mu = \frac{1}{2\pi}\tau^*\Omega \wedge \alpha$ is a volume form on Q. Indeed α is a contact form for the contact manifold Q. As far as the integration is concerned we get

$$\int_Q (\tau^* f)\mu = \int_M f\,\Omega, \qquad \forall f \in C^\infty(M). \tag{5.3}$$

Recall that Ω is the Liouville volume form on M.

With respect to μ we take the L^2-completion $\mathrm{L}^2(Q,\mu)$ of the space of functions on Q. By the natural circle action the bundle Q is an S^1-bundle and the tensor powers of U can be viewed as associated line bundles. Sections of $L^m = U^{-m}$ can be identified with functions ψ on Q which satisfy the equivariance condition $\psi(c\lambda) = c^m\psi(\lambda)$, i.e. which are homogeneous of degree m. This identification is given via the map

$$\gamma_m : \mathrm{L}^2(M, L^m) \to \mathrm{L}^2(Q,\mu), \quad s \mapsto \psi_s \quad \text{where} \quad \psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))), \tag{5.4}$$

which turns out to be an isometry onto its image, where on $\mathrm{L}^2(M, L^m)$ we take the scalar product (3.2).

[1] As the connection ∇ will not be needed anymore, I will drop it in the notation.

The generalized *Hardy space* $\mathcal{H}$ is the closure of the space of those functions in $\mathrm{L}^2(Q,\mu)$ which can be extended to holomorphic functions on the whole disc bundle $\overline{D}$. The generalized *Szegö projector* is the projection

$$\Pi : \mathrm{L}^2(Q,\mu) \to \mathcal{H}\ . \tag{5.5}$$

The space $\mathcal{H}$ is preserved by the S^1-action. It can be decomposed into eigenspaces $\mathcal{H} = \prod_{m=0}^{\infty} \mathcal{H}^{(m)}$ where $c \in S^1$ acts on $\mathcal{H}^{(m)}$ as multiplication by c^m. The Szegö projector is S^1 invariant and can be decomposed into its components, the Bergman projectors

$$\hat{\Pi}^{(m)} : \mathrm{L}^2(Q,\mu) \to \mathcal{H}^{(m)}. \tag{5.6}$$

If we restrict (5.4) on the holomorphic sections we obtain the isometry

$$\gamma_m : \Gamma_{hol}(M, L^m) \cong \mathcal{H}^{(m)}. \tag{5.7}$$

In the case of $\mathbb{P}^N(\mathbb{C})$ this correspondence is nothing else as the identification of the global sections of the m^{th} tensor powers of the hyper plane section bundle with the homogenous polynomial functions of degree m on $\mathbb{C}^{N+1}$.

Remark 5.1. In this set-up the notion of Toeplitz structure (Π, Σ), as developed by Boutet de Monvel and Guillemin in [17, 30] can be applied. After some work this leads to a proof of most of the statements in Theorem 3.3. A sketch of these techniques and of the proof can be found in [53]

5.2 Coherent States

We recall the correspondence (5.4) $\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha)))$ of m-homogeneous functions ψ_s on U with sections of L^m. To obtain this correspondence we fixed the section s and varied a.

Now we do the opposite. We fix $\alpha \in U \setminus 0$ and vary the sections s. Obviously this yields a linear form on $\Gamma_{hol}(M, L^m)$ and hence with the help of the scalar product (3.2) we make the following

Definition 5.2. (a) The *coherent vector (of level m)* associated to the point $\alpha \in U \setminus 0$ is the unique element $e_\alpha^{(m)}$ of $\Gamma_{hol}(M, L^m)$ such that

$$\langle e_\alpha^{(m)}, s\rangle = \psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))) \tag{5.8}$$

for all $s \in \Gamma_{hol}(M, L^m)$.
(b) The *coherent state (of level m)* associated to $x \in M$ is the projective class

$$\mathrm{e}_x^{(m)} := [e_\alpha^{(m)}] \in \mathbb{P}(\Gamma_{hol}(M, L^m)), \qquad \alpha \in \tau^{-1}(x), \alpha \neq 0. \tag{5.9}$$

Of course, we have to show that the object in (b) is well-defined. Recall that $\langle .,. \rangle$ denotes the scalar product on the space of global sections $\Gamma_\infty(M, L^m)$. In our convention it will be anti-linear in the first argument and linear in the second argument. The coherent vectors are antiholomorphic in α and fulfill

$$e^{(m)}_{c\alpha} = \bar{c}^m \cdot e^{(m)}_\alpha, \qquad c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} \ . \tag{5.10}$$

Note that $e^{(m)}_\alpha \equiv 0$ would imply that all sections will vanish at the point $x = \tau(\alpha)$. Hence, the sections of L cannot be used to embed M into projective space, which is a contradiction to the very-ampleness of L. Hence, $e^{(m)}_\alpha \not\equiv 0$ and due to (5.10) the class

$$[e^{(m)}_\alpha] := \{ s \in \Gamma_{hol}(M, L^m) \mid \exists c \in \mathbb{C}^* : s = c \cdot e^{(m)}_\alpha \}$$

is a well-defined element of the projective space $\mathbb{P}(\Gamma_{hol}(M, L^m))$, only depending on $x = \tau(\alpha) \in M$.

This kind of coherent states go back to Berezin. A coordinate independent version and extensions to line bundles were given by Rawnsley [42]. They also exist in the non-compact setting, as the linear form given by the evaluation of the sections is continuous, see again [42].

The coherent states play an important role in the work of Cahen, Gutt, and Rawnsley on the quantization of Kähler manifolds [18, 19, 20, 21], via Berezin's covariant symbols. In these works the coherent vectors are parameterized by the elements of $L \setminus 0$. The definition here uses the points of the total space of the dual bundle U. It has the advantage that one can consider all tensor powers of L together on an equal footing.

Remark 5.3. The *coherent state embedding* is the antiholomorphic embedding

$$M \quad \to \quad \mathbb{P}(\Gamma_{hol}(M, L^m)) \ \cong \ \mathbb{P}^N(\mathbb{C}), \qquad x \mapsto [e^{(m)}_{\tau^{-1}(x)}]. \tag{5.11}$$

Here $N = \dim \Gamma_{hol}(M, L^m) - 1$. Here we will understand under $\tau^{-1}(x)$ always a non-zero element of the fiber over x. The coherent state embedding is up to conjugation the embedding (2.15) with respect to an orthonormal basis of the sections.

5.3 Berezin symbols

In this subsection I will be rather short, but details and complete proofs can be found in [53]. We start with the

Definition 5.4. The *covariant Berezin symbol* $\sigma^{(m)}(A)$ *(of level m)* of an operator $A \in \operatorname{End}(\Gamma_{hol}(M, L^m))$ is defined as

$$\sigma^{(m)}(A) : M \to \mathbb{C}, \qquad x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e^{(m)}_\alpha, A e^{(m)}_\alpha \rangle}{\langle e^{(m)}_\alpha, e^{(m)}_\alpha \rangle}, \quad \alpha \in \tau^{-1}(x). \tag{5.12}$$

As the factors appearing in (5.10) will cancel, it is a well-defined function on M.

We introduce the the *coherent projectors* used by Rawnsley

$$P_x^{(m)} = \frac{|e_\alpha^{(m)}\rangle\langle e_\alpha^{(m)}|}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}, \qquad \alpha \in \tau^{-1}(x) \ . \tag{5.13}$$

in the convenient bra-ket notation of physicists. With their help the covariant symbol can be expressed as

$$\sigma^{(m)}(A)(x) = \mathrm{Tr}(AP_x^{(m)}). \tag{5.14}$$

From the definition of the symbol it follows that $\sigma^{(m)}(A)$ is real analytic and that $\sigma^{(m)}(A^*) = \overline{\sigma^{(m)}(A)}$.

Rawnsley [42] introduced a very helpful function on the manifold M relating the local metric in the bundle with the scalar product on coherent states.

In our dual description we define it in the following way.

Definition 5.5. *Rawnsley's epsilon function* is the function

$$M \to C^\infty(M), \qquad x \mapsto \epsilon^{(m)}(x) := \frac{h^{(m)}(e_\alpha^{(m)}, e_\alpha^{(m)})}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}\,(x), \quad \alpha \in \tau^{-1}(x). \tag{5.15}$$

Indeed it is an extremely interesting function encoding geometric information. In [53, Prop. 6.6] it is shown that for any orthonormal basis $s_1, s_2, \ldots, s_k$ of $\Gamma_{hol}(M, L^m)$ it calculates to

$$\epsilon^{(m)}(x) = \sum_{j=1}^{k} h^{(m)}(s_j, s_j)(x). \tag{5.16}$$

The function $\epsilon^{(m)}$ is strictly positive. Hence, we can define the modified measure

$$\Omega_\epsilon^{(m)}(x) := \epsilon^{(m)}(x)\Omega(x) \tag{5.17}$$

for the space of functions on M and obtain a modified scalar product $\langle .,. \rangle_\epsilon^{(m)}$ for $C^\infty(M)$.

In the case that the functions $\epsilon^{(m)}$ will be constant as function of the points of the manifold it calculates as

$$\epsilon^{(m)} = \frac{\dim \Gamma_{hol}(M, L^m)}{\mathrm{vol}\, M}. \tag{5.18}$$

Here $\mathrm{vol}\, M$ denotes the volume of the manifold with respect to the Liouville measure. Now the question arises when $\epsilon^{(m)}$ will be constant, resp. when the measure $\Omega_\epsilon^{(m)}$ will be the standard measure (up to a scalar). If there is a transitive group action on the manifold and everything, e.g. Kähler form, bundle, metric, is homogenous with respect to the action this will be the case. An example is given by $M = \mathbb{P}^N(\mathbb{C})$. By a result of Rawnsley [42], resp. Cahen, Gutt and Rawnsley [18], $\epsilon^{(m)} \equiv$ *const* if and only if the quantization is projectively induced. This means that under the conjugate of the coherent state embedding (2.15), the Kähler form ω of M coincides with the pull-back of the Fubini-Study form. Note that in general situations this is not the case.

Definition 5.6. Given an operator $A \in \mathrm{End}(\Gamma_{hol}(M, L^m))$ then a *contravariant Berezin symbol* $\check{\sigma}^{(m)}(A) \in C^\infty(M)$ of A is defined by the representation of the operator A as integral

$$A = \int_M \check{\sigma}^{(m)}(A)(x) P_x^{(m)}\, \Omega_\epsilon^{(m)}(x), \tag{5.19}$$

if such a representation exists.

Very important is that we put "a" and not "the" in the definition, as in general the contravariant symbol will not be unique. But

Proposition 5.7. *The Toeplitz operator $T_f^{(m)}$ admits a representation (5.19) with*

$$\check{\sigma}^{(m)}(T_f^{(m)}) = f\ , \tag{5.20}$$

i.e. the function f is a contravariant symbol of the Toeplitz operator $T_f^{(m)}$. Moreover, every operator $A \in \mathrm{End}(\Gamma_{hol}(M, L^m))$ has a contravariant symbol.

As the Toeplitz map is surjective the last statement in the proposition is clear.

We introduce on $\mathrm{End}(\Gamma_{hol}(M, L^m))$ the Hilbert-Schmidt norm

$$\langle A, C\rangle_{HS} = Tr(A^* \cdot C)\ . \tag{5.21}$$

Theorem 5.8. *The Toeplitz map $f \to T_f^{(m)}$ and the covariant symbol map $A \to \sigma^{(m)}(A)$ are adjoint:*

$$\langle A, T_f^{(m)}\rangle_{HS} = \langle \sigma^{(m)}(A), f\rangle_\epsilon^{(m)}\ . \tag{5.22}$$

Let us collect some related results

Proposition 5.9.
(a)

$$\langle A, B\rangle_{HS} = \langle \sigma^{(m)}(A), \check{\sigma}^{(m)}(B)\rangle_\epsilon^{(m)}. \tag{5.23}$$

(b) The covariant symbol map $\sigma^{(m)}$ is injective.
(c)

$$\mathrm{Tr}\, A = \int_M \sigma^{(m)}(A)\, \Omega_\epsilon^{(m)} = \int_M \check{\sigma}^{(m)}(A)\, \Omega_\epsilon^{(m)}. \tag{5.24}$$

Remark 5.10. Under certain very restrictive conditions Berezin covariant symbols can be used to construct a star product, called the *Berezin star product*. As said above the symbol map

$$\sigma^{(m)} : \mathrm{End}(\Gamma_{hol}(M, L^m)) \to C^\infty(M) \tag{5.25}$$

is injective. Its image is a subspace $\mathcal{A}^{(m)}$ of $C^\infty(M)$, called the subspace of covariant symbols of level m. If $\sigma^{(m)}(A)$ and $\sigma^{(m)}(B)$ are elements of this subspace the operators A and B will be uniquely fixed. Hence also $\sigma^{(m)}(A \cdot B)$. Now one takes

$$\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B) \tag{5.26}$$

as definition for an associative and noncommutative product $\star_{(m)}$ on $\mathcal{A}^{(m)}$. The crucial problem is how to relate different levels m to define for all possible symbols a unique product not depending on m. In certain special situations like those studied by Berezin himself [11] and Cahen, Gutt, and Rawnsley [18] the subspaces are nested into each other and the union $\mathcal{A} = \bigcup_{m \in \mathbb{N}} \mathcal{A}^{(m)}$ is a dense subalgebra of $C^\infty(M)$. Indeed, in the cases considered, the manifold is a homogenous manifold and the epsilon function $\epsilon^{(m)}$ is a constant. A detailed analysis shows that then a star product can be given.

For further examples, for which this method works (not necessarily compact) see other articles by Cahen, Gutt, and Rawnsley [19, 20, 21]. For related results see also work of Moreno and Ortega-Navarro [38], [37]. In particular, also the work of Engliš [27, 26, 25, 24]. Reshetikhin and Takhtajan [43] gave a construction of a (formal) star product using formal integrals in the spirit of the Berezin's covariant symbol construction.

6 Berezin transform

Starting from $f \in C^\infty(M)$ we can assign to it its Toeplitz operator $T_f^{(m)} \in \mathrm{End}(\Gamma_{hol}(M, L^m))$ and then assign to $T_f^{(m)}$ the covariant symbol $\sigma^{(m)}(T_f^{(m)})$. It is again an element of $C^\infty(M)$.

Definition 6.1. The map

$$C^\infty(M) \to C^\infty(M), \qquad f \mapsto I^{(m)}(f) := \sigma^{(m)}(T_f^{(m)}) \tag{6.1}$$

is called the *Berezin transform (of level m)*.

From the point of view of Berezin's approach the operator $T_f^{(m)}$ has as a contravariant symbol f. Hence $I^{(m)}$ gives a correspondence between contravariant symbols and covariant symbols of operators. The Berezin transform was introduced and studied by Berezin [11] for certain classical symmetric domains in $\mathbb{C}^n$.

These results where extended by Unterberger and Upmeier [57], see also Engliš [25, 26, 27] and Engliš and Peetre [28]. Obviously, the Berezin transform makes perfect sense in the compact Kähler case which we consider here.

Theorem 6.2. *[34] Given $x \in M$ then the Berezin transform $I^{(m)}(f)$ evaluated at the point x has a complete asymptotic expansion in powers of $1/m$ as $m \to \infty$*

$$I^{(m)}(f)(x) \quad \sim \quad \sum_{i=0}^{\infty} I_i(f)(x)\frac{1}{m^i}, \tag{6.2}$$

where $I_i : C^\infty(M) \to C^\infty(M)$ are maps with

$$I_0(f) = f, \qquad I_1(f) = \Delta f. \tag{6.3}$$

Here the Δ is the usual Laplacian with respect to the metric given by the Kähler form ω.

Complete asymptotic expansion means the following. Given $f \in C^\infty(M)$, $x \in M$ and an $r \in \mathbb{N}$ then there exists a positive constant A such that

$$\left| I^{(m)}(f)(x) - \sum_{i=0}^{r-1} I_i(f)(x)\frac{1}{m^i} \right|_\infty \quad \leq \quad \frac{A}{m^r}\ . \tag{6.4}$$

Remark 6.3. The asymptotic of the Berezin transform is rather useful. It contains a lot of geometric information about the manifold. Moreover, the asymptotic expansion of the Berezin transform supplies a different proof of Theorem 3.3, part (a), using the relation

$$|I^{(m)}(f)|_\infty = |\sigma^{(m)}(T_f^{(m)})|_\infty \quad \leq \quad ||T_f^{(m)}|| \quad \leq \quad |f|_\infty\ . \tag{6.5}$$

(see [53].

Remark 6.4. The Berezin transform can be expressed by the Bergman kernels. Recall from Section 5 the Szegö projectors $\Pi : \mathrm{L}^2(Q,\mu) \to \mathcal{H}$ and its components $\hat{\Pi}^{(m)} : \mathrm{L}^2(Q,\mu) \to \mathcal{H}^{(m)}$, the Bergman projectors. The Bergman projectors have smooth integral kernels, the *Bergman kernels* $\mathcal{B}_m(\alpha,\beta)$ defined on $Q \times Q$, i.e.

$$\hat{\Pi}^{(m)}(\psi)(\alpha) = \int_Q \mathcal{B}_m(\alpha,\beta)\psi(\beta)\mu(\beta). \tag{6.6}$$

The Bergman kernels can be expressed with the help of the coherent vectors.

$$\mathcal{B}_m(\alpha,\beta) = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle, \quad \alpha,\beta \in Q. \tag{6.7}$$

Let $x, y \in M$ and choose $\alpha, \beta \in Q$ with $\tau(\alpha) = x$ and $\tau(\beta) = y$ then the functions

$$u_m(x) := \mathcal{B}_m(\alpha,\alpha) = \langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle, \tag{6.8}$$

$$v_m(x,y) := \mathcal{B}_m(\alpha,\beta)\cdot\mathcal{B}_m(\beta,\alpha) = \langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle \cdot \langle e_\beta^{(m)}, e_\alpha^{(m)}\rangle \tag{6.9}$$

are well-defined on M and on $M\times M$ respectively. An integral representation of the Berezin transform is obtained by

$$\begin{aligned}\left(I^{(m)}(f)\right)(x) &= \frac{1}{\mathcal{B}_m(\alpha,\alpha)}\int_Q \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\alpha)\tau^* f(\beta)\mu(\beta)\\ &= \frac{1}{u_m(x)}\int_M v_m(x,y)f(y)\Omega(y)\ .\end{aligned}\tag{6.10}$$

In [34] an asymptotic expansion of the Bergman kernel is shown. The above formula is then the starting point in [34] for the proof of the existence of the asymptotic expansion of the Berezin transform. Again I refer to [53] for more details and more arguments.

Remark 6.5. As everything is ready now I like to close with a result of the pullback of the Fubini-Study form. Starting from the Kähler manifold (M,ω) and after choosing an orthonormal basis of the space $\Gamma_{hol}(M,L^m)$ we obtain an embedding

$$\phi^{(m)}: M \to \mathbb{P}^{N(m)}$$

of M into projective space of dimension $N(m)$. On $\mathbb{P}^{N(m)}$ we have the standard Kähler form, the Fubini-Study form ω_{FS}. The pull-back $(\phi^{(m)})^*\omega_{FS}$ will not depend on the orthogonal basis chosen for the embedding. But in general it will not coincide with a scalar multiple of the Kähler form ω we started with.

It was shown by Zelditch [59], by generalizing a result of Tian [55], that $(\Phi^{(m)})^*\omega_{FS}$ admits a complete asymptotic expansion in powers of $\frac{1}{m}$ as $m\to\infty$. In fact it is related to the asymptotic expansion of the Bergman kernel (6.8) along the diagonal. The pull-back can be given as [59, Prop.9]

$$\left(\phi^{(m)}\right)^*\omega_{FS} = m\omega + \mathrm{i}\,\partial\bar\partial\log u_m(x)\ . \tag{6.11}$$

If we replace in the asymptotic expansion $1/m$ by the formal variable ν, and denote the resulting formal series by $\mathbb{F}(.)$, we obtain the Karabegov form of the star product "dual" to the Berezin-Toeplitz star product:

$$\widehat{\omega} = \mathbb{F}(\left(\phi^{(m)}\right)^*\omega_{FS}). \tag{6.12}$$

References

[1] Abraham, R., and Marsden, J.E., *Foundations of Mechanics.* Addison Wesley, Redwood City, 1985.

[2] Ali, S.T., Engliš, M., *Quantization methods: A guide for physicists and analysts.* Rev. Math. Phys. **17**(4) (2005), 391–490.

[3] Andersen, J.E., *Asymptotic faithfulness of the quantum $SU(n)$ representations of the mapping class group*. Ann. of Math. **163** (2006), 347–368.

[4] Andersen, J.E., *Deformation quantization and geometric quantization of abelian moduli spaces*. Comm. Math. Phys. **255** (2005), 727–745.

[5] Andersen, J.E., *Mapping class groups do not have property T*. arXiv:0706.2184.

[6] Andersen, J.E., and Gammelgaard, N.L., *Hitchin's projectively flat connection, Toeplitz operators and the asymptotic expansion of TQFT curve operators*. arXiv:0903.4091.

[7] Andersen, J.E., Ueno, K., *Abelian Conformal field theories and the determinant bundles*. Internat. Jour. of Math. **18** (2007), 919–993.

[8] Andersen, J.E., Ueno, K., *Constructing modular functors from conformal field theories*. Jour. of Knot Theories and its Ramifications **16** (2007), 127–202.

[9] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., and Sternheimer, D., *Deformation theory and quantization, Part I*. Lett. Math. Phys. **1** (1977), 521–530: *Deformation theory and quantization, Part II and III*. Ann. Phys. **111** (1978), 61–110, 111–151.

[10] Berezin, F.A., *Quantization*. Math. USSR-Izv. **8** (1974), 1109–1165.

[11] Berezin, F.A., *Quantization in complex symmetric spaces*. Math. USSR-Izv. **9** (1975), 341–379.

[12] Bertelson, M., Cahen, M., and Gutt, S., *Equivalence of star products*. Classical Quantum Gravity **14** (1997), A93–A107.

[13] Bordemann, M., Hoppe, J., Schaller, P., and Schlichenmaier, M., *$gl(\infty)$ and geometric quantization*. Commun. Math. Phys. **138** (1991), 209–244.

[14] Bordemann, M., Meinrenken, E., and Schlichenmaier, M., *Toeplitz quantization of Kähler manifolds and $gl(n), n \to \infty$ limits*. Commun. Math. Phys. **165** (1994), 281–296.

[15] Bordemann, M., and Waldmann, St., *A Fedosov star product of the Wick type for Kähler manifolds*. Lett. Math. Phys. **41** (1997), 243–253.

[16] Borthwick, D., Paul, D., and Uribe, A.: *Semiclassical spectral estimates for Toeplitz operators*. Ann. Inst. Fourier, Grenoble **48** (1998), 1189–1229.

[17] Boutet de Monvel, L., and Guillemin, V, *The spectral theory of Toeplitz operators*. Ann. Math. Studies, Nr.99, Princeton University Press, Princeton, 1981.

[18] Cahen, M., Gutt, S., and Rawnsley, J., *Quantization of Kähler manifolds I: Geometric interpretation of Berezin's quantization*. JGP **7** (1990), 45–62.

[19] Cahen, M., Gutt, S., and Rawnsley, J., *Quantization of Kähler manifolds II.* Trans. Amer. Math. Soc. **337** (1993), 73–98.

[20] Cahen, M., Gutt, S., and Rawnsley, J., *Quantization of Kähler manifolds III.* Lett. Math. Phys. **30** (1994), 291–305.

[21] Cahen, M., Gutt, S., and Rawnsley, J., *Quantization of Kähler manifolds IV.* Lett. Math. Phys. **34** (1995), 159–168.

[22] De Wilde, M., and Lecomte, P.B.A., *Existence of star products and of formal deformations of the Poisson-Lie algebra of arbitrary symplectic manifolds.* Lett. Math. Phys. **7** (1983), 487–496.

[23] Deligne, P., *Deformation de l'algebre des fonctions d'une variete symplectique: Comparaison entre Fedosov et De Wilde,Lecomte*. Sel. Math., New Ser. **1** (1995), 667–697.

[24] Engliš, M, *The asymptotics of a Laplace integral on a Kähler manifold*. J. Reine Angew. Math. **528** (2000), 1-39.

[25] Engliš, M., *Asymptotics of the Berezin transform and quantization on planar domains*. Duke Math. J. **79** (1995), 57–76.

[26] Engliš, M., *Berezin quantization and reproducing kernels on complex domains*. Trans. Amer. Math. Soc. **348** (1996), 411–479.

[27] Engliš, M., *Weighted Bergman kernels and quantization*. Commun. Math. Phys. **227** (2002), 211–241.

[28] Engliš, M., and Peetre, J., *On the correspondence principle for the quantized annulus*. Math. Scand. **78** (1996), 183–206.

[29] Fedosov, B.V., *Deformation quantization and asymptotic operator representation*, Funktional Anal. i. Prilozhen. **25** (1990), 184–194; *A simple geometric construction of deformation quantization*. J. Diff. Geo. **40** (1994), 213–238.

[30] Guillemin, V., *Some classical theorems in spectral theory revisited*. Seminars on singularities of solutions of linear partial differential equations, Ann. Math. Studies, Nr.91 (L. Hörmander, ed.), Princeton University Press, 1979, pp. 219–259.

[31] Gutt, S., Rawnsley, J, *Equivalence of star products on a symplectic manifold: an introduction to Deligne's Čech cohomology classes.* JGP **29**, No.4, 347-392 (1999).

[32] Karabegov, A.V., *Deformation quantization with separation of variables on a Kähler manifold*, Commun. Math. Phys. **180** (1996), 745–755.

[33] Karabegov, A.V., *Cohomological classification of deformation quantizations with separation of variables*. Lett. Math. Phys. **43** (1998), 347–357.

[34] Karabegov, A.V., Schlichenmaier, M., *Identification of Berezin-Toeplitz deformation quantization*. J. reine angew. Math. **540** (2001), 49–76

[35] Kontsevich, M., *Deformation quantization of Poisson manifolds*. Lett. Math. Phys. **66** (2003), 157-216, preprint q-alg/9709040.

[36] Landsman, N.P., *Mathematical topics between classical and quantum mechanics*. Springer, New York, Berlin, Heidelberg, 1998.

[37] Moreno, C., $*$-products on some Kähler manifolds, Lett. Math. Phys. **11** (1986), 361–372.

[38] Moreno, C., and Ortega-Navarro, P., *$*$-products on $D^1(\mathbb{C})$, S^2 and related spectral analysis*, Lett. Math. Phys. **7** (1983), 181–193.

[39] Moyal, J., *Quantum mechanics as a statistical theory*. Proc. Camb. Phil. Soc. **45** (1949), 99-124.

[40] Nest, R., and Tsygan, B., *Algebraic index theory*. Commun. Math. Phys. **172** (1995), 223–262; Algebraic index theory for families. Advances in Math **113** (1995), 151–205.

[41] Omori, H., Maeda, Y., and Yoshioka, A., *Weyl manifolds and deformation quantization*. Advances in Math **85** (1991), 224–255; *Existence of closed star-products*. Lett. Math. Phys. **26** (1992), 284–294.

[42] Rawnsley, J.H., *Coherent states and Kähler manifolds*. Quart. J. Math. Oxford Ser.(2) **28** (1977), 403–415.

[43] Reshetikhin, N., and Takhtajan, L., *Deformation quantization of Kähler manifolds*. Amer. Math. Soc. Transl (2) **201** (2000), 257–276.

[44] Rieffel, M., *Questions on quantization*, Operator algebras and operator theory. Shanghai, China, July 4–9, 1997 (Liming et al., ed.), AMS, 1998, pp. 315–326.

[45] Schlichenmaier, M., *Zwei Anwendungen algebraisch-geometrischer Methoden in der theoretischen Physik: Berezin-Toeplitz-Quantisierung und globale Algebren der zweidimensionalen konformen Feldtheorie*. Habiliationsschrift Universität Mannheim, 1996.

[46] Schlichenmaier, M., *Berezin-Toeplitz quantization of compact Kähler manifolds.* in: Quantization, Coherent States and Poisson Structures, Proc. XIV'th Workshop on Geometric Methods in Physics (Białowieża, Poland, 9-15 July 1995) (A, Strasburger, S.T. Ali, J.-P. Antoine, J.-P. Gazeau, and A, Odzijewicz, eds.), Polish Scientific Publisher PWN, 1998, q-alg/9601016, pp, 101–115.

[47] Schlichenmaier, M., *Deformation quantization of compact Kähler manifolds via Berezin-Toeplitz operators.* Proceedings of the XXI Int. Coll. on Group Theoretical Methods in Physics (15-20 July 1996, Goslar, Germany) (H.-D. Doebner, P. Nattermann, and W. Scherer, eds.), World Scientific, 1997, pp. 396–400.

[48] Schlichenmaier, M., *Berezin-Toeplitz quantization and Berezin symbols for arbitrary compact Kähler manifolds.* in Proceedings of the XVIIth workshop on geometric methods in physics, Białowieża, Poland, July 3 - 10, 1998 (eds. Schlichenmaier, et. al), (math.QA/9902066), Warsaw University Press, 45-56.

[49] Schlichenmaier, M., *Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization*, (in) the Proceedings of the Conference Moshe Flato 1999, (eds. G.Dito, and D. Sternheimer), Kluwer 2000, 289 – 306, math.QA/9910137.

[50] Schlichenmaier, M., *Berezin-Toeplitz quantization and Berezin transform.* (in) Long time behaviour of classical and quantum systems. Proc. of the Bologna APTEX Int. Conf. 13-17 September 1999, eds. S. Graffi, A. Martinez, World-Scientific, 2001, 271-287.

[51] Schlichenmaier, M., *Berezin-Toeplitz quantization of the moduli space of flat $SU(N)$ connections.* J. Geom. Symmetry Phys. **9** (2007), 33-44.

[52] Schlichenmaier, M., *An Introduction to Riemann surfaces, algebraic curves and moduli spaces.* 2nd ed., Springer, Berlin, Heidelberg, 2007.

[53] Schlichenmaier, M., *Berezin-Toeplitz quantization for compact Kähler manifolds. A review of results.* Adv. in Math. Phys. volume 2010, doi:10.1155/2010/927280.

[54] Sternheimer, D., *Deformation quantization. Twenty years after.* In Particles, Fields and Gravitation. Lodz, 1998, AIP Conf. Proc. Vol 453.

[55] Tian, G., *On a set of polarized Kähler metrics on algebraic manifolds.* J. Differential Geom. **32** (1990), 99–130.

[56] Tuynman, G.M., *Generalized Bergman kernels and geometric quantization.* J. Math. Phys. **28**, (1987) 573–583.

[57] Unterberger, A., and Upmeier, H., *The Berezin transform and invariant differential operators*. Commun. Math. Phys. **164** (1994), 563–597.

[58] Weyl, H., *Gruppentheorie und Quantenmechanik*. 1931, Leipzig, Nachdruck Wissenschaftliche Buchgesellschaft, Darmstadt, 1977.

[59] Zelditch, S., *Szegö kernels and a theorem of Tian*. Int. Math. Res. Not. **6** (1998), 317–331.

Martin Schlichenmaier
University of Luxembourg
Mathematics Research Unit, FSTC
6, rue Coudenhove-Kalergi
L-1359 Luxembourg-Kirchberg
Grand Duchy of Luxembourg
martin.schlichenmaier@uni.lu

Travaux mathématiques, Volume 19 (2011), 125–166,

Berezin-Toeplitz quantization and its kernel expansion

by Xiaonan Ma and George Marinescu

Abstract

We survey recent results [33, 34, 35, 36] about the asymptotic expansion of Toeplitz operators and their kernels, as well as Berezin-Toeplitz quantization. We deal in particular with calculation of the first coefficients of these expansions.

Contents

1 Introduction

The aim of the geometric quantization theory of Kostant and Souriau is to relate the classical observables (smooth functions) on a phase space (a symplectic manifold) to the quantum observables (bounded linear operators) on the quantum space (sections of a line bundle). Berezin-Toeplitz quantization is a particularly efficient version of the geometric quantization theory [2, 3, 19, 25, 40]. Toeplitz operators and more generally Toeplitz structures were introduced in geometric quantization by Berezin [3] and Boutet de Monvel-Guillemin [9]. Using the analysis of Toeplitz structures [9], Bordemann-Meinrenken-Schlichenmaier [7] and Schlichenmaier [38] gave asymptotic expansion for the composition of Toeplitz operators in the Kähler case.

The expansions we will be considering are asymptotic expansions relative to the high power p of the quantum line bundle. The limit $p \to \infty$ is interpreted as semi-classical limit process, where the role of the Planck constant is played by $\hbar = 1/p$.

The purpose of this paper is to review some methods and results concerning Berezin-Toeplitz quantization which appeared in the recent articles [34, 35, 36] and in the book [33]. Our approach is based on kernel calculus and the off-diagonal asymptotic expansion of the Bergman kernel. This method allows not only to derive the asymptotic expansions of the Toeplitz operators but also to calculate the first coefficients of the various expansions. Since the formulas for the coefficients encode geometric data of the manifold and prequantum bundle they found extensive and deep applications in the study of Kähler manifolds (see e.g. [16, 17, 18, 20, 21, 26, 43, 44], to quote just a few). We will also twist the powers of the prequantum bundle with a fixed auxiliary bundle, so the formulas for the coefficients also mirror the curvature of the twisting bundle.

The paper is divided in three parts, treating the quantization of Kähler manifolds, of Kähler orbifolds and finally of symplectic manifolds.

In these notes we do not attempt to be exhaustive, neither in the choice of topics, nor in what concerns the references. For previous work on Berezin-Toeplitz star products in special cases see [11, 37]. We also refer the reader to the survey articles [1, 30, 39] for more information for the Berezin-Toeplitz quantization and geometric quantization. The survey [30] gives a review in the context of Kähler and symplectic manifolds and explores the connections to symplectic reduction.

2 Quantization of Kähler manifolds

In this long section we explain our approach to Berezin-Toeplitz quantization of symplectic manifolds by specializing to the Kähler case. The method we use is then easier to follow and the coefficients of the asymptotic expansions have accurate expressions in terms of curvatures of the underlying manifold.

In Section 2.1, we review the definition of the Bergman projector, introduce the Toeplitz operators and their kernels.

In Section 2.2, we describe the spectral gap of the Kodaira-Laplace operator. On one hand, this implies the Kodaira-Serre vanishing theorem and the fact that for high powers of the quantum line bundle the cohomology concentrates in degree zero. On the other hand, the spectral gap provides the natural framework for the asymptotic expansions of the Bergman and Toeplitz kernels.

In Section 2.3 we describe the model operator, its spectrum and the kernel of its spectral projection on the lowest energy level. The expansion of the Bergman kernel, which we study in Section 2.4, is obtained by a localization and rescaling technique due to Bismut-Lebeau [5], and reduces the problem to the model case.

With this expansion at hand, we formulate the expansion of the Toeplitz kernel in Section 2.5. Moreover, we observe in Section 2.6 that these expansion characterizes the Toeplitz operators and this characterization implies the expansion of the product of two Toeplitz operators and the existence of the Berezin-Toeplitz star product.

In Section 2.7, we explain how to apply the previous results when the Riemannian metric used to define the Hilbertian structure on the space of sections is arbitrary.

In Section 2.8, we turn to the general situation of complete Kähler manifolds and show how to apply the introduced method in this case.

2.1 Bergman projections, Toeplitz operators, and their kernels

We consider a complex manifold (X, J) with complex structure J, and complex dimension n. Let L and E be two holomorphic vector bundles on X. We assume that L is a line bundle i.e. $\operatorname{rk} L = 1$. The bundle E is an auxiliary twisting bundle. It is interesting to work with a twisting vector bundle E for several reasons. For example, one has to deal with $(n,0)$-forms with values in L^p, so one sets $E = \Lambda^n(T^{*(1,0)}X)$. From a physical point of view, the presence of E means a quantization of a system with several degrees of internal freedom.

We fix Hermitian metrics h^L, h^E on L, E. Let g^{TX} be a J-invariant Riemannian metric on X, i.e., $g^{TX}(Ju, Jv) = g^{TX}(u, v)$ for all $x \in X$ and $u, v \in T_xX$. The Riemannian volume form of g^{TX} is denoted by dv_X. On the space of smooth sections with compact support $\mathscr{C}^\infty_0(X, L^p \otimes E)$ we introduce the L^2-scalar product associated to the metrics h^L, h^E and the Riemannian volume form dv_X by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1(x), s_2(x) \rangle_{L^p\otimes E}\, dv_X(x)\,. \tag{2.1}$$

The completion of $\mathscr{C}_0^\infty(X, L^p \otimes E)$ with respect to (2.1) is denoted as usual by $L^2(X, L^p \otimes E)$. We consider the space of holomorphic L^2 sections:

$$H^0_{(2)}(X, L^p \otimes E) := \left\{ s \in L^2(X, L^p \otimes E) : s \text{ is holomorphic} \right\}. \tag{2.2}$$

Let us note an important property of the space $H^0_{(2)}(X, L^p \otimes E)$, which follows from the Cauchy estimates for holomorphic functions. Namely, for every compact set $K \Subset X$ there exists $C_K > 0$ such that

$$\sup_{x \in K} |S(x)| \leqslant C_K \|S\|_{L^2}, \quad \text{for all } S \in H^0_{(2)}(X, L^p \otimes E). \tag{2.3}$$

We deduce that $H^0_{(2)}(X, L^p \otimes E)$ is a closed subspace of $L^2(X, L^p \otimes E)$; one can also show that $H^0_{(2)}(X, L^p \otimes E)$ is separable (cf. [45, p. 60]).

Definition 2.1. The *Bergman projection* is the orthogonal projection

$$P_p : L^2(X, L^p \otimes E) \to H^0_{(2)}(X, L^p \otimes E).$$

In view of (2.3), the Riesz representation theorem shows that for a fixed $x \in X$ there exists $P_p(x, \cdot) \in L^2(X, (L^p \otimes E)_x \otimes (L^p \otimes E)^*)$ such that

$$S(x) = \int_X P_p(x, x') S(x') dv_X(x'), \quad \text{for all } S \in H^0_{(2)}(X, L^p \otimes E). \tag{2.4}$$

Definition 2.2. The section $P_p(\cdot, \cdot)$ of $(L^p \otimes E) \boxtimes (L^p \otimes E)^*$ over $X \times X$ is called the *Bergman kernel* of $L^p \otimes E$.

Set $d_p := \dim H^0_{(2)}(X, L^p \otimes E) \in \mathbb{N} \cup \{\infty\}$. Let $\{S_i^p\}_{i=1}^{d_p}$ be any orthonormal basis of $H^0_{(2)}(X, L^p \otimes E)$ with respect to the inner product (2.1). Using the estimate (2.3) we can show that

$$P_p(x, x') = \sum_{i=1}^{d_p} S_i^p(x) \otimes (S_i^p(x'))^* \in (L^p \otimes E)_x \otimes (L^p \otimes E)^*_{x'}, \tag{2.5}$$

where the right-hand side converges on every compact together with all its derivatives (see e.g. [45, p. 62]). Thus $P_p(\cdot, \cdot) \in \mathscr{C}^\infty(X \times X, (L^p \otimes E) \boxtimes (L^p \otimes E)^*)$. It follows that

$$(P_p S)(x) = \int_X P_p(x, x') S(x') dv_X(x'), \quad \text{for all } S \in L^2(X, L^p \otimes E). \tag{2.6}$$

that is, $P_p(\cdot, \cdot)$ is the integral kernel of the Bergman projection P_p. We recall that a bounded linear operator T on $L^2(X, L^p \otimes E)$ is called Carleman operator (see e.g. [22]) if there exists a kernel $T(\cdot, \cdot)$ such that $T(x, \cdot) \in L^2(X, (L^p \otimes E)_x \otimes (L^p \otimes E)^*)$ and

$$(TS)(x) = \int_X T(x, x') S(x') dv_X(x'), \quad \text{for all } S \in L^2(X, L^p \otimes E). \tag{2.7}$$

Hence P_p is a Carleman operator with smooth kernel $P_p(\cdot,\cdot)$.

The Bergman kernel represents the local density of the space of holomorphic sections and is a very efficient tool to study properties of holomorphic sections. It is an "objet souple" in the sense of Pierre Lelong, that is, it interpolates between the rigid objects of complex analysis and the flexible ones of real analysis.

Note that $P_p(x,x) \in \mathrm{End}(E)_x$, since $\mathrm{End}(L^p) = \mathbb{C}$. Using (2.5) and the formula $\mathrm{Tr}_E\left[S_i^p(x)\otimes(S_i^p(x))^*\right] = |S_i^p(x)|^2$, we obtain immediately

$$d_p = \int_X \mathrm{Tr}_E\, P_p(x,x)\, dv_X(x)\,. \tag{2.8}$$

Definition 2.3. For a bounded section $f \in \mathscr{C}^\infty(X,\mathrm{End}(E))$, set

$$T_{f,p} : L^2(X, L^p\otimes E) \longrightarrow L^2(X, L^p\otimes E)\,, \quad T_{f,p} = P_p\, f\, P_p\,, \tag{2.9}$$

where the action of f is the pointwise multiplication by f. The map which associates to $f \in \mathscr{C}^\infty(X,\mathrm{End}(E))$ the family of bounded operators $\{T_{f,p}\}_p$ on $L^2(X, L^p\otimes E)$ is called the *Berezin-Toeplitz quantization*.

Note that $T_{f,p}$ is a Carleman operator with smooth integral kernel given by

$$T_{f,p}(x,x') = \int_X P_p(x,x'')f(x'')P_p(x'',x')\, dv_X(x'')\,. \tag{2.10}$$

For two arbitrary bounded sections $f, g \in \mathscr{C}^\infty(X,\mathrm{End}(E))$ it is easy to see that $T_{f,p}\circ T_{g,p}$ is not in general of the form $T_{fg,p}$. But we have $T_{f,p}\circ T_{g,p} \sim T_{fg,p}$ asymptotically for $p\to\infty$. In order to explain this we introduce the following more general notion of Toeplitz operator.

Definition 2.4. A *Toeplitz operator* is a sequence $\{T_p\}_{p\in\mathbb{N}}$ of linear operators $T_p : L^2(X, L^p\otimes E) \longrightarrow L^2(X, L^p\otimes E)$ verifying $T_p = P_p\, T_p\, P_p$, such that there exist a sequence $g_\ell \in \mathscr{C}^\infty(X,\mathrm{End}(E))$ such that for any $k \geqslant 0$, there exists $C_k > 0$ with

$$\Big\| T_p - \sum_{\ell=0}^{k} T_{g_\ell,p}\, p^{-\ell} \Big\| \leqslant C_k\; p^{-k-1} \quad \text{for any } p \in \mathbb{N}^*, \tag{2.11}$$

where $\|\cdot\|$ denotes the operator norm on the space of bounded operators. The section g_0 is called the *principal symbol* of $\{T_p\}$.

We express (2.11) symbolically by

$$T_p = \sum_{\ell=0}^{k} T_{g_\ell,p}\, p^{-\ell} + \mathcal{O}(p^{-k-1}). \tag{2.12}$$

If (2.11) holds for any $k \in \mathbb{N}$, then we write (2.12) with $k = +\infty$. One of our goals is to show that $T_{f,p}\circ T_{g,p}$ is a Toeplitz operator in the sense of Definition 2.11. This will be achieved by using the asymptotic expansions of the Bergman kernel and of the kernels of the Toeplitz operators.

2.2 Spectral gap and vanishing theorem

In order to have a meaningful theory it is necessary that the spaces $H^0_{(2)}(X, L^p \otimes E)$ are as large as possible. In this section we describe conditions when the growth of $d_p = \dim H^0_{(2)}(X, L^p \otimes E)$ for $p \to \infty$ is maximal.

For this purpose we need Hodge theory, so we introduce the Laplace operator. Let $T^{(1,0)}X$ be the holomorphic tangent bundle on X, $T^{(0,1)}X$ the conjugate of $T^{(1,0)}X$ and $T^{*(0,1)}X$ the dual bundle of $T^{(0,1)}X$. We denote by $\Lambda^q(T^{*(0,1)}X)$ the bundle of $(0,q)$-forms on X and by $\Omega^{0,q}(X, F)$ the space of sections of the bundle $\Lambda^q(T^{*(0,1)}X) \otimes F$ over X, for some vector bundle $F \to X$.

The Dolbeault operator acting on sections of the holomorphic vector bundle $L^p \otimes E$ gives rise to the Dolbeault complex

$$\Big(\Omega^{0,\bullet}(X, L^p \otimes E), \overline{\partial}^{L^p\otimes E}\Big) .$$

Its cohomology, called Dolbeault cohomology, is denoted by $H^{0,\bullet}(X, L^p \otimes E)$. We denote by $\overline{\partial}^{L^p\otimes E,*}$ the formal adjoint of $\overline{\partial}^{L^p\otimes E}$ with respect to the L^2-scalar product (2.1). Set

$$\begin{aligned} D_p &= \sqrt{2}\big(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*}\big)\,, \\ \square^{L^p\otimes E} &= \tfrac{1}{2}D_p^2 = \overline{\partial}^{L^p\otimes E}\,\overline{\partial}^{L^p\otimes E,*} + \overline{\partial}^{L^p\otimes E,*}\,\overline{\partial}^{L^p\otimes E}. \end{aligned} \tag{2.13}$$

The operator $\square^{L^p\otimes E}$ is called the *Kodaira-Laplacian*. It acts on $\Omega^{0,\bullet}(X, L^p \otimes E)$ and preserves its $\mathbb{Z}$-grading.

Let us consider first that X is a *compact Kähler* manifold endowed with a Kähler form ω and L is a *prequantum* line bundle. The latter means that there exists a Hermitian metric h^L such that the curvature $R^L = (\nabla^L)^2$ of the holomorphic Hermitian connection ∇^L on (L, h^L) satisfies

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L\,. \tag{2.14}$$

In particular, L is a positive line bundle.

By Hodge theory, the elements of $\operatorname{Ker}(\square^{L^p\otimes E})$, called *harmonic forms*, represent the Dolbeault cohomology. Namely,

$$\operatorname{Ker}(D_p|_{\Omega^{0,q}}) = \operatorname{Ker}(D_p^2|_{\Omega^{0,q}}) \simeq H^{0,q}(X, L^p \otimes E). \tag{2.15}$$

and the spaces $H^{0,q}(X, L^p \otimes E)$ are finite dimensional. Note that $H^{0,0}(X, L^p \otimes E)$ is the space of holomorphic sections of $L^p \otimes E$, denoted shortly by $H^0(X, L^p \otimes E)$. Since X is compact we have $H^0(X, L^p \otimes E) = H^0_{(2)}(X, L^p \otimes E)$ for any Hermitian metrics on L^p, E and volume form on X. A crucial tool in our analysis of the Bergman kernel is the following spectral gap of the Kodaira-Laplacian.

Theorem 2.5 ([6, Th. 1.1], [33, Th. 1.5.5])**.** *There exist constants positive C_0, C_L such that for any $p \in \mathbb{N}$ and any $s \in \Omega^{0,>0}(X, L^p \otimes E) = \bigoplus_{q\geqslant 1} \Omega^{0,q}(X, L^p \otimes E)$,*

$$\|D_p s\|^2_{L^2} \geqslant (2C_0 p - C_L)\|s\|^2_{L^2}\,. \tag{2.16}$$

The spectrum $\operatorname{Spec}(\square_p)$ *of the Kodaira Laplacian* $\square_p$*, is contained in the set* $\{0\} \cup]p\, C_0 - \frac{1}{2} C_L, +\infty[$.

Theorem 2.5 was first proved by Bismut-Vasserot [6, Th. 1.1] using the non-kählerian Bochner-Kodaira-Nakano formula with torsion due to Demailly, see e.g. [33, Th. 1.4.12] (note that g^{TX} is arbitrary, we don't suppose that it is the metric associated to ω, i.e., $g^{TX}(u, v) = \omega(u, Jv)$ for $u, v \in T_x X$). By Theorem 2.5, we conclude:

Theorem 2.6 (Kodaira–Serre vanishing Theorem)**.** *If L is a positive line bundle, then there exists $p_0 > 0$ such that for any $p \geqslant p_0$,*

$$H^{0,q}(X, L^p \otimes E) = 0 \quad \text{for any } \ q > 0. \tag{2.17}$$

Recall that for a compact manifold X and a holomorphic vector bundle F, the Euler number $\chi(X, F)$ is defined by

$$\chi(X, F) = \sum_{q=0}^{n} (-1)^q \dim H^{0,q}(X, F). \tag{2.18}$$

By the Riemann-Roch-Hirzebruch Theorem [33, Th. 14.6] we have

$$\chi(X, F) = \int_X \operatorname{Td}\left(T^{(1,0)}X\right) \operatorname{ch}(F)\,, \tag{2.19}$$

where Td and ch indicate the Todd class and the Chern character, respectively. By the Kodaira-Serre vanishing (2.17),

$$d_p = \dim H^0(X, L^p \otimes E) = \chi(X, L^p \otimes E)\,, \quad p \geqslant p_0\,. \tag{2.20}$$

Therefore, for $p \geqslant p_0$,

$$\begin{aligned} \dim H^0(X, L^p \otimes E) &= \int_X \operatorname{Td}\left(T^{(1,0)}X\right) \operatorname{ch}(L^p \otimes E) \\ &= \operatorname{rk}(E) \int_X \frac{c_1(L)^n}{n!}\, p^n + \int_X \Big(c_1(E) + \frac{\operatorname{rk}(E)}{2} c_1(T^{(1,0)}X) \Big) \frac{c_1(L)^{n-1}}{(n-1)!}\, p^{n-1} \\ &\quad + \mathscr{O}(p^{n-2}). \end{aligned} \tag{2.21}$$

Note that the first Chern class $c_1(L)$ is represented by ω (see (2.14)) and $c_1(E)$ is represented by $\frac{\sqrt{-1}}{2\pi} \operatorname{Tr}[R^E]$. As a conclusion we have:

Theorem 2.7. *Let (X,ω) be a compact Kähler manifold and let (L,h^L) be a prequantum line bundle satisfying* (2.14). *Then d_p is a polynomial of degree n with positive leading term $\frac{1}{n!}\int_X c_1(L)^n$ (the volume of the manifold (X,ω)).*

Let us consider now the general situation of a (non-compact) complex manifold (X,J). As before we are given a Hermitian metric on X, that is, a J-compatible Riemannian metric g^{TX}. We denote by Θ the associated $(1,1)$-form, i.e., $\Theta(u,v)=g^{TX}(Ju,v)$, for all $x\in X$ and $u,v\in T_xX$. We say that the Hermitian manifold (X,Θ) is complete if the Riemannian metric g^{TX} is complete. Consider further a Hermitian holomorphic vector bundle $(F,h^F)\to X$. Let us denote by $\Omega^{0,q}_{(2)}(X,F):=L^2(X,\Lambda^q(T^{*(0,1)}X)\otimes F)$. We have the complex of closed, densely defined operators

$$\Omega^{0,q-1}_{(2)}(X,F)\xrightarrow{T=\overline{\partial}^F}\Omega^{0,q}_{(2)}(X,F)\xrightarrow{S=\overline{\partial}^F}\Omega^{0,q+1}_{(2)}(X,F)\,, \tag{2.22}$$

where T and S are the maximal extensions of $\overline{\partial}^F$, i.e.,

$$\operatorname{Dom}(\overline{\partial}^F)=\{s\in\Omega^{0,\bullet}_{(2)}(X,F):\overline{\partial}^F s\in\Omega^{0,\bullet}_{(2)}(X,F)\}$$

where $\overline{\partial}^F s$ is calculated in the sense of distributions. Note that $\operatorname{Im}(T)\subset\operatorname{Ker}(S)$, so $ST=0$. The q-th L^2 *Dolbeault cohomology* is defined by

$$H^{0,q}_{(2)}(X,F):=\frac{\operatorname{Ker}(\overline{\partial}^F)\cap\Omega^{0,q}_{(2)}(X,F)}{\operatorname{Im}(\overline{\partial}^F)\cap\Omega^{0,q}_{(2)}(X,F)}\,. \tag{2.23}$$

Consider the quadratic form Q given by

$$\begin{aligned}\operatorname{Dom}(Q)&:=\operatorname{Dom}(S)\cap\operatorname{Dom}(T^*),\\ Q(s_1,s_2)&=\langle Ss_1,Ss_2\rangle+\langle T^*s_1,T^*s_2\rangle\,,\quad\text{for } s_1,s_2\in\operatorname{Dom}(Q).\end{aligned} \tag{2.24}$$

where T^* is the Hilbertian adjoint of T. For the following result due essentially to Gaffney one may consult [33, Prop. 3.1.2, Cor. 3.3.4].

Lemma 2.8. *Assume that the Hermitian manifold (X,Θ) is complete. Then the Kodaira-Laplacian $\square^F:\Omega^{0,\bullet}_0(X,F)\to\Omega^{0,\bullet}_{(2)}(X,F)$ is essentially self-adjoint. Its associated quadratic form is the form Q given by* (2.24).

We denote by $R^{\det}$ the curvature of the holomorphic Hermitian connection $\nabla^{\det}$ on $K_X^*=\det(T^{(1,0)}X)$. We have the following generalization of Theorems 2.5 and 2.6.

Theorem 2.9 ([33, Th. 6.1.1], [34, Th. 3.11])**.** *Assume that (X,Θ) is a complete Hermitian manifold. Let (L,h^L) and (E,h^E) Hermitian holomorphic vector bundles of rank one and r, respectively. Suppose that there exist $\varepsilon>0$, $C>0$ such that:*

$$\sqrt{-1}R^L>\varepsilon\Theta\,,\quad\sqrt{-1}(R^{\det}+R^E)>-C\Theta\operatorname{Id}_E\,,\quad|\partial\Theta|_{g^{TX}}<C, \tag{2.25}$$

Then there exists $C_1 > 0$ *and* $p_0 \in \mathbb{N}$ *such that for* $p \geqslant p_0$ *the quadratic form* Q_p *associated to the Kodaira-Laplacian* $\square_p := \square^{L^p\otimes E}$ *satisfies*

$$Q_p(s,s) \geqslant C_1 p \,\|s\|^2_{L^2}\,, \quad \text{for } s \in \operatorname{Dom}(Q_p) \cap \Omega^{0,q}_{(2)}(X, L^p \otimes E),\ q > 0\,. \tag{2.26}$$

Especially

$$H^{0,q}_{(2)}(X, L^p \otimes E) = 0\,, \quad \text{for } p \geqslant p_0,\ q > 0 \tag{2.27}$$

and the spectrum $\operatorname{Spec}(\square_p)$ *of the Kodaira Laplacian* $\square_p$ *acting on* $L^2(X, L^p \otimes E)$ *is contained in the set* $\{0\} \cup [\,p\,C_1, \infty[\,$.

Thus we are formally in a similar situation as in the compact case, that is, the higher L^2 cohomology groups vanish. But we cannot invoke as in the compact case the index theorem to estimate the dimension of L^2 holomorphic sections of $L^p \otimes E$. Instead we can use an analogue of the local index theorem, namely the asymptotics of the Bergman kernel. Let us denote by $\alpha_1, \dots, \alpha_n$ the eigenvalues of $\frac{\sqrt{-1}}{2\pi} R^L$ with respect to Θ.

Theorem 2.10 ([34, Cor. 3.12])**.** *Under the hypotheses of Theorem 2.9 we have*

$$P_p(x,x) = p^n \boldsymbol{b}_0(x) + \mathcal{O}(p^{n-1})\,, \quad p \to \infty\,, \tag{2.28}$$

uniformly on compact sets, where $\boldsymbol{b}_0 = \alpha_1 \dots \alpha_n \operatorname{Id}_E$. *Hence*

$$\liminf_{p \longrightarrow \infty} p^{-n} \dim H^0_{(2)}(X, L^p \otimes E) \geqslant \frac{\operatorname{rk}(E)}{n!} \int_X \Big(\tfrac{\sqrt{-1}}{2\pi} R^L\Big)^n. \tag{2.29}$$

The asymptotics (2.28) are a particular case of the full asymptotic expansion of the Bergman kernel, see Corollary 2.15. It can be also deduced with the help of L^2 estimates of Hörmander as done by Tian [41]. The estimate (2.29) shows that $\dim H^0_{(2)}(X, L^p \otimes E)$ has at least polynomial growth of degree n. It follows from Fatou's lemma, applied on X with the measure $\Theta^n/n!$ to the sequence $p^{-n} \operatorname{Tr}_E P_p(x,x)$ which converges pointwise to $\operatorname{Tr}_E \boldsymbol{b}_0$ on X.

2.3 Model situation: Bergman kernel on $\mathbb{C}^n$

We introduce here the model operator, a Kodaira-Laplace operator on $\mathbb{C}^n$, and describe explicitly its spectrum. The expansion of the Bergman and Toeplitz kernels will be expressed in terms of the kernel of the projection on $\operatorname{Ker}(\mathscr{L})$. Our whole analysis and calculations are based on the Fourier expansion with respect to the eigenfunctions of $\mathscr{L}$.

Let us consider the canonical real coordinates $(Z_1, \dots, Z_{2n})$ on $\mathbb{R}^{2n}$ and the complex coordinates $(z_1, \dots, z_n)$ on $\mathbb{C}^n$. The two sets of coordinates are linked by the relation $z_j = Z_{2j-1} + \sqrt{-1}Z_{2j}$, $j = 1, \dots, n$. We endow $\mathbb{C}^n$ with the Euclidean metric $g^{T\mathbb{C}^n}$. The associated Kähler form on $\mathbb{C}^n$ is

$$\omega = \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j \, .$$

We are interested in the space $(L^2(\mathbb{R}^{2n}), \|\cdot\|_{L^2})$ of square integrable functions on $\mathbb{R}^{2n}$ with respect to the Lebesgue measure. We denote by $dZ = dZ_1 \cdots dZ_{2n}$ the Euclidean volume form. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $z \in \mathbb{C}^n$, put $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

Let $L = \mathbb{C}$ be the trivial holomorphic line bundle on $\mathbb{C}^n$ with the canonical section $\mathbf{1} : \mathbb{C}^n \to L$, $z \mapsto (z, 1)$. Let h^L be the metric on L defined by

$$|\mathbf{1}|_{h^L}(z) := \exp(-\tfrac{\pi}{2} \textstyle\sum_{j=1}^n |z_j|^2) = \rho(Z) \quad \text{for } z \in \mathbb{C}^n \, . \tag{2.30}$$

The space of L^2-integrable holomorphic sections of L with respect to h^L and dZ is the classical Segal-Bargmann space of L^2-integrable holomorphic functions with respect to the volume form $\rho\, dZ$. It is well-known that $\{z^\beta : \beta \in \mathbb{N}^n\}$ forms an orthogonal basis of this space.

To introduce the model operator $\mathscr{L}$ we set:

$$b_i = -2\frac{\partial}{\partial z_i} + \pi \overline{z}_i \, , \quad b_i^+ = 2\frac{\partial}{\partial \overline{z}_i} + \pi z_i \, , \quad \mathscr{L} = \sum_i b_i \, b_i^+ \, . \tag{2.31}$$

We can interpret the operator $\mathscr{L}$ in terms of complex geometry. Let $\overline{\partial}^L$ be the Dolbeault operator acting on $\Omega^{0,\bullet}(\mathbb{C}^n, L)$ and let $\overline{\partial}^{L,*}$ be its adjoint with respect to the L^2-scalar product induced by $g^{T\mathbb{C}^n}$ and h^L. We have the isometry $\Omega^{0,\bullet}(\mathbb{C}^n, \mathbb{C}) \to \Omega^{0,\bullet}(\mathbb{C}^n, L)$ given by $\alpha \mapsto \rho^{-1}\alpha$. If

$$\square^L = \overline{\partial}^{L,*}\overline{\partial}^L + \overline{\partial}^L\overline{\partial}^{L,*}$$

denotes the Kodaira Laplacian acting on $\Omega^{0,\bullet}(\mathbb{C}^n, L)$, then

$$\begin{aligned} &\rho \, \square^L \rho^{-1} : \Omega^{0,\bullet}(\mathbb{C}^n, \mathbb{C}) \to \Omega^{0,\bullet}(\mathbb{C}^n, \mathbb{C}) \, , \\ &\rho \, \square^L \rho^{-1} = \tfrac{1}{2}\mathscr{L} + \sum_{j=1}^n 2\pi d\overline{z}^j \wedge i_{\frac{\partial}{\partial \overline{z}_j}} \, , \\ &\rho \, \square^L \rho^{-1}|_{\Omega^{0,0}} = \tfrac{1}{2}\mathscr{L} \, . \end{aligned}$$

The operator $\mathscr{L}$ is the complex analogue of the harmonic oscillator, the operators b, b^+ are creation and annihilation operators respectively. Each eigenspace of $\mathscr{L}$ has infinite dimension, but we can still give an explicit description.

Theorem 2.11 ([33, Th. 4.1.20], [34, Th. 1.15])**.** *The spectrum of $\mathscr{L}$ on $L^2(\mathbb{R}^{2n})$ is given by*

$$\operatorname{Spec}(\mathscr{L}) = \Big\{ 4\pi|\alpha| \, : \, \alpha \in \mathbb{N}^n \Big\} \, . \tag{2.32}$$

Each $\lambda \in \operatorname{Spec}(\mathscr{L})$ is an eigenvalue of infinite multiplicity and an orthogonal basis of the corresponding eigenspace is given by

$$B_\lambda = \Big\{ b^\alpha \big(z^\beta e^{-\pi \sum_i |z_i|^2/2} \big) : \alpha \in \mathbb{N}^n \text{ with } 4\pi|\alpha| = \lambda,\ \beta \in \mathbb{N}^n \Big\} \tag{2.33}$$

where $b^\alpha := b_1^{\alpha_1} \cdots b_n^{\alpha_n}$. Moreover, $\bigcup \{B_\lambda : \lambda \in \operatorname{Spec}(\mathscr{L})\}$ forms a complete orthogonal basis of $L^2(\mathbb{R}^{2n})$. In particular, an orthonormal basis of $\operatorname{Ker}(\mathscr{L})$ is

$$\Big\{ \varphi_\beta(z) = \big(\tfrac{\pi^{|\beta|}}{\beta!} \big)^{1/2} z^\beta e^{-\pi \sum_i |z_i|^2/2} \ : \beta \in \mathbb{N}^n \Big\} . \tag{2.34}$$

Let $\mathscr{P} : L^2(\mathbb{R}^{2n}) \longrightarrow \operatorname{Ker}(\mathscr{L})$ be the orthogonal projection and let $\mathscr{P}(Z, Z')$ denote its kernel with respect to dZ'. We call $\mathscr{P}(\cdot,\cdot)$ the Bergman kernel of $\mathscr{L}$. Obviously $\mathscr{P}(Z, Z') = \sum_\beta \varphi_\beta(z)\, \overline{\varphi}_\beta(z')$ so we infer from (2.34) that

$$\mathscr{P}(Z, Z') = \exp\Big(-\tfrac{\pi}{2} \textstyle\sum_{i=1}^n \big(|z_i|^2 + |z_i'|^2 - 2 z_i \overline{z}_i' \big) \Big) . \tag{2.35}$$

2.4 Asymptotic expansion of Bergman kernel

In Sections 2.4-2.6 we assume that (X, ω) is a compact Kähler manifold and (L, h^L) is a Hermitian holomorphic line bundle satisfying (2.14). For the sake of simplicity, we suppose that Riemannian metric g^{TX} is the metric associated to ω, that is, $g^{TX}(u, v) = \omega(u, Jv)$ (or, equivalently, $\Theta = \omega$).

In order to state the result about the asymptotic expansion we start by describing our identifications and notations.

Normal coordinates. Let a^X be the injectivity radius of (X, g^{TX}). We denote by $B^X(x, \varepsilon)$ and $B^{T_xX}(0, \varepsilon)$ the open balls in X and T_xX with center x and radius ε, respectively. Then the exponential map $T_xX \ni Z \to \exp_x^X(Z) \in X$ is a diffeomorphism from $B^{T_xX}(0, \varepsilon)$ onto $B^X(x, \varepsilon)$ for $\varepsilon \leqslant a^X$. From now on, we identify $B^{T_xX}(0, \varepsilon)$ with $B^X(x, \varepsilon)$ via the exponential map for $\varepsilon \leqslant a^X$. Throughout what follows, ε runs in the fixed interval $]0, a^X/4[$.

Basic trivialization. We fix $x_0 \in X$. For $Z \in B^{T_{x_0}X}(0, \varepsilon)$ we identify (L_Z, h_Z^L), (E_Z, h_Z^E) and $(L^p \otimes E)_Z$ to $(L_{x_0}, h_{x_0}^L)$, $(E_{x_0}, h_{x_0}^E)$ and $(L^p \otimes E)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and $\nabla^{L^p \otimes E}$ along the curve

$$\gamma_Z : [0, 1] \ni u \to \exp_{x_0}^X(uZ) .$$

This is the basic trivialization we use in this paper.

Using this trivialization we identify $f \in \mathscr{C}^\infty(X, \operatorname{End}(E))$ to a family $\{f_{x_0}\}_{x_0 \in X}$ where f_{x_0} is the function f in normal coordinates near x_0, i.e.,

$$f_{x_0} : B^{T_{x_0}X}(0, \varepsilon) \to \operatorname{End}(E_{x_0}), \quad f_{x_0}(Z) = f \circ \exp_{x_0}^X(Z) .$$

In general, for functions in the normal coordinates, we will add a subscript x_0 to indicate the base point $x_0 \in X$. Similarly, $P_p(x, x')$ induces in terms of the basic trivialization a smooth section $(Z,Z') \mapsto P_{p,x_0}(Z,Z')$ of $\pi^* \operatorname{End}(E)$ over $\{(Z,Z') \in TX \times_X TX : |Z|,|Z'| < \varepsilon\}$, which depends smoothly on x_0. Here we identify a section $S \in \mathscr{C}^\infty\big(TX \times_X TX, \pi^* \operatorname{End}(E)\big)$ with the family $(S_x)_{x\in X}$, where $S_x = S|_{\pi^{-1}(x)}$.

Coordinates on $T_{x_0}X$***.*** Let us choose an orthonormal basis $\{w_i\}_{i=1}^n$ of $T_{x_0}^{(1,0)}X$. Then $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$, $j = 1, \ldots, n$ form an orthonormal basis of $T_{x_0}X$. We use coordinates on $T_{x_0}X \simeq \mathbb{R}^{2n}$ given by the identification

$$\mathbb{R}^{2n} \ni (Z_1, \ldots, Z_{2n}) \longmapsto \sum_i Z_i e_i \in T_{x_0}X. \tag{2.36}$$

In what follows we also use complex coordinates $z = (z_1, \ldots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

Volume form on $T_{x_0}X$***.*** Let us denote by dv_{TX} the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$, there exists a smooth positive function $\kappa_{x_0} : T_{x_0}X \to \mathbb{R}$, satisfying

$$dv_X(Z) = \kappa_{x_0}(Z) dv_{TX}(Z), \quad \kappa_{x_0}(0) = 1, \tag{2.37}$$

where the subscript x_0 of $\kappa_{x_0}(Z)$ indicates the base point $x_0 \in X$.

Sequences of operators. Let $\Theta_p : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E)$ be a sequence of continuous linear operators with smooth kernel $\Theta_p(\cdot,\cdot)$ with respect to dv_X (e.g. $\Theta_p = T_{f,p}$). Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of TX on X. In terms of our basic trivialization, $\Theta_p(x,y)$ induces a family of smooth sections $Z, Z' \mapsto \Theta_{p,x_0}(Z,Z')$ of $\pi^* \operatorname{End}(E)$ over $\{(Z,Z') \in TX \times_X TX : |Z|,|Z'| < \varepsilon\}$, which depends smoothly on x_0.

We denote by $|\Theta_{p,x_0}(Z,Z')|_{\mathscr{C}^l(X)}$ the $\mathscr{C}^l$ norm with respect to the parameter $x_0 \in X$. We say that

$$\Theta_{p,x_0}(Z,Z') = \mathcal{O}(p^{-\infty}), \quad p \to \infty$$

if for any $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that $|\Theta_{p,x_0}(Z,Z')|_{\mathscr{C}^m(X)} \leqslant C_{l,m}\, p^{-l}$.

The asymptotics of the Bergman kernel will be described in terms of the Bergman kernel $\mathscr{P}_{x_0}(\cdot,\cdot) = \mathscr{P}(\cdot,\cdot)$ of the model operator $\mathscr{L}$ on $T_{x_0}X \cong \mathbb{R}^{2n}$. Recall that $\mathscr{P}(\cdot,\cdot)$ was defined in (2.35).

Notation 2.12. Fix $k \in \mathbb{N}$ and $\varepsilon' \in\,]0, a^X[$. Let

$$\{Q_{r,x_0} \in \operatorname{End}(E)_{x_0}[Z,Z'] \,:\, 0 \leqslant r \leqslant k,\, x_0 \in X\}$$

be a family of polynomials in Z, Z', which is smooth with respect to the parameter $x_0 \in X$. We say that

$$p^{-n}\Theta_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k} (Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + \mathcal{O}(p^{-(k+1)/2}), \tag{2.38}$$

on $\{(Z,Z')\in TX\times_X TX : |Z|,|Z'|<\varepsilon'\}$ if there exist $C_0>0$ and a decomposition

$$p^{-n}\Theta_{p,x_0}(Z,Z')-\sum_{r=0}^{k}(Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')\kappa_{x_0}^{-1/2}(Z)\kappa_{x_0}^{-1/2}(Z')p^{-r/2} = \Psi_{p,k,x_0}(Z,Z')+\mathcal{O}(p^{-\infty})\,, \tag{2.39}$$

where Ψ_{p,k,x_0} satisfies the following estimate: for every $l\in\mathbb{N}$ there exist $C_{k,l}>0$, $M>0$ such that for all $p\in\mathbb{N}^*$

$$|\Psi_{p,k,x_0}(Z,Z')|_{\mathscr{C}^l(X)}\leqslant C_{k,l}\,p^{-(k+1)/2}(1+\sqrt{p}\,|Z|+\sqrt{p}\,|Z'|)^M\,e^{-C_0\sqrt{p}\,|Z-Z'|}\,, \tag{2.40}$$

on $\{(Z,Z')\in TX\times_X TX : |Z|,|Z'|<\varepsilon'\}$.

The sequence P_p **.** We can now state the asymptotics of the Bergman kernel. First we observe that the Bergman kernel decays very fast outside the diagonal of $X\times X$.

Let $\mathbf{f}:\mathbb{R}\to[0,1]$ be a smooth even function such that $\mathbf{f}(v)=1$ for $|v|\leqslant\varepsilon/2$, and $\mathbf{f}(v)=0$ for $|v|\geqslant\varepsilon$. Set

$$F(a)=\Big(\int_{-\infty}^{+\infty}\mathbf{f}(v)dv\Big)^{-1}\int_{-\infty}^{+\infty}e^{iva}\,\mathbf{f}(v)dv. \tag{2.41}$$

Then $F(a)$ is an even function and lies in the Schwartz space $\mathcal{S}(\mathbb{R})$ and $F(0)=1$. We have the *far off-diagonal* behavior of the Bergman kernel:

Theorem 2.13 ([15, Prop. 4.1])**.** *For any $l,m\in\mathbb{N}$ and $\varepsilon>0$, there exists a positive constant $C_{l,m,\varepsilon}>0$ such that for any $p\geqslant1$, $x,x'\in X$, the following estimate holds:*

$$|F(D_p)(x,x')-P_p(x,x')|_{\mathscr{C}^m(X\times X)}\leqslant C_{l,m,\varepsilon}p^{-l}. \tag{2.42}$$

Especially,

$$|P_p(x,x')|_{\mathscr{C}^m(X\times X)}\leqslant C_{l,m,\varepsilon}\,p^{-l}\,,\quad \text{on } \{(x,x')\in X\times X : d(x,x')\geqslant\varepsilon\}\,. \tag{2.43}$$

The $\mathscr{C}^m$ norm in (2.42) *and* (2.43) *is induced by ∇^L, ∇^E, h^L, h^E and g^{TX}.*

Next we formulate the *near off-diagonal* expansion of the Bergman kernel.

Theorem 2.14 ([15, Th. 4.18′])**.** *There exist polynomials $J_{r,x_0}\in\operatorname{End}(E)_{x_0}[Z,Z']$ in Z,Z' with the same parity as r, such that for any $k\in\mathbb{N}$, $\varepsilon\in]0,a^X/4[$, we have*

$$p^{-n}P_{p,x_0}(Z,Z')\cong\sum_{r=0}^{k}(J_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^{-\frac{r}{2}}+\mathcal{O}(p^{-\frac{k+1}{2}})\,, \tag{2.44}$$

on the set $\{(Z,Z')\in TX\times_X TX : |Z|,|Z'|<2\varepsilon\}$, in the sense of Notation 2.12.

Let us briefly explain the idea of the proof for Theorems 2.13 – 2.14. Using the spectral gap property from Theorem 2.5, we obtain (2.42). By finite propagation speed of solutions of hyperbolic equations, we obtain that $F(D_p)(x,x') = 0$ if $d(x,x') \geqslant \varepsilon$ and $F(D_p)(x,\cdot)$ depends only on the restriction $D_p|_{B(x,\varepsilon)}$, so (2.43) follows. This shows that we can localize the asymptotics of $P_p(x_0,x')$ in the neighborhood of x_0. By pulling back all our objects by the exponential map to the tangential space and suitably extending them we can work on $\mathbb{R}^{2n}$. Thus we can use the explicit description of the Bergman kernel of the model operator $\mathscr{L}$ given in Section 2.3. To conclude the proof, we combine the spectral gap property, the rescaling of the coordinates and functional analytic techniques inspired by Bismut-Lebeau [5, §11].

By setting $\boldsymbol{b}_r(x_0) = (J_{2r,\,x_0}\mathscr{P}_{x_0})(0,0)$, we get from (2.44) the following diagonal expansion of the Bergman kernel.

Corollary 2.15. *For any $k,l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $p \in \mathbb{N}$,*

$$\Big| P_p(x,x) - \sum_{r=0}^{k} \boldsymbol{b}_r(x) p^{n-r} \Big|_{\mathscr{C}^l(X)} \leqslant C_{k,l}\, p^{n-k-1}\,, \tag{2.45}$$

where $\boldsymbol{b}_0(x) = \mathrm{Id}_E$.

The existence of the expansion (2.45) and the form of the leading term was proved by [41, 12, 46].

The calculation of the coefficients $\boldsymbol{b}_r$ is of great importance. For this we need $J_{r,\,x_0}$, which are obtained by computing the operators $\mathscr{F}_{r,\,x_0}$ defined by the smooth kernels

$$\mathscr{F}_{r,\,x_0}(Z,Z') = J_{r,\,x_0}(Z,Z')\mathscr{P}(Z,Z') \tag{2.46}$$

with respect to dZ'. Our strategy (already used in [33, 34]) is to rescale the Kodaira-Laplace operator, take the Taylor expansion of the rescaled operator and apply resolvent analysis.

Rescaling $\square_p$ and Taylor expansion. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, E_{x_0})$, $Z \in \mathbb{R}^{2n}$, $|Z| \leq 2\varepsilon$, and for $t = \frac{1}{\sqrt{p}}$, set

$$\begin{aligned} &(S_t s)(Z) := s(Z/t), \\ &\mathscr{L}_t := S_t^{-1}\kappa^{1/2}\, t^2 (2\,\square_p)\kappa^{-1/2} S_t\,. \end{aligned} \tag{2.47}$$

Then by [33, Th. 4.1.7], there exist second order differential operators $\mathcal{O}_r$ such that we have an asymptotic expansion in t when $t \to 0$,

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^{m} t^r \mathcal{O}_r + \mathscr{O}(t^{m+1}). \tag{2.48}$$

From [33, Th. 4.1.21, 4.1.25], we obtain

$$\mathscr{L}_0 = \sum_j b_j b_j^+ = \mathscr{L}, \qquad \mathcal{O}_1 = 0. \tag{2.49}$$

Resolvent analysis. We define by recurrence $f_r(\lambda) \in \operatorname{End}(L^2(\mathbb{R}^{2n}, E_{x_0}))$ by

$$f_0(\lambda) = (\lambda - \mathscr{L}_0)^{-1}, \quad f_r(\lambda) = (\lambda - \mathscr{L}_0)^{-1} \sum_{j=1}^{r} \mathcal{O}_j f_{r-j}(\lambda). \tag{2.50}$$

Let δ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\pi/2$. Then by [34, (1.110)] (cf. also [33, (4.1.91)])

$$\mathscr{F}_{r,\,x_0} = \frac{1}{2\pi\sqrt{-1}} \int_\delta f_r(\lambda) d\lambda. \tag{2.51}$$

Since the spectrum of $\mathscr{L}$ is well understood we can calculate the coefficients $\mathscr{F}_{r,\,x_0}$. Set $\mathscr{P}^\perp = \operatorname{Id} - \mathscr{P}$. From Theorem 2.11, (2.49) and (2.51), we get

$$\begin{aligned} \mathscr{F}_{0,\,x_0} &= \mathscr{P}, \quad \mathscr{F}_{1,\,x_0} = 0, \\ \mathscr{F}_{2,\,x_0} &= -\mathscr{L}^{-1}\mathscr{P}^\perp \mathcal{O}_2 \mathscr{P} - \mathscr{P}\mathcal{O}_2 \mathscr{L}^{-1}\mathscr{P}^\perp, \\ \mathscr{F}_{3,\,x_0} &= -\mathscr{L}^{-1}\mathscr{P}^\perp \mathcal{O}_3 \mathscr{P} - \mathscr{P}\mathcal{O}_3 \mathscr{L}^{-1}\mathscr{P}^\perp, \end{aligned} \tag{2.52}$$

and

$$\begin{aligned} \mathscr{F}_{4,\,x_0} = {} & \mathscr{L}^{-1}\mathscr{P}^\perp \mathcal{O}_2 \mathscr{L}^{-1}\mathscr{P}^\perp \mathcal{O}_2 \mathscr{P} - \mathscr{L}^{-1}\mathscr{P}^\perp \mathcal{O}_4 \mathscr{P} \\ & + \mathscr{P}\mathcal{O}_2 \mathscr{L}^{-1}\mathscr{P}^\perp \mathcal{O}_2 \mathscr{L}^{-1}\mathscr{P}^\perp - \mathscr{P}\mathcal{O}_4 \mathscr{L}^{-1}\mathscr{P}^\perp \\ & + \mathscr{L}^{-1}\mathscr{P}^\perp \mathcal{O}_2 \mathscr{P}\mathcal{O}_2 \mathscr{L}^{-1}\mathscr{P}^\perp - \mathscr{P}\mathcal{O}_2 \mathscr{L}^{-2}\mathscr{P}^\perp \mathcal{O}_2 \mathscr{P} \\ & - \mathscr{P}\mathcal{O}_2 \mathscr{P}\mathcal{O}_2 \mathscr{L}^{-2}\mathscr{P}^\perp - \mathscr{P}^\perp \mathscr{L}^{-2}\mathcal{O}_2 \mathscr{P}\mathcal{O}_2 \mathscr{P}. \end{aligned} \tag{2.53}$$

In particular, the first two identities of (2.52) imply

$$J_{0,\,x_0} = 1, \quad J_{1,\,x_0} = 0. \tag{2.54}$$

In order to formulate the formulas for $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ we introduce now more notations. Let ∇^{TX} be the Levi-Civita connection on (X, g^{TX}). We denote by $R^{TX} = (\nabla^{TX})^2$ the curvature, by Ric the Ricci curvature and by r^X the scalar curvature of ∇^{TX}.

We still denote by ∇^E the connection on $\operatorname{End}(E)$ induced by ∇^E. Consider the (positive) Laplacian Δ acting on the functions on (X, g^{TX}) and the Bochner Laplacian Δ^E on $\mathscr{C}^\infty(X, E)$ and on $\mathscr{C}^\infty(X, \operatorname{End}(E))$. Let $\{e_k\}$ be a (local) orthonormal frame of (TX, g^{TX}). Then

$$\Delta^E = -\sum_k (\nabla^E_{e_k} \nabla^E_{e_k} - \nabla^E_{\nabla^{TX}_{e_k} e_k}). \tag{2.55}$$

Let $\Omega^{q,r}(X, \mathrm{End}(E))$ be the space of (q,r)-forms on X with values in $\mathrm{End}(E)$, and let

$$\nabla^{1,0} : \Omega^{q,\bullet}(X, \mathrm{End}(E)) \to \Omega^{q+1,\bullet}(X, \mathrm{End}(E)) \tag{2.56}$$

be the $(1,0)$-component of the connection ∇^E. Let $(\nabla^E)^*$, $\nabla^{1,0*}, \overline{\partial}^{E*}$ be the adjoints of ∇^E, $\nabla^{1,0}, \overline{\partial}^E$, respectively. Let $D^{1,0}, D^{0,1}$ be the $(1,0)$ and $(0,1)$ components of the connection $\nabla^{T^*X} : \mathscr{C}^\infty(X, T^*X) \to \mathscr{C}^\infty(X, T^*X \otimes T^*X)$ induced by ∇^{TX}. In the following, we denote by

$$\langle \cdot\,, \cdot \rangle_\omega : \Omega^{\bullet,\bullet}(X, \mathrm{End}(E)) \times \Omega^{\bullet,\bullet}(X, \mathrm{End}(E)) \to \mathscr{C}^\infty(X, \mathrm{End}(E))$$

the $\mathbb{C}$-bilinear pairing $\langle \alpha \otimes f, \beta \otimes g \rangle_\omega = \langle \alpha, \beta \rangle f \cdot g$, for forms $\alpha, \beta \in \Omega^{\bullet,\bullet}(X)$ and sections $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. Put

$$R^E_\Lambda = \left\langle R^E, \omega \right\rangle_\omega . \tag{2.57}$$

Let $\mathrm{Ric}_\omega = \mathrm{Ric}(J\cdot, \cdot)$ be the $(1,1)$-form associated to Ric. Set

$$|\,\mathrm{Ric}_\omega|^2 = \sum_{i<j} \mathrm{Ric}_\omega(e_i, e_j)^2 , \quad |R^{TX}|^2 = \sum_{i<j}\sum_{k<l} \langle R^{TX}(e_i, e_j)e_k, e_l\rangle^2,$$

Theorem 2.16. *We have*

$$\boldsymbol{b}_1 = \frac{1}{8\pi} r^X + \frac{\sqrt{-1}}{2\pi} R^E_\Lambda\,, \tag{2.58}$$

$$\begin{aligned} \pi^2 \boldsymbol{b}_2 = &-\frac{\Delta r^X}{48} + \frac{1}{96}|R^{TX}|^2 - \frac{1}{24}|\,\mathrm{Ric}_\omega|^2 + \frac{1}{128}(r^X)^2 \\ &+ \frac{\sqrt{-1}}{32}\Big(2r^X R^E_\Lambda - 4\langle \mathrm{Ric}_\omega, R^E\rangle_\omega + \Delta^E R^E_\Lambda\Big) \\ &- \frac{1}{8}(R^E_\Lambda)^2 + \frac{1}{8}\langle R^E, R^E\rangle_\omega + \frac{3}{16}\overline{\partial}^{E*}\nabla^{1,0*}R^E\,. \end{aligned} \tag{2.59}$$

The terms $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ were computed by Lu [28] (for $E = \mathbb{C}$, the trivial line bundle with trivial metric), X. Wang [44], L. Wang [43], in various degree of generality. The method of these authors is to construct appropriate peak sections as in [41], using Hörmander's L^2 $\overline{\partial}$-method. In [15, §5.1], Dai-Liu-Ma computed $\boldsymbol{b}_1$ by using the heat kernel, and in [34, §2], [32, §2] (cf. also [33, §4.1.8, §8.3.4]), we computed $\boldsymbol{b}_1$ in the symplectic case. A new method for calculating $\boldsymbol{b}_2$ was given in [36].

2.5 Asymptotic expansion of Toeplitz operators

We stick to the situation studied in the previous Section, namely, (X,ω) is a compact Kähler manifold and (L,h^L) is a Hermitian holomorphic line bundle satisfying (2.14), and g^{TX} is the Riemannian metric associated to ω.
In order to develop the calculus of Toeplitz kernels we use the Bergman kernel expansion (2.44) and the Taylor expansion of the symbol. We are thus led to a kernel calculus on $\mathbb{C}^n$ with kernels of the form $F\mathscr{P}$, where F is a polynomial. This calculus can be completely described in terms of the spectral decomposition (2.32)-(2.33) of the model operator $\mathscr{L}$.

For a polynomial F in Z, Z', we denote by $F\mathscr{P}$ the operator on $L^2(\mathbb{R}^{2n})$ defined by the kernel $F(Z,Z')\mathscr{P}(Z,Z')$ and the volume form dZ according to (2.7).

The following very useful Lemma [33, Lemma 7.1.1] describes the calculus of the kernels $(F\mathscr{P})(Z,Z') := F(Z,Z')\mathscr{P}(Z,Z')$.

Lemma 2.17. *For any $F,G \in \mathbb{C}[Z,Z']$ there exists a polynomial $\mathscr{K}[F,G] \in \mathbb{C}[Z,Z']$ with degree $\deg \mathscr{K}[F,G]$ of the same parity as $\deg F + \deg G$, such that*

$$((F\mathscr{P})\circ(G\mathscr{P}))(Z,Z') = \mathscr{K}[F,G](Z,Z')\mathscr{P}(Z,Z'). \tag{2.60}$$

Let us illustrate how Lemma 2.17 works. First observe that from (2.31) and (2.35), for any polynomial $g(z,\overline{z}) \in \mathbb{C}[z,\overline{z}]$, we get

$$\begin{aligned} &b_{j,z}\,\mathscr{P}(Z,Z') = 2\pi(\overline{z}_j - \overline{z}'_j)\mathscr{P}(Z,Z'),\\ &[g(z,\overline{z}), b_{j,z}] = 2\frac{\partial}{\partial z_j} g(z,\overline{z})\,. \end{aligned} \tag{2.61}$$

Now (2.61) entails

$$\overline{z}_j\,\mathscr{P}(Z,Z') = \frac{b_{j,z}}{2\pi}\,\mathscr{P}(Z,Z') + \overline{z}'_j\mathscr{P}(Z,Z'). \tag{2.62}$$

Specializing (2.61) for $g(z,\overline{z}) = z_i$ we get

$$z_i\, b_{j,z}\mathscr{P}(Z,Z') = b_{j,z}(z_i\mathscr{P})(Z,Z') + 2\delta_{ij}\mathscr{P}(Z,Z'), \tag{2.63}$$

Formulas (2.62) and (2.63) give

$$z_i\overline{z}_j\,\mathscr{P}(Z,Z') = \frac{1}{2\pi}\, b_{j,z}\, z_i\mathscr{P}(Z,Z') + \frac{1}{\pi}\,\delta_{ij}\mathscr{P}(Z,Z') + z_i\overline{z}'_j\,\mathscr{P}(Z,Z')\,. \tag{2.64}$$

Using the preceding formula we calculate further some examples for the expression $\mathscr{K}[F,G]$ introduced (2.60). We use the spectral decomposition of $\mathscr{L}$ in the following way. If $\varphi(Z) = b^\alpha z^\beta \exp\big(-\frac{\pi}{2}\sum_{j=1}^n |z_j|^2\big)$ with $\alpha,\beta \in \mathbb{N}^n$, then Theorem 2.11 implies immediately that

$$(\mathscr{P}\varphi)(Z) = \begin{cases} z^\beta \exp\Big(-\dfrac{\pi}{2}\displaystyle\sum_{j=1}^n |z_j|^2\Big) & \text{if } |\alpha| = 0,\\ 0 & \text{if } |\alpha| > 0. \end{cases} \tag{2.65}$$

The identities (2.62), (2.64) and (2.65) imply that

$$\begin{aligned}
&\mathscr{K}[1,\overline{z}_j]\mathscr{P} = \mathscr{P}\circ(\overline{z}_j\mathscr{P}) = \overline{z}'_j\mathscr{P}, \quad \mathscr{K}[1,z_j]\mathscr{P} = \mathscr{P}\circ(z_j\mathscr{P}) = z_j\mathscr{P},\\
&\mathscr{K}[z_i,\overline{z}_j]\mathscr{P} = (z_i\mathscr{P})\circ(\overline{z}_j\mathscr{P}) = z_i\mathscr{P}\circ(\overline{z}_j\mathscr{P}) = z_i\overline{z}'_j\mathscr{P},\\
&\mathscr{K}[\overline{z}_i,z_j]\mathscr{P} = (\overline{z}_i\mathscr{P})\circ(z_j\mathscr{P}) = \overline{z}_i\mathscr{P}\circ(z_j\mathscr{P}) = \overline{z}_i z_j\mathscr{P},\\
&\mathscr{K}[z'_i,\overline{z}_j]\mathscr{P} = (z'_i\mathscr{P})\circ(\overline{z}_j\mathscr{P}) = \mathscr{P}\circ(z_i\overline{z}_j\mathscr{P}) = \tfrac{1}{\pi}\delta_{ij}\mathscr{P} + z_i\overline{z}'_j\mathscr{P},\\
&\mathscr{K}[\overline{z}'_i,z_j]\mathscr{P} = (\overline{z}'_i\mathscr{P})\circ(z_j\mathscr{P}) = \mathscr{P}\circ(\overline{z}_i z_j\mathscr{P}) = \tfrac{1}{\pi}\delta_{ij}\mathscr{P} + \overline{z}'_i z_j\mathscr{P}.
\end{aligned} \tag{2.66}$$

Thus we get

$$\begin{aligned}
&\mathscr{K}[1,\overline{z}_j] = \overline{z}'_j, \quad \mathscr{K}[1,z_j] = z_j,\\
&\mathscr{K}[z_i,\overline{z}_j] = z_i\overline{z}'_j, \quad \mathscr{K}[\overline{z}_i,z_j] = \overline{z}_i z_j,\\
&\mathscr{K}[\overline{z}'_i,z_j] = \mathscr{K}[z'_j,\overline{z}_i] = \tfrac{1}{\pi}\delta_{ij} + \overline{z}'_i z_j.
\end{aligned} \tag{2.67}$$

To simplify our calculations, we introduce the following notation. For any polynomial $F\in\mathbb{C}[Z,Z']$ we denote by $(F\mathscr{P})_p$ the operator defined by the kernel $p^n(F\mathscr{P})(\sqrt{p}Z,\sqrt{p}Z')$, that is,

$$((F\mathscr{P})_p\varphi)(Z) = \int_{\mathbb{R}^{2n}} p^n(F\mathscr{P})(\sqrt{p}Z,\sqrt{p}Z')\varphi(Z')\,dZ'\,, \quad \text{for } \varphi\in L^2(\mathbb{R}^{2n}). \tag{2.68}$$

Let $F,G\in\mathbb{C}[Z,Z']$. By a change of variables we obtain

$$((F\mathscr{P})_p\circ(G\mathscr{P})_p)(Z,Z') = p^n((F\mathscr{P})\circ(G\mathscr{P}))(\sqrt{p}Z,\sqrt{p}Z'). \tag{2.69}$$

We examine now the asymptotic expansion of the kernel of the Toeplitz operators $T_{f,p}$. The first observation is that outside the diagonal of $X\times X$, the kernel of $T_{f,p}$ has the growth $\mathscr{O}(p^{-\infty})$, as $p\to\infty$.

Lemma 2.18 ([35, Lemma 4.2]). *For every $\varepsilon>0$ and every $l,m\in\mathbb{N}$, there exists $C_{l,m,\varepsilon}>0$ such that*

$$|T_{f,p}(x,x')|_{\mathscr{C}^m(X\times X)} \leqslant C_{l,m,\varepsilon}p^{-l} \tag{2.70}$$

for all $p\geqslant 1$ and all $(x,x')\in X\times X$ with $d(x,x')>\varepsilon$, where the $\mathscr{C}^m$-norm is induced by ∇^L,∇^E and h^L,h^E,g^{TX}.

Proof. Due to (2.43), (2.70) holds if we replace $T_{f,p}$ by P_p. Moreover, from (2.44), for any $m\in\mathbb{N}$, there exist $C_m>0, M_m>0$ such that $|P_p(x,x')|_{\mathscr{C}^m(X\times X)}<Cp^{M_m}$ for all $(x,x')\in X\times X$. These two facts and formula (2.10) imply the Lemma. □

The near off-diagonal expansion of the Bergman kernel (2.44) and the kernel calculus on $\mathbb{C}^n$ presented above imply the near off-diagonal expansion of the Toeplitz kernels. (cf. [35, Lemma 4.6], [33, Lemma 7.2.4])

Theorem 2.19. *Let $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. There exists a family*

$$\{Q_{r,x_0}(f) \in \mathrm{End}(E)_{x_0}[Z, Z'] : r \in \mathbb{N},\, x_0 \in X\}\,,$$

depending smoothly on the parameter $x_0 \in X$, where $Q_{r,x_0}(f)$ are polynomials with the same parity as r and such that for every $k \in \mathbb{N}$, $\varepsilon \in]0, a^X/4[$,

$$p^{-n}T_{f,p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (Q_{r,x_0}(f)\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + \mathcal{O}(p^{-(k+1)/2})\,, \tag{2.71}$$

on the set $\{(Z, Z') \in TX \times_X TX : |Z|, |Z'| < 2\varepsilon\}$, in the sense of Notation 2.12. Moreover, $Q_{r,x_0}(f)$ are expressed by

$$Q_{r,x_0}(f) = \sum_{r_1+r_2+|\alpha|=r} \mathscr{K}\Big[J_{r_1,x_0}\,,\ \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}J_{r_2,x_0}\Big]\,. \tag{2.72}$$

where $\mathscr{K}[\cdot,\cdot]$ was introduced in (2.60)*. We have,*

$$Q_{0,x_0}(f) = f(x_0) \in \mathrm{End}(E_{x_0}). \tag{2.73}$$

Proof. Estimates (2.10) and (2.70) learn that for $|Z|, |Z'| < \varepsilon/2$, $T_{f,p,x_0}(Z, Z')$ is determined up to terms of order $\mathscr{O}(p^{-\infty})$ by the behavior of f in $B^X(x_0, \varepsilon)$. Let $\rho : \mathbb{R} \to [0, 1]$ be a smooth even function such that

$$\rho(v) = 1 \ \text{ if } \ |v| < 2; \quad \rho(v) = 0 \ \text{ if } |v| > 4. \tag{2.74}$$

For $|Z|, |Z'| < \varepsilon/2$, we get

$$\begin{aligned} &T_{f,p,x_0}(Z, Z') = \mathscr{O}(p^{-\infty}) \\ &+ \int_{T_{x_0}X} P_{p,x_0}(Z, Z'')\rho(2|Z''|/\varepsilon)f_{x_0}(Z'')P_{p,x_0}(Z'', Z')\kappa_{x_0}(Z'')\, dv_{TX}(Z'')\,. \end{aligned} \tag{2.75}$$

We consider the Taylor expansion of f_{x_0}:

$$\begin{aligned} f_{x_0}(Z) &= \sum_{|\alpha|\leqslant k} \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!} + \mathscr{O}(|Z|^{k+1}) \\ &= \sum_{|\alpha|\leqslant k} p^{-|\alpha|/2}\frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{(\sqrt{p}Z)^\alpha}{\alpha!} + p^{-\frac{k+1}{2}}\mathscr{O}(|\sqrt{p}Z|^{k+1}). \end{aligned} \tag{2.76}$$

We multiply now the expansions given in (2.76) and (2.44). Note the presence of κ_{x_0} in the definition (2.39) of (2.38). Hence we obtain the expansion of

$$\kappa_{x_0}^{1/2}(Z)P_{p,x_0}(Z, Z'')(\kappa_{x_0}f_{x_0})(Z'')P_{p,x_0}(Z'', Z')\kappa_{x_0}^{1/2}(Z')$$

which we substitute in (2.75). We integrate then on $T_{x_0}X$ by using the change of variable $\sqrt{p}\,Z''=W$ and conclude (2.71) and (2.72) by using formulas (2.60) and (2.69).
From (2.54) and (2.72), we get

$$Q_{0,\,x_0}(f)=\mathscr{K}[1,f_{x_0}(0)]=f_{x_0}(0)=f(x_0)\,. \tag{2.77}$$

The proof of Lemma 2.19 is complete. □

As an example, we compute $Q_{1,\,x_0}(f)$. By (2.48), (2.67) and (2.72) we obtain

$$Q_{1,\,x_0}(f)=\mathscr{K}\Big[1,\frac{\partial f_{x_0}}{\partial Z_j}(0)Z_j\Big]=\frac{\partial f_{x_0}}{\partial z_j}(0)z_j+\frac{\partial f_{x_0}}{\partial \overline{z}_j}(0)\overline{z}'_j\,. \tag{2.78}$$

Corollary 2.20. *For any $f\in\mathscr{C}^\infty(X,\mathrm{End}(E))$, we have*

$$T_{f,\,p}(x,x)=\sum_{r=0}^{\infty}\boldsymbol{b}_{r,f}(x)p^{n-r}+\mathscr{O}(p^{-\infty})\,,\quad \boldsymbol{b}_{r,f}\in\mathscr{C}^\infty(X,\mathrm{End}(E))\,. \tag{2.79}$$

Proof. By taking $Z=Z'=0$ in (2.71) we obtain (2.79), with $\boldsymbol{b}_{r,f}(x)=Q_{2r,x}(f)$. □

Since we have the precise formula (2.71) for $Q_{2r,x}(f)$ we can give a closed formula for the first coefficients $\boldsymbol{b}_{r,f}$. In [36], we computed the coefficients $\boldsymbol{b}_{1,f},\boldsymbol{b}_{2,f}$, from (2.79). These computations are also relevant in Kähler geometry (cf. [20], [21], [26]).

Theorem 2.21 ([36, Th. 0.1])**.** *For any $f\in\mathscr{C}^\infty(X,\mathrm{End}(E))$, we have*:

$$\boldsymbol{b}_{0,\,f}=f,\quad \boldsymbol{b}_{1,\,f}=\frac{r^X}{8\pi}f+\frac{\sqrt{-1}}{4\pi}\left(R^E_\Lambda f+fR^E_\Lambda\right)-\frac{1}{4\pi}\Delta^E f\,. \tag{2.80}$$

If $f\in\mathscr{C}^\infty(X)$, then

$$\begin{aligned}\pi^2\boldsymbol{b}_{2,\,f}=&\,\pi^2\boldsymbol{b}_2f+\frac{1}{32}\Delta^2f-\frac{1}{32}r^X\Delta f-\frac{\sqrt{-1}}{8}\big\langle\,\mathrm{Ric}_\omega,\partial\overline{\partial}f\big\rangle\\&+\frac{\sqrt{-1}}{24}\big\langle df,\nabla^ER^E_\Lambda\big\rangle_\omega+\frac{1}{24}\big\langle\partial f,\nabla^{1,0*}R^E\big\rangle_\omega-\frac{1}{24}\big\langle\overline{\partial}f,\overline{\partial}^{E*}R^E\big\rangle_\omega\\&-\frac{\sqrt{-1}}{8}(\Delta f)R^E_\Lambda+\frac{1}{4}\big\langle\partial\overline{\partial}f,R^E\big\rangle_\omega\,.\end{aligned} \tag{2.81}$$

2.6 Algebra of Toeplitz operators, Berezin-Toeplitz star-product

Lemma 2.18 and Theorem 2.19 provide the asymptotic expansion of the kernel of a Toeplitz operator $T_{f,p}$. Using this lemma we can for example easily obtain the expansion of the kernel of the composition $T_{f,p}T_{g,p}$, for two sections $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. The result will be an asymptotic expansion of the type (2.71). Luckily we can show that the existence of a such asymptotic expansion *characterizes* Toeplitz operators (in the sense of Definition 2.4). We have the following useful criterion which ensures that a given family is a Toeplitz operator.

Theorem 2.22. *Let $\{T_p : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E)\}$ be a family of bounded linear operators. Then $\{T_p\}$ is a Toeplitz operator if and only if satisfies the following three conditions:*
(i) For any $p \in \mathbb{N}$, $P_p\, T_p\, P_p = T_p$.
(ii) For any $\varepsilon_0 > 0$ and any $l \in \mathbb{N}$, there exists $C_{l,\varepsilon_0} > 0$ such that for all $p \geqslant 1$ and all $(x, x') \in X \times X$ with $d(x, x') > \varepsilon_0$,

$$|T_p(x, x')| \leqslant C_{l,\varepsilon_0} p^{-l}. \tag{2.82}$$

(iii) There exists a family of polynomials $\{\mathcal{Q}_{r,x_0} \in \mathrm{End}(E)_{x_0}[Z, Z']\}_{x_0 \in X}$ such that:

- *(a) each $\mathcal{Q}_{r,x_0}$ has the same parity as r,*
- *(b) the family is smooth in $x_0 \in X$ and*
- *(c) there exists $0 < \varepsilon' < a^X/4$ such that for every $x_0 \in X$, every $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| < \varepsilon'$ and every $k \in \mathbb{N}$ we have*

$$p^{-n} T_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (\mathcal{Q}_{r,x_0} \mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}), \tag{2.83}$$

in the sense of Notation 2.12.

Proof. In view of Lemma 2.18 and Theorem 2.19 it is easy to see that conditions (i)-(iii) are necessary. To prove the sufficiency we use the following strategy. We define inductively the sequence $(g_l)_{l \geqslant 0}$, $g_l \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ such that

$$T_p = \sum_{l=0}^{m} P_p\, g_l\, p^{-l}\, P_p + \mathcal{O}(p^{-m-1}), \quad \text{for every } m \geqslant 0. \tag{2.84}$$

Let us start with the case $m = 0$ of (2.84). For an arbitrary but fixed $x_0 \in X$, we set

$$g_0(x_0) = \mathcal{Q}_{0,x_0}(0,0) \in \mathrm{End}(E_{x_0}). \tag{2.85}$$

Then show that

$$p^{-n}(T_p - T_{g_0,p})_{x_0}(Z,Z') \cong \mathcal{O}(p^{-1})\,, \tag{2.86}$$

which implies the case $m = 0$ of (2.84), namely,

$$T_p = P_p\, g_0\, P_p + \mathcal{O}(p^{-1}). \tag{2.87}$$

A crucial point here is the following result.

Proposition 2.23 ([35, Prop. 4.11])**.** *In the conditions of Theorem 2.22 we have* $\mathscr{Q}_{0,x_0}(Z,Z') = \mathscr{Q}_{0,x_0}(0,0)$ *for all* $x_0 \in X$ *and all* $Z, Z' \in T_{x_0}X$*.*

The proof is quite technical, so we refer to [35, p. 585-90].

Coming back to the proof of (2.86), let us compare the asymptotic expansion of T_p and $T_{g_0,p} = P_p\, g_0\, P_p$. Using the Notation 2.12, the expansion (2.71) (for $k = 1$) reads

$$p^{-n}T_{g_0,p,x_0}(Z,Z') \cong (g_0(x_0)\mathscr{P}_{x_0} + Q_{1,x_0}(g_0)\mathscr{P}_{x_0}\, p^{-1/2})(\sqrt{p}Z, \sqrt{p}Z') + \mathcal{O}(p^{-1})\,, \tag{2.88}$$

since $\mathscr{Q}_{0,x_0}(g_0) = g_0(x_0)$ by (2.73). The expansion (2.83) (also for $k = 1$) takes the form

$$p^{-n}T_{p,x_0} \cong (g_0(x_0)\mathscr{P}_{x_0} + \mathscr{Q}_{1,x_0}\mathscr{P}_{x_0}\, p^{-1/2})(\sqrt{p}Z, \sqrt{p}Z') + \mathcal{O}(p^{-1})\,, \tag{2.89}$$

where we have used Proposition 2.23 and the definition (2.85) of g_0. Thus, subtracting (2.88) from (2.89) we obtain

$$p^{-n}(T_p - T_{g_0,p})_{x_0}(Z,Z') \cong \big((\mathscr{Q}_{1,x_0} - Q_{1,x_0}(g_0))\mathscr{P}_{x_0}\big)(\sqrt{p}Z, \sqrt{p}Z')\, p^{-1/2} + \mathcal{O}(p^{-1})\,. \tag{2.90}$$

Thus it suffices to prove:

$$F_{1,x} := \mathscr{Q}_{1,x} - Q_{1,x}(g_0) \equiv 0\,. \tag{2.91}$$

which is done in [35, Lemma 4.18]. This finishes the proof of (2.86) and (2.87). Hence the expansion (2.84) of T_p holds for $m = 0$. Moreover, if T_p is self-adjoint, then from (4.70), (4.71) follows that g_0 is also self-adjoint.

We show inductively that (2.84) holds for every $m \in \mathbb{N}$. To handle (2.84) for $m = 1$ let us consider the operator $p(T_p - P_p g_0 P_p)$. The task is to show that $p\big(T_p - T_{g_0,p}\big)$ satisfies the hypotheses of Theorem 2.22. The first two conditions are easily verified. To prove the third, just subtract the asymptotics of $T_{p,x_0}(Z,Z')$ (given by (2.83)) and $T_{g_0,p,x_0}(Z,Z')$ (given by (2.71)). Taking into account Proposition 2.23 and (2.91), the coefficients of p^0 and $p^{-1/2}$ in the difference vanish, which yields the desired conclusion. Proposition 2.23 and (2.87) applied to $p(T_p - P_p g_0 P_p)$ yield $g_1 \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ such that (2.84) holds true for $m = 1$.

We continue in this way the induction process to get (2.84) for any m. This completes the proof of Theorem 2.22. □

Recall that the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on $(X, 2\pi\omega)$ is defined as follows. For $f, g \in \mathscr{C}^\infty(X)$, let ξ_f be the Hamiltonian vector field generated by f, which is defined by $2\pi i_{\xi_f}\omega = df$. Then

$$\{f, g\} := \xi_f(dg). \tag{2.92}$$

Theorem 2.24 ([35, Th. 1.1], [33, Th. 7.4.1])**.** *The product of the Toeplitz operators $T_{f,\,p}$ and $T_{g,\,p}$, with $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, is a Toeplitz operator, i.e., it admits the asymptotic expansion in the sense of* (2.12):

$$T_{f,\,p}\,T_{g,\,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),\,p} + \mathcal{O}(p^{-\infty}), \tag{2.93}$$

where C_r are bi-differential operators, $C_r(f, g) \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, $C_0(f, g) = fg$. If $f, g \in (\mathscr{C}^\infty(X), \{\cdot,\cdot\})$ with the Poisson bracket defined in (2.92)*, we have*

$$\big[T_{f,\,p}\,, T_{g,\,p}\big] = \frac{\sqrt{-1}}{p} T_{\{f,\,g\},\,p} + \mathcal{O}(p^{-2}). \tag{2.94}$$

Proof. Firstly, it is obvious that $P_p\,T_{f,\,p}\,T_{g,\,p}\,P_p = T_{f,\,p}\,T_{g,\,p}$. Lemmas 2.18 and 2.19 imply $T_{f,\,p}\,T_{g,\,p}$ verifies (2.82). Like in (2.75), we have for $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon/4$:

$$\begin{aligned}(T_{f,\,p}\,T_{g,\,p})_{x_0}(Z, Z') = \int_{T_{x_0}X} & T_{f,\,p,\,x_0}(Z, Z'')\rho(4|Z''|/\varepsilon)T_{g,\,p,\,x_0}(Z'', Z') \\ & \times \kappa_{x_0}(Z'')\, dv_{TX}(Z'') + \mathscr{O}(p^{-\infty}).\end{aligned} \tag{2.95}$$

By Lemma 2.19 and (2.95), we deduce as in the proof of Lemma 2.19, that for $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon/4$, we have

$$p^{-n}(T_{f,\,p}\,T_{g,\,p})_{x_0}(Z, Z') \cong \sum_{r=0}^{k} (Q_{r,\,x_0}(f, g)\mathscr{P}_{x_0})(\sqrt{p}\,Z, \sqrt{p}\,Z')p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}), \tag{2.96}$$

and with the notation (2.60),

$$Q_{r,\,x_0}(f, g) = \sum_{r_1+r_2=r} \mathscr{K}[Q_{r_1,\,x_0}(f), Q_{r_2,\,x_0}(g)]. \tag{2.97}$$

Thus $T_{f,\,p}\,T_{g,\,p}$ is a Toeplitz operator by Theorem 2.22. Moreover, it follows from the proofs of Lemma 2.19 and Theorem 2.22 that $g_l = C_l(f, g)$, where C_l are bi-differential operators.

From (2.60), (2.73) and (2.97), we get

$$C_0(f, g)(x) = Q_{0,\,x}(f, g) = \mathscr{K}[Q_{0,\,x}(f), Q_{0,\,x}(g)] = f(x)g(x)\,. \tag{2.98}$$

The commutation relation (2.94) follows from

$$C_1(f,g)(x) - C_1(g,f)(x) = \sqrt{-1}\{f,g\}\ \mathrm{Id}_E\ . \tag{2.99}$$

There are two ways to prove (2.99). One is to compute directly the difference and to use some of the identities (2.66). This method works also for symplectic manifolds, see [35, p. 593-4], [33, p. 311]. On the other hand, in the Kähler case one can compute explicitly each coefficient $C_1(f,g)$ (which in the general symplectic case is more involved), and then take the difference. The formula for $C_1(f,g)$ is given in the next theorem. This finishes the proof of Theorem 4.2. □

Theorem 2.25 ([36, Th. 0.3])**.** *Let $f,g\in\mathscr{C}^\infty(X,\mathrm{End}(E))$. We have*

$$\begin{aligned} C_0(f,g) =& fg,\\ C_1(f,g) =& -\frac{1}{2\pi}\big\langle\nabla^{1,0}f,\overline{\partial}^E g\big\rangle_\omega\in\mathscr{C}^\infty(X,\mathrm{End}(E)),\\ C_2(f,g) =& \boldsymbol{b}_{2,\,f,\,g}-\boldsymbol{b}_{2,\,fg}-\boldsymbol{b}_{1,\,C_1(f,\,g)}. \end{aligned} \tag{2.100}$$

If $f,g\in\mathscr{C}^\infty(X)$, then

$$\begin{aligned} C_2(f,g) =& \frac{1}{8\pi^2}\left\langle D^{1,0}\partial f, D^{0,1}\overline{\partial}g\right\rangle+\frac{\sqrt{-1}}{4\pi^2}\left\langle \mathrm{Ric}_\omega,\partial f\wedge\overline{\partial}g\right\rangle\\ &-\frac{1}{4\pi^2}\left\langle\partial f\wedge\overline{\partial}g,R^E\right\rangle_\omega\ . \end{aligned} \tag{2.101}$$

The next result and Theorem 2.24 show that the Berezin-Toeplitz quantization has the correct semi-classical behavior.

Theorem 2.26. *For $f\in\mathscr{C}^\infty(X,\mathrm{End}(E))$, the norm of $T_{f,p}$ satisfies*

$$\lim_{p\to\infty}\|T_{f,p}\| = \|f\|_\infty := \sup_{0\neq u\in E_x,\,x\in X}|f(x)(u)|_{h^E}/|u|_{h^E}. \tag{2.102}$$

Proof. Take a point $x_0\in X$ and $u_0\in E_{x_0}$ with $|u_0|_{h^E}=1$ such that $|f(x_0)(u_0)|=\|f\|_\infty$. Recall that in Section 2.4, we trivialized the bundles L, E in normal coordinates near x_0, and e_L is the unit frame of L which trivializes L. Moreover, in this normal coordinates, u_0 is a trivial section of E. Considering the sequence of sections $S^p_{x_0}=p^{-n/2}P_p(e_L^{\otimes p}\otimes u_0)$, we have by (2.44),

$$\left\|T_{f,p}\,S^p_{x_0}-f(x_0)S^p_{x_0}\right\|_{L^2}\leqslant\frac{C}{\sqrt{p}}\|S^p_{x_0}\|_{L^2}\,, \tag{2.103}$$

which immediately implies (2.102). □

Note that if f is a real function, then $df(x_0)=0$, so we can improve the bound $Cp^{-1/2}$ in (2.103) to Cp^{-1}.

Remark 2.27. **(i)** Relations (2.94) and (2.102) were first proved in some special cases: in [24] for Riemann surfaces, in [14] for $\mathbb{C}^n$ and in [8] for bounded symmetric domains in $\mathbb{C}^n$, by using explicit calculations. Then Bordemann, Meinrenken and Schlichenmaier [7] treated the case of a compact Kähler manifold (with $E = \mathbb{C}$) using the theory of Toeplitz structures (generalized Szegö operators) by Boutet de Monvel and Guillemin [9]. Moreover, Schlichenmaier [38] (cf. also [23], [13]) continued this train of thought and showed that for any $f, g \in \mathscr{C}^\infty(X)$, the product $T_{f,p}\,T_{g,p}$ has an asymptotic expansion (4.5) and constructed geometrically an associative star product.
(ii) The construction of the star-product can be carried out even in the presence of a twisting vector bundle E. Let $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. Set

$$f *_\hbar g := \sum_{k=0}^{\infty} C_k(f,g)\hbar^k \in \mathscr{C}^\infty(X, \mathrm{End}(E))[[\hbar]]\,, \tag{2.104}$$

where $C_r(f,g)$ are determined by (4.5). Then (2.104) defines an associative star-product on $\mathscr{C}^\infty(X, \mathrm{End}(E))$ called *Berezin-Toeplitz star-product* (cf. [23, 38] for the Kähler case with $E = \mathbb{C}$ and [33, 35] for the symplectic case and arbitrary twisting bundle E). The associativity of the star-product (2.104) follows immediately from the associativity rule for the composition of Toeplitz operators, $(T_{f,p} \circ T_{g,p}) \circ T_{k,p} = T_{f,p} \circ (T_{g,p} \circ T_{k,p})$ for any $f, g, k \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, and from the asymptotic expansion (4.5) applied to both sides of the latter equality.

The coefficients $C_r(f,g)$, $r = 0, 1, 2$ are given by (2.100). Set

$$\{\{f,g\}\} := \frac{1}{2\pi\sqrt{-1}}\big(\langle \nabla^{1,0} g, \overline{\partial}^E f\rangle_\omega - \langle \nabla^{1,0} f, \overline{\partial}^E g\rangle_\omega\big)\,. \tag{2.105}$$

If $fg = gf$ on X we have

$$\big[T_{f,p}\,, T_{g,p}\big] = \tfrac{\sqrt{-1}}{p}\, T_{\{\{f,g\}\},p} + \mathcal{O}\big(p^{-2}\big)\,, \quad p \to \infty. \tag{2.106}$$

Due to the fact that $\{\{f,g\}\} = \{f,g\}$ if E is trivial and comparing (2.94) to (2.106), one can regard $\{\{f,g\}\}$ defined in (2.105) as a non-commutative Poisson bracket.

2.7 Quantization of compact Hermitian manifolds

Throughout Sections 2.4-2.6 we supposed that the Riemannian metric g^{TX} was the metric associated to ω, that is, $g^{TX}(u,v) = \omega(u, Jv)$ (or, equivalently, $\Theta = \omega$). The results presented so far still hold for a general non-Kähler Riemannian metric g^{TX}.

Let us denote the metric associated to ω by $g^{TX}_\omega := \omega(\cdot, J\cdot)$. The volume form of g^{TX}_ω is given by $dv_{X,\omega} = (2\pi)^{-n}\det(\dot{R}^L)dv_X$ (where dv_X is the volume form of g^{TX}). Moreover, $h^E_\omega := \det(\frac{\dot{R}^L}{2\pi})^{-1}h^E$ defines a metric on E. We add a subscript

ω to indicate the objects associated to g^{TX}_ω, h^L and h^E_ω. Hence $\langle\cdot,\cdot\rangle_\omega$ denotes the L^2 Hermitian product on $\mathscr{C}^\infty(X, L^p\otimes E)$ induced by g^{TX}_ω, h^L, h^E_ω. This product is equivalent to the product $\langle\cdot,\cdot\rangle$ induced by g^{TX}, h^L, h^E.

Moreover, $H^0(X, L^p\otimes E)$ does not depend on the Riemannian metric on X or on the Hermitian metrics on L, E. Therefore, the orthogonal projections from $(\mathscr{C}^\infty(X, L^p\otimes E), \langle\cdot,\cdot\rangle_\omega)$ and $(\mathscr{C}^\infty(X, L^p\otimes E), \langle\cdot,\cdot\rangle)$ onto $H^0(X, L^p\otimes E)$ are the same. Hence $P_p = P_{p,\omega}$ and therefore $T_{f,p} = T_{f,p,\omega}$ as operators. However, their kernels are different. If $P_{p,\omega}(x,x')$, $T_{f,\omega,p}(x,x')$, $(x,x'\in X)$, denote the smooth kernels of $P_{p,\omega}$, $T_{f,p,\omega}$ with respect to $dv_{X,\omega}(x')$, we have

$$
\begin{aligned}
P_p(x,x') &= (2\pi)^{-n}\det(\dot{R}^L)(x')P_{p,\omega}(x,x')\,,\\
T_{f,p}(x,x') &= (2\pi)^{-n}\det(\dot{R}^L)(x')T_{f,p,\omega}(x,x')\,.
\end{aligned}
\tag{2.107}
$$

Now, for the kernel $P_{p,\omega}(x,x')$, we can apply Theorem 2.14 since $g^{TX}_\omega(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is a Kähler metric on TX. We obtain in this way the expansion of the Bergman kernel for a non-Kähler Riemannian metric g^{TX} on X, see [33, Th. 4.1.1, 4.1.3]. Of course, the coefficients $\boldsymbol{b}_r$ reflect in this case the presence of g^{TX}. For example

$$
\boldsymbol{b}_0 = \det(\dot{R}^L/(2\pi))\,\mathrm{Id}_E,
\tag{2.108}
$$

and

$$
\boldsymbol{b}_1 = \frac{1}{8\pi}\det\Big(\frac{\dot{R}^L}{2\pi}\Big)\Big[r^X_\omega - 2\Delta_\omega\Big(\log(\det(\dot{R}^L))\Big) + 4\sqrt{-1}\langle R^E,\omega\rangle_\omega\Big]\,.
\tag{2.109}
$$

Using the expansion of the Bergman kernel $P_{p,\omega}(\cdot,\cdot)$ we can deduce the expansion of the Toeplitz operators $T_{f,p,\omega}$ and their kernels, analogous to Theorem 2.19, Corollary 2.20 and Theorem 2.24. By (2.107), the coefficients of these expansion satisfy

$$
\begin{aligned}
\boldsymbol{b}_{f,r} &= (2\pi)^{-n}\det(\dot{R}^L)\boldsymbol{b}_{f,r,\omega}\,,\\
C_r(f,g) &= C_{r,\omega}(f,g)\,.
\end{aligned}
\tag{2.110}
$$

Since X is compact, (2.107) allowed to reduce the general situation considered here to the case $\omega = \Theta$ and apply Theorem 2.16. However, if X is not compact, the trick of using (2.107) does not work anymore, because the operator associated to g^{TX}_ω, h^L, h^E_ω might not have a spectral gap. But under the hypotheses of Theorem 2.9 the spectral gap for D_p exists, so we can extend these results to certain complete Hermitian manifolds in the next section.

2.8 Quantization of complete Hermitian manifolds

We return to the general situation of a complete manifold already considered in §2.2. The following result, obtained in [34, Th. 3.11], extends the asymptotic expansion of the Bergman kernel to complete manifolds.

Theorem 2.28. *Let (X, Θ) be a complete Hermitian manifold, (L, h^L), (E, h^E) be Hermitian holomorphic vector bundles of rank one and r, respectively. Assume that the hypotheses of Theorem 2.9 are fulfilled. Then the kernel $P_p(x, x')$ has a full off–diagonal asymptotic expansion analogous to that of Theorem 2.14 uniformly for any $x, x' \in K$, a compact set of X. If $L = K_X := \det(T^{*(1,0)}X)$ is the canonical line bundle on X, the first two conditions in (5.5) are to be replaced by*

$$h^L \text{ is induced by } \Theta \text{ and } \sqrt{-1}R^{\det} < -\varepsilon\Theta,\ \sqrt{-1}R^E > -C\Theta\,\mathrm{Id}_E\,.$$

The idea of the proof is that the spectral gap property (2.26) of Theorem 2.9 allows to generalize the analysis leading to the expansion in the compact case (Theorems 2.13 and 2.14) to the situation at hand.

Remark 2.29. Consider for the moment that in Theorem 2.28 we have $\Theta = \frac{\sqrt{-1}}{2\pi}R^L$. Since in the proof of Theorem 2.28 we use the same localization technique as in the compact case, the coefficients $J_{r,x}$ in the expansion of the Bergman kernel (cf. (2.44)), in particular the coefficients $\boldsymbol{b}_r(x) = J_{2r,x}(x)$ of the diagonal expansion have the same universal formulas as in the compact case. Thus the explicit formulas from Theorem 2.16 for $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ remain valid in the case of the situation considered in Theorem 2.28. Moreover, in the general case when $\frac{\sqrt{-1}}{2\pi}R^L > \varepsilon\Theta$ (for some constant $\varepsilon > 0$), the first formulas in (2.107) and (2.108), (2.109) are still valid.

Let $\mathscr{C}^\infty_{\mathrm{const}}(X, \mathrm{End}(E))$ denote the algebra of smooth sections of X which are constant map outside a compact set. For any $f \in \mathscr{C}^\infty_{\mathrm{const}}(X, \mathrm{End}(E))$, we consider the Toeplitz operator $(T_{f,p})_{p\in\mathbb{N}}$ as in (2.9). The following result generalizes Theorems 4.2 and 2.26 to complete manifolds.

Theorem 2.30 ([35, Th. 5.3])**.** *Let (X, Θ) be a complete Hermitian manifold, let (L, h^L) and (E, h^E) be Hermitian holomorphic vector bundles on X of rank one and r, respectively. Assume that the hypotheses of Theorem 2.9 are fulfilled. Let $f, g \in \mathscr{C}^\infty_{\mathrm{const}}(X, \mathrm{End}(E))$. Then the following assertions hold:*

(i) The product of the two corresponding Toeplitz operators admits the asymptotic expansion (4.5) *in the sense of* (2.12)*, where C_r are bi-differential operators, especially,* $\mathrm{supp}(C_r(f, g)) \subset \mathrm{supp}(f) \cap \mathrm{supp}(g)$*, and $C_0(f, g) = fg$.*

(ii) If $f, g \in \mathscr{C}^\infty_{\mathrm{const}}(X)$, then (2.94) *holds.*

(iii) Relation (2.102) *also holds for any $f \in \mathscr{C}^\infty_{\mathrm{const}}(X, \mathrm{End}(E))$.*

(iv) The coefficients $C_r(f, g)$ are given by $C_r(f, g) = C_{r,\omega}(f, g)$, where $\omega = \frac{\sqrt{-1}}{2\pi}R^L$ (compare (2.110)*).*

3 Berezin-Toeplitz quantization on Kähler orbifolds

In this Section we review the theory of Berezin-Toeplitz quantization on Kähler orbifolds, especially we show that set of Toeplitz operators forms an algebra. Note that the problem of quantization of orbifolds appears naturally in the study of the phenomenon of "quantization commutes to reduction", since the reduced spaces are often orbifolds, see e.g. [30], or in the problem of quantization of moduli spaces.

Complete explanations and references for Sections 3.1 and 3.2 are contained in [33, §5.4], [35, §6]. Moreover, we treat there also the case of symplectic orbifolds.

This Section is organized as follows. In Section 3.1 we recall the basic definitions about orbifolds. In Section 3.2 we explain the asymptotic expansion of Bergman kernel on complex orbifolds [15, §5.2], which we apply in Section 3.3 to derive the Berezin-Toeplitz quantization on Kähler orbifolds.

3.1 Preliminaries about orbifolds

We begin by the definition of orbifolds. We define at first a category $\mathcal{M}_s$ as follows : The objects of $\mathcal{M}_s$ are the class of pairs (G, M) where M is a connected smooth manifold and G is a finite group acting effectively on M (i.e., if $g \in G$ such that $gx = x$ for any $x \in M$, then g is the unit element of G). Consider two objects (G, M) and (G', M'). For $g \in G', \varphi \in \Phi$, we define $g\varphi : M \to M'$ by $(g\varphi)(x) = g(\varphi(x))$ for $x \in M$. A morphism $\Phi : (G, M) \to (G', M')$ is a family of open embeddings $\varphi : M \to M'$ satisfying :

i) For each $\varphi \in \Phi$, there is an injective group homomorphism $\lambda_\varphi : G \to G'$ that makes φ be λ_φ-equivariant.

ii) If $(g\varphi)(M) \cap \varphi(M) \neq \emptyset$, then $g \in \lambda_\varphi(G)$.

iii) For $\varphi \in \Phi$, we have $\Phi = \{g\varphi, g \in G'\}$.

Definition 3.1 (Orbifold chart, atlas, structure)**.** Let X be a paracompact Hausdorff space. An m-dimensional *orbifold chart* on X consists of a connected open set U of X, an object $(G_U, \widetilde{U})$ of $\mathcal{M}_s$ with $\dim \widetilde{U} = m$, and a ramified covering $\tau_U : \widetilde{U} \to U$ which is G_U-invariant and induces a homeomorphism $U \simeq \widetilde{U}/G_U$. We denote the chart by $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$.

An m-dimensional *orbifold atlas* $\mathcal{V}$ on X consists of a family of m-dimensional orbifold charts $\mathcal{V}(U) = ((G_U, \widetilde{U}) \xrightarrow{\tau_U} U)$ satisfying the following conditions:

(i) The open sets $U \subset X$ form a covering $\mathcal{U}$ with the property:

$$\begin{aligned}&\text{For any } U, U' \in \mathcal{U} \text{ and } x \in U \cap U', \text{ there exists } U'' \in \mathcal{U}\\ &\text{such that } x \in U'' \subset U \cap U'.\end{aligned} \tag{3.1}$$

(ii) For any $U, V \in \mathcal{U}, U \subset V$ there exists a morphism $\varphi_{VU} : (G_U, \widetilde{U}) \to (G_V, \widetilde{V})$, which covers the inclusion $U \subset V$ and satisfies $\varphi_{WU} = \varphi_{WV} \circ \varphi_{VU}$ for any $U, V, W \in \mathcal{U}$, with $U \subset V \subset W$.

It is easy to see that there exists a unique maximal orbifold atlas $\mathcal{V}_{\max}$ containing $\mathcal{V}$; $\mathcal{V}_{\max}$ consists of all orbifold charts $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$, which are locally isomorphic to charts from $\mathcal{V}$ in the neighborhood of each point of U. A maximal orbifold atlas $\mathcal{V}_{\max}$ is called an *orbifold structure* and the pair $(X, \mathcal{V}_{\max})$ is called an orbifold. As usual, once we have an orbifold atlas $\mathcal{V}$ on X we denote the orbifold by $(X, \mathcal{V})$, since $\mathcal{V}$ uniquely determines $\mathcal{V}_{\max}$.

In Definition 3.1 we can replace $\mathcal{M}_s$ by a category of manifolds with an additional structure such as orientation, Riemannian metric, almost-complex structure or complex structure. We impose that the morphisms (and the groups) preserve the specified structure. So we can define oriented, Riemannian, almost-complex or complex orbifolds.

Definition 3.2 (regular and singular set)**.** Let $(X, \mathcal{V})$ be an orbifold. For each $x \in X$, we can choose a small neighborhood $(G_x, \widetilde{U}_x) \to U_x$ such that $x \in \widetilde{U}_x$ is a fixed point of G_x (it follows from the definition that such a G_x is unique up to isomorphisms for each $x \in X$). We denote by $|G_x|$ the cardinal of G_x. If $|G_x| = 1$, then X has a smooth manifold structure in the neighborhood of x, which is called a smooth point of X. If $|G_x| > 1$, then X is not a smooth manifold in the neighborhood of x, which is called a singular point of X. We denote by $X_{\text{sing}} = \{x \in X; |G_x| > 1\}$ the singular set of X, and $X_{\text{reg}} = \{x \in X; |G_x| = 1\}$ the regular set of X.

It is useful to note that on an orbifold $(X, \mathcal{V})$ we can construct partitions of unity. First, let us call a function on X smooth, if its lift to any chart of the orbifold atlas $\mathcal{V}$ is smooth in the usual sense. Then the definition and construction of a smooth partition of unity associated to a locally finite covering carries over easily from the manifold case. The point is to construct smooth G_U-invariant functions with compact support on $(G_U, \widetilde{U})$.

Definition 3.3 (Orbifold Riemannian metric)**.** Let $(X, \mathcal{V})$ be an arbitrary orbifold. A *Riemannian metric* on X is a Riemannian metric g^{TX} on X_{reg} such that the lift of g^{TX} to any chart of the orbifold atlas $\mathcal{V}$ can be extended to a smooth Riemannian metric.

Certainly, for any $(G_U, \widetilde{U}) \in \mathcal{V}$, we can always construct a G_U-invariant Riemannian metric on $\widetilde{U}$. By a partition of unity argument, we see that there exist Riemannian metrics on the orbifold $(X, \mathcal{V})$.

Definition 3.4. An *orbifold vector bundle* E over an orbifold $(X, \mathcal{V})$ is defined as follows: E is an orbifold and for any $U \in \mathcal{U}$, $(G_U^E, \widetilde{p}_U : \widetilde{E}_U \to \widetilde{U})$ is a G_U^E-equivariant vector bundle and $(G_U^E, \widetilde{E}_U)$ (resp. $(G_U = G_U^E/K_U^E, \widetilde{U})$, where

$K_U^E = \ker(G_U^E \to \mathrm{Diffeo}(\widetilde{U}))$ is the orbifold structure of E (resp. X). If G_U^E acts effectively on $\widetilde{U}$ for $U \in \mathcal{U}$, i.e. $K_U^E = \{1\}$, we call E a proper orbifold vector bundle.

Note that any structure on X or E is locally G_x or $G_{U_x}^E$-equivariant.

Let E be an orbifold vector bundle on $(X, \mathcal{V})$. For $U \in \mathcal{U}$, let $\widetilde{E_U^{\mathrm{pr}}}$ be the maximal K_U^E-invariant sub-bundle of $\widetilde{E}_U$ on $\widetilde{U}$. Then $(G_U, \widetilde{E_U^{\mathrm{pr}}})$ defines a proper orbifold vector bundle on $(X, \mathcal{V})$, denoted by E^{pr}.

The (proper) orbifold tangent bundle TX on an orbifold X is defined by $(G_U, T\widetilde{U} \to \widetilde{U})$, for $U \in \mathcal{U}$. In the same vein we introduce the cotangent bundle T^*X. We can form tensor products of bundles by taking the tensor products of their local expressions in the charts of an orbifold atlas.

Let $E \to X$ be an orbifold vector bundle and $k \in \mathbb{N} \cup \{\infty\}$. A section $s : X \to E$ is called $\mathscr{C}^k$ if for each $U \in \mathcal{U}$, $s|_U$ is covered by a G_U^E-invariant $\mathscr{C}^k$ section $\widetilde{s}_U : \widetilde{U} \to \widetilde{E}_U$. We denote by $\mathscr{C}^k(X, E)$ the space of $\mathscr{C}^k$ sections of E on X.

Integration on orbifolds. If X is oriented, we define the integral $\int_X \alpha$ for a form α over X (i.e. a section of $\Lambda(T^*X)$ over X) as follows. If $\mathrm{supp}(\alpha) \subset U \in \mathcal{U}$ set

$$\int_X \alpha := \frac{1}{|G_U|} \int_{\widetilde{U}} \widetilde{\alpha}_U. \tag{3.2}$$

It is easy to see that the definition is independent of the chart. For general α we extend the definition by using a partition of unity.

If X is an oriented Riemannian orbifold, there exists a canonical volume element dv_X on X, which is a section of $\Lambda^m(T^*X)$, $m = \dim X$. Hence, we can also integrate functions on X.

Metric structure on orbifolds. Assume now that the Riemannian orbifold $(X, \mathcal{V})$ is compact. We define a metric on X by setting for $x, y \in X$,

$$\begin{aligned} d(x,y) = \mathrm{Inf}_\gamma \Big\{ \textstyle\sum_i \int_{t_{i-1}}^{t_i} |\frac{\partial}{\partial t}\widetilde{\gamma}_i(t)| dt \Big| & \gamma : [0,1] \to X, \gamma(0) = x, \gamma(1) = y, \\ & \text{such that there exist } t_0 = 0 < t_1 < \cdots < t_k = 1, \gamma([t_{i-1}, t_i]) \subset U_i, \\ & U_i \in \mathcal{U}, \text{ and a } \mathscr{C}^\infty \text{ map } \widetilde{\gamma}_i : [t_{i-1}, t_i] \to \widetilde{U}_i \text{ that covers } \gamma|_{[t_{i-1}, t_i]} \Big\}. \end{aligned}$$

Then (X, d) is a metric space. For $x \in X$, set $d(x, X_{\mathrm{sing}}) := \inf_{y \in X_{\mathrm{sing}}} d(x, y)$.

Kernels on orbifolds. Let us discuss briefly kernels and operators on orbifolds. For any open set $U \subset X$ and orbifold chart $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$, we will add a superscript $\sim$ to indicate the corresponding objects on $\widetilde{U}$. Assume that $\widetilde{\mathcal{K}}(\widetilde{x}, \widetilde{x}') \in \mathscr{C}^\infty(\widetilde{U} \times \widetilde{U}, \pi_1^* \widetilde{E} \otimes \pi_2^* \widetilde{E}^*)$ verifies

$$(g,1)\widetilde{\mathcal{K}}(g^{-1}\widetilde{x}, \widetilde{x}') = (1, g^{-1})\widetilde{\mathcal{K}}(\widetilde{x}, g\widetilde{x}') \quad \text{for any } g \in G_U, \tag{3.3}$$

where (g_1, g_2) acts on $\widetilde{E}_{\widetilde{x}} \times \widetilde{E}^*_{\widetilde{x}'}$ by $(g_1, g_2)(\xi_1, \xi_2) = (g_1\xi_1, g_2\xi_2)$.

We define the operator $\widetilde{\mathcal{K}} : \mathscr{C}^\infty_0(\widetilde{U}, \widetilde{E}) \to \mathscr{C}^\infty(\widetilde{U}, \widetilde{E})$ by

$$(\widetilde{\mathcal{K}}\,\widetilde{s})(\widetilde{x}) = \int_{\widetilde{U}} \widetilde{\mathcal{K}}(\widetilde{x}, \widetilde{x}')\widetilde{s}(\widetilde{x}')dv_{\widetilde{U}}(\widetilde{x}') \quad \text{for } \widetilde{s} \in \mathscr{C}^\infty_0(\widetilde{U}, \widetilde{E})\,. \tag{3.4}$$

For $\widetilde{s} \in \mathscr{C}^\infty(\widetilde{U}, \widetilde{E})$ and $g \in G_U$, g acts on $\mathscr{C}^\infty(\widetilde{U}, \widetilde{E})$ by: $(g \cdot \widetilde{s})(\widetilde{x}) := g \cdot \widetilde{s}(g^{-1}\widetilde{x})$. We can then identify an element $s \in \mathscr{C}^\infty(U, E)$ with an element $\widetilde{s} \in \mathscr{C}^\infty(\widetilde{U}, \widetilde{E})$ verifying $g \cdot \widetilde{s} = \widetilde{s}$ for any $g \in G_U$.

With this identification, we define the operator $\mathcal{K} : \mathscr{C}^\infty_0(U, E) \to \mathscr{C}^\infty(U, E)$ by

$$(\mathcal{K}s)(x) = \frac{1}{|G_U|} \int_{\widetilde{U}} \widetilde{\mathcal{K}}(\widetilde{x}, \widetilde{x}')\widetilde{s}(\widetilde{x}')dv_{\widetilde{U}}(\widetilde{x}') \quad \text{for } s \in \mathscr{C}^\infty_0(U, E)\,, \tag{3.5}$$

where $\widetilde{x} \in \tau_U^{-1}(x)$. Then the smooth kernel $\mathcal{K}(x, x')$ of the operator $\mathcal{K}$ with respect to dv_X is

$$\mathcal{K}(x, x') = \sum_{g \in G_U} (g, 1)\widetilde{\mathcal{K}}(g^{-1}\widetilde{x}, \widetilde{x}'). \tag{3.6}$$

Let $\mathcal{K}_1, \mathcal{K}_2$ be two operators as above and assume that the kernel of one of $\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2$ has compact support. By (3.2), (3.3) and (3.5), the kernel of $\mathcal{K}_1 \circ \mathcal{K}_2$ is given by

$$(\mathcal{K}_1 \circ \mathcal{K}_2)(x, x') = \sum_{g \in G_U} (g, 1)(\widetilde{\mathcal{K}}_1 \circ \widetilde{\mathcal{K}}_2)(g^{-1}\widetilde{x}, \widetilde{x}'). \tag{3.7}$$

3.2 Bergman kernel on Kähler orbifolds

In this section we study the asymptotics of the Bergman kernel on orbifolds.

Dolbeault cohomology of orbifolds. Let X be a compact complex orbifold of complex dimension n with complex structure J. Let E be a holomorphic orbifold vector bundle on X.

Let $\mathscr{O}_X$ be the sheaf over X of local G_U-invariant holomorphic functions over $\widetilde{U}$, for $U \in \mathcal{U}$. The local G^E_U -invariant holomorphic sections of $\widetilde{E} \to \widetilde{U}$ define a sheaf $\mathscr{O}_X(E)$ over X. Let $H^\bullet(X, \mathscr{O}_X(E))$ be the cohomology of the sheaf $\mathscr{O}_X(E)$ over X. Notice that by Definition, we have $\mathscr{O}_X(E) = \mathscr{O}_X(E^{\rm pr})$. Thus without lost generality, we may and will assume that E is a proper orbifold vector bundle on X.

Consider a section $s \in \mathscr{C}^\infty(X, E)$ and a local section $\widetilde{s} \in \mathscr{C}^\infty(\widetilde{U}, \widetilde{E}_U)$ covering s. Then $\overline{\partial}^{\widetilde{E}_U}\widetilde{s}$ covers a section of $T^{*(0,1)}X \otimes E$ over U, denoted $\overline{\partial}^E s|_U$. The family of sections $\{\overline{\partial}^E s|_U \ : \ U \in \mathcal{U}\}$ patch together to define a global section $\overline{\partial}^E s$ of $T^{*(0,1)}X \otimes E$ over X. In a similar manner we define $\overline{\partial}^E\alpha$ for a $\mathscr{C}^\infty$ section α of $\Lambda(T^{*(0,1)}X) \otimes E$ over X. We obtain thus the Dolbeault complex $(\Omega^{0,\bullet}(X, E), \overline{\partial}^E)$:

$$0 \longrightarrow \Omega^{0,0}(X, E) \xrightarrow{\overline{\partial}^E} \cdots \xrightarrow{\overline{\partial}^E} \Omega^{0,n}(X, E) \longrightarrow 0. \tag{3.8}$$

From the abstract de Rham theorem there exists a canonical isomorphism

$$H^\bullet(\Omega^{0,\bullet}(X,E),\overline{\partial}^E)\simeq H^\bullet(X,\mathscr{O}_X(E)). \tag{3.9}$$

In the sequel, we also denote $H^\bullet(X,\mathscr{O}_X(E))$ by $H^\bullet(X,E)$.

Prequantum line bundles. We consider a complex orbifold (X,J) endowed with the complex structure J. Let g^{TX} be a Riemannian metric on TX compatible with J. There is then an associated $(1,1)$-form Θ given by $\Theta(U,V)=g^{TX}(JU,V)$. The metric g^{TX} is called a Kähler metric and the orbifold (X,J) is called a *Kähler orbifold* if Θ is a closed form, that is, $d\Theta=0$. In this case Θ is a symplectic form, called Kähler form. We will denote the Kähler orbifold by (X,J,Θ) or shortly by (X,Θ).

Let (L,h^L) be a holomorphic Hermitian proper orbifold line bundle on an orbifold X, and let (E,h^E) be a holomorphic Hermitian proper orbifold vector bundle on X.

We assume that the associated curvature R^L of (L,h^L) verifies (2.14), i.e., (L,h^L) is a positive proper orbifold line bundle on X. This implies that $\omega:=\frac{\sqrt{-1}}{2\pi}R^L$ is a Kähler form on X, (X,ω) is a Kähler orbifold and (L,h^L,∇^L) is a prequantum line bundle on (X,ω).

Note that the existence of a positive line bundle L on a compact complex orbifold X implies that the Kodaira map associated to high powers of L gives a holomorphic embedding of X in the projective space. This is the generalization due to Baily of the Kodaira embedding theorem (see e.g. [33, Theorem 5.4.20]).

Hodge theory. Let $g^{TX}=\omega(\cdot,J\cdot)$ be the Riemannian metric on X induced by $\omega=\frac{\sqrt{-1}}{2\pi}R^L$. Using the Hermitian product along the fibers of L^p, E, $\Lambda(T^{*(0,1)}X)$, the Riemannian volume form dv_X and the definition (3.2) of the integral on an orbifold, we introduce an L^2-Hermitian product on $\Omega^{0,\bullet}(X,L^p\otimes E)$ similar to (2.1). This allows to define the formal adjoint $\overline{\partial}^{L^p\otimes E,*}$ of $\overline{\partial}^{L^p\otimes E}$ and the operators D_p and $\square_p$ as in (2.13). Then D_p^2 preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X,L^p\otimes E)$. We note that Hodge theory extends to compact orbifolds and delivers a canonical isomorphism

$$H^q(X,L^p\otimes E)\simeq \operatorname{Ker}(D_p^2|_{\Omega^{0,q}}). \tag{3.10}$$

Spectral gap. By the same proof as in [31, Theorems 1.1, 2.5], [6, Theorem 1], we get vanishing results and the spectral gap property.

Theorem 3.5. *Let (X,ω) be a compact Kähler orbifold, (L,h^L) be a prequantum holomorphic Hermitian proper orbifold line bundle on (X,ω) and (E,h^E) be an arbitrary holomorphic Hermitian proper orbifold vector bundle on X.*
Then there exists $C>0$ such that for any $p\in\mathbb{N}$

$$\operatorname{Spec}(D_p^2)\subset\{0\}\cup\,]4\pi p-C,+\infty[, \tag{3.11}$$

and $D_p^2|_{\Omega^{0,>0}}$ is invertible for p large enough. Consequently, we have the Kodaira-Serre vanishing theorem, namely, for p large enough,

$$H^q(X, L^p \otimes E) = 0\,, \quad \text{for every } q > 0. \tag{3.12}$$

Bergman kernel. As in §2.1, we define the Bergman kernel as the smooth kernel with respect to the Riemannian volume form $dv_X(x')$ of the orthogonal projection (Bergman projection) P_p from $\mathscr{C}^\infty(X, L^p \otimes E)$ onto $H^0(X, L^p \otimes E)$.

Let $d_p = \dim H^0(X, L^p \otimes E)$ and consider an arbitrary orthonormal basis $\{S_i^p\}_{i=1}^{d_p}$ of $H^0(X, L^p \otimes E)$ with respect to the Hermitian product (2.1) and (3.2). In fact, in the local coordinate above, $\widetilde{S}_i^p(\widetilde{z})$ are G_x-invariant on $\widetilde{U}_x$, and

$$P_p(y, y') = \sum_{i=1}^{d_p} \widetilde{S}_i^p(\widetilde{y}) \otimes (\widetilde{S}_i^p(\widetilde{y}'))^*, \tag{3.13}$$

where we use $\widetilde{y}$ to denote the point in $\widetilde{U}_x$ representing $y \in U_x$.

Asymptotics of the Bergman kernel. The Bergman kernel on orbifolds has an asymptotic expansion, which we now describe. We follow the same pattern as in the smooth case. The spectral gap property (3.11) shows that we have the analogue of Theorem 2.13, with the same F as given in (2.41):

$$|P_p(x, x') - F(D_p)(x, x')|_{\mathscr{C}^m(X \times X)} \leqslant C_{l,m,\varepsilon} p^{-l}. \tag{3.14}$$

As pointed out in [29], the property of the finite propagation speed of solutions of hyperbolic equations still holds on an orbifold (see the proof in [33, Appendix D.2]). Thus $F(D_p)(x, x') = 0$ for every for $x, x' \in X$ satisfying $d(x, x') \geqslant \varepsilon$. Likewise, given $x \in X$, $F(D_p)(x, \cdot)$ only depends on the restriction of D_p to $B^X(x, \varepsilon)$. Thus the problem of the asymptotic expansion of $P_p(x, \cdot)$ is local.

For any compact set $K \subset X_{\mathrm{reg}}$, the Bergman kernel $P_p(x, x')$ has an asymptotic expansion as in Theorem 2.14 by the same argument as in Theorem 2.13.

Let now $x \in X_{\mathrm{sing}}$ and let $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$ be an orbifold chart near x. We recall that for every open set $U \subset X$ and orbifold chart $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$, we add a superscript $\widetilde{\ }$ to indicate the corresponding objects on $\widetilde{U}$. Let $\partial U = \overline{U} \setminus U$, $U_1 = \{x \in U, d(x, \partial U) \geqslant \varepsilon\}$. Then $F(\widetilde{D}_p)(\widetilde{x}, \widetilde{x}')$ is well defined for $\widetilde{x}, \widetilde{x}' \in \widetilde{U}_1 = \tau_U^{-1}(U_1)$. Since $g \cdot F(\widetilde{D}_p) = F(\widetilde{D}_p) g$, we get

$$(g, 1) F(\widetilde{D}_p)(g^{-1}\widetilde{x}, \widetilde{x}') = (1, g^{-1}) F(\widetilde{D}_p)(\widetilde{x}, g\widetilde{x}')\,, \tag{3.15}$$

for every $g \in G_U$, $\widetilde{x}, \widetilde{x}' \in \widetilde{U}_1$. Formula (3.6) shows that for every $x, x' \in U_1$ and $\widetilde{x}, \widetilde{x}' \in \widetilde{U}_1$ representing x, x', we have

$$F(D_p)(x, x') = \sum_{g \in G_U} (g, 1) F(\widetilde{D}_p)(g^{-1}\widetilde{x}, \widetilde{x}'). \tag{3.16}$$

In view of (3.16), the strategy is to use the expansion for $F(\widetilde{D}_p)(\cdot,\cdot)$ in order to deduce the expansion for $F(D_p)(\cdot,\cdot)$ and then for $P_p(\cdot,\cdot)$, due to (3.14). In the present situation the kernel $\mathscr{P}$ takes the form

$$\mathscr{P}(\widetilde{Z},\widetilde{Z}') = \exp\Big(-\frac{\pi}{2}\sum_i \big(|\widetilde{z}_i|^2 + |\widetilde{z}'_i|^2 - 2\widetilde{z}_i\overline{\widetilde{z}}'_i\big)\Big). \tag{3.17}$$

For details we refer to [33, § 5.4.3].

3.3 Berezin-Toeplitz quantization on Kähler orbifolds

We apply now the results of Section 3.2 to establish the Berezin-Toeplitz quantization on Kähler orbifolds. We use the notations and assumptions of that Section.

Toeplitz operators on orbifolds. We define Toeplitz operators as a family $\{T_p\}$ of linear operators $T_p : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E)$ satisfying the conditions from Definition 2.4.

For any section $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the *Berezin-Toeplitz quantization* of f is defined by

$$T_{f,\,p} : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E)\,, \quad T_{f,\,p} = P_p\, f\, P_p\,. \tag{3.18}$$

Now, by the same argument as in Lemma 2.18, we get

Lemma 3.6. *For any $\varepsilon > 0$ and any $l, m \in \mathbb{N}$ there exists $C_{l,m,\varepsilon} > 0$ such that*

$$|T_{f,\,p}(x,x')|_{\mathscr{C}^m(X\times X)} \leqslant C_{l,m,\varepsilon}\, p^{-l} \tag{3.19}$$

for all $p \geqslant 1$ and all $(x,x') \in X \times X$ with $d(x,x') > \varepsilon$, where the $\mathscr{C}^m$-norm is induced by ∇^L, ∇^E and h^L, h^E, g^{TX}.

As in Section 2.5 we obtain next the asymptotic expansion of the kernel $T_{f,\,p}(x,x')$ in a neighborhood of the diagonal.

We need to introduce the appropriate analogue of the condition introduced in the Notation 2.12 in the orbifold case, in order to take into account the group action associated to an orbifold chart. Let $\{\Theta_p\}_{p\in\mathbb{N}}$ be a sequence of linear operators $\Theta_p : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E)$ with smooth kernel $\Theta_p(x,y)$ with respect to $dv_X(y)$.

Condition 3.7. Let $k \in \mathbb{N}$. Assume that for every open set $U \in \mathcal{U}$ and every orbifold chart $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$, there exists a sequence of kernels $\{\widetilde{\Theta}_{p,U}(\widetilde{x},\widetilde{x}')\}_{p\in\mathbb{N}}$ and a family $\{Q_{r,\,x_0}\}_{0\leqslant r\leqslant k,\, x_0\in X}$ such that

(a) $Q_{r,\,x_0} \in \mathrm{End}(E)_{x_0}[\widetilde{Z},\widetilde{Z}']\,,$

(b) $\{Q_{r,\,x_0}\}_{r\in\mathbb{N},\, x_0\in X}$ is smooth with respect to the parameter $x_0 \in X$,

(c) for every fixed $\varepsilon'' > 0$ and every $\widetilde{x}, \widetilde{x}' \in \widetilde{U}$ the following holds

(3.20)
$$\begin{aligned}
&(g,1)\widetilde{\Theta}_{p,U}(g^{-1}\widetilde{x},\widetilde{x}') = (1,g^{-1})\widetilde{\Theta}_{p,U}(\widetilde{x},g\widetilde{x}') \quad \text{for any} \quad g \in G_U \ (\text{cf. } (3.15)),\\
&\widetilde{\Theta}_{p,U}(\widetilde{x},\widetilde{x}') = \mathscr{O}(p^{-\infty}) \qquad \text{for } d(x,x') > \varepsilon'',\\
&\Theta_p(x,x') = \sum_{g\in G_U} (g,1)\widetilde{\Theta}_{p,U}(g^{-1}\widetilde{x},\widetilde{x}') + \mathscr{O}(p^{-\infty}),
\end{aligned}$$

and moreover, for every relatively compact open subset $\widetilde{V} \subset \widetilde{U}$, the relation (3.21)
$$p^{-n}\,\widetilde{\Theta}_{p,U,\widetilde{x}_0}(\widetilde{Z},\widetilde{Z}') \cong \sum_{r=0}^{k} (Q_{r,\widetilde{x}_0}\mathscr{P}_{\widetilde{x}_0})(\sqrt{p}\widetilde{Z},\sqrt{p}\widetilde{Z}')p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}), \text{ for } \widetilde{x}_0 \in \widetilde{V},$$

holds in the sense of (2.38).

Notation 3.8. If the sequence $\{\Theta_p\}_{p\in\mathbb{N}}$ satisfies Condition 3.7, we write

$$p^{-n}\,\Theta_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k} (Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}})\,. \tag{3.22}$$

Note that although the Notations 3.8 and 2.12 are formally similar, they have different meaning.

Lemma 3.9. *The smooth family* $Q_{r,x_0} \in \mathrm{End}(E)_{x_0}[\widetilde{Z},\widetilde{Z}']$ *in Condition 3.7 is uniquely determined by* Θ_p.

Proof. Clearly, for $W \subset U$, the restriction of $\widetilde{\Theta}_{p,U}$ to $\widetilde{W} \times \widetilde{W}$ verifies (3.20), thus we can take $\widetilde{\Theta}_{p,W} = \widetilde{\Theta}_{p,U}|_{\widetilde{W}\times\widetilde{W}}$. Since G_U acts freely on $\tau_U^{-1}(U_{\mathrm{reg}}) \subset \widetilde{U}$, we deduce from (3.20) and (3.21) that

$$\Theta_{p,x_0}(Z,Z') = \widetilde{\Theta}_{p,U,\widetilde{x}_0}(\widetilde{Z},\widetilde{Z}') + \mathscr{O}(p^{-\infty})\,, \tag{3.23}$$

for every $x_0 \in U_{\mathrm{reg}}$ and $|\widetilde{Z}|, |\widetilde{Z}'|$ small enough. We infer from (3.21) and (3.23) that $Q_{r,x_0} \in \mathrm{End}(E)_{x_0}[\widetilde{Z},\widetilde{Z}']$ is uniquely determined for $x_0 \in X_{\mathrm{reg}}$. Since Q_{r,x_0} depends smoothly on x_0, its lift to $\widetilde{U}$ is smooth. Since the set $\tau_U^{-1}(U_{\mathrm{reg}})$ is dense in $\widetilde{U}$, we see that the smooth family Q_{r,x_0} is uniquely determined by Θ_p. □

Lemma 3.10. *There exist polynomials* $J_{r,x_0}, Q_{r,x_0}(f) \in \mathrm{End}(E)_{x_0}[\widetilde{Z},\widetilde{Z}']$ *so that Theorem 2.14, Lemmas 2.18, 2.19 and* (2.78) *still hold under the notation* (3.22). *Moreover,*

$$J_{0,x_0} = \mathrm{Id}_E, \quad J_{1,x_0} = 0. \tag{3.24}$$

Proof. The analogues of Theorems 2.13-2.14 for the current situation and (3.15), (3.16) show that Theorem 2.14 and Lemmas 2.18, 2.19 still hold under the notation (3.22). By (2.49), we have $\mathcal{O}_1 = 0$. Hence (2.52) entails (3.24). Moreover, (3.14) implies

$$T_{f,p}(x,x') = \int_X F(D_p)(x,x'')f(x'')F(D_p)(x'',x')dv_X(x'') + \mathscr{O}(p^{-\infty}). \tag{3.25}$$

Therefore, we deduce from (3.7), (3.15), (3.16) and (3.25) that Lemmas 2.19 and (2.78) still hold under the notation (3.22). □

We have therefore orbifold asymptotic expansions for the Bergman and Toeplitz kernels, analogues to those for smooth manifolds. Following the strategy used in §2.6 we can prove a characterization of Toeplitz operators as in Theorem 2.22 (see [35, Th. 6.11]).

Proceeding as in §2.6 we can show that the set of Toeplitz operators on a compact orbifold is closed under the composition of operators, so forms an algebra.

Theorem 3.11 ([35, Th. 6.13]). *Let (X,ω) be a compact Kähler orbifold and let (L,h^L) be a holomorphic Hermitian proper orbifold line bundle satisfying the prequantization condition* (2.14). *Let (E,h^E) be an arbitrary holomorphic Hermitian proper orbifold vector bundle on X.*

Consider $f,g \in \mathscr{C}^\infty(X,\mathrm{End}(E))$. Then the product of the Toeplitz operators $T_{f,p}$ and $T_{g,p}$ is a Toeplitz operator, more precisely, it admits an asymptotic expansion in the sense of (2.11), *where $C_r(f,g) \in \mathscr{C}^\infty(X,\mathrm{End}(E))$ and C_r are bi-differential operators defined locally as in* (2.93) *on each covering $\widetilde{U}$ of an orbifold chart $(G_U,\widetilde{U}) \xrightarrow{\tau_U} U$. In particular $C_0(f,g) = fg$.*

If $f,g \in \mathscr{C}^\infty(X)$, then (2.94) *holds.*

Relation (2.102) *also holds for any $f \in \mathscr{C}^\infty(X,\mathrm{End}(E))$.*

Remark 3.12. As in Remark 2.27, Theorem 3.11 shows that on every compact Kähler orbifold X admitting a prequantum line bundle (L,h^L), we can define in a canonical way an associative star-product

$$f *_\hbar g = \sum_{l=0}^{\infty} \hbar^l C_l(f,g) \in \mathscr{C}^\infty(X,\mathrm{End}(E))[[\hbar]] \tag{3.26}$$

for every $f,g \in \mathscr{C}^\infty(X,\mathrm{End}(E))$, called the *Berezin-Toeplitz star-product*. Moreover, $C_l(f,g)$ are bi-differential operators defined locally as in the smooth case.

4 Quantization of symplectic manifolds

We will briefly describe in this Section how to generalize the ideas used before in the Kähler case in order to study the Toeplitz operators and Berezin-Toeplitz quantization for symplectic manifolds. For details we refer the reader to [33, 35]. We recall in Section 4.1 the definition of the spin^c Dirac operator and formulate the spectral gap property for prequantum line bundles. In Section 4.2 we state the asymptotic expansion of the composition of Toeplitz operators.

4.1 Spectral gap of the spin^c Dirac operator

We will first show that in the general symplectic case the kernel of the spin^c operator is a good substitute for the space of holomorphic sections used in Kähler quantization.

Let (X, ω) be a compact symplectic manifold, $\dim_{\mathbb{R}} X = 2n$, with compatible almost complex structure $J : TX \to TX$. Let g^{TX} be the associated Riemannian metric compatible with ω, i.e., $g^{TX}(u, v) = \omega(u, Jv)$. Let $(L, h^L, \nabla^L) \to X$ be Hermitian line bundle, endowed with a Hermitian metric h^L and a Hermitian connection ∇^L, whose curvature is $R^L = (\nabla^L)^2$. We assume that the *prequantization condition* (2.14) is fulfilled. Let $(E, h^E, \nabla^E) \to X$ be a Hermitian vector bundle. We will be concerned with asymptotics in terms of high tensor powers $L^p \otimes E$, when $p \to \infty$, that is, we consider the semi-classical limit $\hbar = 1/p \to 0$.

Let $\nabla^{\det}$ be the connection on $\det(T^{(1,0)}X)$ induced by the projection of the Levi-Civita connection ∇^{TX} on $T^{(1,0)}X$. Let us consider the Clifford connection ∇^{Cliff} on $\Lambda^\bullet(T^{*(0,1)}X)$ associated to ∇^{TX} and to the connection $\nabla^{\det}$ on $\det(T^{(1,0)}X)$ (see e.g. [33, § 1.3]). The connections ∇^L, ∇^E and ∇^{Cliff} induce the connection

$$\nabla_p = \nabla^{\text{Cliff}} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{L^p \otimes E} \quad \text{on } \Lambda^\bullet(T^{*(0,1)}X) \otimes L^p \otimes E.$$

The *spin^c Dirac operator* is defined by

$$D_p = \sum_{j=1}^{2n} \mathbf{c}(e_j) \nabla_{p,e_j} : \Omega^{0,\bullet}(X, L^p \otimes E) \longrightarrow \Omega^{0,\bullet}(X, L^p \otimes E)\,. \tag{4.1}$$

where $\{e_j\}_{j=1}^{2n}$ local orthonormal frame of TX and $\mathbf{c}(v) = \sqrt{2}(\overline{v}^*_{1,0} \wedge -i_{v_{0,1}})$ is the Clifford action of $v \in TX$. Here we use the decomposition $v = v_{1,0} + v_{0,1}$, $v_{1,0} \in T^{(1,0)}X$, $v_{0,1} \in T^{(0,1)}X$.

If (X, J, ω) is Kähler then $D_p = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$ so $\mathrm{Ker}(D_p) = H^0(X, L^p \otimes E)$ for $p \gg 1$. The following result shows that $\mathrm{Ker}(D_p)$ has all semi-classical properties of $H^0(X, L^p \otimes E)$. The proof is based on a direct application of the Lichnerowicz formula for D_p^2. Note that the metrics g^{TX}, h^L and h^E induce an L^2-scalar product on $\Omega^{0,\bullet}(X, L^p \otimes E)$, whose completion is denoted $(\Omega^{0,\bullet}_{(2)}(X, L^p \otimes E), \|\cdot\|_{L^2})$.

Theorem 4.1 ([31, Th. 1.1, 2.5], [33, Th. 1.5.5])**.** *There exists $C > 0$ such that for any $p \in \mathbb{N}$ and any $s \in \bigoplus_{k>0} \Omega^{0,k}(X, L^p \otimes E)$ we have*

$$\|D_p s\|^2_{L^2} \geq (4\pi p - C)\|s\|^2_{L^2}\,. \tag{4.2}$$

Moreover, the spectrum of D_p^2 verifies

$$\operatorname{Spec}(D_p^2) \subset \{0\} \cup [4\pi p - C, +\infty[\,. \tag{4.3}$$

By the Atiyah-Singer index theorem we have for $p \gg 1$

$$\dim \operatorname{Ker}(D_p) = \int_X \operatorname{Td}(T^{(1,0)}X)\operatorname{ch}(L^p \otimes E) = \operatorname{rk}(E)\,\frac{p^n}{n!}\int_X \omega^n + \mathcal{O}(p^{n-1})\,. \tag{4.4}$$

Theorem 4.1 shows the forms in $\operatorname{Ker}(D_p)$ concentrate asymptotically in the L^2 sense on their zero-degree component and (4.4) shows that $\dim \operatorname{Ker}(D_p)$ is a polynomial in p of degree n, as in the holomorphic case.

4.2 Toeplitz operators in spin^c quantization

Let us introduce the orthogonal projection $P_p : \Omega^{0,\bullet}_{(2)}(X, L^p \otimes E) \longrightarrow \operatorname{Ker}(D_p)$, called the Bergman projection in analogy to the Kähler case. Its integral kernel is called *Bergman kernel*. The *Toeplitz operator* with symbol $f \in \mathscr{C}^\infty(X, \operatorname{End}(E))$ is

$$T_{f,p} : \Omega^{0,\bullet}_{(2)}(X, L^p \otimes E) \to \Omega^{0,\bullet}_{(2)}(X, L^p \otimes E)\,, \quad T_{f,p} = P_p f P_p$$

In analogy to the Kähler case we define a (generalized) *Toeplitz operator* is a sequence (T_p) of linear operators $T_p \in \operatorname{End}(\Omega^{0,\bullet}_{(2)}(X, L^p \otimes E))$ verifying $T_p = P_p\, T_p\, P_p$, such that there exist a sequence $g_l \in \mathscr{C}^\infty(X, \operatorname{End}(E))$ with the property that for all $k \geq 0$, there exists $C_k > 0$ so that (2.11) is fulfilled.

A basic fact is that the Bergman kernel $P_p(\cdot,\cdot)$ of the Dirac operator has an asymptotic expansion similar to Theorems 2.13 and 2.14. This was shown by Dai-Liu-Ma in [15, Prop. 4.1 and Th. 4.18′] (see also [33, Th. 8.1.4]). By the Bergman kernel expansion of Dai-Liu-Ma we obtain the expansion of the integral kernels of $T_{f,p}$, similar to Theorem 2.19. Moreover, the characterization of Toeplitz operators in terms of the off-diagonal asymptotic expansion of their integral kernels, formulated in Theorem 2.22, holds also in the symplectic case (cf. [35, Th. 4.9], [33, Lemmas 7.2.2, 7.2.4, Th. 7.3.1]). We obtain thus the symplectic analogue of Theorem 2.24.

Theorem 4.2 ([35, Th. 1.1], [33, Th. 8.1.10])**.** *Let $f, g \in \mathscr{C}^\infty(X, \operatorname{End}(E))$. The composition $(T_{f,p} \circ T_{g,p})$ is a Toeplitz operator, i.e.,*

$$T_{f,p} \circ T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + \mathcal{O}(p^{-\infty}), \tag{4.5}$$

where C_r are bi-differential operators, $C_0(f,g) = fg$ and $C_r(f,g) \in \mathscr{C}^\infty(X, \operatorname{End}(E))$. Let $f, g \in \mathscr{C}^\infty(X)$ and let $\{\cdot,\cdot\}$ be the Poisson bracket on $(X, 2\pi\omega)$, defined as in (2.92). Then

$$C_1(f,g) - C_1(g,f) = \sqrt{-1}\{f,g\}\operatorname{Id}_E, \tag{4.6}$$

and therefore

$$[T_{f,p}\,, T_{g,p}] = \frac{\sqrt{-1}}{p} T_{\{f,g\},p} + \mathcal{O}(p^{-2}). \tag{4.7}$$

Thus the construction of the Berezin-Toeplitz star-product can be carried out also in the case of symplectic manifolds. Namely, for $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ we set $f *_\hbar g := \sum_{k=0}^\infty C_k(f,g)\hbar^k \in \mathscr{C}^\infty(X, \mathrm{End}(E))[[\hbar]]$, where $C_r(f,g)$ are determined by (4.5). Then $*_\hbar$ is an associative star product.

References

[1] S. T. Ali, M. Englis, *Quantization methods: a guide for physicists and analysts*, Rev. Math. Phys. **17** (2005), 391-490.

[2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantization*, Part I, Lett. Math. Phys. **1** (1977), 521–530; Part II and III, Ann. Phys. **111** (1978), 61–110, 111–151.

[3] F. A. Berezin, *Quantization*, Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 1116–1175.

[4] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundl. Math. Wiss. Band 298, Springer-Verlag, Berlin, 1992.

[5] J.-M. Bismut and G. Lebeau, *Complex immersions and Quillen metrics*, Inst. Hautes Études Sci. Publ. Math. (1991), no. 74, ii+298 pp.

[6] J.-M. Bismut and E. Vasserot, *The asymptotics of the Ray–Singer analytic torsion associated with high powers of a positive line bundle*, Comm. Math. Phys. **125** (1989), 355–367.

[7] M. Bordemann and E. Meinrenken and M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and* gl(N)*,* $N \to \infty$ *limits*, Comm. Math. Phys. **165** (1994), no. 2, 281–296.

[8] D. Borthwick, A. Lesniewski, and H. Upmeier, *Nonperturbative deformation quantization of Cartan domains*, J. Funct. Anal. **113** (1993), no. 1, 153–176.

[9] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, Annals of Mathematics Studies, vol. 99, Princeton University Press, Princeton, NJ, 1981.

[10] L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegö*, Astérisque, **34–35** (1976), 123–164.

[11] M. Cahen, S. Gutt, and J. Rawnsley, *Quantization of Kähler manifolds. I-IV*, Part I, J. Geom. Phys., **7**, 1990, 1, 45–62; Part II, Trans. Amer. Math. Soc., **337**, 1993, 1, 73–98; Part III, Lett. Math. Phys., **30**, 1994, 291–305; Part IV, Lett. Math. Phys., **34**, 1995, 2, 159–168.

[12] D. Catlin, *The Bergman kernel and a theorem of Tian*, Analysis and geometry in several complex variables (Katata, 1997), Trends Math., Birkhäuser Boston, Boston, MA, 1999, pp. 1– 23.

[13] L. Charles, *Berezin-Toeplitz operators, a semi-classical approach*, Comm. Math. Phys. **239** (2003), 1–28.

[14] L. A. Coburn, *Deformation estimates for the Berezin-Toeplitz quantization*, Comm. Math. Phys. **149** (1992), no. 2, 415–424.

[15] X. Dai, K. Liu, and X. Ma, *On the asymptotic expansion of Bergman kernel*, J. Differential Geom. **72** (2006), 1– 41; announced in C. R. Math. Acad. Sci. Paris **339** (2004), no. 3, 193–198.

[16] S. K. Donaldson, *Some numerical results in complex differential geometry*, Pure Appl. Math. Q. **5** (2009), No. 2, 571– 618.

[17] M. R. Douglas, S. Klevtsov, *Bergman kernel from path integral*, Comm. Math. Phys. **293** (2010), 205–230.

[18] ______, *Black holes and balanced metrics*, arXiv: 0811.0367.

[19] B. V. Fedosov, *Deformation quantization and index theory*, Mathematical Topics, 9, Akademie Verlag, Berlin, 1996.

[20] J. Fine, *Calabi flow and projective embeddings*, with an Appendix by K. Liu and X. Ma, J. Differential Geom. **84** (2010), 489-523.

[21] ______, *Quantisation and the Hessian of Mabuchi energy*, arXiv: 1009.4543.

[22] P. Halmos, V. S. Sunder, Bounded integral operators on L^2 spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, 96, Springer-Verlag, Berlin-New York, 1978.

[23] A. V. Karabegov and M. Schlichenmaier, *Identification of Berezin-Toeplitz deformation quantization*, J. Reine Angew. Math. **540** (2001), 49–76.

[24] S. Klimek and A. Lesniewski, *Quantum Riemann surfaces. I. The unit disc*, Comm. Math. Phys. **146** (1992), no. 1, 103–122.

[25] B. Kostant, *Quantization and unitary representations. I. Prequantization*, Lectures in modern analysis and applications, III, Springer, Berlin, 1970, pp. 87–208. Lecture Notes in Math., Vol. 170.

[26] K. Liu and X. Ma, *A remark on 'some numerical results in complex differential geometry'*, Math. Res. Lett. **14** (2007), no. 2, 165–171.

[27] ———, *Asymptotic of the operators Q_k*, (2008), Appendix to [20].

[28] Z. Lu, *On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch*, Amer. J. Math. **122** (2000), no. 2, 235–273.

[29] X. Ma, *Orbifolds and analytic torsions*, Trans. Amer. Math. Soc. **357** (2005), no. 6, 2205–2233.

[30] X. Ma, *Geometric quantization on Kähler and symplectic manifolds*, International Congress of Mathematicians, vol. II, Hyderabad, India, August 19-27 (2010), 785–810.

[31] X. Ma and G. Marinescu, *The spin^c Dirac operator on high tensor powers of a line bundle*, Math. Z. **240** (2002), no. 3, 651–664.

[32] ———, *The first coefficients of the asymptotic expansion of the Bergman kernel of the spin^c Dirac operator*, Internat. J. Math. **17** (2006), no. 6, 737–759.

[33] ———, *Holomorphic Morse inequalities and Bergman kernels*, Progress in Mathematics, vol. 254, Birkhäuser Boston Inc., Boston, MA, 2007, 422 pp.

[34] ———, *Generalized Bergman kernels on symplectic manifolds*, Adv. Math. **217** (2008), no. 4, 1756–1815; announced in C. R. Math. Acad. Sci. Paris **339** (2004), no. 7, 493–498.

[35] ———, *Toeplitz operators on symplectic manifolds*, J. Geom. Anal. **18** (2008), no. 2, 565–611.

[36] ———, *Berezin-Toeplitz quantization on Kähler manifolds*, arXiv:1009.4405, to appear in Journal de Crelle.

[37] C. Moreno and P. Ortega-Navarro, *Deformations of the algebra of functions on Hermitian symmetric spaces resulting from quantization*, Ann. Inst. H. Poincaré Sect. A (N.S.) **38** (1983), no. 3, 215–241.

[38] M. Schlichenmaier, *Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization*, Conférence Moshé Flato 1999, Vol. II (Dijon), Math. Phys. Stud., vol. 22, Kluwer Acad. Publ., Dordrecht, 2000, pp. 289–306.

[39] ———, *Berezin-Toeplitz Quantization for Compact Kähler Manifolds. A Review of Results*, Advances in Mathematical Physics, vol. 2010, Article ID 927280, 38 pages, 2010.

[40] J.-M. Souriau, *Structure des systèmes dynamiques*, Mâtrises de mathématiques, Dunod, Paris, 1970.

[41] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990), 99–130.

[42] M. E. Taylor, *Partial differential equations. 1: Basic theory*, Applied Mathematical Sciences, vol. 115, Springer-Verlag, Berlin, 1996.

[43] L. Wang, *Bergman kernel and stability of holomorphic vector bundles with sections*, MIT Ph.D. Dissertation (2003), 85 pages.

[44] X. Wang, *Canonical metrics on stable vector bundles*, Comm. Anal. Geom. **13** (2005), 253–285.

[45] A. Weil, *Introduction à l'étude des variétés kählériennes*, Actualités scientifiques et industrielles, vol. 1267, Hermann, Paris, 1958.

[46] S. Zelditch, *Szegö kernels and a theorem of Tian*, Internat. Math. Res. Notices (1998), 317–331.

Xiaonan Ma
Université Paris Diderot - Paris 7, UFR de Mathématiques,
Case 7012, Site Chevaleret, 75205 Paris Cedex 13, France
ma@math.jussieu.fr

George Marinescu
Universität zu Köln, Mathematisches Institut,
Weyertal 86-90, 50931 Köln, Germany
& Institute of Mathematics 'Simion Stoilow',
Romanian Academy, Bucharest, Romania
gmarines@math.uni-koeln.de

Travaux mathématiques, Volume 19 (2011), 167–201,

Asymptotics of Toeplitz operators and applications in TQFT

by Jørgen Ellegaard Andersen and Jakob Lindblad Blaavand

Abstract

In this paper we provide a review of asymptotic results of Toeplitz operators and their applications in TQFT. To do this we review the differential geometric construction of the Hitchin connection on a prequantizable compact symplectic manifold. We use asymptotic results relating the Hitchin connection and Toeplitz operators, to, in the special case of the moduli space of flat SU(n)-connections on a surface, prove asymptotic faithfulness of the SU(n) quantum representations of the mapping class group. We then go on to review formal Hitchin connections and formal trivializations of these. We discuss how these fit together to produce a Berezin–Toeplitz star product, which is independent of the complex structure. Finally we give explicit examples of all the above objects in the case of the abelian moduli space. We furthermore discuss an approach to curve operators in the TQFT associated to abelian Chern–Simons theory.

1 Introduction

Witten constructed, via path integral techniques, a quantization of Chern-Simons theory in $2+1$ dimensions, and he argued in [Wi] that this produced a TQFT, indexed by a compact simple Lie group and an integer level k. For the group SU(n) and level k, let us denote this TQFT by $Z_k^{(n)}$. Combinatorially, this theory was first constructed by Reshetikhin and Turaev, using representation theory of $U_q(\mathrm{sl}(n,\mathbb{C}))$ at $q = e^{(2\pi i)/(k+n)}$, in [RT1] and [RT2]. Subsequently, the TQFT's $Z_k^{(n)}$ were constructed using skein theory by Blanchet, Habegger, Masbaum and Vogel in [BHMV1], [BHMV2] and [B1].

The two-dimensional part of the TQFT $Z_k^{(n)}$ is a modular functor with a certain label set. For this TQFT, the label set $\Lambda_k^{(n)}$ is a finite subset (depending on k) of the set of finite dimensional irreducible representations of SU(n). We use the usual labeling of irreducible representations by Young diagrams, so in particular $\Box \in \Lambda_k^{(n)}$ is the defining representation of SU(n). Let further $\lambda_0^{(d)} \in \Lambda_k^{(n)}$ be the Young diagram consisting of d columns of length k. The label set is also equipped with an involution, which is simply induced by taking the dual representation. The trivial representation is a special element in the label set which is clearly preserved by the involution.

$$Z_k^{(n)} : \left\{ \begin{array}{l} \text{Category of (ex-} \\ \text{tended) closed} \\ \text{oriented sur-} \\ \text{faces with } \Lambda_k^{(n)}\text{-} \\ \text{labeled marked} \\ \text{points with pro-} \\ \text{jective tangent} \\ \text{vectors} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{Category of fi-} \\ \text{nite dimensional} \\ \text{vector spaces} \\ \text{over } \mathbb{C} \end{array} \right\}$$

The three-dimensional part of $Z_k^{(n)}$ is an association of a vector,

$$Z_k^{(n)}(M, L, \lambda) \in Z_k^{(n)}(\partial M, \partial L, \partial\lambda),$$

to any compact, oriented, framed 3–manifold M together with an oriented, framed link $(L, \partial L) \subseteq (M, \partial M)$ and a $\Lambda_k^{(n)}$-labeling $\lambda : \pi_0(L) \to \Lambda_k^{(n)}$.

This association has to satisfy the Atiyah-Segal-Witten TQFT axioms (see e.g. [At], [Se] and [Wi]). For a more comprehensive presentation of the axioms, see Turaev's book [T].

The geometric construction of these TQFTs was proposed by Witten in [Wi] where he derived, via the Hamiltonian approach to quantum Chern-Simons theory, that the geometric quantization of the moduli spaces of flat connections should give the two-dimensional part of the theory. Further, he proposed an alternative construction of the two-dimensional part of the theory via WZW-conformal field theory. This theory has been studied intensively. In particular, the work of Tsuchiya, Ueno and Yamada in [TUY] provided the major geometric constructions and results needed. In [BK], their results were used to show that the category of integrable highest weight modules of level k for the affine Lie algebra associated to any simple Lie algebra is a modular tensor category. Further, in [BK], this result is combined with the work of Kazhdan and Lusztig [KL] and the work of Finkelberg [Fi] to argue that this category is isomorphic to the modular tensor category associated to the corresponding quantum group, from which Reshetikhin and Turaev constructed their TQFT. Unfortunately, these results do not allow one to conclude the validity of the geometric constructions of the two-dimensional part of the TQFT proposed by Witten. However, in joint work with Ueno, [AU1], [AU2], [AU3] and [AU4], the first author have given a proof, based mainly on the results of [TUY], that the TUY-construction of the WZW-conformal field theory, after twist by a fractional power of an abelian theory, satisfies all the axioms of a modular functor. Furthermore, we have proved that the full $2+1$-dimensional TQFT resulting from this is isomorphic to the aforementioned one, constructed by BHMV via skein theory. Combining this with the theorem of Laszlo [La1], which identifies (projectively) the representations of the mapping class groups obtained from the geometric quantization of the moduli space of flat connections with the ones obtained from the TUY-constructions, one gets a proof of the validity of the construction proposed by Witten in [Wi].

Another part of this TQFT is the quantum $\mathrm{SU}(n)$ representations of the mapping class groups. Namely, if Σ is a closed oriented surfaces of genus g, Γ is the mapping class group of Σ, and p is a point on Σ, then the modular functor induces a representation

$$Z_k^{(n,d)} : \Gamma \to \mathbb{P}\operatorname{Aut}\big(Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)})\big). \tag{1.1}$$

For a general label of p, we would need to choose a projective tangent vector $v_p \in T_p\Sigma/\mathbb{R}_+$, and we would get a representation of the mapping class group of (Σ, p, v_p). But for the special labels $\lambda_0^{(d)}$, the dependence on v_p is trivial and in fact we get a representation of Γ.

Let us now briefly recall the geometric construction of the representations $Z_k^{(n,d)}$ of the mapping class group, as proposed by Witten, using geometric quantization of moduli spaces.

We assume from now on that the genus of the closed oriented surface Σ is at least two. Let M be the moduli space of flat $\mathrm{SU}(n)$ connections on $\Sigma - p$ with holonomy around p equal to $\exp(2\pi i d/n)\,\mathrm{Id} \in \mathrm{SU}(n)$. When (n,d) are coprime, the moduli space is smooth. In all cases, the smooth part of the moduli space has a natural symplectic structure ω. There is a natural smooth symplectic action of the mapping class group Γ of Σ on M. Moreover, there is a unique prequantum line bundle $(\mathcal{L}, \nabla, (\cdot,\cdot))$ over (M, ω). The Teichmüller space $\mathcal{T}$ of complex structures on Σ naturally, and Γ-equivariantly, parametrizes Kähler structures on (M, ω). For $\sigma \in \mathcal{T}$, we denote by M_σ the manifold (M, ω) with its corresponding Kähler structure. The complex structure on M_σ and the connection ∇ in $\mathcal{L}$ induce the structure of a holomorphic line bundle on $\mathcal{L}$. This holomorphic line bundle is simply the determinant line bundle over the moduli space, and it is an ample generator of the Picard group [DN].

By applying geometric quantization to the moduli space M, one gets, for any positive integer k, a certain finite rank bundle over Teichmüller space $\mathcal{T}$ which we will call the *Verlinde bundle* $\mathcal{V}^{(k)}$ at level k. The fiber of this bundle over a point $\sigma \in \mathcal{T}$ is $\mathcal{V}^{(k)}_\sigma = H^0(M_\sigma, \mathcal{L}^k)$. We observe that there is a natural Hermitian structure $\langle\cdot,\cdot\rangle$ on $H^0(M_\sigma, \mathcal{L}^k)$ by restricting the L_2-inner product on global L_2 sections of $\mathcal{L}^k$ to $H^0(M_\sigma, \mathcal{L}^k)$.

The main result pertaining to this bundle is:

Theorem 1.1 (Axelrod, Della Pietra and Witten; Hitchin)**.** *The projectivization of the bundle $\mathcal{V}^{(k)}$ supports a natural flat Γ-invariant connection $\hat{\nabla}$.*

This is a result proved independently by Axelrod, Della Pietra and Witten [ADW] and by Hitchin [H]. In section 2, we review our differential geometric construction of the connection $\hat{\nabla}$ in the general setting discussed in [A6]. We obtain as a corollary that the connection constructed by Axelrod, Della Pietra and Witten projectively agrees with Hitchin's.

Because of the existence of this connection, the 2-dimensional part of the modular functor $Z_k^{(n)}$ is the vector space $\mathbb{P}(V^{(k)})$ of covariant constant sections of $\mathbb{P}(\mathcal{V}^{(k)})$ over Teichmüller space $\mathcal{T}$.

Definition 1.2. We denote by $Z_k^{(n,d)}$ the representation,

$$Z_k^{(n,d)} : \Gamma \to \operatorname{Aut}\big(\mathbb{P}(V^{(k)})\big),$$

obtained from the action of the mapping class group on the covariant constant sections of $\mathbb{P}(\mathcal{V}^{(k)})$ over $\mathcal{T}$.

The projectively flat connection $\hat{\nabla}$ induces a *flat* connection $\hat{\nabla}^e$ in $\operatorname{End}(\mathcal{V}^{(k)})$. This flat connection can be used to show asymptotically flatness of the quantum representations $Z_k^{(n,d)}$,

Theorem 1.3 (Andersen [A3])**.** *Assume that $g \geq 2$, n and d are coprime or that $(n,d) = (2,0)$ when $g = 2$. Then, we have that*

$$\bigcap_{k=1}^{\infty} \ker(Z_k^{(n,d)}) = \begin{cases} \{1, H\} & g = 2,\ n = 2 \text{ and } d = 0 \\ \{1\} & \text{otherwise}, \end{cases}$$

where H is the hyperelliptic involution.

In Section 4 we discuss the proof of this Theorem, and how it relies on the asymptotics of *Toeplitz operators* $T_f^{(k)}$ associated a smooth function f on M. For each $f \in C^\infty(M)$ and each point $\sigma \in \mathcal{T}$ we have the Toeplitz operator,

$$T_{f,\sigma}^{(k)} : H^0(M_\sigma, \mathcal{L}_\sigma^k) \to H^0(M_\sigma, \mathcal{L}_\sigma^k),$$

which is given by

$$T_{f,\sigma}^{(k)} s = \pi_\sigma^{(k)}(fs)$$

for all $s \in H^0(M_\sigma, \mathcal{L}_\sigma^k)$. Here $\pi_\sigma^{(k)}$ is the orthogonal projection onto $H^0(M_\sigma, \mathcal{L}_\sigma^k)$ induced from the L_2-inner product on $C^\infty(M, \mathcal{L}^k)$. We get a smooth section of $\operatorname{End}(\mathcal{V}^{(k)})$,

$$T_f^{(k)} \in C^\infty(\mathcal{T}, \operatorname{End}(\mathcal{V}^{(k)})),$$

by letting $T_f^{(k)}(\sigma) = T_{f,\sigma}^{(k)}$. See Section 3 for a discussion of the Toeplitz operators and their connection to deformation quantization. The sections $T_f^{(k)}$ of $\operatorname{End}\mathcal{V}^{(k)}$ over $\mathcal{T}$ are not covariant constant with respect to $\hat{\nabla}^e$. However, they are asymptotically as k goes to infinity. This is made precise when we discuss the formal Hitchin Connection below.

The existence of a connection as above is not a unique thing for the moduli spaces, the construction can be generalized to a general compact prequantizable symplectic manifold (M, ω) with prequantum line bundle $(\mathcal{L}, (\cdot, \cdot), \nabla)$. We assume that $\mathcal{T}$ is a complex manifold which holomorphically and rigidly (see Definition 2.3) parameterizes Kähler structures on (M, ω). Then, the following theorem, proved in [A6], establishes the existence of the Hitchin connection (see Definition 2.4) under a mild cohomological condition.

Theorem 1.4 (Andersen)**.** *Suppose that I is a rigid family of Kähler structures on the compact, prequantizable symplectic manifold (M, ω) which satisfies that there exists an $n \in \mathbb{Z}$ such that the first Chern class of (M, ω) is $n[\frac{\omega}{2\pi}] \in H^2(M, \mathbb{Z})$ and $H^1(M, \mathbb{R}) = 0$. Then, the Hitchin connection $\hat{\nabla}$ in the trivial bundle $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$ preserves the subbundle $H^{(k)}$ with fibers $H^0(M_\sigma, \mathcal{L}^k)$. It is given by*

$$\hat{\nabla}_V = \hat{\nabla}^t_V + \frac{1}{4k+2n} \left\{ \Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} + 4kV'[F] \right\},$$

where $\hat{\nabla}^t$ is the trivial connection in $\mathcal{H}^{(k)}$, and V is any smooth vector field on $\mathcal{T}$.

This result is discussed in much greater detail in Section 2, where all ingredients are introduced.

In Section 5, we study the formal Hitchin connection which was introduced in [A6]. Let $\mathcal{D}(M)$ be the space of smooth differential operators on M acting on smooth functions on M. Let $\mathbb{C}_h$ be the trivial $C^\infty_h(M)$-bundle over $\mathcal{T}$, where $C^\infty_h(M)$ is formal power series with coefficients in $C^\infty(M)$.

Definition 1.5. A formal connection D is a connection in $\mathbb{C}_h$ over $\mathcal{T}$ of the form

$$D_V f = V[f] + \tilde{D}(V)(f),$$

where $\tilde{D}$ is a smooth one-form on $\mathcal{T}$ with values in $\mathcal{D}_h(M) = \mathcal{D}(M)[[h]]$, f is any smooth section of $\mathbb{C}_h$, V is any smooth vector field on $\mathcal{T}$ and $V[f]$ is the derivative of f in the direction of V.

Thus, a formal connection is given by a formal series of differential operators

$$\tilde{D}(V) = \sum_{l=0}^{\infty} \tilde{D}^{(l)}(V) h^l.$$

From Hitchin's connection in $H^{(k)}$, we get an induced connection $\hat{\nabla}^e$ in the endomorphism bundle $\mathrm{End}(H^{(k)})$. As previously mentioned, the Toeplitz operators are not covariant constant sections with respect to $\hat{\nabla}^e$, but asymptotically in k they are. This follows from the properties of the formal Hitchin connection, which is the formal connection D defined through the following theorem (proved in [A6]).

Theorem 1.6. *(Andersen) There is a unique formal connection D which satisfies that*

$$\hat{\nabla}^e_V T^{(k)}_f \sim T^{(k)}_{(D_V f)(1/(k+n/2))} \tag{1.2}$$

for all smooth section f of $\mathbb{C}_h$ and all smooth vector fields V on $\mathcal{T}$. Moreover,

$$\tilde{D} = 0 \mod h.$$

Here $\sim$ means the following: For all $L \in \mathbb{Z}_+$ we have that

$$\left\| \hat{\nabla}^e_V T^{(k)}_f - \left(T^{(k)}_{V[f]} + \sum_{l=1}^{L} T^{(k)}_{\tilde{D}^{(l)}_V f} \frac{1}{(k+n/2)^l} \right) \right\| = O(k^{-(L+1)}),$$

uniformly over compact subsets of $\mathcal{T}$, for all smooth maps $f : \mathcal{T} \to C^\infty(M)$.

Now fix an $f \in C^\infty(M)$, which does not depend on $\sigma \in \mathcal{T}$, and notice how the fact that $\tilde{D} = 0$ mod h implies that

$$\left\| \hat{\nabla}^e_V T^{(k)}_f \right\| = O(k^{-1}).$$

This expresses the fact that the Toeplitz operators are asymptotically flat with respect to the Hitchin connection.

We define a mapping class group equivariant formal trivialization of D as follows.

Definition 1.7. A formal trivialization of a formal connection D is a smooth map $P : \mathcal{T} \to \mathcal{D}_h(M)$ which modulo h is the identity, for all $\sigma \in \mathcal{T}$, and which satisfies

$$D_V(P(f)) = 0,$$

for all vector fields V on $\mathcal{T}$ and all $f \in C^\infty_h(M)$. Such a formal trivialization is mapping class group equivariant if $P(\phi(\sigma)) = \phi^* P(\sigma)$ for all $\sigma \in \mathcal{T}$ and $\phi \in \Gamma$.

Since the only mapping class group invariant functions on the moduli space are the constant ones (see [Go1]), we see that in the case where M is the moduli space, such a P, if it exists, must be unique up to multiplication by a formal constant, i.e. an element of $\mathbb{C}_h = \mathbb{C}[[h]]$.

Clearly if D is not flat, such a formal trivialization cannot exist even locally on $\mathcal{T}$. However, if D is flat and its zero-order term is just given by the trivial connection in C_h, then a local formal trivialization exists, as proved in [A6].

Furthermore, it is proved in [A6] that flatness of the formal Hitchin connection is implied by projective flatness of the Hitchin connection. As was proved by Hitchin in [H], and stated above in Theorem 1.1, this is the case when M is the moduli space.

In Section 5 we discuss how this formal trivialization of a formal connection give a way of defining a star product from the Berezin–Topelitz star product, which turn out not to depend on the complex structure σ. In Section 5 we furthermore discuss the lower order terms of formal trivialization and the star product.

In Section 6 we consider the all of the above objects in the case where the manifold M is a principal polarized abelian variety. We furthermore discuss abelian Chern–Simons theory and the moduli space of $U(1)$-connections on a closed surface Σ. We find a flat Hitchin connection on the $U(1)$-moduli space M and find a formal trivialization P of the formal Hitchin connection. With this formal trivialization we define the curve operators to a cylinder $\Sigma \times [0,1]$ with a link γ inside, to be the Toeplitz operator associated to the corresponding holonomy function h_γ on M, $Z^{(k)} = T^{(k)}_{h_\gamma}$. With this definition of a curve operator we show that

$$\langle h_{\gamma_1}, h_{\gamma_2} \rangle = \lim_{k\to\infty} \left\langle Z^{(k)}(\Sigma,\gamma_1), Z^{(k)}(\Sigma,\gamma_2) \right\rangle$$

and as required by the TQFT axioms that

$$Z^{(k)}(\Sigma \times S^1) = \dim(Z^{(k)}(\Sigma)).$$

2 The Hitchin connection

In this section, we review our construction of the Hitchin connection using the global differential geometric setting of [A6]. This approach is close in spirit to Axelrod, Della Pietra and Witten's in [ADW], however we do not use any infinite dimensional gauge theory. In fact, the setting is more general than the gauge theory setting in which Hitchin in [H] constructed his original connection. But when applied to the gauge theory situation, we get the corollary that Hitchin's connection agrees with Axelrod, Della Pietra and Witten's.

Hence, we start in the general setting and let (M,ω) be any compact symplectic manifold.

Definition 2.1. A prequantum line bundle $(\mathcal{L}, (\cdot,\cdot), \nabla)$ over the symplectic manifold (M,ω) consist of a complex line bundle $\mathcal{L}$ with a Hermitian structure $(\cdot,\cdot)$ and a compatible connection ∇ whose curvature is

$$F_\nabla(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = -i\omega(X,Y).$$

We say that the symplectic manifold (M,ω) is prequantizable if there exist a prequantum line bundle over it.

Recall that the condition for the existence of a prequantum line bundle is that $[\frac{\omega}{2\pi}] \in \operatorname{Im}(H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R}))$. Furthermore, the inequivalent choices of prequantum line bundles (if they exist) are parametriced by $H^1(M,U(1))$ (see e.g. [Wo]).

We shall assume that (M,ω) is prequantizable and fix a prequantum line bundle $(\mathcal{L},(\cdot,\cdot),\nabla)$.

Before dwelving into the details we discuss general facts about families of Kähler structures on a symplectic manifold.

Families of Kähler structures

From now on we assume $\mathcal{T}$ is a smooth manifold. Later we impose extra structure.

A *family of Kähler structures* on a symplectic manifold (M,ω) parametrized by $\mathcal{T}$ is a map

$$I : \mathcal{T} \to C^\infty(M, \operatorname{End} TM),$$

that to each element $\sigma \in \mathcal{T}$ associates an integrable and compatible almost complex structure. I is said to be smooth if I defines a smooth section of $\pi_M^* \operatorname{End}(TM) \to \mathcal{T} \times M$.

For each point $\sigma \in \mathcal{T}$ we define M_σ to be M with the Kähler structure defined by ω and $I_\sigma := I(\sigma)$, and the Kähler metric is denoted by g_σ.

Every I_σ is an almost complex structure and hence induce a splitting of the complexified tangent bundle $TM_\mathbb{C}$, denoted by $TM_\mathbb{C} = T_\sigma \oplus \bar{T}_\sigma$, and the projection to each factor is given by

$$\pi_\sigma^{1,0} = \frac{1}{2}(Id - iI_\sigma) \quad \text{and} \quad \pi_\sigma^{0,1} = \frac{1}{2}(Id + iI_\sigma).$$

If $I_\sigma^2 = -Id$ is differentiated along a vector field V on $\mathcal{T}$, we get

$$V[I]_\sigma I_\sigma + I_\sigma V[I]_\sigma = 0,$$

and hence $V[I]_\sigma$ changes types on M_σ. Then for each σ, $V[I]_\sigma$ give an element of

$$C^\infty(M, ((\bar{T}_\sigma)^* \otimes T_\sigma) \oplus ((T_\sigma)^* \otimes \bar{T}_\sigma)),$$

and we have a splitting $V[I]_\sigma = V[I]'_\sigma + V[I]''_\sigma$ where

$$V[I]'_\sigma \in C^\infty(M, (\bar{T}_\sigma)^* \otimes T_\sigma) \quad \text{and} \quad V[I]''_\sigma \in C^\infty(M, (T_\sigma)^* \otimes \bar{T}_\sigma).$$

This splitting of $V[I]$ happens for every vector field on $\mathcal{T}$ and actually induce an almost complex structure on $\mathcal{T}$.

Since $V[I]_\sigma$ is a smooth section of $TM_\mathbb{C} \otimes T^*M_\mathbb{C}$ and the symplectic structure is a smooth section of $T^*M_\mathbb{C} \otimes T^*M_\mathbb{C}$ we can define a bivector field $\tilde{G}(V)$ by contraction with the symplectic form

$$\tilde{G}(V) \cdot \omega = V[I].$$

$\tilde{G}(V)$ is unique since ω is non-degenerate. By definition of the Kähler metric, g is the contraction of ω and I, $g = \omega \cdot I$. We use the $\cdot$-notation for contraction in the following way

$$g(X,Y) = (\omega{\cdot}I)(X,Y) = \omega(X,IY) \quad \text{and} \quad g(X,Y) = -(I{\cdot}\omega)(X,Y) = -\omega(IX,Y).$$

Since ω is independent of σ taking the derivative of this identity in the direction of a vector field V on $\mathcal{T}$ we obtain

$$V[g] = \omega \cdot V[I] = \omega \cdot \tilde{G}(V) \cdot \omega.$$

Since g is symmetric so is $V[g]$, and with ω being anti-symmetric $\tilde{G}(V)$ is symmetric. We furthermore have that ω is of type $(1,1)$ when regarded as a Kähler form on M_σ, using this and the fact that $V[I]_\sigma$ changes types on M_σ we get that $\tilde{G}(V)$ splits as $\tilde{G}(V) = G(V) + \bar{G}(V)$, where

$$G(V) \in C^\infty(M, S^2(T')) \quad \text{and} \quad \bar{G}(V) \in C^\infty(M, S^2(\bar{T})).$$

In the above we have suppressed the dependence on σ, since this is valid for any σ.

Holomorphic families of Kähler structures

Let us now assume that $\mathcal{T}$ furthermore is a complex manifold. We can then ask $I : \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ to be holomorphic. By using the splitting of $V[I]$ we make the following definition.

Definition 2.2. Let $\mathcal{T}$ be a complex manifold and I a smooth family of complex structures on M parametrized by $\mathcal{T}$. Then I is holomorphic if

$$V'[I] = V[I]' \quad \text{and} \quad V''[I] = V[I]''$$

for all vector fields V on $\mathcal{T}$.

Assume J is an integrable almost complex structure on $\mathcal{T}$ induced by the complex structure on $\mathcal{T}$. J induces an almost complex structure, $\hat{I}$ on $\mathcal{T} \times M$ by

$$\hat{I}(V \oplus X) = JV \oplus I_\sigma X,$$

where $V + X \in T_{(\sigma,p)(\mathcal{T}\times M)}$. In [AGL] a simple calculation shows that the Nijenhuis tensor on $\mathcal{T} \times M$ vanish exactly when $\pi^{0,1}V'[I]X = 0$ and $\pi^{1,0}V''[I]X = 0$, which by the Newlander–Nirenberg theorem shows that $\hat{I}$ is integrable if and only if I is holomorphic, hence the name.

Remark that for a holomorphic family of Kähler structures on (M, ω) we have

$$\tilde{G}(V') \cdot \omega = V'[I] = V[I]' = G(V) \cdot \omega,$$

which implies $\tilde{G}(V') = G(V)$. We can in the same way show that $\bar{G}(V) = \tilde{G}(V'')$.

Rigid families of Kähler structures

In constructing an explicit formula for the Hitchin connection we need the following rather restrictive assumption on our family of Kähler structures.

Definition 2.3. A family of Kähler structures I on M is called *rigid* if

$$\nabla_{X''} G(V) = 0$$

for all vector fields V on $\mathcal{T}$ and X on M_σ.

Equivalently we could give the above equation in terms of the induced $\bar{\partial}_\sigma$-operator on M_σ,

$$\bar{\partial}_\sigma(G(V)_\sigma) = 0,$$

for all $\sigma \in \mathcal{T}$ and all vector fields V on $\mathcal{T}$.

There are several examples of rigid families of Kähler structures, see e.g. [AGL]. It should be remarked that this condition is also built into the arguments of [H].

The Hitchin connection

Now all tools are defined and we can construct the Hitchin connection. In Theorem 2.6 we need M to be compact, so let us assume this. Recall the quantum spaces

$$H_\sigma^{(k)} = H^0(M_\sigma, \mathcal{L}_\sigma^k)\{s \in C^\infty(M, \mathcal{L}^k) \mid \nabla_\sigma^{0,1} s = 0\},$$

where $\nabla_\sigma^{0,1} = \frac{1}{2}(Id + iI_\sigma)\nabla$.

It is not clear that these spaces form a vector bundle over $\mathcal{T}$. But by constructing a bundle, where these sit as subspaces of each of the fibers, and a connection in this bundle preserving $H_\sigma^{(k)}$, $H^{(k)}$ will be a subbundle over $\mathcal{T}$.

Define the trivial bundle $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$ of infinite rank. The finite dimensional subspaces $H_\sigma^{(k)}$ sits inside each of the fibers. This bundle has of course the trivial connection ∇^t, but we seek a connection preserving $H_\sigma^{(k)}$.

Definition 2.4. A *Hitchin connection* is a connection $\hat{\nabla}$ in $\mathcal{H}^{(k)}$, which preserves the subspaces $H_\sigma^{(k)}$, and is of the form

$$\hat{\nabla} = \nabla^t + u,$$

where $u \in \Omega^1(\mathcal{T}, \mathcal{D}(M, \mathcal{L}^k))$ is a 1-form on $\mathcal{T}$ with values in differential operators acting on sections of $\mathcal{L}^k$.

By analyzing the condition $\nabla_\sigma^{0,1} \hat{\nabla}_V s = 0$ for every vector field V on $\mathcal{T}$, we hope to find an explicit expression for u. If we express the above condition in terms of u, u should satisfy

$$0 = \nabla_\sigma^{0,1} V[s] + \nabla_\sigma^{0,1} u(V) s,$$

and if we differentiate $\nabla^{0,1}_\sigma s = 0$ along the a vector field V on $\mathcal{T}$ we get

$$0 = V[\nabla^{0,1}_\sigma s] = V[\frac{1}{2}(Id + iI_\sigma)\nabla s] = \frac{i}{2}V[I_\sigma]\nabla s + \nabla^{0,1}_\sigma V[s].$$

If we combine the previous two equations we get the following

Lemma 2.5. *The connection $\hat{\nabla} = \nabla^t + u$ preserves $H^{(k)}_\sigma$ for all $\sigma \in \mathcal{T}$ if and only if u satisfy the equation*

$$\nabla^{0,1}_\sigma u(V)s = \frac{i}{2}V[I]_\sigma \nabla^{1,0}_\sigma s \tag{2.1}$$

for all $\sigma \in \mathcal{T}$ and all vector fields V on $\mathcal{T}$.

If the conclusion is true the collection of subspaces $H^{(k)}_\sigma \subset C^\infty(M, \mathcal{L}^k)$ constitute a subbundle $H^{(k)}$ of $\mathcal{H}^{(k)}$.

Let us now assume that $\mathcal{T}$ is a complex manifold, and that the family I is holomorphic. First of all, $\nabla^{1,0}_\sigma s$ is a section of $(T_\sigma)^* \otimes \mathcal{L}^k$, so it is constant in the $\bar{T}_\sigma$-direction, which is why $V[I]''_\sigma \nabla^{1,0}_\sigma s = 0$, and by holomorphicity $V''[I] = V[I]''$, so $V''[I]_\sigma \nabla^{1,0}_\sigma s = 0$. Hence we can choose $u(V'') = 0$, and we therefore only need to focus on u in the V'-direction.

$u(V)$ should be a differential operator acting on sections of $\mathcal{L}^k$, and be related to I, so let us construct an operator from I.

Given a smooth symmetric bivector field B on M we define a differential operator on smooth sections of $\mathcal{L}^k$ by

$$\Delta_B = \nabla^2_B + \nabla_{\delta B},$$

where δB is the divergence of a symmetric bivector field

$$\delta_\sigma(B) = \operatorname{Tr} \nabla_\sigma B.$$

∇^2_B is defined by

$$\nabla^2_{X,Y} = \nabla_X \nabla_Y s - \nabla_{\nabla_X Y} s,$$

which is tensorial in the vector fields X and Y. Thus we can evaluate it on a bivector field, and have thus defined Δ_B.

Recall the bivector field $G(V)$ defined by $G(V) \cdot \omega = V'[I]$. Using the above construction give a differential operator $\Delta_{G(V)} : C^\infty(M, \mathcal{L}^k) \to C^\infty(M, \mathcal{L}^k)$. Locally $G(V) = \sum_j X_j \otimes Y_j$, and

$$\Delta_{G(V)} = \nabla^2_{G(V)} + \nabla_{\delta G(V)} = \sum_j \nabla_{X_j} \nabla_{Y_j} + \nabla_{\delta(X_j) Y_j}, \tag{2.2}$$

since $\delta(X_j \otimes Y_j) = \delta(X_j)Y_j + \nabla_{X_j} Y_j$, where $\delta(X)$ is the usual divergence of a vector field, which can be defined in many ways e.g. in terms of the Levi-Cevitta connection on M_σ by $\delta(X) = \operatorname{Tr} \nabla_\sigma X$.

The second order differential operator $\Delta_{G(V)}$ is the cornerstone in the construction of $u(V)$. The idea in Andersen's construction is to calculate $\nabla^{0,1}\Delta_{G(V)}s$ and find remainder terms, which cancel other terms such that $\Delta_{G(V)}$ with these correction terms satisfy equation (2.1). When calculating $\nabla^{0,1}\Delta_{G(V)}s$ the trace of the curvature of M_σ show up – that is the Ricci curvature Ric_σ. From Hodge decomposition $\mathrm{Ric}_\sigma = \mathrm{Ric}_\sigma^H + 2i\partial_\sigma\bar\partial_\sigma F_\sigma$ where Ric^H is harmonic and F_σ is the Ricci potential. As with the family of Kähler structures the Ricci potentials F_σ is a family of Ricci potentials parametrized by $\mathcal{T}$ and can therefore be differentiated along a vector field on $\mathcal{T}$.

Define $u \in \Omega^1(\mathcal{T}, \mathcal{D}(M, \mathcal{L}^k))$ by

$$u(V) = \frac{1}{4k+2n}(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} + 4kV'[F]). \tag{2.3}$$

Theorem 2.6 (Andersen [A6])**.** *Let (M,ω) be a compact prequantizable symplectic manifold with $H^1(M,\mathbb{R}) = 0$ and first Chern class $c_1(M,\omega) = n[\frac{\omega}{2\pi}]$. Let I be a rigid holomorphic family of Kähler structures on M parametrized by a complex manifold $\mathcal{T}$. Then*

$$\hat\nabla_V = \nabla_V^t + \frac{1}{4k+2n}(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} + 4kV'[F])$$

is a Hitchin connection in the bundle $H^{(k)}$ over $\mathcal{T}$.

Remark 2.7. The condition $c_1(M,\omega) = n[\frac{\omega}{2\pi}] = nc_1(\mathcal{L})$ can be removed by switching to the metaplectic correction. Here we make the same construction but now a square root of the canonical bundle of (M,ω) is tensored onto $\mathcal{L}^k$. Such a square root exists exactly if the second Stiefel–Whitney class is 0 – that is if M is spin, see [AGL] and also [Ch].

Remark 2.8. The condition $H^1(M,\mathbb{R}) = 0$ is used to make the calculations in the proof easier, but there is no known examples of manifolds with $H^1(M,\mathbb{R}) \neq 0$, which satisfy the remaining conditions where the Hitchin connection cannot be built in this way. An example is the torus T^{2n} which we will study in much greater detail in Section 6.

Remark 2.9. By using Toeplitz operator theory, it can be shown that under some further assumptions on the family of Kähler structures the Hitchin connection is actually projectively flat. A proof of this can be found in [G].

Suppose Γ is a group which acts by bundle automorphisms of $\mathcal{L}$ over M preserving both the Hermitian structure and the connection in $\mathcal{L}$. Then there is an induced action of Γ on (M,ω). We will further assume that Γ acts on $\mathcal{T}$ and that I is Γ-equivariant. In this case we immediately get the following invariance.

Lemma 2.10. *The natural induced action of Γ on $\mathcal{H}^{(k)}$ preserves the subbundle $H^{(k)}$ and the Hitchin connection.*

We are actually interested in the induced connection $\hat{\nabla}^e$ in the endomorphism bundle $\mathrm{End}(H^{(k)})$. Suppose Φ is a section of $\mathrm{End}(H^{(k)})$. Then for all sections s of $H^{(k)}$ and all vector fields V on $\mathcal{T}$, we have that

$$(\hat{\nabla}^e_V \Phi)(s) = \hat{\nabla}_V \Phi(s) - \Phi(\hat{\nabla}_V(s)).$$

Assume now that we have extended Φ to a section of $\mathrm{Hom}(\mathcal{H}^{(k)}, H^{(k)})$ over $\mathcal{T}$. Then

$$\hat{\nabla}^e_V \Phi = \hat{\nabla}^{e,t}_V \Phi + [\Phi, u(V)], \tag{2.4}$$

where $\hat{\nabla}^{e,t}$ is the trivial connection in the trivial bundle $\mathrm{End}(\mathcal{H}^{(k)})$ over $\mathcal{T}$.

3 Toeplitz operators on compact Kähler manifolds

In this section we discuss the Toeplitz operators on compact Kähler manifolds (M, ω) with Kähler structures parametrized by a smooth manifold $\mathcal{T}$ and their asymptotics as the level k goes to infinity.

For each $f \in C^\infty(M)$ we consider the differential operator $M_f^{(k)} : C^\infty(M, \mathcal{L}^k) \to C^\infty(M, \mathcal{L}^k)$ given by

$$M_f^{(k)}(s) = fs$$

for all $s \in H^0(M, \mathcal{L}^k)$.

These operators act on $C^\infty(M, \mathcal{L}^k)$ and therefore also on the trivial bundle $\mathcal{H}^{(k)}$, however they do not preserve the subbundle $H^{(k)}$. There is however a solution to this, which is given by the Hilbert space structure. Integrating the inner product of two sections of $\mathcal{L}^k$ against the volume form associated to the symplectic form gives the pre-Hilbert space structure on $C^\infty(M)$

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_M (s_1, s_2) \omega^m.$$

This is not only a pre-Hilbert space structure on $C^\infty(M, \mathcal{L}^k)$ but also on the trivial bundle $\mathcal{H}^{(k)}$ which is of course compatible with the trivial connection in this bundle. This pre-Hilbert space structure induces a Hermitian structure $\langle \cdot, \cdot \rangle$ on the finite rank subbundle $H^{(k)}$ of $\mathcal{H}^{(k)}$. The Hermitian structure $\langle \cdot, \cdot \rangle$ on $H^{(k)}$ also induces the operator norm on $\mathrm{End}(H^{(k)})$. By the finite dimensionality of $H^{(k)}_\sigma$ in $\mathcal{H}^{(k)}_\sigma$ we have the orthogonal projection $\pi^{(k)}_\sigma : \mathcal{H}^{(k)}_\sigma \to H^{(k)}_\sigma$. From these projections we can construct the Toeplitz operators associated to any smooth function $f \in C^\infty(M)$. It is the operator $T^{(k)}_{f,\sigma} : \mathcal{H}^{(k)}_\sigma \to H^{(k)}_\sigma$ defined by

$$T^{(k)}_{f,\sigma}(s) = \pi^{(k)}_\sigma(fs)$$

for any element $s \in \mathcal{H}^{(k)}_\sigma$ and any point $\sigma \in \mathcal{T}$. Since the projections form a smooth map $\pi^{(k)}$ from $\mathcal{T}$ to the space of bounded operators in the L_2-completion of $C^\infty(M, \mathcal{L}^k)$ the Toeplitz operators are smooth sections $T^{(k)}_f$ of the bundle of homomorphisms $\mathrm{Hom}(\mathcal{H}^{(k)}, H^{(k)})$ and restrict to smooth sections of $\mathrm{End}(H^{(k)})$.

Remark 3.1. It should be remarked that the above construction could be used for any Pseudo-differential operator A on M with coefficients in $\mathcal{L}^k$ – it can even depend on σ, and we will then consider it as a section of $\mathrm{Hom}(\mathcal{H}^{(k)}, H^{(k)})$. However when we consider their asymptotic expansions or operator norms, we implicitly restrict them to $H^{(k)}$ and consider them as sections of $\mathrm{End}(H^{(k)})$ – or as $\pi^{(k)} A \pi^{(k)}$.

We need the following two theorems on Toeplitz operators to proceed. The first is due to Bordemann, Meinrenken and Schlichenmaier (see [BMS]).

Theorem 3.2 (Bordemann, Meinrenken and Schlichenmaier)**.** *For any $f \in C^\infty(M)$ we have that*

$$\lim_{k \to \infty} \|T^{(k)}_f\| = \sup_{x \in M} |f(x)|.$$

Since the association of the sequence of Toeplitz operators $T^{(k)}_f$, $k \in \mathbb{Z}_+$ is linear in f, we see from this Theorem, that this association is faithful.

The product of two Toeplitz operators associated to two smooth functions will in general not be a Toeplitz operator associated to a smooth function again. But by Schlichenmaier [Sch], there is an asymptotic expansion of the product in terms of Toeplitz operators associated to smooth functions on a compact Kähler manifold.

Theorem 3.3 (Schlichenmaier)**.** *For any pair of smooth functions $f_1, f_2 \in \mathbb{C}^\infty(M)$, we have an asymptotic expansion*

$$T^{(k)}_{f_1} T^{(k)}_{f_2} \sim \sum_{l=0}^\infty T^{(k)}_{c_l(f_1,f_2)} k^{-l},$$

where $c_l(f_1, f_2) \in C^\infty(M)$ are uniquely determined since $\sim$ means the following: For all $L \in \mathbb{Z}_+$ we have that

$$\|T^{(k)}_{f_1} T^{(k)}_{f_2} - \sum_{l=0}^L T^{(k)}_{c_l(f_1,f_2)} k^{-l}\| = O(k^{-(L+1)}) \tag{3.1}$$

uniformly over compact subsets of $\mathcal{T}$. Moreover, $c_0(f_1, f_2) = f_1 f_2$.

Remark 3.4. In Section 5 it will be useful for us to define new coefficients $\tilde{c}^{(l)}_\sigma(f, g) \in C^\infty(M)$ which correspond to the expansion of the product in $1/(k + n/2)$ (where n is some fixed integer):

$$T^{(k)}_{f_1,\sigma} T^{(k)}_{f_2,\sigma} \sim \sum_{l=0}^\infty T^{(k)}_{\tilde{c}^{(l)}_\sigma(f_1,f_2),\sigma} (k + n/2)^{-l}.$$

Note that the first three coefficients are given by $\tilde{c}^{(0)}_\sigma(f_1,f_2) = c^{(0)}_\sigma(f_1,f_2)$, $\tilde{c}^{(1)}_\sigma(f_1,f_2) = c^{(1)}_\sigma(f_1,f_2)$ and $\tilde{c}^{(2)}_\sigma(f_1,f_2) = c^{(2)}_\sigma(f_1,f_2) + \frac{n}{2}c^{(1)}_\sigma(f_1,f_2)$.

This Theorem was proved in [Sch] where it is also proved that the formal generating series for the $c_l(f_1,f_2)$'s gives a formal deformation quantization of the Poisson structure on M induced by ω. An English version is available in [Sch1] see [Sch2] for further developments. We return to this in Section 5 where we discuss formal Hitchin connections.

4 Asymptotic faithfulness

In this section we will concentrate on the case where M is the moduli space of flat $\mathrm{SU}(n)$-connections on $\Sigma - p$ with holonomy d around p. As in the introduction Σ is a closed oriented surface of genus $g \geq 2$, p a point in Σ and $d \in \mathbb{Z}/n\mathbb{Z} \simeq Z_{\mathrm{SU}(n)}$ in the center of $\mathrm{SU}(n)$ is fixed.

As mentioned in the introduction the main result about the Verlinde bundle $\mathcal{V}^{(k)}$ from geometrically quantizing the moduli space M is that its projectivization $\mathbb{P}(\mathcal{V}^{(k)})$ carries a flat connection $\hat{\nabla}$. This flat connection induces a flat connection in $\hat{\nabla}^e$ in the endomorphism bundle $\mathrm{End}(\mathcal{V}^{(k)})$ as described in Section 2.

An important ingredient in proving asymptotic faithfulness is the corollary to Theorem 1.6 saying that Toeplitz operators viewed as a section of $\mathrm{End}(\mathcal{V}^{(k)})$ is in some sense asymptotically flat,

$$\|\hat{\nabla}^e_V T^{(k)}_f\| = O(k^{-1}).$$

This can be reformulated in terms of the induced parallel transport between the fibers of $\mathrm{End}(\mathcal{V}^{(k)})$. Let σ_0, σ_1 be two points in Teichmüller space $\mathcal{T}$, and P_{σ_0,σ_1} the parallel transport from σ_0 to σ_1. Then

$$\|P_{\sigma_0,\sigma_1} T^{(k)}_{f,\sigma_0} - T^{(k)}_{f,\sigma_1}\| = O(k^{-1}), \tag{4.1}$$

where $\|\cdot\|$ is the operator norm on $H^0(M_{\sigma_1}, \mathcal{L}^k_{\sigma_1})$.

Equation (4.1) and Theorem 3.2 together prove asymptotic faithfulness. Below we explain how.

Recall that the flat connection in the bundle $\mathbb{P}(\mathcal{V}^{(k)})$ gives the projective representation of the mapping class group

$$Z^{(n,d)}_k : \Gamma \to \mathrm{Aut}(\mathbb{P}(V^k))$$

where $\mathbb{P}(V^{(k)})$ are the covariant constant sections of $\mathbb{P}(\mathcal{V}^{(k)})$ over Teichmüller space with respect to the Hitchin connection $\hat{\nabla}$.

Proof of Theorem 1.3. Suppose we have a $\phi \in \Gamma$. Then ϕ induces a symplectomorphism of M which we also just denote ϕ and we get the following commutative diagram for any $f \in \mathbb{C}^\infty(M)$

$$\begin{array}{ccccc} H^0(M_\sigma, \mathcal{L}^k_\sigma) & \xrightarrow{\phi^*} & H^0(M_{\phi(\sigma)}, \mathcal{L}^k_{\phi(\sigma)}) & \xrightarrow{P_{\phi(\sigma),\sigma}} & H^0(M_\sigma, \mathcal{L}^k_\sigma) \\ \Big\downarrow T^{(k)}_{f,\sigma} & & \Big\downarrow T^{(k)}_{f\circ\phi,\phi(\sigma)} & & \Big\downarrow P_{\phi(\sigma),\sigma} T^{(k)}_{f\circ\phi,\phi(\sigma)} \\ H^0(M_\sigma, \mathcal{L}^k_\sigma) & \xrightarrow{\phi^*} & H^0(M_{\phi(\sigma)}, \mathcal{L}^k_{\phi(\sigma)}) & \xrightarrow{P_{\phi(\sigma),\sigma}} & H^0(M_\sigma, \mathcal{L}^k_\sigma), \end{array}$$

where $P_{\phi(\sigma),\sigma} : H^0(M_{\phi(\sigma)}, \mathcal{L}^k_{\phi(\sigma)}) \to H^0(M_\sigma, \mathcal{L}^k_\sigma)$ on the horizontal arrows refer to parallel transport in the Verlinde bundle itself, whereas $P_{\phi(\sigma),\sigma}$ refers to the parallel transport in the endomorphism bundle $\mathrm{End}(\mathcal{V}_k)$ in the last vertical arrow. Suppose now $\phi \in \bigcap_{k=1}^\infty \ker Z_k^{(n,d)}$, then $P_{\phi(\sigma),\sigma} \circ \phi^* = Z_k^{(n,d)}(\phi) \in \mathbb{C}\,\mathrm{Id}$ and we get that $T^{(k)}_{f,\sigma} = P_{\phi(\sigma),\sigma} T^{(k)}_{f\circ\phi,\phi(\sigma)}$. By Theorem 4.1 we get that

$$\begin{aligned} \lim_{k\to\infty} \|T^{(k)}_{f - f\circ\phi,\sigma}\| &= \lim_{k\to\infty} \|T^{(k)}_{f,\sigma} - T^{(k)}_{f\circ\phi,\sigma}\| \\ &= \lim_{k\to\infty} \|P_{\phi(\sigma),\sigma} T^{(k)}_{f\circ\phi,\phi(\sigma)} - T^{(k)}_{f\circ\phi,\sigma}\| = 0. \end{aligned}$$

By Bordemann, Meinrenken and Schlichenmaier's Theorem 3.2, we must have that $f = f \circ \phi$. Since this holds for any $f \in \mathbb{C}^\infty(M)$, we must have that ϕ acts by the identity on M. □

5 The Formal Hitchin connection and Berezin–Toeplitz Deformation Quantization

In this section we discuss the formal Hitchin connection. We return to the general setup of compact Kähler manifolds, where we impose conditions on (M, ω, I) as in Theorem 2.6, thus providing us with a Hitchin connection $\hat{\nabla}$ in $H^{(k)}$ over $\mathcal{T}$ and the associated connection $\hat{\nabla}^e$ in $\mathrm{End}(H^{(k)})$. Firstly werecall the definition of a formal deformation quantization and the results about star products from [Sch] and [KS]. We introduce the space of formal functions $C_h^\infty(M) = C^\infty(M)[[h]]$ as the space for formal power series in the variable h with coefficients in $C^\infty(M)$, and let $\mathbb{C}_h = \mathbb{C}[[h]]$ denote the formal constants.

Definition 5.1. A deformation quantization of (M, ω) is an associative product $\star$ on $C_h^\infty(M)$ which respects the $\mathbb{C}_h$-module structure. For $f, g \in C^\infty(M)$, it is defined as

$$f \star g = \sum_{l=0}^\infty c^{(l)}(f, g) h^l,$$

through a sequence of bilinear operators

$$c^{(l)} : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M),$$

which musth satisfy

$$c^{(0)}(f,g) = fg \qquad \text{and} \qquad c^{(1)}(f,g) - c^{(1)}(g,f) = -i\{f,g\}.$$

The deformation quantization is said to be *differential* if the operators $c^{(l)}$ are bidifferential operators. Considering the symplectic action of Γ on (M,ω), we say that a star product is Γ-invariant if

$$\gamma^*(f \star g) = \gamma^*(f) \star \gamma^*(g)$$

for all $f, g \in C^\infty(M)$ and all $\gamma \in \Gamma$.

Recall Theorem 3.3 where the asymptotic expansion of the product of two Toeplitz operators associated to smooth functions f_1, f_2 on M create maps $c_i(f_1, f_2) \in C^\infty(M)$. In [Sch] Schlichenmaier also showed that these maps generate a star product. It was first in [KS] Karabegov and Schlichenmaier showed that it was a differentiable star product.

Theorem 5.2 (Karabegov & Schlichenmaier)**.** *The product $\star_\sigma^{BT}$ given by*

$$f \star_\sigma^{BT} g = \sum_{l=0}^{\infty} c_\sigma^{(l)}(f,g) h^l,$$

where $f, g \in C^\infty(M)$ and $c_\sigma^{(l)}(f,g)$ are determined by Theorem 3.3, is a differentiable deformation quantization of (M,ω).

Definition 5.3. The Berezin-Toeplitz deformation quantization of the compact Kähler manifold (M_σ, ω) is the product $\star_\sigma^{\mathrm{BT}}$.

For the remaining part of this paper we let Γ be a symmetry group as in Section 2, that is a group which acts by bundle automorphisms on $\mathcal{L}$ over M preserving both the Hermitian structure and the connection in $\mathcal{L}$. Such a group has an induced action on (M,ω). Note that Γ in the case of moduli spaces is the mapping class group of the surface.

Remark 5.4. Let Γ_σ be the σ-stabilizer subgroup of Γ. For any element $\gamma \in \Gamma_\sigma$, we have that

$$\gamma^*(T_{f,\sigma}^{(k)}) = T_{\gamma^* f,\sigma}^{(k)}.$$

This implies the invariance of $\star_\sigma^{\mathrm{BT}}$ under the σ-stabilizer Γ_σ.

Remark 5.5. Using the coefficients from Remark 3.4, we define a new star product by

$$f \tilde{\star}_\sigma^{\mathrm{BT}} g = \sum_{l=0}^{\infty} \tilde{c}_\sigma^{(l)}(f,g) h^l.$$

Then

$$f \tilde{\star}_\sigma^{\mathrm{BT}} g = \big((f \circ \phi^{-1}) \star_\sigma^{\mathrm{BT}} (g \circ \phi^{-1})\big) \circ \phi$$

for all $f, g \in C_h^\infty(M)$, where $\phi(h) = \frac{2h}{2+nh}$.

Recall from the introduction the definition of a formal connection in the trivial bundle of formal functions. Theorem 1.6, establishes the existence of a unique formal Hitchin connection, expressing asymptotically the interplay between the Hitchin connection and the Toeplitz operators.

We want to give an explicit formula for the formal Hitchin connection in terms of the star product $\tilde{\star}^{BT}$. We recall that in the proof of Theorem 1.6, given in [A6], it is shown that the formal Hitchin connection is given by

$$\tilde{D}(V)(f) = -V[F]f + V[F]\tilde{\star}^{\mathrm{BT}}f + h(E(V)(f) - H(V)\tilde{\star}^{\mathrm{BT}}f), \tag{5.1}$$

where E is the one-form on $\mathcal{T}$ with values in $\mathcal{D}(M)$ such that

$$T^{(k)}_{E(V)f} = \pi^{(k)}o(V)^* f\pi^{(k)} + \pi^{(k)}fo(V)\pi^{(k)}, \tag{5.2}$$

and H is the one form on $\mathcal{T}$ with values in $C^\infty(M)$ such that $H(V) = E(V)(1)$. In [AG] an explicit expression for the operator $E(V)$ is found by calculating the adjoint of

$$o(V) = -\frac{1}{4}(\Delta_{G(V)} + 2\hat{\nabla}_{G(V)\cdot dF} - 2nV'[F]).$$

This operator is essential in the proof of Theorem 2.6 since by comparing the above equation with Equation 2.3 we see that $u(V) = \frac{1}{k+n/2}o(V) - V'[F]$.

Theorem 5.6. *The formal Hitchin connection is given by*

$$\begin{aligned} D_V f = V[f] - \frac{1}{4}h\Delta_{\tilde{G}(V)}(f) + \frac{1}{2}h\nabla_{\tilde{G}(V)dF}(f) + V[F]\tilde{\star}^{BT}f - V[F]f \\ - \frac{1}{2}h(\Delta_{\tilde{G}(V)}(F)\tilde{\star}^{BT}f + nV[F]\tilde{\star}^{BT}f - \Delta_{\tilde{G}(V)}(F)f - nV[F]f) \end{aligned}$$

for any vector field V and any section f of C_h.

When we geometrically quantize a symplectic manifold, we have to choose a polarization of the complexified tangent bundle, to reduce the space upon the quantum operators act. This is equivalent to choosing a compatible complex structure on the symplectic manifold, hence making it Kähler. It is however quite unfortunate that the quantum space then depend on the choice of Kähler structure. The solution to this is the projectively flat Hitchin Connection, which by parallel transport between the fibers of $H^{(k)}$ give us a space of quantum states as the covariant constant sections of $\mathbb{P}H^{(k)}$, which does not depend on the chosen complex structure. Instead of doing geometric quantization we could do Berezin–Toeplitz deformation quantization. The created star product $\star^{\mathrm{BT}}_\sigma$ depend on the complex structure, and in the same spirit as above we want to make all these star products equivalent to a star product which does not depend on σ. This is the purpose of the formal Hitchin connection.

If the Hitchin connection is projectively flat, then the induced connection in the endomorphism bundle is flat and hence so is the formal Hitchin connection by Proposition 3 of [A6].

Recall from Definition 1.7 in the introduction the definition of a formal trivialization. As mentioned there, such a formal trivialization will not exist even locally on $\mathcal{T}$, if D is not flat. However, if D is flat, then we have the following result from [A6].

Proposition 5.7. *Assume that D is flat and that $\tilde{D} = 0$ mod h. Then locally around any point in $\mathcal{T}$, there exists a formal trivialization. If $H^1(\mathcal{T}, \mathbb{R}) = 0$, then there exists a formal trivialization defined globally on $\mathcal{T}$. If further $H^1_\Gamma(\mathcal{T}, D(M)) = 0$, then we can construct P such that it is Γ-equivariant.*

An immediate corollary of Proposition 5.7 is

Corollary 5.8. *If $\mathcal{T}$ is contractible, then any* flat *formal connection admits a global formal trivialization that is Γ-equivariant.*

In the proposition, $H^1_\Gamma(\mathcal{T}, D(M))$ refers to the Γ-equivariant first de Rham cohomology of $\mathcal{T}$ with coefficients in the real vector space $D(M)$ of differential operators on M. The first steps towards proving that this cohomology group vanishes in the case where M is the moduli space have been taken in [AV1, AV2, AV3, Vi].

In [AG] an explicit formula for P up to first order is found.

Theorem 5.9. *The Γ-equivariant formal trivialization of the formal Hitchin connection exists to first order, and we have the following explicit formula for the first order term of P*

$$P^{(1)}_\sigma(f) = \frac{1}{4}\Delta_\sigma(f) + i\nabla_{X''_F}(f),$$

where X''_F denotes the (0,1)-part of the Hamiltonian vector field for the Ricci potential, F.

Now suppose we have a formal trivialization P of the formal Hitchin connection D. We can then define a new smooth family of star products, parametrized by $\mathcal{T}$, by

$$f \star_\sigma g = P^{-1}_\sigma(P_\sigma(f) \tilde{\star}^{\mathrm{BT}}_\sigma P_\sigma(g))$$

for all $f, g \in C^\infty(M)$ and all $\sigma \in \mathcal{T}$. Using the fact that P is a trivialization, it is not hard to prove

Proposition 5.10. *The star products $\star_\sigma$ are independent of $\sigma \in \mathcal{T}$.*

This is done by simply differentiating $\star_\sigma$ along a vector field on $\mathcal{T}$, see [A6].

Then, we have the following which is proved in [A6].

Theorem 5.11 (Andersen). *Assume that the formal Hitchin connection D is flat and*

$$H^1_\Gamma(\mathcal{T}, D(M)) = 0,$$

then there is a Γ-invariant trivialization P of D and the star product

$$f \star g = P_\sigma^{-1}(P_\sigma(f) \tilde{\star}_\sigma^{BT} P_\sigma(g))$$

is independent of $\sigma \in \mathcal{T}$ and Γ-invariant. If $H^1_\Gamma(\mathcal{T}, C^\infty(M)) = 0$ and the commutant of Γ in $D(M)$ is trivial, then a Γ-invariant differential star product on M is unique.

In [AG] the star product of Theorem 5.11 is identified up to second order in h.

Theorem 5.12. *The star product $\star$ has the form*

$$f \star g = fg - \frac{i}{2}\{f, g\}h + O(h^2).$$

We observe that this formula for the first-order term of $\star$ agrees with the first-order term of the star product constructed by Andersen, Mattes and Reshetikhin in [AMR2], when we apply the formula in Theorem 5.12 to two holonomy functions h_{γ_1,λ_1} and h_{γ_2,λ_2}:

$$h_{\gamma_1,\lambda_1} \star h_{\gamma_2,\lambda_2} = h_{\gamma_1\gamma_2,\lambda_1\cup\lambda_2} - \frac{i}{2} h_{\{\gamma_1,\gamma_2\},\lambda_1\cup\lambda_2} + O(h^2).$$

We recall that $\{\gamma_1, \gamma_2\}$ is the Goldman bracket (see [Go2]) of the two simple closed curves γ_1 and γ_2.

A similar result was obtained for the abelian case, i.e. in the case where M is the moduli space of flat $U(1)$-connections, by the first author in [A2], where the agreement between the star product defined in differential geometric terms and the star product of Andersen, Mattes and Reshetikhin was proved to all orders.

6 Abelian varieties and $U(1)$-moduli space

In this chapter we will investigate all the previous mentioned objects in the setting of principally polarized abelian varieties $M = V/\Lambda$, where V is a real vector space with a symplectic form ω, Λ a discrete lattice of maximal rank such that ω is integral and unimodular when restricted to Λ. Let now $\mathcal{T}$ be the space of complex structures on V, which are compatible with ω. Then for any $I \in \mathcal{T}$, $M_I = (M, \omega, I)$ is an abelian variety. A prime example of an abelian variety is the abelian moduli space. Here we let Σ be a closed surface of genus g, and M be the moduli space of flat U(1)-connections on Σ. Then

$$M = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{U}(1)) = H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z}).$$

There is the usual symplectic structure ω on $H^1(M,\mathbb{R})$ which is of course integral and unimodular over the lattice $H^1(M,\mathbb{Z})$. We will return to this example below when we consider abelian Chern–Simons theory.

In the following we will focus on M_I being a principal polarized abelian variety, where the compatible complex structures are parametrized by $\mathcal{T}$. There exists a symplectic basis $(\lambda_1,\dots,\lambda_{2n})$ over the integers for Λ (e.g. [GH, p. 304]). Let $(x_1,\dots,x_n,y_1,\dots,y_{2n})$ be the dual coordinates on V. Then

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Let A be the automorphism group of (Λ,ω). Then A injects into the symplectomorphism group of (M,ω), and by using the symplectic basis $(\lambda_1,\dots,\lambda_{2n})$ we get an identification $A \simeq \mathrm{Sp}(2n,\mathbb{Z})$. Notice that A acts on the principal polarized variety M_I.

Using the symplectic basis we can identify $\mathcal{T}$ with the Siegel Upper Half Space

$$\mathbb{H} = \{Z \in M_{n,n}(\mathbb{C}) \mid Z = Z^T,\, \mathrm{Im}(Z) > 0\}.$$

For any $I \in \mathcal{T}$ we have that $(\lambda_1,\dots,\lambda_n)$ is a basis over $\mathbb{C}$ for V with respect to I. Let $(z_1,\dots,z_n)$ be the dual complex coordinates on V relative to the basis $(\lambda_1,\dots,\lambda_n)$. The complex structure I determines and is determined by a unique $Z \in \mathbb{H}$ such that

$$z = x + Zy.$$

Since any $Z \in \mathbb{H}$ gives a complex structure, say $I(Z)$, compatible with the symplectic form, we have a bijective map $I : \mathbb{H} \to \mathcal{T}$ given by sending $Z \in \mathbb{H}$ to $I(Z)$. For $Z \in \mathbb{H}$ we use the notation $X = \mathrm{Re}(Z)$ and $Y = \mathrm{Im}(Z)$.

For each $Z \in \mathbb{H}$ we explicitly construct a prequantum line bundle on $M_{I(Z)}$. We do that by providing a lift of the action Λ action on V to the trivial bundle $\tilde{\mathcal{L}} = V \times \mathbb{C}$, such that the quotient is the prequantum line bundle $\mathcal{L}_Z$. We only need to specify a set of multipliers $\{e_\lambda\}_{\lambda\in\Lambda}$ and a Hermitian structure h. The multipliers are non-vanishing functions on V that are holomorphic with respect to $I(Z)$ and depend on Z. They should furthermore satisfy the following functional equation

$$e_{\lambda'}(v+\lambda)e_\lambda(v) = e_{\lambda'}(v)e_\lambda(v+\lambda') = e_{\lambda+\lambda'}(v),$$

for all $\lambda,\lambda' \in \Lambda$. The action of Λ on $\tilde{\mathcal{L}}$ is given by

$$\lambda \cdot (v,z) = (v+\lambda, e_\lambda(z)),$$

for all $\lambda \in \Lambda$ and $(v,z) \in \tilde{\mathcal{L}}$. For a fixed basis of Λ the functional equations determine the multipliers for all $\lambda \in \Lambda$. For $I(Z)$ we choose the multipliers

$$e_{\lambda_i}(z) = 1, \qquad\qquad i = 1,\dots,n,$$

$$e_{\lambda_{n+i}}(z) = e^{-2\pi i z_i - \pi i Z_{ii}}, \qquad i = 1, \dots, n.$$

The constructed line bundle is denoted $\mathcal{L}_Z$. If we define $h(z) = e^{-2\pi y \cdot Y y}$, where $Z = X + iY$, it will define a Hermitian structure on $V \times \mathbb{C}$ by $h(z) \langle \cdot, \cdot \rangle_{\mathbb{C}}$ where $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ is the standard inner product on $\mathbb{C}^n$. This function satisfies the functional equation

$$h(z + \lambda) = \frac{1}{|e_\lambda(z)|^2} h(z),$$

and the inner product on $V \times \mathbb{C}$ is invariant under the action of Λ and hence induces a Hermitian structure $\langle \cdot, \cdot \rangle$ on $\mathcal{L}_Z$. By general theory of abelian varieties, e.g. [GH, Sect. 2.6], a line bundle with the above multipliers and Hermitian metric $(\mathcal{L}_Z, \langle \cdot, \cdot \rangle)$ has curvature $-2\pi i \omega$, and hence is a prequantum line bundle. Note taht the prequantum condition in Definition 2.1 is scaled with 2π. We could just have used $2\pi\omega$ as the symplectic structure. We choose the normalization at hand to make later equations nicer.

The space of holomorphic sections of $\mathcal{L}_Z^k$, $H^0(M_Z, \mathcal{L}_Z^k)$ has dimension k^n, and as in the general theory they give a vector bundle $H^{(k)}$ over $\mathbb{H}$ by letting $H_Z^{(k)} = H^0(M_Z, \mathcal{L}_Z^k)$.

The L^2-inner product on $H^0(M_Z, \mathcal{L}_Z^k)$ is given by

$$(s_1, s_2) = \int_{M_Z} s_1(z) \overline{s_2(z)} h(z) dx dy,$$

for $s_1, s_2 \in H^0(M_Z, \mathcal{L}_Z^k)$.

A basis for the space of sections are the *Theta functions*,

$$\theta_\alpha^{(k)}(z, Z) = \sum_{l \in \mathbb{Z}^n} e^{\pi i k (l+\alpha) \cdot Z (l+\alpha)} e^{2\pi i k (l+\alpha) \cdot z},$$

where $\alpha \in \frac{1}{k}\mathbb{Z}^n / \mathbb{Z}^n$. The Theta functions satisfies the following heat equation,

$$\frac{\partial \theta_\alpha^{(k)}}{\partial Z_{ij}} = \frac{1}{4\pi i k} \frac{\partial^2 \theta_\alpha^{(k)}}{\partial z_i \partial z_j}.$$

The geometric interpretation of this differential equation is a definition of a connection $\tilde{\nabla}$ in the trivial $C^\infty(\mathbb{C}^n)$-bundle over $\mathbb{H}$, by

$$\tilde{\nabla}_{\frac{\partial}{\partial Z_{ij}}} = \frac{\partial}{\partial Z_{ij}} - \frac{1}{4\pi i k} \frac{\partial^2}{\partial z_i \partial z_j}.$$

The coordinates $z = x + Zy$ identify $H^0(M_Z, \mathcal{L}_Z^k)$ as a subspace of $C^\infty(\mathbb{C}^n)$ and $H^{(k)}$ as a subbundle of the trivial $C^\infty(\mathbb{C}^n)$-bundle on $\mathbb{H}$. This bundle is preserved by $\tilde{\nabla}$ and hence induces a connection ∇ in $H^{(k)}$. The covariant constant sections of $H^{(k)}$ with respect to ∇ will, under the embedding induced by the coordinates, be identified with the Theta functions. Since now ∇ has a global frame of covariant constant sections it is flat. Remember that $\mathbb{H}$ is contractible, so since parallel transport with a flat connection only depend on the homotopy class of the curve transported along, we get a canonical way to identify all $H^0(M_Z, \mathcal{L}_Z^k)$, and hence there is no ambiguity in defining the quantum space of geometric quantization to be $H^0(M_Z, \mathcal{L}_Z^k)$. Since the Theta functions are covariant constant, they explicitly realize this identification. The usual action of $\mathrm{Sp}(2n, \mathbb{Z})$ on Theta functions induce an action of $A' = \ker(\mathrm{Sp}(2n, \mathbb{Z}) \to \mathrm{Sp}(2n, \mathbb{Z}/2\mathbb{Z}))$ on the bundle $H^{(k)}$ which covers the A'-action on $\mathbb{H} \simeq \mathcal{T}$. This is the subgroup of A acting trivially on $\Lambda/2\Lambda$.

Remark 6.1. Instead of the above connection ∇ in $H^{(k)}$ over $\mathbb{H}$, we could have rolled out the machinery of Theorem 2.6 to get another connection in the same bundle. This can be done even though $H^1(M, \mathbb{R}) \neq 0$. Since the torus is flat the Ricci potential F is 0 as is the Chern class of M. Lemma 7.1 in the appendix shows that $I(Z)$ is constant on M and thus is a rigid family of Kähler structures. Thus we have a rather nice formula for the Hitchin connection

$$\hat{\nabla}_V = \nabla_V^t + \frac{1}{8\pi k}\Delta_{G(V)}.$$

The extra factor of 2π is from the different prequantum condition. It should be noted that explicit computations show that $\hat{\nabla}$ is not flat like ∇ induced by the heat equation, but rather projectively flat.

In [A2] the inner product of two Theta functions are explicitly calculated.

Lemma 6.2. *The theta functions $\theta_\alpha^{(k)}(z, Z)$, $\alpha \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n$, define an orthonormal basis with respect to the inner product on $H^0(M_Z, \mathcal{L}_Z^k)$ defined by*

$$(s_1, s_2)_Y = (s_1, s_2)\sqrt{2^n k^n \det Y},$$

where $Y = \operatorname{Im} Z$. This is a Hermitian structure on $H^{(k)}$ compatible with ∇.

Let $(r, s) \in \mathbb{Z}^n \times \mathbb{Z}^n$ and consider the function $F_{r,s} \in \mathbb{C}^\infty(M)$ given in (x, y)-coordinates by

$$F_{r,s}(x, y) = e^{2\pi i(x\cdot r + s\cdot y)}.$$

We have previously defined Toeplitz operators associated to a function $f \in C^\infty(M)$, $T_f^{(k)} : H^0(M_Z, \mathcal{L}_Z^k) \to H^0(M_Z, \mathcal{L}_Z^k)$. We shall now explicitly compute the matrix coefficients of these operators in terms of the basis consisting of Theta functions.

To get our hands on the matrix coefficients $(T_{F_{r,s}}^{(k)})_{\beta\alpha}$ we only need to calculate $(F_{r,s}\theta_\alpha^{(k)}, \theta_\beta^{(k)})$, since this indeed is the coefficient. This is also calculated in [A2] and is done in the exact same way as in Lemma 6.2,

$$(F_{r,s}\theta_\alpha^{(k)}, \theta_\beta^{(k)})_Y = \delta_{\alpha-\beta,-[\frac{r}{k}]} e^{-\frac{\pi i}{k} r\cdot\bar{Z}r} e^{-2\pi i s\cdot\alpha} e^{-\pi^2(s-\bar{Z}r)\cdot(2\pi k Y)^{-1}(s-\bar{Z}r)}, \tag{6.1}$$

where $[\frac{r}{k}]$ is the residue class of $\frac{r}{k}$ mod $\mathbb{Z}^n$. A simple rewriting gives

$$(T^{(k)}_{f(r,s,Z)(k)F_{r,s}})_{\beta\alpha} = \delta_{\alpha-\beta,-[\frac{r}{k}]} e^{-\frac{\pi i}{k} r\cdot s} e^{-2\pi i s\cdot\alpha},$$

where

$$f(r,s,Z)(k) = e^{\frac{\pi}{2k}(s-Xr)\cdot Y^{-1}(s-Xr)} e^{\frac{\pi}{2k} r\cdot Yr}.$$

Remark 6.3. The Toeplitz operators $T^{(k)}_{F_{r,s}}$ are sections of $\mathrm{End}(H^{(k)})$ over $\mathbb{H}$. The flat connection ∇ induces a flat connection ∇^e in the bundle $\mathrm{End}(H^{(k)})$, with respect to which we see that $T^{(k)}_{F_{r,s}}$ is *not* covariant constant. However the operators $T^{(k)}_{f(r,s,Z)(k)F_{r,s}}$ are covariant constant. Since the pure phases $F_{r,s}$, $r, s \in \mathbb{Z}^n$ is a Fourier basis for $C^\infty(M)$, we have that $T^{(k)}_{f(r,s,Z)(k)F_{r,s}}$ is covariant constant with respect to $\hat{\nabla}^e$ for all $f \in C^\infty(M)$.

It should also be noted that the coefficient $f(r,s,Z)(k)$ is not so arbitrary as it looks. This is the content of the following

Proposition 6.4. *Let $\Delta_{I(Z)}$ be the Laplace operator with respect to the metric*

$$g_{I(Z)}(\cdot,\cdot) = 2\pi\omega(\cdot, I(Z)\cdot)$$

on M. Then

$$e^{-\frac{1}{4k}\Delta_{I(Z)}} F_{r,s} = f(r,s,Z)(k) F_{r,s}.$$

Proof. Recall that

$$\Delta_{I(Z)} = \frac{1}{2\pi}\left((\frac{\partial}{\partial y} - X\frac{\partial}{\partial x}) \cdot Y^{-1}(\frac{\partial}{\partial y} - X\frac{\partial}{\partial x}) + \frac{\partial}{\partial x}\cdot Y\frac{\partial}{\partial x}\right).$$

Now it is a simple calculation, which we will omit, to show the equality. □

As remarked in Remark 6.3, $T^{(k)}_{f(r,s,Z)(k)F_{r,s}}$ is covariant constant with respect to $\hat{\nabla}$. If we define

$$E_{I(Z)} = e^{-\frac{h}{4}\Delta_{I(Z)}} : C^\infty_h(M) \to C^\infty_h(M)$$

we see that

$$\hat{\nabla}^e_V T^{(k)}_{E_I(f)(1/k)} = 0,$$

for all vector fields V on $\mathbb{H}$ and all functions $f \in C^\infty(M)$ since the pure phase functions consitute a Fourier basis. We furthermore see that E_I is $\mathrm{Sp}(2n,\mathbb{Z})$-equivariant, since for all $\Psi \in \mathrm{Sp}(2n,\mathbb{Z})$ we have that

$$\Psi^* \circ E_I = E_{\Psi(I)}\Psi^*.$$

That $T^{(k)}_{E_I(f)}$ is covariant constant with respect to $\hat{\nabla}^e$ can be interpreted as E_I is a formal parametrization for the formal Hitchin Connection which we know exists by Theorem 1.6. If $\hat{\nabla}^e_V T^{(k)}_{E_I(f)} = 0$ Equation (1.2) and Theorem 3.2 imply that

$$D_V(E_I(f)) = 0$$

for all vector fields on $\mathbb{H}$ and all $f \in C^\infty_h(M)$, so by Definition 1.7, E_I is a formal trivialization of the formal connection D. We compare this with the explicit formula for the first order term of P in Theorem 5.9, and see that they agree since the Ricci potential F is 0.

Now since the Ricci potential is 0 we reduce the formula in Theorem 5.6 for the formal Hitchin connection.

Theorem 6.5. *Let $(M, \omega, I(Z))$ be a principal polarized variety, then the formal Hitchin connection is given by*

$$D_V f = V[f] - \frac{1}{8\pi} h \Delta_{\tilde{G}(V)}(f),$$

and if Z is normal, we get explicit formulas for $\Delta_{\tilde{G}(V)}$. If $i \neq j$

$$\Delta_{\tilde{G}(\frac{\partial}{\partial Z_{ij}})} = 2i\nabla_{\frac{\partial}{\partial z_i}}\nabla_{\frac{\partial}{\partial z_j}} + 2i\nabla_{\frac{\partial}{\partial z_j}}\nabla_{\frac{\partial}{\partial z_i}} \quad \textit{and} \quad \Delta_{\tilde{G}(\frac{\partial}{\partial \bar{Z}_{ij}})} = 2i\nabla_{\frac{\partial}{\partial \bar{z}_i}}\nabla_{\frac{\partial}{\partial \bar{z}_j}} + 2i\nabla_{\frac{\partial}{\partial \bar{z}_j}}\nabla_{\frac{\partial}{\partial \bar{z}_i}},$$

and if $i = j$

$$\Delta_{\tilde{G}(\frac{\partial}{\partial Z_{ii}})} = 2i\nabla_{\frac{\partial}{\partial z_i}}\nabla_{\frac{\partial}{\partial z_i}} \quad \textit{and} \quad \Delta_{\tilde{G}(\frac{\partial}{\partial \bar{Z}_{ii}})} = 2i\nabla_{\frac{\partial}{\partial \bar{z}_i}}\nabla_{\frac{\partial}{\partial \bar{z}_i}}.$$

This theorem is proved in the appendix. It should be noted that the requirement on Z to be normal, only is to ease the calculations, and it will not be used anywhere else in the rest of this paper.

With this formal trivialization we use Theorem 5.11 and create an I independent star product on $C^\infty(M)$ which all Berezin–Toeplitz star products are equivalent to. This is done in [A2] Theorem 5 where it is shown that the I independent star product actually is the Moyal–Weyl product

$$f \star g = \mu \circ \exp(-\frac{i}{2} h Q)(f \otimes g),$$

where $\mu : C^\infty(\mathbb{R}^{2n}) \otimes C^\infty(\mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n})$ given by multiplication $f \otimes g \mapsto fg$ and

$$Q = \sum_i \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_i} \otimes \frac{\partial}{\partial x_i}.$$

Again we see that this is exactly as in Theorem 5.12.

Abelian Chern–Simons Theory

In 2+1 dimensional Chern–Simons theory, the 2-dimensional part of the theory is a modular functor, which is a functor from the category of compact smooth oriented surfaces to the category of finite dimensional complex vector spaces, which satisfy certain properties. In the gauge-theoretic construction of this functor one first fixes a compact Lie group G and an invariant non-degenerate inner product on its Lie algebra. The functor then associates to a closed oriented surface the finite dimensional vector space one obtains by applying geometric quantization to the moduli space of flat G-connections on the surface (see e.g. [Wi] and [At]). In the abelian case $G = U(1)$ at hand this concretely means the following. For a closed oriented surface Σ the moduli space of flat $U(1)$-connections

$$M = \mathrm{Hom}(\pi_1(\Sigma), U(1)) = H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})$$

has a symplectic structure given by the cup product followed by evaluation on the fundamental class of Σ. This symplectic structure is by Poincar??? duality integral and is unimodular over the lattice $H^1(\Sigma, \mathbb{Z})$. A subgroup of the mapping class group Γ of Σ acts on M via the induced homomorphism

$$\rho : \Gamma \to \mathrm{Aut}(H^1(\Sigma, \mathbb{Z}), \omega) = \mathrm{Sp}(2n, \mathbb{Z}).$$

Define $\Gamma' = \rho^{-1}(A')$ and

$$\rho' = \rho|_{\Gamma'} : \Gamma' \to A'.$$

The homomorphism ρ' is surjective and has the Torelli subgroup of Γ as its kernel.

If we use the above theory we construct a Hermitian vector bundle $H^{(k)}$ over the space of complex structures $\mathcal{T}$ on $H^1(\Sigma, \mathbb{R})$. As discussed this bundle has a flat connection, and an action of action of $\mathrm{Aut}(H^1(\Sigma, \mathbb{Z}), \omega)$ that preserves the Hermitian structure and the flat connection. In this case the modular functor is defined by associating to Σ, the vector space $Z^{(k)}(\Sigma)$ consisting of covariant constant sections of $H^{(k)}$ over $\mathcal{T}$. So through the representation ρ, we get a representation ρ_k of the mapping class group Γ of Σ on $Z^{(k)}(\Sigma)$. In the SU(n)-case in the introduction this representation was denoted $Z_k^{(n,d)}$.

The $2 + 1$ dimensional Chern–Simons theory also fits into a TQFT setup. Suppose Y is a compact oriented 3-manifold such that $\partial Y = (-\Sigma_1) \cup \Sigma_2$, where Σ_1 and Σ_2 are closed oriented surfaces and $-\Sigma_1$ is Σ_1 with reversed orientation. Assume furthermore that γ is a link inside $Y - \partial Y$. Then the TQFT-axioms states that there should be a linear morphism $Z^{(k)}(Y, \gamma) : Z^{(k)}(\Sigma_1) \to Z^{(k)}(\Sigma_2)$, which satisfies that gluing along boundary components goes to the corresponding composition of linear maps.

Definition 6.6. The curve operator

$$Z^{(k)}(Y, \gamma) : Z^{(k)}(\Sigma_1) \to Z^{(k)}(\Sigma_2),$$

is defined to be

$$Z^{(k)}(Y,\gamma) := T^{(k)}_{E_{I(Z)}(h_\gamma),I(Z)},$$

where h_γ is the holonomy function associated to γ.

To a simple closed curve γ on Σ the holonomy function $h_\gamma \in C^\infty(M)$ is a pure phase function, i.e. $h_\gamma = F_{r,s}$ where $r, s \in \mathbb{Z}^n$. Note that we do not label γ with an irreducible $U(1)$-representation λ.

Using this definition we could give the exact same proof as of Theorem 1.3 and obtain a classical theorem from the theory of theta functions.

Theorem 6.7. *Elements in the Torelli subgroup* $\ker\rho'$ *are exactly those who are in the kernels of all* $\rho'_k s$*,*

$$\bigcap_{k=1}^{\infty} \ker\rho_k = \ker\rho'.$$

To this end we want to give a proof of Theorem 1 from [A9] in the case of abelian moduli spaces. We do this by studying the Hilbert–Schmidt norm of the curve operators.

Definition 6.8. The Hilbert–Schmidt inner product of two operator A, B is

$$\langle A, B\rangle = \mathrm{Tr}(AB^*).$$

If we introduce the notation

$$\begin{aligned}\eta_k(r,s) &= \mathrm{Re}(e^{-\frac{\pi i}{k} r\cdot\bar{Z}r} e^{-2\pi i s\cdot\alpha} e^{-\pi^2(s-\bar{Z}r)\cdot(2\pi kY)^{-1}(s-\bar{Z}r)}) \\ &= e^{-\frac{\pi}{2k}((s-Xr)\cdot Y^{-1}(s-Xr)+r\cdot Yr)}\end{aligned}$$

and recall the matrix coefficients of the Toeplitz operators $T^{(k)}_{F_{r,s}}$ in terms of the basis of theta functions then

$$(F_{r,s}\theta^{(k)}_\alpha, \theta^{(k)}_\beta)_Y = \delta_{\alpha-\beta,-[\frac{r}{k}]} e^{-2\pi i s\cdot\alpha} e^{-\frac{\pi i}{k} r\cdot s}\eta_k(r,s).$$

Note that $f(r,s,Z)(k) = \eta_k(r,s)^{-1}$. Here we suppress the Z dependence in $\eta_k(r,s)$ since we from now only consider fixed Kähler structure.

Lemma 6.9.

$$\mathrm{Tr}(T^{(k)}_{F_{r,s}}(T^{(k)}_{F_{t,u}})^*) = \begin{cases} k^n\eta_k(r,s)\eta_k(t,u)\epsilon(r,s,t,u) & (r,s)\equiv(t,u) \ \mathit{mod}\ k \\ 0 & \mathit{else}\end{cases}$$

where $\epsilon(r,s,t,u) \in \{\pm1\}$ *and is* 1 *for* $(r,s) = (t,u)$*.*

Proof. We start by calculating the matrix coefficients of the product of the Toeplitz operators

$$\begin{aligned}(T^{(k)}_{F_{r,s}}(T^{(k)}_{F_{t,u}})^*)_{\beta\alpha} &= \sum_{\phi}(T^{(k)}_{F_{r,s}})_{\beta\phi}\overline{(T^{(k)}_{F_{t,u}})_{\alpha\phi}} \\ &= \delta_{\alpha-\beta,-[\frac{r-t}{k}]}e^{-2\pi i\alpha\cdot(s-u)}e^{-\frac{\pi i}{k}(r\cdot s-2s\cdot t+t\cdot u)}\eta_k(r,s)\eta_k(t,u).\end{aligned}$$

Now when taking the trace $\alpha=\beta$ and to get something non-zero we must have $r\equiv t \mod k$. In that case

$$\operatorname{Tr}(T^{(k)}_{F_{r,s}}(T^{(k)}_{F_{t,u}})^*) = \epsilon(r,s,t,u)\eta_k(r,s)\eta_k(t,u)e^{\frac{\pi i}{k}r\cdot(s-u)}\sum_{\alpha}e^{-2\pi i\alpha(s-u)},$$

the ϵ is obtained since $t=r+kv$ only determines the equality

$$e^{-\frac{\pi i}{k}(r\cdot s-2s\cdot t+t\cdot u)} = \pm e^{\frac{\pi i}{k}(r\cdot(s-u))}$$

up to a sign. Now if $s\not\equiv u$ the last term is zero since it is n sums of all k'th roots of unity, and hence 0. If $s\equiv u$ each term in the sum is 1, and we get the desired result. □

Using the above lemma and the following limits

$$\lim_{k\to\infty}\eta_k(r,s)=1 \qquad \text{and} \qquad \lim_{k\to\infty}\eta_k(r+kt,s+ku)=0, \tag{6.2}$$

for all $r,s\in\mathbb{Z}^n$, we can prove the following

Theorem 6.10. *For any two smooth functions $f,g\in C^\infty(M)$ and any $Z\in\mathbb{H}$ one has that*

$$\langle f,g\rangle = \lim_{k\to\infty}k^{-n}\left\langle T^{(k)}_{f,I(Z)},T^{(k)}_{g,I(Z)}\right\rangle,$$

where the real dimension of M is $2n$.

Proof. From Lemma 6.9 we get in particular

$$\|T^{(k)}_{F_{r,s}}\|_k = k^{-n/2}\sqrt{\operatorname{Tr}(T^{(k)}_{F_{r,s}}(T^{(k)}_{F_{r,s}})^*)} = \eta_k(r,s),$$

and

$$\|T^{(k)}_{E_I(F_{r,s})}\|_k = 1,$$

where $\|\cdot\|_k = k^{-n/2}\sqrt{\langle\cdot,\cdot\rangle}$ is the k-scaled Hilbert–Schmidt norm.

Let $f,g\in C^\infty(M)$ be an arbitrary elements and expand them in Fourier series

$$f=\sum_{(r,s)\in\mathbb{Z}^{2n}}\lambda_{r,s}F_{r,s} \qquad \text{and} \qquad g=\sum_{(t,u)\in\mathbb{Z}^{2n}}\mu_{t,u}F_{t,u}.$$

$\eta_k(r,s)$ and $\eta_k(t,u)$ decays very fast for increasing $r,s \in \mathbb{Z}^n$ and we have

$$\begin{aligned} k^{-n}\operatorname{Tr}(T_f^{(k)}(T_g^{(k)})^*) &= k^{-n} \sum_{(r,s),(t,u)\in\mathbb{Z}^{2n}} \lambda_{r,s}\bar{\mu}_{t,u}\operatorname{Tr}(T_{F_{r,s}}^{(k)}(T_{F_{t,u}}^{(k)})^*) \\ &= \sum_{(r,s)\in\mathbb{Z}^{2n}} \lambda_{r,s}\bar{\mu}_{t,u}\eta_k(r,s)^2 \\ &\quad + \sum_{\substack{(r,s),(t,u)\in\mathbb{Z}^{2n}\\(t,u)\neq(0,0)}} \lambda_{r,s}\bar{\mu}_{r+kt,s+ku}\eta_k(r,s)\eta_k(r+kt,s+ku)\epsilon(r,s,t,u). \end{aligned}$$

This sum converges uniformly so if we take the large k limit we can interchange limit and summation. Now by Equation 6.2 and since

$$\lim_{k\to\infty} \mu_{r+kt,s+ku} = 0$$

by pointwise convergence of the Fourier series we finally get

$$\lim_{k\to\infty} k^{-n}\operatorname{Tr}(T_f^{(k)}(T_g^{(k)})^*) = \sum_{(r,s)\in\mathbb{Z}^{2n}} \lambda_{r,s}\bar{\mu}_{r,s}.$$

Now since the pure phase functions are orthogonal we get to desired result. □

It should be remarked that Theorem 6.10 just is a particular case of a theorem of the same wording, with M being a compact Kähler manifold, see e.g. [A9]. Theorem 6.10 was also proved in [BHSS] but only for a small class of principal polarized abelian varieties.

As a corollary to the proof of Theorem 6.10 we have

Corollary 6.11.

$$\langle f,g\rangle = \lim_{k\to\infty} k^{-n}\left\langle T_{E_I(f)}^{(k)}, T_{E_I(g)}^{(k)}\right\rangle.$$

We can interpret Corollary 6.11 in terms of TQFT curve operators. Since we defined a curve operator $Z^{(k)}(\Sigma,\gamma)$ to be $T_{E_I(h_\gamma)}^{(k)}$ where h_γ is the corresponding holonomy function of γ we immediately get

$$\langle h_{\gamma_1}, h_{\gamma_2}\rangle = \lim_{k\to\infty} k^{-n}\left\langle Z^{(k)}(\Sigma,\gamma_1), Z^{(k)}(\Sigma,\gamma_2)\right\rangle,$$

which was proved in [A9] and [MN].

Another interpretation is that gluing two cylinders $(\Sigma\times[0,1],\gamma_1)$ and $(\Sigma\times[0,1],\gamma_2)$ along $\Sigma\times\{0\}$ and $-\Sigma\times\{0\}$ and again at the top $\Sigma\times\{1\}$ along $-\Sigma\times\{1\}$, we obtain the closed three manifold $\Sigma\times S^1$ with the link $\gamma_1\cup\gamma_2^*$ embedded. Here γ_2^* means γ_2 with reversed orientation. The TQFT gluing axioms now say that

$$Z^{(k)}(\Sigma\times S^1,\gamma_1\cup\gamma_2^*) = \operatorname{Tr}(Z^{(k)}(\Sigma\times[0,1],\gamma_1)Z^{(k)}(\Sigma\times[0,1],\gamma_2)^*).$$

If we now define $Z^{(k)}(\Sigma\times S^1,\gamma_1\cup\gamma_2^*)$ to be exactly this, we see that if we take γ_1 and γ_2 to be the empty links we have

$$Z^{(k)}(\Sigma\times S^1) = \operatorname{Tr}(T_1^{(k)}(T_1^{(k)})^*) = k^n = \dim(Z^{(k)}(\Sigma)),$$

as it should be according to the axioms.

7 Appendix

In this appendix we provide the calculations needed to prove the explicit formuli for the formal Hitchin connection given in Theorem 6.5.

We first observe that the theorem will follow from Equation 2.2 if we can show that for $i \neq j$

$$\tilde{G}(\frac{\partial}{\partial Z_{ij}}) = 2i\frac{\partial}{\partial z_i}\otimes\frac{\partial}{\partial z_j} + 2i\frac{\partial}{\partial z_j}\otimes\frac{\partial}{\partial z_i} \quad \text{and} \quad \tilde{G}(\frac{\partial}{\partial \bar{Z}_{ij}}) = 2i\frac{\partial}{\partial \bar{z}_i}\otimes\frac{\partial}{\partial \bar{z}_j} + 2i\frac{\partial}{\partial \bar{z}_j}\otimes\frac{\partial}{\partial \bar{z}_i}$$

and for $i = j$

$$\tilde{G}(\frac{\partial}{\partial Z_{ii}}) = 2i\frac{\partial}{\partial z_i}\otimes\frac{\partial}{\partial z_i} \quad \text{and} \quad \tilde{G}(\frac{\partial}{\partial \bar{Z}_{ii}}) = 2i\frac{\partial}{\partial \bar{z}_i}\otimes\frac{\partial}{\partial \bar{z}_i}.$$

Since the family of Kähler structures parametrized by $\mathbb{H}$ is holomorphic we just have to solve the equations

$$G(\frac{\partial}{\partial Z_{ij}})\cdot\omega = \frac{\partial I(Z)}{\partial Z_{ij}} \quad \text{and} \quad \bar{G}(\frac{\partial}{\partial \bar{Z}_{ij}})\cdot\omega = \frac{\partial I(Z)}{\partial \bar{Z}_{ij}}.$$

Lemma 7.1. *The Kähler structure associated to a $Z = X + iY \in \mathbb{H}$ is*

$$I(Z) = \begin{pmatrix} -Y^{-1}X & -(Y + XY^{-1}X) \\ Y^{-1} & XY^{-1} \end{pmatrix},$$

where we have written it as tensor in the frame $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}$ of the tangent bundle TM.

Proof. This follows from the fact that the complex frame of TM are eigenvectors for $I(Z)$, that is

$$I(Z)(\frac{\partial}{\partial z_i}) = i\frac{\partial}{\partial z_i} \quad \text{and} \quad I(Z)(\frac{\partial}{\partial \bar{z}_i}) = -i\frac{\partial}{\partial \bar{z}_i},$$

and that

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{i}{2}Y^{-1}\bar{Z}\frac{\partial}{\partial x} + \frac{1}{2i}Y^{-1}\frac{\partial}{\partial y} = \frac{i}{2}(Y^{-1}X - i)\frac{\partial}{\partial x} + \frac{1}{2i}Y^{-1}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2i}Y^{-1}Z\frac{\partial}{\partial x} + \frac{i}{2}Y^{-1}\frac{\partial}{\partial y} = \frac{1}{2i}(Y^{-1}X + i)\frac{\partial}{\partial x} + \frac{i}{2}Y^{-1}\frac{\partial}{\partial y}. \end{aligned}$$

□

In the above we used the vector notation $\frac{\partial}{\partial z}$ meaning an n-tuple of vectors $\frac{\partial}{\partial z_i}$. This convention eases the following calculations and will be used in the following.

Proof of Theorem 6.5. To this end we also need to recall the following derivation property for matrices. If $A = (a_{ij})$ is a symmetric invertible $n \times n$-matrix then

$$\frac{\partial A^{-1}}{\partial a_{ij}} = -A^{-1}\frac{\partial A}{\partial a_{ij}}A^{-1} = -A^{-1}\Delta_{ij}A^{-1},$$

where Δ_{ij} is an $n \times n$-matrix with all entries 0 except the ij'th and ji'th which is 1, if $i \neq j$ and Δ_{ii} is an $n \times n$-matrix with all entries 0 except the ii'th diagonal entry which is 1. This follows easily from $A^{-1}A = Id$. Using this rule and that $Y^{-1} = 2i(Z - \bar{Z})^{-1}$ we get

$$\frac{\partial Y^{-1}}{\partial Z_{ij}} = -\frac{1}{2i}Y^{-1}\Delta_{ij}Y^{-1}.$$

Derivation of the above equations with respect to Z_{ij} becomes rather messy if we do not also require Z to be normal, that is since Z is symmetric $[Z, \bar{Z}] = 0$, which is equivalent to $[X, Y] = 0$, or $[X, Y^{-1}] = 0$. A consequence of this is, that everything will commute even $[Y^{-1}, \Delta_{ij}] = 0$ since the imaginary part of derivation of $Z\bar{Z} = \bar{Z}Z$ with respect to Z_{ij} give $Y\Delta_{ij} = \Delta_{ij}Y$, and hence $[Y^{-1}, \Delta_{ij}] = 0$. Written as a tensor

$$\frac{\partial I(Z)}{\partial Z_{ij}} = \frac{1}{2i}Y^{-1}\Delta_{ij}Y^{-1}\begin{pmatrix} \bar{Z} & \bar{Z}^2 \\ -1 & -\bar{Z} \end{pmatrix}.$$

The symplectic form $\omega = -\frac{1}{2i}\sum_{ij=1}^{n} w_{ij}dz_i \wedge d\bar{z}_j$ where $Y^{-1} = W = (w_{ij})$, should be contracted with $G(\frac{\partial}{\partial Z_{ij}})$ we want to know its appearance in the Z-dependent $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar{z}}$ frame. It is clear from above that

$$\frac{\partial I(Z)}{\partial Z_{ij}}(\frac{\partial}{\partial z}) = \frac{i}{2}Y^{-1}\bar{Z}\frac{\partial I(Z)}{\partial Z_{ij}}(\frac{\partial}{\partial x}) + \frac{1}{2i}Y^{-1}\frac{\partial I(Z)}{\partial Z_{ij}}(\frac{\partial}{\partial y}) = 0,$$

and an easy calculation shows that

$$\frac{\partial I(Z)}{\partial Z_{ij}}(\frac{\partial}{\partial \bar{z}}) = -Y^{-1}\Delta_{ij}\frac{\partial}{\partial z}.$$

In other words

$$\frac{\partial I(Z)}{\partial Z_{ij}} = \begin{cases} -\sum_{k=1}^{n}(w_{ki}\frac{\partial}{\partial z_j} \otimes d\bar{z}_k + w_{kj}\frac{\partial}{\partial z_i} \otimes d\bar{z}_k) & \text{for } i \neq j \\ -\sum_{k=1}^{n} w_{ki}\frac{\partial}{\partial z_i} \otimes d\bar{z}_k & \text{for } i = j \end{cases}$$

Remark that since $I(Z)^2 = -Id$, $\frac{\partial I(Z)}{\partial Z_{ij}}$ and $I(Z)$ anti-commute. This is clearly reflected in the above expressions for $\frac{\partial I(Z)}{\partial Z_{ij}}$. Now since $G(\frac{\partial}{\partial Z_{ij}})$ is defined by

$$-G(\frac{\partial}{\partial Z_{ij}}) \cdot \frac{1}{2i}\sum_{kl=1}^{n} w_{kl}dz_k \wedge d\bar{z}_l = \frac{\partial I(Z)}{\partial Z_{ij}}$$

it is

$$G(\frac{\partial}{\partial Z_{ij}}) = \begin{cases} 2i\frac{\partial}{\partial z_i}\otimes\frac{\partial}{\partial z_j} + 2i\frac{\partial}{\partial z_j}\otimes\frac{\partial}{\partial z_i} & \text{for } i\neq j \\ 2i\frac{\partial}{\partial z_i}\otimes\frac{\partial}{\partial z_i} & \text{for } i=j. \end{cases}$$

With $G(\frac{\partial}{\partial Z_{ij}})$ being expressed in complex coordinates, we should mentioned that the family of Kähler structures parametrized by $\mathbb{H}$ in the way described above, actually is rigid, i.e. $\bar\partial_Z(G(V)_Z) = 0$ for all vector field V on $\mathbb{H}$. This is clear since $G(\frac{\partial}{\partial Z_{ij}})$ is zero in $\bar z_i$ directions and $G(\frac{\partial}{\partial \bar Z_{ij}}) = 0$.

We could do exactly the same thing with $\frac{\partial}{\partial \bar Z_{ij}}$ and obtain

$$\frac{\partial I(Z)}{\partial \bar Z_{ij}} = -\frac{1}{2i}Y^{-1}\Delta_{ij}Y^{-1}\begin{pmatrix} Z & Z^2 \\ -1 & -Z\end{pmatrix}.$$

Again it is clear that

$$\frac{\partial I(Z)}{\partial \bar Z_{ij}}(\frac{\partial}{\partial z}) = 0 \quad \text{and} \quad \frac{\partial I(Z)}{\partial \bar Z_{ij}} = -Y^{-1}\Delta_{ij}\frac{\partial}{\partial \bar z}.$$

In a similar way as above we obtain

$$\bar G(\frac{\partial}{\partial \bar Z_{ij}}) = \begin{cases} 2i\frac{\partial}{\partial \bar z_i}\otimes\frac{\partial}{\partial \bar z_j} + 2i\frac{\partial}{\partial \bar z_j}\otimes\frac{\partial}{\partial \bar z_i} & \text{for } i\neq j \\ 2i\frac{\partial}{\partial \bar z_i}\otimes\frac{\partial}{\partial \bar z_i} & \text{for } i=j. \end{cases} \qquad \square$$

References

[A2] J.E. Andersen, *Geometric quantization and deformation quantization of abelian moduli spaces*, Commun. Math. Phys. **255** (2005), 727–745.

[A3] J. E. Andersen, *Asymptotic faithfulness of the quantum* SU(n) *representations of the mapping class groups*. Annals of Mathematics, **163** (2006), 347–368.

[A6] J.E. Andersen, *Hitchin's connection, Toeplitz operators and symmetry invariant deformation quantization*, math.DG/0611126.

[A9] J.E. Andersen, *Asymptotics of the Hilbert–Schmidt norm in TQFT*, Lett Math Phys (2010) 91:205-214.

[AG] J.E. Andersen & N. L. Gammelgaard, *Hitchin's Projectively Flat Connection, Toeplitz Operators And The Asymptotic Expansion of TQFT Curve Operators*, arXiv:0903.4091v1.

[AGL] J.E. Andersen, M. Lauritsen & N. L. Gammelgaard, *Hitchin's Connection in Half-form Quantization*, arXiv:0711.3995.

[AMR2] J.E. Andersen, J. Mattes & N. Reshetikhin, *Quantization of the Algebra of Chord Diagrams*. Math. Proc. Camb. Phil. Soc. **124** pp.451–467 (1998).

[AU1] J.E. Andersen & K. Ueno, *Abelian Conformal Field theories and Determinant Bundles*, International Journal of Mathematics. **18**, (2007) 919–993.

[AU2] J.E. Andersen & K. Ueno, *Constructing modular functors from conformal field theories*, Journal of Knot theory and its Ramifications. **16** 2 (2007), 127–202.

[AU3] J.E. Andersen & K. Ueno, *Modular functors are determined by their genus zero data*, math.QA/0611087.

[AU4] J.E. Andersen & K. Ueno, *Construction of the Reshetikhin-Turaev TQFT from conformal field theory*, In preparation.

[AV1] J.E. Andersen & R. Villemoes, *Degree One Cohomology with Twisted Coefficients of the Mapping Class Group*, arXiv:0710.2203v1.

[AV2] J.E. Andersen & R. Villemoes, *The first cohomology of the mapping class group with coefficients in algebraic functions on the SL(2, C) moduli space*, arXiv:0802.4372v1.

[AV3] J.E. Andersen & R. Villemoes, *Cohomology of mapping class groups and the abelian moduli space*, In preparation.

[At] M. Atiyah, *The Jones-Witten invariants of knots*. Séminaire Bourbaki, Vol. 1989/90. Astérisque No. **189-190** (1990), Exp. No. 715, 7–16.

[ADW] S. Axelrod, S. Della Pietra, E. Witten, *Geometric quantization of Chern Simons gauge theory*, J.Diff.Geom. **33** (1991) 787–902.

[BK] B. Bakalov and A. Kirillov, *Lectures on tensor categories and modular functors*, AMS University Lecture Series, **21** (2000).

[B1] C. Blanchet, *Hecke algebras, modular categories and* 3*-manifolds quantum invariants*, Topology **39** (2000), no. 1, 193–223.

[BHMV1] C. Blanchet, N. Habegger, G. Masbaum & P. Vogel, *Three-manifold invariants derived from the Kauffman Bracket*. Topology **31** (1992), 685–699.

[BHMV2] C. Blanchet, N. Habegger, G. Masbaum & P. Vogel, *Topological Quantum Field Theories derived from the Kauffman bracket*. Topology **34** (1995), 883–927.

[BHSS] M. Bordemann, J. Hoppe, P. Schaller & M. Schlichenmaier, *gl(*∞*) and Geometric Quantization*, Commun. Math. Phys. 138, 209–244 (1991).

[BMS] M. Bordeman, E. Meinrenken & M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and* $gl(N), N \to \infty$ *limit*, Comm. Math. Phys. **165** (1994), 281–296.

[Ch] L. Charles, *Asymptotic properties of the quantum representations of the mapping class group*, arXiv:1005.3452v2.

[DN] J.-M. Drezet & M.S. Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques*, Invent. math. **97** (1989) 53–94.

[Fi] M. Finkelberg, *An equivalence of fusion categories*, Geom. Funct. Anal. **6** (1996), 249–267.

[G] N. L. Gammelgaard, *Kähler Quantization and Hitchin Connections*, Thesis, Aarhus University, 2010.

[GH] P. Griffiths & J. Harris, *Principles of algebraic geometry*, New York: Wiley & Sons Inc, 1978.

[Go1] W. M. Goldman, *Ergodic theory on moduli spaces*, Ann. of Math. (2) 146 (1997), no. 3, 475–507.

[Go2] W. M. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, Invent. Math. 85 (1986), no. 2, 263–302.

[H] N. Hitchin, *Flat connections and geometric quantization*, Comm.Math.Phys., **131** (1990) 347–380.

[KS] A. V. Karabegov & M. Schlichenmaier, *Identification of Berezin-Toeplitz deformation quantization*, J. Reine Angew. Math. **540** (2001), 49–76.

[KL] D. Kazhdan & G. Lusztig, *Tensor structures arising from affine Lie algebras I*, J. AMS, **6** (1993), 905–947; II J. AMS, **6** (1993), 949–1011; III J. AMS, **7** (1994), 335–381; IV, J. AMS, **7** (1994), 383–453.

[La1] Y. Laszlo, *Hitchin's and WZW connections are the same*, J. Diff. Geom. **49** (1998), no. 3, 547–576.

[MN] J. March??? & M. Narimannejad, *Some asymptotics of TQFT via skein theory*. Duke Math. J. 141(3), 572–587 (2008).

[RT1] N. Reshetikhin & V. Turaev, *Ribbon graphs and their invariants derived fron quantum groups*, Comm. Math. Phys. **127** (1990), 1–26.

[RT2] N. Reshetikhin & V. Turaev, *Invariants of* 3*-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), 547–597.

[Sch] M. Schlichenmaier, *Berezin-Toeplitz quantization and conformal field theory*, Habilitation Thesis.

[Sch1] M. Schlichenmaier, *Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization.* In *Conférence Moshé Flato 1999, Vol. II (Dijon)*, 289–306, Math. Phys. Stud., **22**, Kluwer Acad. Publ., Dordrecht, (2000), 289–306.

[Sch2] M. Schlichenmaier, *Berezin–Toeplitz quantizatino of compact Kähler manifolds.*, Quantization, coherent states and Poisson structures (Bialoweiza, 1995), 101–115, PWN, Warsaw, 1998.

[Se] G. Segal, *The Definition of Conformal Field Theory*, Oxford University Preprint (1992).

[TUY] A. Tsuchiya, K. Ueno & Y. Yamada, *Conformal Field Theory on Universal Family of Stable Curves with Gauge Symmetries*, *Advanced Studies in Pure Mathmatics*, **19** (1989), 459–566.

[T] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, 1994. x+588 pp. ISBN: 3-11-013704-6

[Vi] R. Villemoes, *The mapping class group orbit of a multicurve*, arXiv:0802.3000v2

[Wi] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys **121** (1989) 351–98.

[Wo] N.J. Woodhouse, *Geometric Quantization*, Oxford University Press, Oxford (1992).

Jørgen Ellegaard Andersen
Centre for Quantum Geometry of Moduli Spaces
Faculty of Science and Technology
Aarhus University, DK-8000, Denmark
andersen@imf.au.dk

Jakob Lindblad Blaavand
Centre for Quantum Geometry of Moduli Spaces
Faculty of Science and Technology
Aarhus University, DK-8000, Denmark
blaavand@imf.au.dk

Travaux mathématiques, Volume 19 (2011), 203–271, © Université du Luxembourg

Quantum spectral curve method

by Dmitry V. Talalaev

Contents

1 Introduction

The main focus of these lectures is of direct relevance in two of the most important directions of developments in geometry and topology of the 20th century, the applications of the theory of integrable systems and the applications of the ideas of quantum physics. The most visible result of the first direction is the solution of the Schottky problem [1], based on the conjectures of S.P. Novikov. The challenge of characterizing Jacobians among other principally-polarized abelian varieties has been resolved in terms of non-linear equations: as expression in the θ-function satisfies the KP equation if and only if the corresponding abelian variety is the Jacobian of an algebraic curve. A further development of this direction was the proof of the Welters conjecture [2] on Jacobian matrices in terms of trisecant of Kummer variety. A second major layer of results is associated to applications of quantum field theory in the challenge to construct topological invariants. Jones-Witten invariants or more generally – quantum topological field theory – generalizes the traditional invariants: Alexander polynomials and Jones polynomials. The invariants in these cases are constructed as correlation functions for some quantum field theory [3]. The theory of Donaldson invariants [4] and its development by Seiberg and Witten is another important example of an application of quantum physics in topology.

This work is devoted to constructing quantum analogues of algebraic-geometric methods which are applicable in solving classical integrable systems. These methods are based on the spectral curve concept and the Abel transform. In addition to applications in topology, the explicit description of solutions for quantum integrable systems is directly linked to such problems as the calculation of the cohomology of the θ-divisor for abelian varieties [6], calculation of cohomology and characteristic classes for moduli spaces of stable holomorphic bundles [7], and further generalizations [8].

In these lectures we propose a quantum analog of the spectral curve method for the rational and elliptic Gaudin models [9]. These cases correspond to genus 0 and 1 base curves in Hitchin classification. The material is related to topological invariants of quantum field theory type, as well as it is closely connected with geometric properties of the moduli spaces, to a certain part with the goal to describe the spectrum of quantum systems. The results are based on the methodological approach based on the concept of the quantum spectral curve. They show up in the explicit construction of discrete group symmetries for the corresponding spectral systems.

Classical integrable systems.

Interactions between the theory of integrable systems and algebraic geometry appeared quite early. A pioneering work, linking these areas of mathematics, was due to Jacoby [12]. It solved the problem of geodesics on ellipsoid in terms of the Abel transform for some algebraic curve. The extent of this observation was recognized in the 1970s by the S.P. Novikov school [10, 13]. Later the universal geometric description of the phase space of a wide class of finite-dimensional integrable systems in terms of the cotangent bundle to some moduli space of holomorphic bundles on an algebraic curve was given in the work of N. Hitchin [14].

The algebraic point of view on integrable systems, which evolved in parallel, was based on the principles of Hamiltonian Dynamics and Poisson geometry. The significant progress made within the classical theory of integrable systems was related to the invention of the inverse scattering method in the 60s of last century [16]. It turned out that the Lax representation is an extremely effective description of dynamical systems [17]. This language relates Hamiltonian flows with a corresponding Lie algebra action. This point of view allows to introduce the notion of the spectral curve and use methods of algebraic geometry to construct explicit solutions [18], to solve dynamical systems in algebraic terms by the projection method [19], or by a little bit more general construction of the Sato grassmannian and the corresponding τ-function [20].

Further, we use the term "the spectral curve method" for the method of solving dynamical systems having Lax representation in terms of the Abel transform for the curve defined by the characteristic polynomial of the Lax operator.

The first part of the work is dedicated to a construction of a generalization of the Hitchin type systems in case that the base curves has singularities and fixed points. The main example of the here proposed quantization technique, the Gaudin model, is a particular case of a Hitchin system of generalized nature.

Quantization.

The examples of quantum integrable models discussed here, have an independent physical meaning as spin chain quantum-mechanical systems describing one-dimensional magnets.

However, the main focus of these lectures is to study the structural role of integrable systems including the quantum level, where their role as symmetries of more complex objects is also evident. In particular, spin chains that describe one-dimensional physical systems are associated to 2D problems of statistical physics [9]. A principal method of quantum systems called quantum inverse scattering method (QISM) was established in the 70s of the 20th century by the school of L. D. Faddeev [21]. In many aspects this method relies on the classical inverse problem method, in particular with respect to the Hamiltonian description. Using QISM, several examples of quantum integrable systems were constructed: quantum nonlinear Schroedinger equation, the Heisenberg magnet and the sine-Gordon model (it is equivalent to the massive Thirring model). The asymptotic correlation functions for these models were found in [47]. Many of the results regarding QISM were aware of the earlier framework of the Bethe ansatz method discovered in 1931 [22].

QISM was considerably generalized by the theory of quantum groups developed by Drinfeld [23]. The language of Hopf algebras is very convenient for working with algebraic structures of the theory of quantum integrable systems, specifically for generalizations of the ring of invariant polynomials on the group.

One can consider the QISM as the quantum analog of the algebraic part of the theory of integrable systems. The second part of the lectures concerns the quantum spectral curve method, whose central object is the quantum characteristic polynomial for the quantum Lax operator. We propose a construction for the $\mathfrak{sl}_n$ Hitchin-type systems for base curves of genus 0 and 1 with marked points. The elliptic spin Calogero-Moser system is a particular case of the considered families. The quantum characteristic polynomial is a generating function for the quantum Hamiltonians. The construction is based on quantum group methods, in particular, the theory of Yangians and the Felder's elliptic dynamical quantum algebras.

As noted above QISM has not provided substantial progress in solving quantum systems on the finite scale level. Despite the fact that separated variables were found for some models, the analogue of the Abel transform as transition from the divisor space to the Jacobian has not been found in the quantum case. In part three a family of geometric symmetries on the set of quantum system solutions is constructed, essentially using the quantum characteristic polynomial of the model. The alternative formulation of the Bethe system is used to construct this family. The formulation is given in terms of a family of special Fuchsian systems with restricted monodromy representations. In turn, these differential operators are scalar analogues of the quantum characteristic polynomial. This permits to realize quantum symmetries in terms of well-known Schlesinger transformations in the theory of isomonodromic deformations [24], and apply known solutions of the differential equations of Painleve type to describe variations of the spectrum of the quantum systems, changing the inhomogeneity parameters. In a sense to build a family of symmetries of the spectrum is an analogue of the Abel transform.

Quantum method of the spectral curve and other areas of modern mathematics.

The study of the quantum characteristic polynomial for the Gaudin models was systematized and gave much more efficient methods for solving quantum integrable systems. The constructed discrete symmetries of the spectral systems provide generalized angle operators, meaning that one can build eigenvectors of the model recurrently. The significance of the results in geometry and topology is the possibility to apply this technique to field theoretic models arising in topological quantum field theories and field theories used in the construction of Donaldson and Seiberg-Witten invariants. In addition, the results on the solutions of quantum systems have direct application for the description of cohomologies of moduli spaces of holomorphic bundles, analogues of the Laumon spaces, as well as affine Jacobians.

The method have got influences in numerous relations and application in other areas of modern mathematics and mathematical physics. In the representation theory of Lie algebras the results are related to the effectivization of the multiplicity formula. Applications of this type occur thanks to special limits of the Gaudin commutative subalgebras which are interpreted as subalgebras of central elements in $U(\mathfrak{sl}_n)^{\otimes N}$ [25]. Another result of this technique is an explicit description of the center of the universal enveloping algebra of the affine algebra at the critical level for $\mathfrak{sl}_n$. It is also worth noting that the quantum spectral curve method is also important in the geometrical Langlands program over $\mathbb{C}$ [26], in the booming field of Noncommutative Geometry, mathematical physics and condensed matter theory. Some of the applications are presented in Section 5.

Thanks The author is very grateful to the staff of the Chair of Higher Geometry and Topology of the Mechanics and Mathematics Faculty of the Moscow State University for the fruitful atmosphere and valuable observations during the preparation of the lectures. The author is grateful to the 170-th and 197-th laboratories of the Institute for Theoretical and Experimental Physics for stimulating conversations. The author expresses special thanks to O. Babelon, V.M. Buhshtaber, A.P. Veselov, A. M. Levin, S.A. Loktev, M.A. Olshanetsky, I.E. Panov, V.N. Rubtsov, A.V. Silantiev, A.V. Chervov, G.I. Sharygin. This work is partially supported by the Foundation "Dynasty", RFBR grant 09-01-00239 and the grant for support of scientific school 5413.2010.1.

2 The classical spectral curve method

2.1 Lax representation

This topic describes the classical spectral curve method for finite-dimensional integrable systems. The explanation begins with the Lax representation [17], which has led to the formulation of the inverse problem method in the theory of integrable systems. It turns out that a very wide class of integrable systems is of Lax type

$$\dot{L}(z) = [M(z), L(z)] \tag{2.1}$$

where $M(z), L(z)$ are matrix-valued functions of the formal variable z, those matrix elements are, in turn, the functions on the phase space of the model. In other words, the phase space of a system may be embedded into some space of matrix-valued functions where the dynamics is described by the Lax equation (2.1).

Locally, this property is fulfilled for all integrable systems due to the existence of local "action-angle" variables ([28], 2 4 Example 1). In general, the Lax expression is known for: harmonic oscillator, integrable tops, the Newman model, the problem of geodesics on an ellipsoid, the open and periodic Toda chains, the Calogero-Moser systems for all types of root systems, the Gaudin model, nonlinear hierarchies: KdV, KP, Toda, as well as their famous matrix generalizations. The Lax representation demonstrates that the Hamiltonian vector field $\dot{L} = \{h, L\}$ can be expressed in terms of the Lie algebra structure on the space of matrices. This property is at the heart of many algebraic analytic techniques, in particular of the r-matrix approach and the decomposition problem [29].

The Lax representation means that the characteristic polynomial of the Lax operator is preserved by the dynamics. The spectral curve is defined by the equation

$$det(L(z) - \lambda) = 0. \tag{2.2}$$

It turns out that the solution of equations that allow the Lax representation simplifies using the so-called linear problem

$$L(z)\Psi(z) = \lambda\Psi(z). \tag{2.3}$$

The Lax equation is equivalent to the compatibility condition of the following equations:

$$\begin{aligned} \lambda\Psi(z) &= L(z)\Psi(z), \\ \dot{\Psi}(z) &= M(z)\Psi(z). \end{aligned}$$

If we interpret this auxiliary linear problem as a way of specifying a line bundle on a spectral curve, the system can be solved by means of linear coordinates on the moduli space of line bundles on the spectral curve identified with the associated Jacobian.

Further on, a Hitchin scheme and some of its generalizations sets out pretending to the classification description in the theory of finite-dimensional integrable systems. In this section we also define the Gaudin model, and give details of the classical method of spectral curve for the system and separation variables technique.

2.2 The Hitchin description

Let Σ_0 be an algebraic curve and $\mathcal{M} = \mathcal{M}_{r,d}(\Sigma_0)$ be the moduli space of holomorphic stable bundles over Σ_0 of rank r and the determinant bundle d [30]. Let us consider the canonical holomorphic symplectic form on the cotangent bundle to the moduli space $T^*\mathcal{M}$.

The deformation theory [31] allows to explicitly describe fibers of the cotangent bundle. A tangent vector to the moduli space at E corresponding to the infinitesimal deformation in terms of the Cech cocycle can be realized by an element of $H^1(\mathrm{End}(E))$, in turn the cotangent vector at E to the moduli space $\mathcal{M}$ through the Serre duality is an element of the cohomology space $\Phi \in H^0(\mathrm{End}(E) \otimes \mathcal{K})$;

here $\mathcal{K}$ denotes the canonical bundle on Σ_0. In this description the following family of functions can be defined on $T^*\mathcal{M}$

$$h_i : T^*\mathcal{M} \to H^0(\mathcal{K}^{\otimes i}); \qquad h_i(E, \Phi) = \frac{1}{i} tr\Phi^i. \tag{2.4}$$

The direct sum of the collection of mappings h_i

$$h : T^*\mathcal{M} \longrightarrow \oplus_{i=1}^r H^0(\mathcal{K}^{\otimes i})$$

is called the Hitchin map [14] and defines a Lagrangian fibration of the phase space of the integrable system.

2.2.1 Spectral curve

The spectral curve method provides an explicit method of solution in terms of some geometric objects on a certain algebraic curve. Consider the (nonlinear) bundle map

$$char(\Phi) : \mathcal{K} \to \mathcal{K}^{\otimes r}, \tag{2.5}$$

defined by the expression

$$char(\Phi)(\mu) = det(\Phi - \mu * Id) \tag{2.6}$$

where μ is a point of $\mathcal{K}$, and $Id \in End(E)$ is the unit. The spectral curve is defined as the preimage of the zero section of $\mathcal{K}^{\otimes r}$. The preimage defines an algebraic curve Σ in the projectivization of the total space of $\mathcal{K}$.

2.2.2 Line bundle

Solution to the Hitchin type system can be constructed in terms of the following line bundle. Consider the projection map π corresponding to the canonical bundle $\mathcal{K}$

$$\pi : \mathcal{K} \to \Sigma_0$$

and the inverse image map

$$\pi^* E \stackrel{\Phi - \tilde{\mu} * Id}{\longrightarrow} \pi^*(E \otimes \mathcal{K}),$$

where $\tilde{\mu}$ is the tautological section $\pi^*\mathcal{K}$. Let us also consider the quotient $\mathcal{F}$, corresponding to the inclusion

$$0 \longrightarrow \pi^* E \stackrel{\Phi - \mu * Id}{\longrightarrow} \pi^*(E \otimes \mathcal{K}) \longrightarrow \mathcal{F} \longrightarrow 0. \tag{2.7}$$

The support of $\mathcal{F}$ coincides with the spectral curve Σ defined below. Let us restrict the exact sequence (2.7) to Σ

$$0 \longrightarrow \mathcal{L} \longrightarrow \pi^* E|_\Sigma \stackrel{\Phi-\mu * Id}{\longrightarrow} \pi^*(E\otimes\mathcal{K})|_\Sigma \longrightarrow \mathcal{F}|_\Sigma \longrightarrow 0.$$

It turns out that $\mathcal{L}$ specifies a line bundle on the spectral curve associated with eigenvectors of the Lax operator.

Let us define the Abel transform as follows: let $\{a_1,\dots,a_g,b_1,\dots,b_g\}$ be a basis in $H_1(\Sigma_0,\mathbb{Z})$ with the intersection indexes $(a_i,b_j)=\delta_{ij}$, $\{\omega_i\}$ be the basis of holomorphic differentials in $H^0(\mathcal{K})$ normalized by the condition $\oint_{a_i}\omega_j=\delta_{ij}$, and let $B_{ij}=\oint_{b_i}\omega_j$ be the matrix of b-periods. Then we define the lattice Λ in $\mathbb{C}^g$ generated by the $\mathbb{Z}^g$ and the lattice generated by the columns of the matrix B. Fixing a point $P_0\in\Sigma$ one can define the Abel transform by the formula

$$A:\Sigma\to Jac_\Sigma=\mathbb{C}^g/\Lambda;\qquad A(P)=\begin{pmatrix}\int_{P_0}^P\omega_1\\ \vdots\\ \int_{P_0}^P\omega_g\end{pmatrix}. \tag{2.8}$$

This definition does not depend on the integrating path due to the factorization and generalizes to the map from the space of divisor classes to the moduli space of line bundles.

Theorem 2.1 ([14])**.** *The linear coordinates on the Jacobian $Jac(\Sigma)$ applied to the image of the Abel transform $A(\mathcal{L})$ are the "'angle"' variables for the Hitchin system.*

2.3 The Hitchin systems on singular curves

2.3.1 Generalizations

The Hitchin construction can be generalized to the case of singular curves and curves with fixed points [32], [33]. This generalization permits to give explicit parametrization to the wide class of integrable systems preserving the geometric analogy with the intrinsic ingredients of the original Hitchin system.

- Fixed points: It can be considered the moduli space of holomorphic bundles on an algebraic curve with additional structures, namely with trivializations at fixed points. This moduli space can be obtained as the quotient of the space of gluing functions by the trivialization change group with the condition of preserving trivializations at fixed points. Let us denote this moduli space by $\mathcal{M}_{r,d}(z_1,\dots,z_k)$. The tangent vector to the space $\mathcal{M}_{r,d}(z_1,\dots,z_k)$ at the point E in an element of the space

$$T_E\mathcal{M}_{r,d}(z_1,\dots,z_k)\simeq H^1(\mathrm{End}(E)\otimes\mathcal{O}(-\sum_{i=1}^k z_i)).$$

The cotangent vector can be identified with the following element

$$\Phi \in H^0(\mathrm{End}(E) \otimes \mathcal{K} \otimes \mathcal{O}(\sum_{i=1}^{k} z_i)).$$

- Singular points: The moduli space of bundles can be considered on curves with singularities of the types: double point, cusp or the so-called scheme double point. In this situation the consistent Hitchin system formalism can be established. This results in constructing a large class of interesting integrable systems. The description of the dualizing sheaf and the moduli space of bundles in this case turns out to be more explicit then in the case of nonsingular curve of the same algebraic genus.

2.3.2 Scheme points

Let us describe in detail the Hitchin formalism on curves with double scheme points.

The singularities class Let us consider a curve Σ^{proj} obtained by gluing 2 subschemes $A(\epsilon), B(\epsilon)$ of $\mathbb{C}P^1$ (i.e. a curve obtained by adding the point ∞ to the affine curve $\Sigma^{aff} = Spec\{f \in \mathbb{C}[z] : \ f(A(\epsilon)) = f(B(\epsilon))\}$, where $\epsilon^N = 0$). Calculating the algebraic genus ($dim H^1(\mathcal{O})$) we obtain:

- Nilpotent elements: $A(\epsilon) = \epsilon, B(\epsilon) = 0$, $g = N - 1$.
- Roots of unity: $A(\epsilon) = \epsilon, B(\epsilon) = \alpha\epsilon$, where $\alpha^k = 1$, $g = N - 1 - [(N-1)/k]$.
- Different points:

  $$A(\epsilon) = a_0 + a_1\epsilon + ... + a_{N-1}\epsilon^{N-1}, \quad B(\epsilon) = b_0 + b_1\epsilon + ... + b_{N-1}\epsilon^{N-1},$$

  supposing $a_0 \neq b_0$, $g = N$.

Holomorphic bundles The most convenient way to describe the moduli space of holomorphic bundles for singular curves is an algebraic language, due to the duality between a bundle and the sheaf of its sections, which is a sheaf of locally-free and thus projective modules over the structure sheaf of an algebraic curve.

The geometrical characterisation of a projective module in the affine chart without ∞ of the normalized curve is made in terms of the submodule M_Λ of rank r in the trivial module of vector-functions $s(z)$ on $\mathbb{C}$ satisfying the condition:

$$s(A(\epsilon)) = \Lambda(\epsilon)s(B(\epsilon)),$$

where $\Lambda(\epsilon) = \sum_{i=0,\ldots,N-1} \Lambda_i \epsilon^i$ is a matrix-valued polynomial. The projectivity condition of this module M_Λ is expressed as follows:

- Nilpotent elements: $A(\epsilon)=\epsilon, B(\epsilon)=0$, condition: $\Lambda_0 = Id$.
- Roots of unity: $A(\epsilon)=\epsilon, B(\epsilon)=\alpha\epsilon$, where $\alpha^k=1$, condition:
$$\Lambda(\epsilon)\Lambda(\alpha\epsilon)...\Lambda(\alpha^{k-1}\epsilon)=Id.$$
- Different points:
$$A(\epsilon)=a_0+a_1\epsilon+...+a_{N-1}\epsilon^{N-1}, B(\epsilon)=b_0+b_1\epsilon+...+b_{N-1}\epsilon^{N-1},$$
condition: Λ_0 is invertible.

The open cell of the moduli space of holomorphic bundles for Σ^{proj} is the quotient space of the space of $\Lambda(\epsilon)$ in generic position satisfying the above condition with respect to the adjoint action of GL_r.

Dualizing sheaf and global section In smooth situation the canonical class $\mathcal{K}$ is determined by the line bundle of the highest order forms on complex analytical variety M. To reconstruct this object in the singular case we axiomatize the Serre duality condition
$$H^n(\mathcal{F})\times H^{m-n}(\mathcal{F}^*\otimes\mathcal{K})\to\mathbb{C}$$
for a coherent sheaf $\mathcal{F}$. In the present case the dualizing sheaf can be defined by its global sections. The global sections of the dualizing sheaf on Σ^{proj} can be described in terms of meromorphic differentials on $\mathbb{C}$ of the form

$$\omega_\phi = Res_\epsilon\left(\frac{\phi(\epsilon)dz}{z-A(\epsilon)}-\frac{\phi(\epsilon)dz}{z-B(\epsilon)}\right), \tag{2.9}$$

for an element $\phi(\epsilon)=\sum_{i=0,...,N-1}\phi_i\frac{1}{\epsilon^{i+1}}$. In (2.9), fractions should be understood as geometric progression:
$$\frac{1}{z-A(\epsilon)}=\frac{1}{z-a_0-a_1\epsilon-a_2\epsilon^2-...}=\frac{1}{(z-a_0)(1-\frac{a_1\epsilon+a_2\epsilon^2+...}{z-a_0})}$$
$$=\frac{1}{(z-a_0)}(1+\frac{a_1\epsilon+a_2\epsilon^2+...}{z-a_0}+(\frac{a_1\epsilon+a_2\epsilon^2+...}{z-a_0})^2+...).$$
The symbol Res_ϵ means the coefficient at $\frac{1}{\epsilon}$. It turns out that for an arbitrary $\phi(\epsilon)$ the above expression gives a holomorphic differential on the singular curve Σ^{proj}, and in addition any differential is obtained in this way. Let us describe the Serre pairing for the structure sheaf. Let us consider the covering consisting of two opens $U_0=\Sigma^{aff}$ and U_∞ - an open disk centered at ∞. The intersection $U_0\cap U_\infty$ can be identified with the punctured disk $U_\infty^\bullet$ also centered at ∞. Let $s\in\mathcal{O}_{U_\infty^\bullet}$ be a representative of $H^1(\mathcal{O})$. The pairing is determined by the formula:

$$<\omega_\phi, s> = \oint_{\delta U_0\cap U_\infty}\omega_\phi s. \tag{2.10}$$

It is easy to see that the pairing is correctly defined on classes of cohomology.

The endomorphisms of the module M_Λ are described by polynomial matrix-valued functions $\Phi(z)$ satisfying the condition

$$\Phi(A(\epsilon)) = \Lambda(\epsilon)\Phi(B(\epsilon))\Lambda(\epsilon)^{-1}.$$

The action of $\Phi(z)$ on a section $s(z)$ is given by the formula : $s(z) \mapsto \Phi(z)s(z)$. The space $H^1(\mathrm{End}(M_\Lambda))$ is described as the quotient of the space of matrix-valued polynomial functions by two subspaces:

$$\mathrm{End}_{out} = \{\chi(z) \in Mat_n[z] | \chi(z) = const\}$$

and

$$\mathrm{End}_{in} = \{\chi(z) \in Mat_n[z] | \chi(A(\epsilon))) = \Lambda(\epsilon)\chi(B(\epsilon))\Lambda(\epsilon)^{-1}\}.$$

Elements of $H^1(\mathrm{End}(M_\Lambda))$ are treated as tangential vectors to the moduli space of holomorphic bundles at M_Λ. The infinitesimal deformation corresponding to an element $\chi(z)$ is defined by the formula

$$\delta_{\chi(z)}\Lambda(\epsilon) = \chi(A(\epsilon))\Lambda(\epsilon) - \Lambda(\epsilon)\chi(B(\epsilon)). \tag{2.11}$$

Global sections $H^0(\mathrm{End}(M_\Lambda) \otimes \mathcal{K})$ are described by the expressions:

$$\Phi(z) = Res_\epsilon \left(\frac{\Phi(\epsilon)}{z - A(\epsilon)} dz - \frac{\Lambda(\epsilon)^{-1}\Phi(\epsilon)\Lambda(\epsilon)}{z - B(\epsilon)} dz \right), \tag{2.12}$$

where

$$Res_\epsilon(\Lambda(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)^{-1} - \Phi(\epsilon)) = 0$$

and $\Phi(\epsilon) = \sum_i \Phi_i \frac{1}{\epsilon^{i+1}}$ is a polynomial matrix-valued function. The expression (2.12) also implies a decomposition of the denominator in the geometric progression. It turns out that all global sections of $H^0(\mathrm{End}(M_\Lambda) \otimes \mathcal{K})$ are of this form.

Symplectic form on the cotangent bundle to the moduli space of holomorphic bundles can be described in terms of Hamiltonian reduction with respect to the adjoint action of GL_n of the symplectic form on the space of pairs $\Lambda(\epsilon), \Phi(\epsilon)$, given by the expression:

$$Res_\epsilon Trd(\Lambda(\epsilon)^{-1}\Phi(\epsilon)) \wedge d\Lambda(\epsilon). \tag{2.13}$$

Integrability The Hitchin system on Σ^{proj} in now defined as a system with phase space which is the Hamiltonian quotient of the space of pairs $\Lambda(\epsilon), \Phi(\epsilon)$. The symplectic form is given by the formula (2.13). The reduction is considered with respect to the adjoint action of the group GL_n. The Lax operator is defined by the formula 3.42. The Hamiltonians are defined by the coefficients of the function $Tr(\Phi(z)^k)$ subject to some basis of holomorphic k-differentials (i.e. sections of $H^0(\mathcal{K}^k)$). Let us remark that $\forall z, w, k, l$ the following commutativity condition is fulfilled: $Tr(\Phi(z)^k)$ and $Tr(\Phi(w)^l)$ commute on the nonreduced space.

The integrability proof realizes the r-matrix technique.

Example 2.2. Consider a rational curve with a double point $z_1 \leftrightarrow z_2$ (the ring of rational functions on a curve is a subring of rational functions f on $\mathbb{C}P^1$ satisfying the condition $f(z_1) = f(z_2)$) with one marked point z_3. The dualizing sheaf has the global section $dz(\frac{1}{z-z_1} - \frac{1}{z-z_2})$. Consider the moduli space $\mathcal{M}$ of holomorphic bundles E of rank n on Σ_{node} with fixed trivialization at z_3. There is the following isomorphism of linear spaces

$$T_E\mathcal{M} = H^1(\mathrm{End}(E) \otimes \mathcal{O}(-p)).$$

Let us restrict ourself to the open sell of the moduli space of equivalence classes of matrices Λ with different eigenvalues. The cotangent space is isomorphic to the space of holomorphic sections of $\mathrm{End}^*(E) \otimes \mathcal{K} \otimes \mathcal{O}(p)$. This space can be realized by the space of rational matrix-valued functions on z of the following type

$$\Phi(z) = \left(\frac{\Phi_1}{z-z_1} - \frac{\Phi_2}{z-z_2} + \frac{\Phi_3}{z-z_3}\right) dz,$$

with the following conditions on residues

$$\Phi_1\Lambda = \Lambda\Phi_2 \quad ??? \quad \Phi_1 - \Phi_2 + \Phi_3 = 0.$$

The phase space of the system is parameterized by elements of $U \in GL(n)$ giving a trivialization at z_3, matrix Λ describing the projective module over $\mathcal{O}(\Sigma_{mode})$, residues of the Higgs field Φ_i. In these coordinates the canonical symplectic form on $\mathcal{T}^*\mathcal{M}$ can be expressed as follows

$$\omega = Tr(d(\Lambda^{-1}\Phi_1) \wedge d\Lambda) + Tr(d(U^{-1}\Phi_3) \wedge dU).$$

After Hamiltonian reduction with respect to the group $GL(n)$ action (the right action on U and the adjoint action on $\Phi_i,\ \Lambda$) one obtains the space parameterized by the matrix elements $(\Phi_3)_{ij} = f_{ij}, i \neq j$; eigenvalues e^{2x_i} of the matrix Λ and the diagonal elements of the matrix $(\Phi_1)_{ii} = p_i$ with the following Poisson structure

$$\{x_i, p_j\} = \delta_{ij}, \qquad \{f_{ij}, f_{kl}\} = \delta_{jk} f_{il} - \delta_{il} f_{kj}.$$

The Hamiltonian of the trigonometric spin Calogero-Moser system related with the finite-zone solutions of the matrix generalization for the KP equation [34] can be obtained as the coefficient of $Tr\Phi^2(z)$ at $1/(z-z_1)^2$

$$H = Tr\Phi_1^2 = \sum_{i=1}^n p_i^2 - 4\sum_{i\neq j} \frac{f_{ij} f_{ji}}{\sinh^2(x_i - x_j)}.$$

2.4 The Gaudin model

2.4.1 The Lax operator

The Gaudin model was proposed in [9] (section 13.2.2) as a limit of the XXX Heisenberg magnet. It describes a one-dimensional chain of interacting particles with spin. The Gaudin model can be considered as a generalization of the Hitchin system for the rational curve $\Sigma = \mathbb{C}P^1$ with N marked points $z_1, \ldots, z_N$. The Higgs field (the Lax operator) can be represented by a rational section $\Phi = L(z)dz$ where

$$L(z) = \sum_{i=1\ldots N} \frac{\Phi_i}{z - z_i}. \tag{2.14}$$

The residues of the Lax operator Φ_i are matrices $n \times n$ whose matrix elements lie in $\mathfrak{gl}_n \oplus \ldots \oplus \mathfrak{gl}_n$. $(\Phi_i)_{kl}$ coincides with the kl-th generator of the i-th copy of $\mathfrak{gl}_n$. The generators of the Lie algebra are interpreted as functions on the dual space $\mathfrak{gl}_n^*$. The symmetric algebra $S(\mathfrak{gl}_n)^{\otimes N} \simeq \mathbb{C}[\mathfrak{gl}_n^* \oplus \ldots \oplus \mathfrak{gl}_n^*]$ is equipped with the Poisson structure given by the Kirillov-Kostant bracket:

$$\{(\Phi_i)_{kl}, (\Phi_j)_{mn}\} = \delta_{ij}(\delta_{lm}(\Phi_i)_{kn} - \delta_{nk}(\Phi_i)_{ml}).$$

2.4.2 R-matrix bracket

R- matrix representations of Poisson structures turned out to be a key element of the theory of quantum groups. In some sense the existence of an R-matrix structure is equivalent to integrability. It should be noted that in the theory of quantum groups [35] an important concept of quasitriangular or braided bialgebra arises. Let us introduce the notation:

- $\{e_i\}$ - the standard basis in $\mathbb{C}^n$;
- $\{E_{ij}\}$ - the standard basis in $\mathrm{End}(\mathbb{C}^n)$, $(E_{ij}e_k = \delta_k^j e_i)$;
- $e_{ij}^{(s)}$ - generators of the s-th copy $\mathfrak{gl}_n \subset \oplus^N \mathfrak{gl}_n$.

The Lax operator can be represented as

$$L(z) = \sum_{ij} E_{ij} \otimes \sum_{s=1}^{N} \frac{e_{ij}^{(s)}}{z - z_s}.$$

The Poisson structure can be described in terms of generating functions:

$$\{L(z) \otimes L(u)\} = [R_{12}(z-u), L(z) \otimes 1 + 1 \otimes L(u)] \in \mathrm{End}(\mathbb{C}^n)^{\otimes 2} \otimes S(\mathfrak{gl}_n)^{\otimes N},$$

with the classical Yang R-matrix

$$R(z) = \frac{P_{12}}{z}, \qquad P_{12} v_1 \otimes v_2 = v_2 \otimes v_1, \qquad P_{12} = \sum_{ij} E_{ij} \otimes E_{ji}.$$

2.4.3 The integrals

The integrals of motion can be retrieved as the characteristic polynomial coefficients

$$det(L(z)-\lambda)=\sum_{k=0}^{n} I_k(z)\lambda^{n-k}. \tag{2.15}$$

It is often used the alternative basis of symmetric functions of eigenvalues of the Lax operators

$$J_k(z)=TrL^k(z), \qquad k=1,\dots,n.$$

Traditional quadratic Hamiltonians can be obtained as follows

$$H_{2,k}=Res_{z=z_k}TrL^2(z)=\sum_{j\neq k}\frac{2Tr\Phi_k\Phi_j}{(z_k-z_j)}=2\sum_{j\neq k}\frac{\sum_{lm}e^{(k)}_{lm}e^{(j)}_{ml}}{z_k-z_j}.$$

They describe the magnet model that consists of a set of pairs of interacting particles. It is known that

Proposition 2.3. *The coefficients of the characteristic polynomial of $L(z)$ commute with respect to the Kirillov-Kostant bracket*

$$\{I_k(z),I_m(u)\}=0.$$

Let us present here the baseline of the proof.

Proof

Let $L_1(z)=L(z)\otimes 1$ and $L_2(u)=1\otimes L(u)$.

$$\begin{aligned}
&\{J_k(z),J_m(u)\}=Tr_{12}\{L^k(z)\otimes L^m(u)\}\\
=\;& Tr_{12}\sum_{ij}L_1^i(z)L_2^j(u)\{L(z)\otimes L(u)\}L_1^{k-i-1}(z)L_2^{m-j-1}(u)\\
=\;& Tr_{12}\sum_{ij}L_1^i(z)L_2^j(u)R_{12}(z-u)L_1^{k-i}(z)L_2^{m-j-1}(u) \qquad (2.16)\\
+\;& Tr_{12}\sum_{ij}L_1^i(z)L_2^j(u)R_{12}(z-u)L_1^{k-i-1}(z)L_2^{m-j}(u)\\
-\;& Tr_{12}\sum_{ij}L_1^{i+1}(z)L_2^j(u)R_{12}(z-u)L_1^{k-i-1}(z)L_2^{m-j-1}(u) \qquad (2.17)\\
-\;& Tr_{12}\sum_{ij}L_1^i(z)L_2^{j+1}(u)R_{12}(z-u)L_1^{k-i-1}(z)L_2^{m-j-1}(u).
\end{aligned}$$

In particular,

$$(2.16)+(2.17)=Tr_{12}[\sum_{ij}L_1^i(z)L_2^j(u)R_{12}(z-u)L_1^{k-i-1}(z)L_2^{m-j-1}(u),L_1(z)].$$

The last expression is zero because it is trace of a commutator.

2.4.4 Algebraic-geometric description

This section describes the basic algebraic-geometrical components of the generalized Hitchin system for curves with marked points. Namely it is constructed a pair $\{\Sigma, \mathcal{L}\}$ — the spectral curve and the line bundle on it, which makes it possible to resolve the classical Gaudin model.

Spectral curve The spectral curve of the Gaudin system $\tilde{\Sigma}$ is described by the equation

$$det(L(z) - \lambda) = 0. \tag{2.18}$$

To build a nonsingular compactification of that curve one should consider the total space of the bundle where the Lax operator takes values

$$\Phi(z) = L(z)dz \in H^0(\mathbb{C}P^1, \mathrm{End}(\mathcal{O}^n) \otimes \Lambda) \tag{2.19}$$

where $\Lambda = \mathcal{K}(k) = \mathcal{O}(k-2)$. We define a compactification of Σ by the equation

$$det(\Phi(z) - \lambda) = 0. \tag{2.20}$$

This curve is a subvariety of the rational surface S_{k-2}, obtained by compactification of the total space of the line bundle $\mathcal{O}(k-2)$ over $\mathbb{C}P^1$, or as a projectivisation $P(\mathcal{O}(k-2) \oplus \mathcal{O})$ over the rational curve. This rational surface contains three types of divisors: E_∞ — the infinite divisor, C — the fiber of the bundle and E_0 — the base curve, with the following intersections

$$\begin{aligned}
&E_0 \cdot E_0 = k-2, \\
&E_0 \cdot C = 1, \\
&C \cdot C = 0, \\
&E_\infty \cdot C = 1.
\end{aligned}$$

To determine the genus of the curve Σ we use the adjunction formula. First let us calculate the canonical class of S_{k-2}. It corresponds to the class of divisors

$$\mathcal{K}_{S_{k-2}} = -2E_0 + (k-4)C.$$

Let the class of Σ be equal to $[\Sigma] = n_1 E_0 + n_2 C$. Σ is n-folded covering of $\mathbb{C}P^1$. Hence $[\Sigma] \cdot C = n$ and $n_1 = n$. To calculate n_2 it is sufficient to use the fact that Σ is a spectral curve of a holomorphic section of $\mathrm{End}(\mathcal{O}^n) \otimes \Lambda$ and hence does not intersect E_∞. We obtain

$$[\Sigma] = nE_0.$$

By the adjunction formula we have

$$2g - 2 \;=\; \mathcal{K}_{S_{k-2}} \cdot [\Sigma] + [\Sigma] \cdot [\Sigma]$$

$$\begin{aligned} &= (-2E_0 + (k-4)C) \cdot nE_0 + n^2 E_0 \cdot E_0 \\ &= -2(k-2)n + (k-4)n + (k-2)n^2. \end{aligned}$$

This allows to calculate the genus of the spectral curve

$$g(\Sigma) = \frac{(k-2)n(n-1)}{2} - (n-1). \tag{2.21}$$

Line bundle Let us recall the sequence defining the line bundle on the spectral curve

$$0 \to \mathcal{L} \to \mathcal{O}_\Sigma^n \to \mathcal{O}_S^n((k-2)C + E_\infty)|_\Sigma \to \mathcal{F}_\Sigma \to 0, \tag{2.22}$$

where $\mathcal{L}$ and $\mathcal{F}_\Sigma$ are line bundles. We also obtain the following

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_\Sigma^n) - \chi(\mathcal{O}_S^n((k-2)C + E_\infty)|_\Sigma) + \chi(\mathcal{F}_\Sigma)$$

Let us denote the divisor $(k-2)C + E_\infty \subset S$ as D. Then

$$\begin{aligned} \chi(\mathcal{O}_\Sigma^n) &= n(1-g), \\ \chi(\mathcal{O}_S^n(D)|_\Sigma) &= nD \cdot [\Sigma] + n(1-g) \\ &= n^2(k-2) + n(1-g), \\ \chi(\mathcal{F}_\Sigma) &= \chi(\pi^*\mathcal{O}^n(k-2) \otimes \mathcal{O}_S(E_\infty)) - \chi(\mathcal{O}_S^n) \\ &= n\frac{1}{2}(D \cdot D - D \cdot \mathcal{K}_S) = n(k-1). \end{aligned} \tag{2.23}$$

Hence $\chi(\mathcal{L}) = -n^2(k-2) + n(k-1)$. Calculating the number of branching points $\nu = 2(g + n - 1) = (k-2)(n^2 - n)$ we obtain

Lemma 2.4.

$$deg(\mathcal{L}) = g + n - 1 - \nu.$$

The dimension of the commutative family On the affine chart without $\{z_i\}$ and ∞ the spectral curve is given by the equation

$$R(z, \lambda) = 0, \qquad R(z, \lambda) = (-1)^n \lambda^n + \sum_{m=0}^{n-1} \lambda^m R_m(z), \tag{2.24}$$

where

$$R_m(z) = \sum_{i=1}^{k} \sum_{l=1}^{n-m} \frac{R_{m,i}^{(l)}}{(z - z_i)^l}.$$

The number of free coefficients is equal $\sum_{m=0}^{n-1} k(n-m) = k\frac{n(n+1)}{2}$. The central functions (the symmetric polynomials of eigenvalues for corresponding orbits) are the highest coefficients of $R_m(z)$ of the total number $kn-1$. The Lax operator has double zero at infinity

$$L(z) = \frac{1}{z^2}\sum_i \Phi_i z_i + O(\frac{1}{z^3}).$$

It follows that $R_m(z)$ has zero of order $2(n-m)$ at infinity. This observation, in turn, imposes additional conditions

$$\sum_{m=0}^{n-1}(2(n-m)-1) = n^2$$

on the values of Hamiltonians. Thus, the dimension of the commutative family is

$$k\frac{n(n+1)}{2} - kn + 1 - n^2 = k\frac{n(n-1)}{2} - n^2 + 1 = g.$$

2.5 Separated variables

For the wide class of integrable systems the separated variables are associated with the divisor of the line bundle $\mathcal{L}$ on the spectral curve. Namely pairs of coordinates of the divisor points are separated. Typically, the divisor is the divisor for the Baker function. A construction of separated variables for some class of integrable systems is given in [36]. In the case of $\mathfrak{sl}_2$-Gaudin model separated variables were known before [37], and can be found even more explicitly.

2.5.1 $\mathfrak{sl}_2$-Gaudin model

Let us remind that $\mathfrak{sl}_2$-Gaudin model is obtained from the $\mathfrak{gl}_2$ model (2.14) choosing orbits with $tr = 0$. The Lax operator in this case is:

$$L = \begin{pmatrix} A(z) & B(z) \\ C(z) & -A(z) \end{pmatrix}.$$

We will consider the characteristic polynomial as a function of parameters z, λ and values of the Hamiltonians:

$$det(L(z) - \lambda) = R(z, \lambda, h_1, \dots, h_d).$$

Let us define the variables y_j as zeroes of $C(z)$. For dual variables we take

$$w_j = A(y_j).$$

This set of variables defines the Darboux coordinates of the phase space:

$$\{y_i, w_j\} = \delta_{ij}.$$

Let us consider the generating function $S(I, y)$ of the canonical transformation from the variables y_j, w_j to the "'action-angle"' variables I_j, ϕ_j

$$w_j = \partial_{y_j} S, \qquad \phi_j = \partial_{I_j} S.$$

The point with coordinates (y_j, w_j) is a point of the spectral curve by definition. The fact that "'action"' variables are functions of Hamiltonians allows to separate variables in the problem of finding the canonical transformation S

$$S(I, y_1, \ldots, y_d) = \prod_i s(I, y_i),$$

where each factor $s(I, z)$ solves the equation

$$R(z, \partial_z s, h_1, \ldots, h_d) = 0.$$

3 The quantization problem

The quantization problem has physical motivation, it is related to the quantum paradigm in modern physics. In mathematical context this problem can be formulated in different ways: in [38] it was considered the problem of deformation of an algebra of functions on a symplectic manifold satisfying the so-called "'correspondence principle"'. The particular case of the deformation quantization for the cotangent bundle to a Lie group used in these lectures was considered in [39]. Further on, the methods of deformation quantization, $*$-product, the Moyal product and the geometric quantization were generalized to wider class of examples. One of the structure results in this field was the formality theorem by M. Kontsevich [40] which demonstrates the existence of the quantization. Another ensemble of important results in this domain are due to Fedosov [41].

In this work it is proposed radically more strong quantization problem, demanding not only the deformation of an algebra of functions but of a pair: Poisson algebra + Poisson commutative subalgebra, representing an integrable system. Let us refer to this problem as to the algebraic part of integrable system quantization. Moreover it is stated a problem of constructing quantum analogs of the essential geometric objects from the point of view of algebraic-geometric methods in integrable systems. In general the problem is to find an associative deformation of a Poisson algebra such that the Poisson-commutative subalgebra remains commutative, and moreover the deformation of the spectral curve provides quantum separated variables. The last part of the quantization problem is called "'algebraic-geometric"' quantization.

3.1 The deformation quantization

3.1.1 Correspondence

The traditional scheme of deformation quantization supposes a construction of an associative algebra starting with a Poisson algebra. A Poisson algebra is a commutative algebra $\mathcal{A}_{cl}$ with multiplication denoted by $\cdot$, furnished by an anti-symmetric bilinear operation called the Poisson bracket $\{\circ,\circ\}$, such that $\mathcal{A}_{cl}$ is a Lie algebra and both structures are compatible by the Leibniz rule:

$$\{f, g\cdot h\} = \{f,g\}\cdot h + g\cdot\{f,h\}.$$

A Poisson algebra is an infinitesimal version of an associative algebra. Due to the so-called Drinfeld ε-construction it is not hard to note that the space $\mathcal{A}_{cl}[\varepsilon]/\varepsilon^2$ with multiplication

$$f * g = f\cdot g + \varepsilon\{f,g\}$$

is an associative algebra. The quantization of the Poisson algebra $\mathcal{A}_{cl}$ with the structure defined by operations $(\cdot,\{\circ,\circ\})$ which is called the algebra of classical observables is an associative algebra $\mathcal{A}$ with multiplication $(*)$, satisfying the following conditions:

$$\mathcal{A} \simeq \mathcal{A}_{cl}[[\mathsf{h}]] \text{ as linear spaces.}$$

Moreover, if the algebra of classical observables and the space of constants in $\mathcal{A}$ are identified, the following structure compatibility is required:

$$\begin{aligned} a*b &= a\cdot b + O(\mathsf{h}),\\ a*b - b*a &= \mathsf{h}\{a,b\} + O(\mathsf{h}^2). \end{aligned}$$

The map

$$lim : \mathcal{A} \longrightarrow \mathcal{A}_{cl} : \qquad \mathsf{h}\mapsto 0$$

is called the classical limit.

Example 3.1. Let us consider the Poisson algebra $S(\mathfrak{gl}_n)$ on the space of symmetric algebra of the Lie algebra $\mathfrak{gl}_n$ defined by the Kirillov-Kostant bracket. This has a canonical quantization, realizing the concept of the deformation quantization: let $U_{\mathsf{h}}(\mathfrak{gl}_n)$ be the deformed universal enveloping algebra

$$U_{\mathsf{h}}(\mathfrak{gl}_n) = T^*(\mathfrak{gl}_n)[[\mathsf{h}]]/\{x\otimes y - y\otimes x - \mathsf{h}[x,y]\}.$$

The classical limit is defined as the limit $\mathsf{h}\to 0$ which is correctly defined on the family of algebras $U_{\mathsf{h}}(\mathfrak{gl}_n)$. The existence of a limit follows from the common Poincare-Birkhoff-Witt basis for this family.

3.1.2 Quantization of an integrable system

An integrable system is a pair: a Poisson algebra $\mathcal{A}_{cl}$ and a Poisson commutative subalgebra $\mathcal{H}_{cl}$ of the dimension $dim(Spec(\mathcal{H}_{cl})) = 1/2dim(Spec(\mathcal{A}_{cl}))$. An algebraic problem of quantization is the following correspondence

$$\mathcal{H}_{cl} \subset \mathcal{A}_{cl} \Leftrightarrow \mathcal{H} \subset \mathcal{A}$$

satisfying the conditions

- $\mathcal{A} \simeq \mathcal{A}_{cl}[[\mathsf{h}]]$ as linear spaces, the map $lim : \mathcal{A} \to \mathcal{A}_{cl}$ is called the classical limit;
- $\mathcal{H}$ is commutative;
- $lim : \mathcal{H} = \mathcal{H}_{cl}$

Remark 3.2. In the case of quantization for the symmetric algebra of the Lie algebra $\mathfrak{gl}_n$ the correspondence can be simplified. Let us consider $U(\mathfrak{gl}_n)$, which is a filtered algebra (the filtration is given by degree) $\{\mathcal{F}_i\}$. The projection map to the associated graded algebra induces a Poisson structure:

$$U(\mathfrak{gl}_n) \to Gr(U(\mathfrak{gl}_n)) = \oplus_i \mathcal{F}_i/\mathcal{F}_{i-1} = S(\mathfrak{gl}_n). \tag{3.1}$$

We will associate this map with the classical limit operation. On generators $a \in \mathcal{F}_i$ and $b \in \mathcal{F}_j$ the induced commutative multiplication and the Poisson bracket are given by the following expressions:

$$a \cdot b = a * b \bmod \mathcal{F}_{i+j-1}, \qquad \{a, b\} = a * b - b * a \bmod \mathcal{F}_{i+j-2}.$$

3.1.3 The Gaudin model quantization problem

The classical part is defined by the following objects

$$\begin{aligned} \mathcal{A}_{cl} &= S(\mathfrak{gl}_n)^{\otimes N} \simeq \mathbb{C}[\mathfrak{gl}_n^* \oplus \ldots \oplus \mathfrak{gl}_n^*], \\ \mathcal{H}_{cl} &- \text{the subalgebra generated by the Gaudin Hamiltonians(2.15).} \end{aligned}$$

The algebraic part of the quantization problem is reduced to constructing a pair with the quantum observables algebra coinciding with the tensor power of the universal enveloping algebra:

$$\mathcal{A} = U(\mathfrak{gl}_n)^{\otimes N},$$

such that the commutative subalgebra $\mathcal{H}$ is a deformation of the subalgebra generated by the classical Gaudin Hamiltonians.

3.2 Quantum spectral curve

3.2.1 Noncommutative determinant

Let us consider a matrix $B = \sum_{ij} E_{ij} \otimes B_{ij}$ whose elements are elements of some generally speaking not commutative associative algebra $B_{ij} \in A$. We will use the following definition for the noncommutative determinant in this case

$$det(B) = \frac{1}{n!} \sum_{\tau,\sigma\in\Sigma_n} (-1)^{\tau\sigma} B_{\tau(1),\sigma(1)} \ldots B_{\tau(n),\sigma(n)}.$$

This definition is the same as the classical one for matrices with commuting elements. There is an equivalent definition. Let us introduce the operator A_n of the antisymmetrization in $(\mathbb{C}^n)^{\otimes n}$

$$A_n v_1 \otimes \ldots \otimes v_n = \frac{1}{n!} \sum_{\sigma\in S_n} (-1)^{\sigma} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.$$

The definition above is equivalent to the following

$$det(B) = Tr_{1\ldots n} A_n B_1 \ldots B_n,$$

where B_k denotes an operator in $\mathrm{End}(\mathbb{C}^n)^{\otimes n} \otimes A$ given by the inclusion

$$B_k = \sum_{ij} 1 \otimes \ldots \otimes \underbrace{E_{ij}}_{k} \otimes \ldots \otimes 1 \otimes B_{ij},$$

the trace is taken on $\mathrm{End}(\mathbb{C}^n)^{\otimes n}$.

3.2.2 Quantum spectral curve

Let us call a quantum Lax operator for the Gaudin system the following expression:

$$L(z) = \sum_{ij} E_{ij} \otimes \sum_{s=1}^{N} \frac{e_{ij}^{(s)}}{z - z_s}.$$

$L(z)$ is a rational function in the variable z with values in $\mathrm{End}(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n)^{\otimes N}$. Let us define a quantum characteristic polynomial of the quantum Lax operator by the formula

$$det(L(z) - \partial_z) = \sum_{k=0}^{n} QI_k(z) \partial_z^{n-k}. \tag{3.2}$$

The following theorem says that this generalization of the classic characteristic polynomial (2.15) makes it possible to construct quantum Hamiltonians.

Theorem 3.3 ([42]). *The coefficients $QI_k(z)$ commute*

$$[QI_k(z), QI_m(u)] = 0$$

and quantize the classical Gaudin Hamiltonians in the following sense

$$lim(QI_k) = I_k.$$

The proof of this fact uses significant results of the theory of quantum groups such as the construction of the Yangian, the Bethe subalgebras and generally fits into the concept of quantum inverse scattering method. The following sections introduce the necessary definitions and provide an outline of the proof of the theorem of quantization of the Gaudin model.

3.2.3 Yangian

This Hopf algebra was constructed in [23] and plays an important role in the problem of description of rational solutions to the Yang-Baxter equation. $Y(\mathfrak{gl}_n)$ first and foremost is an associative algebra generated by the elements $t_{ij}^{(k)}$ (in this section $i = 1, \ldots, n;\ j = 1, \ldots, n;\ k = 1, \ldots, \infty$). Let us introduce the generating function

$$T(u, \mathsf{h}) \in Y(\mathfrak{gl}_n) \otimes \mathrm{End}(\mathbb{C}^n)[[u^{-1}, \mathsf{h}]],$$

which takes the form

$$T(u, \mathsf{h}) = \sum_{i,j} E_{ij} \otimes t_{ij}(u, \mathsf{h}), \qquad t_{ij}(u, \mathsf{h}) = \delta_{ij} + \sum_k t_{ij}^{(k)} \mathsf{h}^k u^{-k},$$

where E_{ij} are the matrix units in $\mathrm{End}(\mathbb{C}^n)$. The relations can be written with the help of the Yang R-matrix

$$R(u) = 1 - \frac{\mathsf{h}}{u} \sum_{i,j} E_{ij} \otimes E_{ji}$$

and take the form

$$R(z-u, \mathsf{h}) T_1(z, \mathsf{h}) T_2(u, \mathsf{h}) = T_2(u, \mathsf{h}) T_1(z, \mathsf{h}) R(z-u, \mathsf{h}). \tag{3.3}$$

Both parts are regarded as elements of

$$\mathrm{End}(\mathbb{C}^n)^{\otimes 2} \otimes Y(\mathfrak{gl}_n)[[z^{-1}, z, u^{-1}, u, \mathsf{h}]],$$

the rational function $\frac{1}{z-u}$ in the R-matrix formula has an expansion

$$\frac{1}{z-u} = \sum_{l=0}^{\infty} \frac{u^l}{z^{l+1}}.$$

We use the following notation

$$T_1(z,\mathsf{h}) = \sum_{i,j} E_{ij}\otimes 1\otimes t_{ij}(z,\mathsf{h}),\quad T_2(u,\mathsf{h}) = \sum_{i,j} 1\otimes E_{ij}\otimes t_{ij}(u,\mathsf{h}).$$

The Yangian is a Hopf algebra whose comultiplication is given in terms of the generating function by the following formula

$$(id\otimes\Delta)T(z,\mathsf{h}) = T^1(z,\mathsf{h})T^2(z,\mathsf{h}),$$

where we use the notation

$$T^1(z,\mathsf{h}) = \sum_{i,j} E_{ij}\otimes t_{ij}(z,\mathsf{h})\otimes 1,\quad T^2(z,\mathsf{h}) = \sum_{i,j} E_{ij}\otimes 1\otimes t_{ij}(z,\mathsf{h}).$$

The evaluation representation Let us remind the construction of the so-called evaluation homomorphism $\rho : Y(\mathfrak{gl}_n)\to U(\mathfrak{gl}_n)$. To do this we consider a rational function on u,h with values in $\mathrm{End}(\mathbb{C}^n)\otimes U(\mathfrak{gl}_n)$ given by the formula

$$T_{ev}(u,\mathsf{h}) = 1+\frac{\mathsf{h}}{u}\sum_{i,j} E_{ij}\otimes e_{ij} \overset{def}{=}: 1+\frac{\mathsf{h}\Phi}{u}, \tag{3.4}$$

where e_{ij} are the generators of $\mathfrak{gl}_n$. $T_{ev}(u,\mathsf{h})$ satisfy RTT relations (3.3), hence the map $\{t_{ij}^{(1)}\mapsto e_{ij};\ t_{ij}^{(k)}\mapsto 0 \text{ with } k>1\}$ determines an algebra homomorphism.

Let us consider the tensor product $U(\mathfrak{gl}_n)^{\otimes N}[[\mathsf{h},\mathsf{h}^{-1}]]$ and the generating function (3.4) for the evaluation representation to the l-th component of the product $T^l_{ev}(u-z_l,\mathsf{h})$. It turns out that for an arbitrary set of complex numbers $(z_1,\ldots,z_N)$, the expression

$$T^\alpha(u,\mathsf{h}) = T^1_{ev}(u-z_1,\mathsf{h})T^2_{ev}(u-z_2,\mathsf{h})\ldots T^k_{ev}(u-z_N,\mathsf{h}), \tag{3.5}$$

which is a rational function on u and h with values in $\mathrm{End}(\mathbb{C}^n)\otimes U(\mathfrak{gl}_n)^{\otimes N}$, determines a homomorphism $\rho_\alpha : Y(\mathfrak{gl}_n)\to U(\mathfrak{gl}_n)^{\otimes N}[[\mathsf{h},\mathsf{h}^{-1}]]$. More precisely, the following lemma is true.

Lemma 3.4. *The map, defined on the Yangian generators $t_{ij}^{(k)}$ as the ij-th matrix element of the expansion coefficient of $T^\alpha(u,\mathsf{h})\mathsf{h}^{-k}$ at u^{-k} in $u=\infty$ gives an algebra homomorphism*

$$\rho_\alpha : Y(\mathfrak{gl}_n)\to U(\mathfrak{gl}_n)^{\otimes N}[[\mathsf{h},\mathsf{h}^{-1}]].$$

This lemma follows from the properties of the comultiplication homomorphism and the evaluation homomorphism.

3.2.4 The Bethe subalgebra

This subalgebra is closely related with Quantum Inverse Scattering Method (QISM) [21, 44, 45], namely its generators are quantum integrals of the Heisenberg XXX model [44, 43]. Here we use the description from [46] (section 2.14): let us consider an $n \times n$-matrix C and $T(u, \mathsf{h})$ - a generating function for the Yangian generators $Y(\mathfrak{gl}_n)$. Let us also use the notation A_n for the antisymmetrization operator in $(\mathbb{C}^n)^{\otimes n}$ and the following elements of $\mathrm{End}(\mathbb{C}^n)^{\otimes n} \otimes Y(\mathfrak{gl}_n)[[u, u^{-1}, \mathsf{h}]]$

$$T_m(u, \mathsf{h}) = \sum_{ij} 1 \otimes \ldots \otimes 1 \otimes \overset{m}{E_{ij}} \otimes 1 \otimes \ldots \otimes 1 \otimes t_{ij}(u, \mathsf{h}).$$

It turns out [46] (section 2.14), that the expressions of the form

$$(3.6)\quad \tau_k(u, \mathsf{h}) = Tr A_n T_1(u, \mathsf{h}) T_2(u - \mathsf{h}, \mathsf{h}) \ldots T_k(u - \mathsf{h}(k-1), \mathsf{h}) C_{k+1} \ldots C_n$$

for $k = 1, \ldots n$, which are called the Bethe generators, constitute a commutative family in $Y(\mathfrak{gl}_n)[[u, u^{-1}, \mathsf{h}]]$ in the following sense:

$$[\tau_i(u, \mathsf{h}), \tau_j(v, \mathsf{h})] = 0.$$

In addition, this family is maximal if the matrix C has simple spectrum. The trace in the formula (3.6) is meant over matrix components $\mathrm{End}(\mathbb{C}^n)^{\otimes n}$, the series expansion of $T_m(u - \mathsf{h}(m-1), \mathsf{h})$ is realized at $u = \infty$, for example

$$\frac{1}{u - \mathsf{h}} = \sum_{m=0}^{\infty} \frac{\mathsf{h}^m}{u^{m+1}},$$

Next we will consider an identity matrix C and images of the Bethe generators with the evaluation homomorphism. For simplicity, we refer to the same letters

$$(3.7)\quad \tau_k(u, \mathsf{h}) = Tr A_n T_1^{\alpha}(u, \mathsf{h}) T_2^{\alpha}(u - \mathsf{h}, \mathsf{h}) \ldots T_k^{\alpha}(u - \mathsf{h}(k-1), \mathsf{h}) \quad k = 1, \ldots n.$$

3.2.5 The commutativity proof

The presence of the comultiplication structure in the theory of quantum groups allows to use the so-called "'fusion"' method to construct non-trivial integrable systems. Literally, the method is as follows: let us consider the image $T(z)$ by the evaluation homomorphism in composition with comultiplication operations $\rho_{z_1} \otimes \ldots \otimes \rho_{z_N} \Delta^{N-1}$

$$T^{\aleph}(z) = T^1_{z_1}(z) \ldots T^N_{z_N}(z) \in \mathrm{End}(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n)^{\otimes N}.$$

The image of the Bethe subalgebra raises to some commutative subalgebra which can be described by the generating function:

$$Q(z, \mathsf{h}) \;=\; Tr A_n (e^{-\mathsf{h}\partial_z} T_1^{\aleph}(z, \mathsf{h}) - 1) \ldots (e^{-\mathsf{h}\partial_z} T_n^{\aleph}(z, \mathsf{h}) - 1)$$

$$= \sum_{j=0}^{n} \tau_j(z-\mathsf{h},\mathsf{h})(-1)^{n-j}C_n^j e^{-j\mathsf{h}\partial_z}$$

$$= det(e^{-\mathsf{h}\partial_z}T^{\aleph}(z,\mathsf{h})-1). \tag{3.8}$$

The expression (3.8) can be represented as a series of ∂_z. From the commutativity of the Bethe generators it follows that the coefficients of this series which are rational functions on u with values in $U(\mathfrak{gl}_n)^{\otimes N}[[\mathsf{h}]]$ also commute at different values of the parameter u. Hence the lowest coefficients on h also commute. These are exactly the coefficients of the characteristic polynomial of the Gaudin model. It turns out that the highest coefficient of the expression (3.8) on h has the form

$$det(e^{-\mathsf{h}\partial_z}T^{\aleph}(z,\mathsf{h})-1) = \mathsf{h}^n det(L(z)-\partial_z) + O(\mathsf{h}^{n+1})$$

in virtue of the expansion:

$$e^{-\mathsf{h}\partial_z}T^{\aleph}(z)-1 = \mathsf{h}(L(z)-\partial_z)+O(\mathsf{h}^2).$$

Remark 3.5. It should be noted that the independence of the quantum Hamiltonians directly follows from the independence of their classic limits, since the algebraic relations on the constructed operators in $U(\mathfrak{gl}_n)^{\otimes N}$ induces a nontrivial relation on their symbols. The maximality follows from the maximality on the classical level.

3.3 Traditional solution methods

The traditional methods solving a quantum integrable system on finite scale are reduced to the Bethe ansatz method or the method of separated variables which in turn allow to express the condition on the quantum model spectrum in terms of the solutions of some system of algebraic equations or the monodromy properties of some Fuchsian system. Those methods do not suppose any way of solving the substituting problems. However there is quite rich material in solving quantum integrable systems in various limits.

Further we explain two basic methods in the case of the simplest Gaudin model.

3.3.1 Bethe ansatz

Let us consider the quantum $\mathfrak{sl}_2$ Gaudin model. The Lax operator in this case takes the form

$$L = \begin{pmatrix} A(z) & B(z) \\ C(z) & -A(z) \end{pmatrix} = \sum_i \frac{\Phi_i}{z-z_i},$$

where

$$\Phi_i = \begin{pmatrix} h_i/2 & e_i \\ f_i & -h_i/2 \end{pmatrix}.$$

The quantum characteristic polynomial is a differential operator of the second order with values in the algebra of quantum observables:

$$det(L(z)-\partial_z)=\partial_z^2-\frac{1}{2}\sum_i\frac{c_i^{(2)}}{(z-z_i)^2}-\sum_i\frac{H_i}{z-z_i}.$$

The Gaudin Hamiltonians are the residues

$$H_i=\sum_{i\neq j}\frac{h_ih_j/2+e_if_j+e_jf_i}{z_i-z_j}.$$

The coefficients at the poles of second order are also elements of the commutative subalgebra but of trivial nature - they are central in the quantum algebra.

The Bethe ansatz method was firstly proposed for the Heisenberg model but fits well for a wide class of systems. The Bethe ansatz method for the Gaudin model was realized in [9]. Let us outline the construction. We consider the $\mathfrak{sl}_2$ Gaudin model in fixed representation $V_\lambda=V_{\lambda_1}\otimes\ldots\otimes V_{\lambda_N}$ where V_{λ_i} are the finite dimensional irreducible representations of highest weights λ_i.

Lemma 3.6. *The vector*

$$\Omega=\prod_{j=1}^{M}C(\mu_j)|vac>$$

is the common eigenvector for the ensemble of Gaudin Hamiltonians if the set of parameters μ_j (called the Bethe roots) satisfies the system of Bethe equations

$$-\frac{1}{2}\sum_i\frac{\lambda_i}{\mu_j-z_i}+\sum_{k\neq j}\frac{1}{\mu_j-\mu_k}=0,\qquad j=1,\ldots,M. \tag{3.9}$$

The eigenvalues of H_i on the vector Ω are expressed as follows

$$H_i^\Omega=-\lambda_i\left(\sum_j\frac{1}{z_i-\mu_j}-\frac{1}{2}\sum_{j\neq i}\frac{\lambda_j}{z_i-z_j}\right).$$

Proof

In this case the quantum characteristic polynomial takes the form:

$$det(L(z)-\partial_z)=\partial_z^2-A^2(z)-C(z)B(z)+A'(z)=\partial_z^2-H(z).$$

The following commutation relations on the matrix elements of the Lax operator are also true:

$$[A(z),B(z)]=-B'(z),\qquad [A(z),C(u)]=\frac{1}{z-u}(C(z)-C(u)),$$
$$[A(z),C(z)]=C'(z),\qquad [B(z),C(u)]=\frac{2}{u-z}(A(z)-A(u)).$$

Using this relation and the condition:

$$H(z)|vac>= \left(\frac{1}{4} (\sum_i \frac{\lambda_i}{z - z_i})^2 - \frac{1}{2} \sum_i \frac{\lambda_i}{(z - z_i)^2} \right) |vac>= h_0(z)|vac>$$

we obtain:

$$\begin{aligned} H(z)\Omega &= \left(h_0(z) + 2 \sum_{j=1}^{M} \frac{1}{\mu_j - z} A(z) + \sum_{j \neq k} \frac{1}{(\mu_j - z)(\mu_k - z)} \right) \Omega \\ &+ 2C(z) \sum_{j=1}^{M} \frac{1}{z - \mu_j} \prod_{l \neq j} C(\mu_l) \left(\sum_{k \neq j} \frac{1}{\mu_k - \mu_j} + A(\mu_j) \right). \end{aligned}$$

Let us remark that the Bethe equations can be rephrased in the form:

$$\sum_{k \neq j} \frac{1}{\mu_k - \mu_j} + A(\mu_j) = 0.$$

This proves the lemma.

3.3.2 Quantum separated variables

Let us consider the quantum $\mathfrak{sl}_2$ Gaudin model as in the previous section. An irreducible representation of this type can be realized as the quotient of the Verma module $\mathbb{C}[t_i]/t_i^{\lambda_i+1}$, such that the generators of $\mathfrak{sl}_2$ act as differential operators:

$$h^{(s)} = -2t_s \frac{\partial}{\partial t_s} + \lambda_s, \ e^{(s)} = -t_s \frac{\partial^2}{\partial t_s^2} + \lambda_s \frac{\partial}{\partial t_s}, \ f^{(s)} = t_s.$$

Let us explore the problem in the tensor product of the Verma modules which is realized in this case on the space of polynomials on N variables $\mathbb{C}[t_1, \dots, t_N]$. Let us introduce the set of variables y_j, defined by the formula:

$$C(z) = C_0 \frac{\prod_j (z - y_j)}{\prod_i (z - z_i)}.$$

They are elements of some algebraic extension of the ring $\mathbb{C}[t_1, \dots, t_N]$. Let us denote by the same symbols functions and operators of multiplication by those functions.

Let Ω be a common eigenvector for the Gaudin Hamiltonians in ??? $\mathbb{C}[t_1, \dots, t_N]$

$$H(z)\Omega = h(z)\Omega. \tag{3.10}$$

Considering both parts of (3.10) as rational functions on z and substituting $z = y_j$ from the left we obtain:

$$H(y_j) = A^2(y_j) - A'(y_j)$$

$$= \frac{1}{4}\sum_{i,k}\frac{1}{(y_j - z_i)(y_j - z_k)}h_i h_k + \frac{1}{2}\sum_k \frac{1}{(y_j - z_k)^2}h_k. \tag{3.11}$$

Using the definition of the separated variables let us express the partial derivatives:

$$\partial_{y_j} = \sum_k \frac{\partial t_k}{\partial y_j}\partial_{t_k} = \sum_k \frac{t_k}{y_j - z_k}\partial_{t_k}. \tag{3.12}$$

Substituting 3.12 in 3.11 we obtain:

$$\left(-\partial_{y_j} + \frac{1}{2}\sum_k \frac{\lambda_k}{y_j - z_k}\right)^2 \Omega = h(y_j)\Omega.$$

Hence the common eigenfunction for the Gaudin Hamiltonians factorizes, its dependence on y_j is separated:

$$\Omega = \prod_j \omega(y_j).$$

Each of the factors $\omega(z)$ is related to the solution to the Sturm-Liouville equation

$$(\partial_z^2 - h(z))\widetilde{\omega}(z) = 0$$

as follows:

$$\widetilde{\omega}(z) = \prod_i (z - z_i)^{-\lambda_i/2}\omega(z).$$

3.3.3 The monodromy of Fuchsian systems

The results of traditional separation of variables in quantum integrable systems discussed above demonstrate that the spectrum description is closely related with the families of Fuchsian equations obeying special monodromy properties. These properties are quite natural in the Heisenberg approach explained in [47], and correspond to existence of globally defined wave-functions.

In the considered $\mathfrak{sl}_2$ Gaudin model it was obtained that if Ω is a common Bethe eigenvector with values H_i^Ω then the equation

$$\left(\partial^2 - \frac{1}{4}\sum_i \frac{\lambda_i(\lambda_i + 2)}{(z - z_i)^2} - \sum_i \frac{H_i^\Omega}{z - z_i}\right)\Psi(z) = 0 \tag{3.13}$$

has a solution of the form

$$\Psi(z) = \prod_i (z - z_i)^{-\lambda_i/2}\prod_j (z - \mu_j),$$

where the set of parameters μ_j satisfy the system of Bethe equations.

This observation was generalized in [48]. Let us consider the quantum characteristic polynomial:

$$det(L(z)-\partial_z) = \partial_z^2 - \sum_i \frac{C_i^{(2)}}{(z-z_i)^2} - \sum_i \frac{H_i}{z-z_i}.$$

Let $\mathcal{H}$ be the algebra generated by the coefficients of the quantum characteristic polynomial. A character χ of the algebra $\mathcal{H}$ is called ???admissible??? if it takes values $\chi(C_i^{(2)}) = \frac{1}{4}(\lambda_i+2)\lambda_i$ on central elements.

Theorem 3.7 ([48]). *There is a one-to-one correspondence between the set of "'admissible"' characters χ for which the differential equation*

$$\chi(det(L(z)-\partial_z))\Psi(z) = 0$$

has monodromy ± 1, and the set of common eigenvectors of the Gaudin model in the representation V_λ.

In contrast with the traditional Bethe ansatz and separation of variables methods this spectrum characterization can be generalized to the $\mathfrak{sl}_n$ case.

3.4 Elliptic case

It turns out that the elements of the algebraic-geometric part of the quantization problem can the constructed also in the case of the Elliptic Gaudin model: the quantum spectral curve and the quantum separated variables. Let us remark that the elliptic Gaudin model can be obtained in generalized Hitchin system framework. This corresponds to the moduli space of holomorphic semistable bundles with the trivial determinant bundle over an elliptic curve with a set of marked points. A modified algebraic structure is applicable for this problem, namely the dynamical $\mathfrak{gl}_n$ RLL equation corresponding to the "'elliptic quantum group"' $E_{\tau,\hbar}(\mathfrak{gl}_n)$, defined in [50].

The commutativity in this case is meant modulo the Cartan subalgebra. To obtain an integrable system one should restrict the constructed family to the zero weight subspace with respect to the diagonal action of the Lie algebra.

3.4.1 The notation

Let us define the so-called odd Riemann θ-functions on an elliptic curve. Let $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$ be the parameter of elliptic curve $\mathbb{C}/\Gamma$, where $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ - is the periods lattice. The odd θ-function $\theta(u) = -\theta(-u)$ is defined by the relations

$$(3.14) \qquad \theta(u+1) = -\theta(u), \qquad \theta(u+\tau) = -e^{-2\pi i u - \pi i \tau}\theta(u), \qquad \theta'(0) = 1.$$

Let us also introduce some matrix notation. Let

$$T = \sum_j t_j \cdot a_{1,j} \otimes \ldots \otimes a_{N,j}$$

be a tensor over an algebra $\mathfrak{R}$, where $t_j \in \mathfrak{R}$ and $a_{i,j}$ are elements of the space $\operatorname{End}\mathbb{C}^n$. Then the notation $T^{(k_1,\ldots,k_N)}$ corresponds to the following element of $\mathfrak{R} \otimes (\operatorname{End} C^n)^{\otimes M}$ for numbers $M \geqslant N$:

$$T^{(k_1,\ldots,k_N)} = \sum_j t_j \cdot 1 \otimes \ldots \otimes a_{1,j} \otimes \ldots \otimes a_{N,j} \otimes \ldots \otimes 1.$$

Here each element $a_{i,j}$ is placed in the k_i-th tensor component, the numbers k_i are pairwise different and the following condition fulfills $1 \leqslant k_i \leqslant M$.

Let $F(\lambda) = F(\lambda_1, \ldots, \lambda_n)$ be a function on n parameters λ_k, taking values in an algebra $\mathfrak{R}$: i.e. $F\colon \mathbb{C}^n \to \mathfrak{R}$. In this case we define special shifts

$$\begin{aligned} F(\lambda + P) &= F(\lambda_1 + P_1, \ldots, \lambda_n + P_n) \\ &= \sum_{i_1,\ldots,i_n=0}^{\infty} \frac{1}{i_1! \cdots i_n!} \frac{\partial^{i_1+\ldots+i_n} F(\lambda_1, \ldots, \lambda_n)}{\partial \lambda_1^{i_1} \cdots \partial \lambda_n^{i_n}} P_1^{i_1} \cdots P_n^{i_n} \end{aligned} \tag{3.15}$$

for some set $P = (P_1, \ldots, P_n)$, $P_k \in \mathfrak{R}$. We do not discuss here the convergency questions, in our context all such expressions will be well defined.

3.4.2 Felder algebra

Let us introduce the notion of the elliptic L-operator, corresponding to the Felder R-matrix.

We use the notations $\{e_i\}$, $\{E_{ij}\}$ from the section 2.4.2. Let $\mathfrak{h}$ be a commutative algebra of dimension n. In [50] it was constructed an element of $\operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n$, meromorphically depending on the parameter u and n dynamical parameters $\lambda_1, \ldots, \lambda_n$:

$$\begin{aligned} R(u;\lambda) = R(u;\lambda_1,\ldots,\lambda_n) = \frac{\theta(u+\hbar)}{\theta(u)} \sum_{i=1}^n E_{ii} \otimes E_{ii} + \\ + \sum_{i \neq j} \Big(\frac{\theta(\lambda_{ij}+\hbar)}{\theta(\lambda_{ij})} E_{ii} \otimes E_{jj} + \frac{\theta(u-\lambda_{ij})\theta(\hbar)}{\theta(u)\theta(-\lambda_{ij})} E_{ij} \otimes E_{ji} \Big), \end{aligned} \tag{3.16}$$

where $\lambda_{ij} = \lambda_i - \lambda_j$. This element is called the dynamical Felder R-matrix. It satisfies the dynamical Yang-Baxter equation

$$\begin{aligned} & R^{(12)}(u_1-u_2;\lambda) R^{(13)}(u_1-u_3;\lambda+\hbar E^{(2)}) R^{(23)}(u_2-u_3;\lambda) = \\ = \ & R^{(23)}(u_2-u_3;\lambda+\hbar E^{(1)}) R^{(13)}(u_1-u_3;\lambda) R^{(12)}(u_1-u_2;\lambda+\hbar E^{(3)}), \end{aligned}$$

and the additional conditions

$$R^{(21)}(-u;\lambda)R^{(12)}(u;\lambda) = \frac{\theta(u+\hbar)\theta(u-\hbar)}{\theta(u)^2},$$
$$(E_{ii}^{(1)} + E_{ii}^{(2)})R(u;\lambda) = R(u;\lambda)(E_{ii}^{(1)} + E_{ii}^{(2)}),$$
$$(\widehat{D}_\lambda^{(1)} + \widehat{D}_\lambda^{(2)})R(u;\lambda) = R(u;\lambda)(\widehat{D}_\lambda^{(1)} + \widehat{D}_\lambda^{(2)}),$$

where

$$\widehat{D}_\lambda = \sum_{k=1}^n E_{kk}\frac{\partial}{\partial \lambda_k}, \quad \widehat{D}_\lambda^{(i)} = \sum_{k=1}^n E_{kk}^{(i)}\frac{\partial}{\partial \lambda_k}.$$

We should mention that, in the above formulas, λ denotes a vector $\lambda_1, \ldots, \lambda_n$, and the expression $\lambda + \hbar E^{(s)}$ implies a shift of the type (3.15) with the parameters values $P_i = \hbar E_{ii}^{(s)}$.

Let $\mathfrak{R}$ be a $\mathbb{C}[[\hbar]]$-algebra, $L(u;\lambda)$ an invertible $n \times n$ matrix over $\mathfrak{R}$ depending on the spectral parameter u and n dynamical parameters $\lambda_1, \ldots, \lambda_n$. Let $h_1, \ldots, h_n$ be a set of pairwise commuting elements of $\mathfrak{R}$. $L(u;\lambda)$ is called an elliptic dynamical L-operator corresponding to the set of Cartan elements h_k if $L(u;\lambda)$ satisfies the dynamical RLL relation

$$\begin{aligned} &R^{(12)}(u-v;\lambda)L^{(1)}(u;\lambda+\hbar E^{(2)})L^{(2)}(v;\lambda) \\ = \; &L^{(2)}(v;\lambda+\hbar E^{(1)})L^{(1)}(u;\lambda)R^{(12)}(u-v;\lambda+\hbar h), \end{aligned} \tag{3.17}$$

and a condition of the form

$$(E_{ii} + h_i)L(u;\lambda) = L(u;\lambda)(E_{ii} + h_i).$$

Let us introduce an equivalent but more symmetric form of RLL relations. For an L-operator we define the expression:

$$L_D(u) = e^{-\hbar \widehat{D}_\lambda} L(u;\lambda). \tag{3.18}$$

The equation (3.17) can be rewritten in the new notation as follows:

$$R^{(12)}(u-v;\lambda)L_D^{(1)}(u)L_D^{(2)}(v) = L_D^{(2)}(v)L_D^{(1)}(u)R^{(12)}(u-v;\lambda+\hbar h). \tag{3.19}$$

The next lemma plays the role analogous to the fusion method in the rational case, namely it describes a method of elliptic L-operators construction.

Lemma 3.8. *If $L_1(u;\lambda) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathfrak{R}_1$ and $L_2(u;\lambda) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathfrak{R}_2$ are two elliptic dynamical L-operators with respect to two sets of Cartan elements: $h^1 = (h_1^1, \ldots, h_n^1)$ and $h^2 = (h_1^2, \ldots, h_n^2)$, then the product $L_2(u;\lambda)L_1(u;\lambda+\hbar h^2) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathfrak{R}_1 \otimes \mathfrak{R}_2$ is also an elliptic dynamical L-operator with respect to the set $h = h^1 + h^2 = (h_1^1 + h_1^2, \ldots, h_n^1 + h_n^2)$. Hence, if $L_1(u;\lambda)$, $\ldots$, $L_m(u;\lambda)$ are* *elliptic dynamical L-operators with the sets of Cartan elements $h^1, \ldots, h^m$, then the matrix*

$$\overleftarrow{\prod_{m\geqslant j\geqslant 1}} L_j\big(u;\lambda+\hbar\sum_{l=j+1}^{m} h^l\big) \tag{3.20}$$

is also an elliptic dynamical L-operator with the following set of Cartan elements $h=\sum\limits_{i=1}^{m} h^i$.

Remark 3.9. The arrow in the above product denotes the order of multipliers with growing indexes: for example, the expression $\overleftarrow{\prod}_{3\geqslant i\geqslant 1} A_i$ means $A_3A_2A_1$.

The main example of elliptic dynamical L-operator is given by the Felder R-matrix: $L(u)=R(u-v;\lambda)$. In this case the second space $\mathrm{End}(\mathbb{C}^n)$ takes the role of the algebra $\mathfrak{R}$. Here v is a complex number and the Cartan elements coincide with the diagonal matrices $h_k=E^{(2)}_{kk}$. Lemma 3.8 makes it possible to generalize this example: let $v_1, \ldots, v_m$ be a set of complex numbers, then the matrix

$$\mathbb{R}^{(0)}(u;\{v_j\};\lambda)=\overleftarrow{\prod_{m\geqslant j\geqslant 1}} R^{(0j)}\big(u-v_j;\lambda+\hbar\sum_{l=j+1}^{m}E^{(l)}\big) \tag{3.21}$$

is a dynamical elliptic L-operator with the Cartan elements $h_k=\sum\limits_{l=1}^{m}E^{(l)}_{kk}$.

A more general class of dynamical elliptic L-operators is related with the so-called *small elliptic quantum group* $e_{\tau,\hbar}(\mathfrak{gl}_n)$ constructed in [51]. This represents a $\mathbb{C}[[\hbar]]((\lambda_1,\ldots,\lambda_n))$-algebra generated by $\tilde{t}_{ij}$ and h_k with relations

$$\begin{gathered}
\tilde{t}_{ij}h_k=(h_k-\delta_{ik}+\delta_{jk})\tilde{t}_{ij},\\
t_{ij}\lambda_k-(\lambda_k-\hbar\delta_{ik})t_{ij}=0,\\
t_{ij}t_{ik}-t_{ik}t_{ij}=0,\\
t_{ik}t_{jk}-\frac{\theta(\lambda^{\{1\}}_{ij}+\hbar)}{\theta(\lambda^{\{1\}}_{ij}-\hbar)}t_{jk}t_{ik}=0,\qquad i\neq j,\\
\frac{\theta(\lambda^{\{2\}}_{jl}+\hbar)}{\theta(\lambda^{\{2\}}_{jl})}t_{ij}t_{kl}-\frac{\theta(\lambda^{\{1\}}_{ik}+\hbar)}{\theta(\lambda^{\{1\}}_{ik})}t_{kl}t_{ij}-\frac{\theta(\lambda^{\{1\}}_{ik}+\lambda^{\{2\}}_{jl})\theta(\hbar)}{\theta(\lambda^{\{1\}}_{ik})\theta(\lambda^{\{2\}}_{jl})}t_{il}t_{kj}=0,
\end{gathered} \tag{3.22}$$

with $i\neq k,\ j\neq l$, where $t_{ij}=\delta_{ij}+\hbar\tilde{t}_{ij}$, $\lambda^{\{1\}}_{ij}=\lambda_i-\lambda_j$, $\lambda^{\{2\}}_{ij}=\lambda_i-\lambda_j-\hbar h_i+\hbar h_j$, it is also supposed that $h_1,\ldots,h_k,\lambda_1,\ldots,\lambda_k$ commute. One constructs a generating function for these generators $T(-u)$

$$T_{ij}(-u)=\theta(-u+\lambda_{ij}-\hbar h_i)t_{ji}. \tag{3.23}$$

Representing this matrix in the form

$$T(-u) = \theta(-u)e^{-\hbar\sum_{k=0}^{n}(h_k+E_{kk})\partial_{\lambda_k}}L_0(u;\lambda)e^{\hbar\sum_{k=0}^{n}h_k\partial_{\lambda_k}} \tag{3.24}$$

we obtain a dynamical elliptic L-operator $L_0(u;\lambda)$ for the algebra $\mathfrak{T} = e_{\tau,\hbar}(\mathfrak{gl}_n)[[\partial_\lambda]]$ with the Cartan elements $h = (h_1, \ldots, h_n)$, where $\mathbb{C}[[\partial_\lambda]] = \mathbb{C}[[\partial_{\lambda_1}, \ldots, \partial_{\lambda_n}]]$. The elements $\partial_{\lambda_k} = \frac{\partial}{\partial\lambda_k}$ commute with h_i and do not commute with $\tilde{t}_{ij}$.

3.4.3 Commutative algebra

Let us consider a dynamical elliptic L-operator $L(u;\lambda)$ with a set of Cartan elements h_k. This function takes values in the algebra $\operatorname{End}\mathbb{C}^n \otimes \mathfrak{R}$.

Let us introduce the operators

$$\mathbb{L}^{[m,N]}(\{u_i\};\lambda) = e^{-\hbar\widehat{D}_\lambda^{(m+1)}}L^{(m+1)}(u_{m+1};\lambda)\cdots e^{-\hbar\widehat{D}_\lambda^{(N)}}L^{(N)}(u_N;\lambda), \tag{3.25}$$

where $m < N$. Let us consider a particular case with the parameters values $u_i = u + \hbar(i-1)$,

$$\mathbb{L}^{[a,b]}(u;\lambda) = \mathbb{L}^{[a,b]}(\{u_i = u + \hbar(i-a-1)\};\lambda)$$

for $a < b$. Let $\mathbb{A}_n = \mathbb{C}((\lambda_1, \ldots, \lambda_n))$ be the completed function space. The operators $\widehat{D}_\lambda$ act on the space $\mathbb{A}_n \otimes \mathbb{C}^n$, in turn the operators $\mathbb{L}^{[a,b]}(u;\lambda)$ act from $\mathbb{A}_n \otimes (\mathbb{C}^n)^{\otimes(b-a)}$ to the space $\mathbb{A}_n \otimes (\mathbb{C}^n)^{\otimes(b-a)} \otimes \mathfrak{R}$: fixing u we obtain $\mathbb{L}^{[a,b]}(u;\lambda) \in \operatorname{End}(\mathbb{C}^n)^{\otimes(b-a)} \otimes \mathfrak{A}_n$, where $\mathfrak{A}_n = \mathbb{A}_n[e^{\pm\hbar\partial_\lambda}] \otimes \mathfrak{R}$. Let us consider a subalgebra $\mathfrak{h} \subset \mathfrak{R} \subset \mathfrak{A}_n$ generated by the elements h_k and its normalizer $\mathfrak{A}_n$:

$$\mathfrak{N}_n = \mathfrak{N}_{\mathfrak{A}_n}(\mathfrak{h}) = \{x \in \mathfrak{A}_n \mid \mathfrak{h}x \subset \mathfrak{A}_n\mathfrak{h}\}. \tag{3.26}$$

Observe that $\mathfrak{A}_n\mathfrak{h}$ is a two-sided ideal in $\mathfrak{N}_n$. In [49] the following statement is proved

Theorem 3.10. *Let us define $\mathfrak{A}_n$-valued functions*

$$t_m(u) = tr\big(A^{[0,m]}\mathbb{L}^{[0,m]}(u;\lambda)\big), \tag{3.27}$$

where we suppose the trace operation over m spaces $\mathbb{C}^n$. These expressions commute with the Cartan elements h_k:

$$h_k t_m(u) = t_m(u)h_k. \tag{3.28}$$

Hence they are elements of the subalgebra $\mathfrak{N}_n$. Moreover these generators commute modulo the ideal $\mathfrak{A}_n\mathfrak{h} \subset \mathfrak{N}_n$:

$$t_m(u)t_s(v) = t_s(v)t_m(u) \qquad \mod \mathfrak{A}_n\mathfrak{h}. \tag{3.29}$$

3.4.4 Characteristic polynomial

As in the rational case the generators $t_m(u)$ can be organized into a generating function called the quantum characteristic polynomial. This generating function is constructed as a "'determinant"' of the corresponding L-operator.

Proposition 3.11. *Let us consider the matrix* $M = e^{-\hbar\widehat{D}_\lambda} L(u;\lambda) e^{\hbar\frac{\partial}{\partial u}}$. *Then the determinant of* $1 - M$ *generates the family* $t_m(u)$ *in the following sense:*

$$P(u, e^{\hbar\partial_u}) = det(1 - e^{-\hbar\widehat{D}_\lambda} L(u;\lambda) e^{\hbar\frac{\partial}{\partial u}}) = \sum_{m=0}^{n} (-1)^m t_m(u) e^{m\hbar\frac{\partial}{\partial u}}, \tag{3.30}$$

where $t_0(u) = 1$. *This property induces the commutativity of the quantum characteristic polynomial with elements* h_k, *and the pairwise commutativity modulo* $\mathfrak{A}_n\mathfrak{h}$ *of the generating functions:*

$$[P(u, e^{\hbar\partial_u}), h_k] = 0, \qquad [P(u, e^{\hbar\partial_u}), P(v, e^{\hbar\partial_v})] = 0 \mod \mathfrak{A}_n\mathfrak{h}. \tag{3.31}$$

3.4.5 The limit and the Gaudin model

Let us consider degenerated elliptic dynamical RLL relations at $\hbar \to 0$. This limit describes the elliptic quantum Gaudin model. To do this we use a shift of the L-operator. The limit of the generating function for the generic family gives the generating function for the Hamiltonians of the elliptic Gaudin model. The result obtained generalizes the works [52],[53].

Let $L(u;\lambda)$ be a dynamical elliptic L-operator of the form

$$L(u;\lambda) = 1 + \hbar\Lambda(u;\lambda) + o(\hbar), \tag{3.32}$$

those matrix elements are elements of the algebra $\mathfrak{R}_0 = \mathfrak{R}/\hbar\mathfrak{R}$. The matrix $\Lambda(u;\lambda)$ is called a classical dynamical elliptic L-operator. It satisfies the rLL-relations

$$\begin{aligned} [\Lambda^{(1)}(u;\lambda) - \widehat{D}^{(1)}_\lambda, \Lambda^{(2)}(v;\lambda) - \widehat{D}^{(2)}_\lambda] - \sum_{k=1}^{n} h_k \frac{\partial}{\partial\lambda} r(u-v;\lambda) = \\ = [\Lambda^{(1)}(u;\lambda) + \Lambda^{(2)}(v;\lambda), r(u-v;\lambda)] \end{aligned} \tag{3.33}$$

with the classical dynamical elliptic r-matrix

$$\begin{aligned} r(u;\lambda) &= \frac{\theta'(u)}{\theta(u)} \sum_{i=1}^{n} E_{ii} \otimes E_{ii} \\ &+ \sum_{i\neq j} \Big(\frac{\theta'(\lambda_{ij})}{\theta(\lambda_{ij})} E_{ii} \otimes E_{jj} + \frac{\theta(u-\lambda_{ij})}{\theta(u)\theta(-\lambda_{ij})} E_{ij} \otimes E_{ji} \Big). \end{aligned} \tag{3.34}$$

The matrix (3.34) is related with the Felder R-matrix (3.16) by the formula

$$R(u;\lambda) = 1 + \hbar r(u;\lambda) + o(\hbar). \tag{3.35}$$

Theorem 3.12. *Let* $\mathcal{A}_n = \mathfrak{R}_0 \otimes \mathbb{A}_n[\partial_\lambda]$ *???* $\mathcal{N}_n = \mathfrak{N}_{\mathcal{A}_n}(\mathfrak{h}) = \{x \in \mathcal{A}_n \mid \mathfrak{h}x \subset \mathcal{A}_n\mathfrak{h}\}$*, where* $\mathbb{A}_n = \mathbb{C}((\lambda_1, \dots, \lambda_n))$*. Let us define a set of* $\mathcal{N}_n$*-valued functions* $s_m(u)$ *by the formula*

$$Q(u, \partial_u) = det\Big(\frac{\partial}{\partial u} - \widehat{D}_\lambda + \Lambda(u;\lambda)\Big) = \sum_{m=0}^{n} s_m(u)\Big(\frac{\partial}{\partial u}\Big)^{n-m}, \tag{3.36}$$

where $s_0(u) = 1$*. They commute with the Cartan elements* h_k*:*

$$h_k s_m(u) = s_m(u) h_k \tag{3.37}$$

and moreover pairwise commute modulo $\mathcal{A}_n\mathfrak{h}$*:*

$$s_m(u)s_l(v) = s_l(v)s_m(u) \mod \mathcal{A}_n\mathfrak{h}. \tag{3.38}$$

The values of the functions $s_1(u)$, $s_2(u)$, $\dots$, $s_n(u)$ generate a commutative subalgebra in $\mathcal{N}_n$ on the level $h_k = 0$. This means that the images of these elements with respect to the canonical homomorphism $\mathcal{N}_n \to \mathcal{N}_n/\mathcal{A}_n\mathfrak{h}$ pairwise commute.

The quantum elliptic Gaudin model is defined with the help of the Lax operator

$$\Lambda_{ij}(u;\lambda) = e_{ji}(u;\lambda), \qquad \Lambda_{ii}(u;\lambda) = e_{ii}(u;\lambda) + \sum_{k\neq i} \frac{\theta'(\lambda_{ik})}{\theta(\lambda_{ik})} h_k, \tag{3.39}$$

with coefficients expressed by the formulas:

$$e_{ii}(u) = \frac{\theta'(u-z)}{\theta(u-z)} e_{ii} = \sum_{m\geqslant 0} \frac{(-1)^m}{m!} \Big(\frac{\theta'(u)}{\theta(u)}\Big)^{(m)} e_{ii} z^m, \tag{3.40}$$

$$e_{ij}(u;\lambda) = \frac{\theta(u-z+\lambda_{ij})}{\theta(u-z)\theta(\lambda_{ij})} e_{ij} = \sum_{m\geqslant 0} \frac{(-1)^m}{m!} \Big(\frac{\theta(u+\lambda_{ij})}{\theta(u)\theta(\lambda_{ij})}\Big)^{(m)} e_{ij} z^m. \tag{3.41}$$

An analog of the evaluation representation is the homomorphism to the small elliptic quantum group defined by the generating function (3.24). Let us consider an expansion on the parameter $\hbar$ of the dynamical L-operator corresponding to the tensor power of the small elliptic group. It turns out that the coefficient at $\hbar$ of this expansion coincides with the elliptic Gaudin model L-operator.

3.4.6 The explicit form of the $\mathfrak{sl}_2$ elliptic Gaudin model

The L-operator of the elliptic $\mathfrak{sl}_2$ Gaudin model considered in [52, 53, 54] has the form

$$\Lambda(u;\lambda) = \begin{pmatrix} h(u)/2 & f_\lambda(u) \\ e_\lambda(u) & -h(u)/2 \end{pmatrix}, \tag{3.42}$$

where $\lambda = \lambda_{12} = \lambda_1 - \lambda_2$ and the currents are expressed by the formulas

$$h(u) = e_{11}(u) - e_{22}(u) = \sum_{s=1}^{N} \frac{\theta'(u - v_s)}{\theta(u - v_s)}(e_{11}^{(s)} - e_{22}^{(s)}),$$
$$e_\lambda(u) = e_{12}(u;\lambda) = \sum_{s=1}^{N} \frac{\theta(u - v_s + \lambda)}{\theta(u - v_s)\theta(\lambda)} e_{12}^{(s)},$$
$$f_\lambda(u) = e_{21}(u;\lambda) = \sum_{s=1}^{N} \frac{\theta(u - v_s - \lambda)}{\theta(u - v_s)\theta(-\lambda)} e_{21}^{(s)}.$$

The fact that the L-operator depends only on the difference $\lambda = \lambda_1 - \lambda_2$ allows to restrict the generating function of the commutative subalgebra $Q(u, \partial_u)$ to the space $\mathbb{A} = \{a \in \mathbb{A}_2 \mid (\partial_{\lambda_1} + \partial_{\lambda_1})a = 0\} \subset \mathbb{A}_2$ coinciding with $\mathbb{C}((\lambda_{12}))$. Let $\mathcal{A} = \mathfrak{R}_0 \otimes \mathbb{A}[\partial_\lambda]$ then the values of $s_m(u)$ are elements of $\mathcal{N} = \mathfrak{N}_{\mathcal{A}}(\mathfrak{h}) = \{x \in \mathcal{A} \mid hx \in \mathcal{A}h\}$. In virtue of the representation ρ: $h_1 + h_2 \to 0$ the operator $\widehat{D}_\lambda$ has the form $H\partial_\lambda$, where $H = E_{11} - E_{22}$.

Let us find the quantum characteristic polynomial in this case:

$$Q(u, \partial_u) = det\Big(\frac{\partial}{\partial u} - \widehat{D}_\lambda + \widetilde{\Lambda}(u;\lambda) - \frac{\theta'(\lambda)}{\theta(\lambda)}\frac{h}{2}\Big) =$$
$$= \ det \begin{pmatrix} \frac{\partial}{\partial u} - \partial_\lambda + h^+(u)/2 - \frac{\theta'(\lambda)}{\theta(\lambda)}h/2 & f_\lambda^+(u) \\ e_\lambda^+(u) & \frac{\partial}{\partial u} + \partial_\lambda - h^+(u)/2 - \frac{\theta'(\lambda)}{\theta(\lambda)}h/2 \end{pmatrix}$$
$$(3.43)= \ \Big(\frac{\partial}{\partial u}\Big)^2 - \frac{\theta'(\lambda)}{\theta(\lambda)}h\frac{\partial}{\partial u} - S_\lambda(u),$$

where $h = h_1 - h_2$. $S_\lambda(u)$ is an $\mathcal{N}$-valued function

$$S_\lambda(u) = \big(\partial_\lambda - h(u)/2\big)^2 + \partial_u h(u)/2 + e_\lambda(u) f_\lambda(u) \qquad \text{mod } \mathcal{A}h.$$

The commutativity condition can be formulated in terms of this generating function as follows:

$$[S_\lambda(u), S_\lambda(v)] = 0 \ \text{ mod } \ \mathcal{A}h.$$

Using the commutation relations

$$[e_\lambda^+(u), f_\lambda^+(u)] = -\frac{\partial}{\partial u}h^+(u) + \big(\frac{\theta'(\lambda)}{\theta(\lambda)}\big)'h$$

one can simplify this generating function:

$$S_\lambda(u) = \big(\partial_\lambda - h(u)/2\big)^2 + \big(e_\lambda(u) f_\lambda(u) + f_\lambda(u) e_\lambda(u)\big)/2 \qquad \text{mod } \mathcal{A}h. \tag{3.44}$$

4 Solution for quantum integrable systems

As was mentioned above the traditional methods of solving quantum integrable systems on the finite scale in some cases allow to solve the Hamiltonian diagonalization problem in terms of solutions of a system of algebraic equations (the Bethe system). However, the system of equations itself, in cases where it can be deduced, turns out to be quite complicated and hypothetically admits no algebraic solution. In this section we use an equivalent formulation for quantum eigenproblem in terms of Fuchsian systems with special monodromy representation. In turn the construction of relevant Fuchsian systems uses the quantum characteristic polynomial of a model. This observation also distinguishes the quantum characteristic polynomial among others generating function for the commutative subalgebra.

4.1 Monodromic formulation

4.1.1 A scalar and a matrix Fuchsian equation

Consider a Fuchsian system defined by a connection in trivial bundle of rank 2 on the disk with punctures:

$$A(z) = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} = \sum_{i=1}^{k} \frac{A_i}{z - z_i} \tag{4.1}$$

with residues satisfying the conditions:

$$Tr(A_i) = 0; \quad Det(A_i) = -d_i^2; \quad \sum_i A_i = \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix}. \tag{4.2}$$

The Fuchsian system is written by the equation

$$(\partial_z - A(z))\Psi(z) = 0. \tag{4.3}$$

The components of this system may be represented as follows

$$\begin{aligned} \psi_1' &= a_{11}\psi_1 + a_{12}\psi_2, \\ \psi_2' &= a_{21}\psi_1 + a_{22}\psi_2. \end{aligned}$$

The first vector component satisfies the second order equation

$$\psi_1'' = (a_{12}'/a_{12})\psi_1' + u\psi_1,$$

where

$$u = a_{11}' + a_{11}^2 - a_{11}(a_{12}'/a_{12}) + a_{12}a_{21}.$$

With the following variable change: $\Phi = \psi_1/\chi$, where $\chi = \sqrt{a_{12}}$, we obtain the equation

$$\Phi'' + U\Phi = 0,$$

with the potential defined by the formula

$$U = \chi''/\chi - (a'_{12}/a_{12})\chi'/\chi - u. \tag{4.4}$$

Introducing the expression for χ to U we obtain:

$$U = \frac{1}{2}\left(\frac{a'_{12}}{a_{12}}\right)' - \frac{1}{4}\left(\frac{a'_{12}}{a_{12}}\right)^2 + a_{11}\frac{a'_{12}}{a_{12}} - a'_{11} - a_{11}^2 - a_{12}a_{21}. \tag{4.5}$$

Let us suppose that $a_{12}(z)$ has no multiple poles

$$a_{12}(z) = c\frac{\prod_{j=1}^{k-2}(z - w_j)}{\prod_{i=1}^{k}(z - z_i)}.$$

We should note that the number of zeros agrees with the normalization (4.2). The expression for the logarithmic derivative can be simplified:

$$\frac{a'_{12}}{a_{12}} = \sum_{j=1}^{k-2}\frac{1}{z - w_j} - \sum_{i=1}^{k}\frac{1}{z - z_i}. \tag{4.6}$$

The potential U takes the form

$$U = \sum_{j=1}^{k-2}\frac{-3/4}{(z - w_j)^2} + \sum_{i=1}^{k}\frac{1/4 + detA_i}{(z - z_i)^2} + \sum_{j=1}^{k-2}\frac{H_{w_j}}{z - w_j} + \sum_{i=1}^{k}\frac{H_{z_i}}{z - z_i}, \tag{4.7}$$

in which

$$\begin{aligned} H_{w_j} &= a_{11}(w_j) + \frac{1}{2}\left(\sum_{i\neq j}\frac{1}{w_j - w_i} - \sum_{i}\frac{1}{w_j - z_i}\right); \\ H_{z_i} &= \left(\frac{1}{2} + a_{11}^i\right)\sum_j \frac{1}{z_i - w_j} - \sum_{j\neq i}\frac{Tr(A_iA_j) + a_{11}^i + a_{11}^j + 1/2}{z_i - z_j}. \end{aligned}$$

Let us remark that the coefficients at $(z - z_i)^{-2}$ take values

$$1/4 + detA_i = (1/2 - d_i)(1/2 + d_i). \tag{4.8}$$

In what follows we identify these factors with the values of the quadratic Casimir elements of the Lie algebras $\mathfrak{sl}_2$ in the representations of highest weights λ_i ($\lambda_i = 2d_i - 1$ in our case).

4.1.2 Dual equation

As was shown in previous calculations, the matrix form of the connection leads to the Sturm-Liouville operator with additional poles at points w_j. A consideration of the second vector component of a solution of the matrix equation Ψ_2 leads to another scalar differential operator with poles at points z_i and additional points $\widetilde{w}_j$, determined by the formula

$$a_{21}(z) = \widetilde{c}\frac{\prod_{j=1}^{k-2}(z-\widetilde{w}_j)}{\prod_{i=1}^{k}(z-z_i)}.$$

Let us call the corresponding Sturm-Liouville operator

$$\partial_z^2 - \widetilde{U}$$

the dual $\mathfrak{sl}_2$-oper. In this case, the potential is expressed by the formula

$$\widetilde{U} = \sum_{j=1}^{k-2}\frac{-3/4}{(z-\widetilde{w}_j)^2} + \sum_{i=1}^{k}\frac{1/4+detA_i}{(z-z_i)^2} + \sum_{j=1}^{k-2}\frac{H_{\widetilde{w}_j}}{z-\widetilde{w}_j} + \sum_{i=1}^{k}\frac{\widetilde{H}_{z_i}}{z-z_i}. \tag{4.9}$$

4.1.3 Backup

In this section we construct an inverse map, namely for a Sturm-Liouville operator that has trivial monodromy we construct a rank 2 connection of the form (4.3) with the monodromy representation in the subgroup $\mathbb{Z}/2\mathbb{Z} \subset GL(2)$ of scalar matrices ± 1.

Let us consider an ansatz for the solution of the matrix linear equation (4.3)

$$(\partial_z - A(z))\Psi = 0$$

of the type

$$\psi_l = \prod_{i=1}^{k}(z-z_i)^{-s_i}\phi_l(z), \qquad l = 1,2; \tag{4.10}$$

which satisfy

$$\phi_1 = \prod_{j=1}^{M}(z-\gamma_j),$$

$$\phi_2/\phi_1 = \sum_{j=1}^{M}\frac{\alpha_j}{z-\gamma_j}. \tag{4.11}$$

Let us rewrite the system (4.3) taking into account the new parameterization (4.11)

$$\partial_z\psi_1/\psi_1 = a_{11} + a_{12}\phi_2/\phi_1, \tag{4.12}$$

$$(4.13) \qquad (\partial_z\psi_1/\psi_1)(\phi_2/\phi_1) + \partial_z(\phi_2/\phi_1) = a_{21} + a_{22}\phi_2/\phi_1.$$

Let us represent these equations more precisely:

$$(4.14) \qquad -\sum_i \frac{s_i}{z-z_i} + \sum_j \frac{1}{z-\gamma_j} = \sum_i \frac{a_{11}^i}{z-z_i} + \sum_i \frac{a_{12}^i}{z-z_i}\sum_j \frac{\alpha_j}{z-\gamma_j},$$

$$\left(-\sum_i \frac{s_i}{z-z_i} + \sum_j \frac{1}{z-\gamma_j}\right)\sum_j \frac{\alpha_j}{z-\gamma_j} - \sum_j \frac{\alpha_j}{(z-\gamma_j)^2} =$$

$$(4.15) \qquad \sum_i \frac{a_{21}^i}{z-z_i} - \sum_i \frac{a_{11}^i}{z-z_i}\sum_j \frac{\alpha_j}{z-\gamma_j}.$$

The comparison of residues in both parts of (4.14), (4.15) at the points $z = z_i$ gives:

$$(4.16) \qquad -s_i = a_{11}^i + a_{12}^i \sum_j \frac{\alpha_j}{z_i-\gamma_j},$$

$$(4.17) \qquad -\sum_j \frac{\alpha_j}{z_i-\gamma_j}s_i = a_{21}^i - a_{11}^i \sum_j \frac{\alpha_j}{z_i-\gamma_j}.$$

These equations coupled with the condition of zero trace $a_{11}^i + a_{22}^i = 0$ lead to a condition that s_i must be one of the eigenvalues of A_i, in particular, can be adopted as $s_i = d_i$. Let us consider the behavior at the poles $z = \gamma_j$. Let us note that the second order poles of the equation (4.15) at these points cancel. Calculating residues of both sides of the equations (4.14) and (4.15) we obtain

$$(4.18) \qquad 1 = \alpha_j \sum_i \frac{a_{12}^i}{\gamma_j - z_i},$$

$$(4.19) \qquad \alpha_j\left(-\sum_i \frac{s_i}{\gamma_j-z_i} + \sum_{i\neq j}\frac{1}{\gamma_j-\gamma_i}\right) + \sum_{i\neq j}\frac{\alpha_i}{\gamma_j-\gamma_i} = -\alpha_j \sum_i \frac{a_{11}^i}{\gamma_j-z_i}.$$

Let us recall that one of the normalization condition controls the diagonal form of the residue at ∞

$$(4.20) \qquad \sum_{i=1}^k a_{12}^i = 0,$$

$$(4.21) \qquad \sum_{i=1}^k a_{21}^i = 0.$$

We also note that the choice of the Sturm-Liouville operator poles involves zeros of the rational function $a_{12}(z)$, which is determined up to a constant:

$$a_{12}(z) = c\frac{\prod_{j=1}^{k-2}(z-w_j)}{\prod_{i=1}^{k}(z-z_i)}.$$

Then the condition (4.20) will be satisfied automatically. The coefficients a^i_{12} are expressed by the formula

$$a^i_{12} = c\frac{\prod_j (z_i - w_j)}{\prod_{j\neq i}(z_i - z_j)}. \tag{4.22}$$

The coefficients a^i_{11} are expressed by the following formula in virtue of (4.16)

$$a^i_{11} = -s_i - c\frac{\prod_{j=1}(z_i - w_j)}{\prod_{j\neq i}(z_i - z_j)}\sum_l \frac{\alpha_l}{z_i - \gamma_l}. \tag{4.23}$$

Let us substitute the expressions for a^i_{12} and a^i_{11} to the equations (4.18), (4.19). Then expressing α_j from the first and substituting to the second we obtain:

$$\begin{aligned} &- \sum_k \frac{2s_k}{\gamma_j - z_k} + \sum_{k\neq j}\frac{1}{\gamma_j - \gamma_k} + \sum_{k,m}\frac{\prod_l (z_k - w_l)\prod_{s\neq k}(\gamma_m - z_s)}{\prod_{l\neq k}(z_k - z_l)\prod_s(\gamma_m - w_s)(\gamma_j - z_k)} \\ &+ \frac{\prod_i(\gamma_j - w_i)}{\prod_i(\gamma_j - z_i)}\sum_{k\neq j}\frac{\prod_i(\gamma_k - z_i)}{\prod_i(\gamma_k - w_i)(\gamma_j - \gamma_k)} = 0. \end{aligned}$$

An equivalent form can be obtained if one divides both sides by $\frac{\prod_i(\gamma_j - z_i)}{\prod_i(\gamma_j - w_i)}$

$$\begin{aligned} &-\sum_k \frac{2s_k}{\gamma_j - z_k} + \sum_{k\neq j}\frac{1}{\gamma_j - \gamma_k} + \sum_{k,m}\frac{\prod_l (z_k - w_l)\prod_{s\neq k}(\gamma_m - z_s)}{\prod_{l\neq k}(z_k - z_l)\prod_s(\gamma_m - w_s)(\gamma_j - z_k)} \\ &+ \sum_{k\neq j}\frac{\prod_i(\gamma_k - z_i)}{\prod_i(\gamma_k - w_i)(\gamma_j - \gamma_k)}\frac{\prod_m(\gamma_j - w_m)}{\prod_m(\gamma_j - z_m)} = 0. \end{aligned}$$

Let us consider the left-hand side of equality as a rational function $F(\gamma_j)$ and calculate its primitive fractions decomposition at poles $z_k,\ w_k,\ \gamma_k$ and ∞. It turns out that this decomposition will look like:

$$F(\gamma_j) = -\sum_k \frac{2s_k - 1}{\gamma_j - z_k} - \sum_k \frac{1}{\gamma_j - w_k} + 2\sum_{i\neq j}\frac{1}{\gamma_j - \gamma_i}. \tag{4.24}$$

Thus, the equality is equivalent to an equation of the Bethe system. Let us demonstrate, for example, the residue calculating at the point $\gamma_j = w_i$

$$\begin{aligned} Res_{\gamma_j = w_i}F(\gamma_j) &= \sum_k \frac{\prod_l (z_k - w_l)\prod_{s\neq k}(w_i - z_s)}{\prod_{l\neq k}(z_k - z_l)\prod_{s\neq i}(w_i - w_s)(w_i - z_k)} \\ &= \frac{\prod_s (w_i - z_s)}{\prod_{s\neq i}(w_i - w_s)}\sum_k \frac{\prod_l (z_k - w_l)}{\prod_{l\neq k}(z_k - z_l)(w_i - z_k)^2}. \end{aligned} \tag{4.25}$$

Let us write down the expression on the right side of the equality

$$(Res_{z=w_i}\Phi(z))^{-1}\sum_k Res_{z=z_k}\Phi(z),$$

where

$$\Phi(z)=\frac{\prod_l(z-w_l)}{\prod_l(z-z_l)(z-w_i)^2},$$

and therefore is -1.

The sufficiency condition was proved in [55].

Theorem 4.1. *If the set of numbers γ_i, where $i=1,\dots,M$, satisfies the system of Bethe equations (3.9) with the set of poles $z_1,\dots,z_k$ and $w_1,\dots,w_{k-2}$, and the set of highest weights $2s_1-1,\dots,2s_k-1$ and $1,\dots,1$ as parameters, then the vector*

$$\Psi=\prod_{i=1}^{k}(z-z_i)^{-s_i}\begin{pmatrix}\phi_1(z)\\ \phi_2(z)\end{pmatrix}, \tag{4.26}$$

where

$$\phi_1=\prod_{j=1}^{M}(z-\gamma_j),$$

$$\phi_2/\phi_1=\sum_{j=1}^{M}\frac{\alpha_j}{z-\gamma_j}, \tag{4.27}$$

and the coefficients α_j are given by the expressions

$$\alpha_j=\frac{\prod_i(\gamma_j-z_i)}{\prod_i(\gamma_j-w_i)}, \tag{4.28}$$

solves the matrix linear problem (4.3), where the connection coefficients are given by

$$a^i_{12} \;=\; \frac{\prod_j(z_i-w_j)}{\prod_{j\neq i}(z_i-z_j)}, \tag{4.29}$$

and the coefficients a^i_{11} and a^i_{21} are determined form (4.16), (4.17). The conditions of normalization (4.2) are fulfilled.

Proof. Actually, we should prove just that normalization condition (4.21) does not depend on the choice of the parameter c, in particular, it may be taken equal to 1. Indeed, on the basis of (4.16), (4.17) we obtain:

$$a^i_{21} = -\left(2s_i \sum_j \frac{\alpha_j}{z_i - \gamma_j} + \frac{\prod_j (z_i - w_j)}{\prod_{j\neq i}(z_i - z_j)} \left(\sum_j \frac{\alpha_j}{z_i - \gamma_j}\right)^2\right). \tag{4.30}$$

We need to prove that

$$\sum_i 2s_i \sum_j \frac{\alpha_j}{z_i - \gamma_j} + \sum_i \frac{\prod_j (z_i - w_j)}{\prod_{j\neq i}(z_i - z_j)} \left(\sum_j \frac{\alpha_j}{z_i - \gamma_j}\right)^2 = 0. \tag{4.31}$$

Then the first summand of (4.31) using the Bethe equations can be converted to the following:

$$\begin{aligned} \sum_i 2s_i \sum_j \frac{\alpha_j}{z_i - \gamma_j} &= \sum_j \alpha_j \sum_i \frac{2s_i}{z_i - \gamma_j} \\ &= \sum_j \alpha_j \left(-\sum_i \frac{1}{\gamma_j - z_i} + \sum_i \frac{1}{\gamma_j - w_i} - 2\sum_{i\neq j} \frac{1}{\gamma_j - \gamma_i}\right). \end{aligned} \tag{4.32}$$

Now we will simplify the second summand (4.31) changing the order

$$\begin{aligned} &\sum_{m\neq l} \alpha_m \alpha_l \sum_i \frac{\prod_j (z_i - w_j)}{\prod_{j\neq i}(z_i - z_j)(z_i - \gamma_m)(z_i - \gamma_l)} \\ + &\sum_m (\alpha_m)^2 \sum_i \frac{\prod_j (z_i - w_j)}{\prod_{j\neq i}(z_i - z_j)(z_i - \gamma_m)^2}. \end{aligned} \tag{4.33}$$

Considering the second summand (4.33) let us note that

$$\sum_i \frac{\prod_j (z_i - w_j)}{\prod_{j\neq i}(z_i - z_j)(z_i - \gamma_m)^2} = -\partial_{\gamma_m} \Phi^1(\gamma_m), \tag{4.34}$$

where

$$\Phi^1(\gamma_m) = \sum_i \frac{\prod_j (z_i - w_j)}{\prod_{j\neq i}(z_i - z_j)(z_i - \gamma_m)} = \frac{\prod_j (\gamma_m - w_j)}{\prod_j (\gamma_m - z_j)}. \tag{4.35}$$

Therefore, the expression (4.34) becomes:

$$-\frac{\prod_j (\gamma_m - w_j)}{\prod_j (\gamma_m - z_j)} \left(\sum_s \frac{1}{\gamma_m - w_s} - \sum_s \frac{1}{\gamma_m - z_s}\right), \tag{4.36}$$

which is reduced with the relevant part of (4.32). Let us consider the first summand (4.33), this also can be simplified:

$$\sum_i \frac{\prod_j (z_i - w_j)}{\prod_{j\neq i}(z_i - z_j)(z_i - \gamma_m)(z_i - \gamma_l)} \tag{4.37}$$
$$= \frac{\prod_j (\gamma_l - w_j)}{\prod_j (\gamma_l - z_j)(\gamma_m - \gamma_l)} - \frac{\prod_j (\gamma_m - w_j)}{\prod_j (\gamma_m - z_j)(\gamma_m - \gamma_l)}.$$

Substituting the expression in (4.33) we finish the proof □

4.2 Schlesinger transformations

There is a discrete group of transformations that preserve the connection form (4.3) and, moreover, do not change the class of monodromy representation. However, these changes shift characteristic exponents at fixed points by half-integer values. Such transformations are called Schlesinger, Hecke or Backlund transformations depending on the context. They have simple geometric interpretation explained in the beginning of this section.

4.2.1 Action on bundles

Let us consider a curve C, a holomorphic bundle F on it, the corresponding sheaf of sections $\mathcal{F}$, the additional set of data $x \in C$ and a point of the dual space to the fiber $l \in F_x^*$. Then the lower Hecke transform $T_{(x,l)}E$ is defined by the subsheaf $\mathcal{F}' = \{s \in \mathcal{F} : (s(x), l) = 0\}$, which in turn corresponds to a certain holomorphic bundle on the curve C.

The equivalent definition can be defined in terms of gluing functions. Let us consider the action on holomorphic bundles on $\mathbb{C}P^1$. In virtue of the Birkhoff-Grothendieck theorem [56] any holomorphic bundle on $\mathbb{C}P^1$ of rank n is isomorphic to the sum of line bundles $\mathcal{O}(k_1) \oplus \ldots \oplus \mathcal{O}(k_n)$ for a specific set of integers $(k_1, \ldots, k_n)$ called the type of a bundle and determined up to the symmetric group action. Let us consider the open covering of $\mathbb{C}P^1$ consisting in: U_∞ - a disk around ∞ which does not contain $z = z_i,\ i = 1, \ldots, N$ and the domain $U_0 = \mathbb{C}P^1 \backslash \{\infty\}$. We consider holomorphic rank 2 bundles and parameterize them by gluing function $G(z)$ which is a holomorphically invertible function on $U_0 \cap U_\infty$ with values in $GL(2)$. Let us say that a pair $S_\infty(z) \in \mathcal{O}^{(2)}(U_\infty)$ and $S_0(z) \in \mathcal{O}^{(2)}(U_0)$ defines a global section if $S_0(z) = G(z)S_\infty(z)$.

We describe the transformation on bundles in terms of actions on corresponding gluing functions defined as a multiplication on the left by an element

$$G_s(z) = G_s \begin{pmatrix} z - z_s & 0 \\ 0 & 1 \end{pmatrix} G_s^{-1} \tag{4.38}$$

for some constant matrix G_s and some point $z_s \in U_0$

Remark 4.2. The action on the space of gluing functions can be reduced to the action on the isomorphism classes of holomorphic bundles if one chooses G_s appropriately. If changing a trivialization in U_0 by $T(z)$ we change also the matrix G_s as follows: $T(z_s)G_s$. This is obviously referring to the invariant definition above.

We will investigate the composition of these changes applied at two points.

Lemma 4.3. *A composition of two transformations specified by an expression $G_i(z)G_j^{-1}(z)$, for a generic choice of matrices $G_i,\ G_j$ preserves the trivial bundle.*

Proof It is sufficient to find a decomposition for $G(z) = G_i(z)G_j^{-1}(z)$ with $G(z) = G_{ij}(z)G_\infty(z)$, where $G_{ij}(z),\ G_\infty(z)$ are invertible at $U_0,\ U_\infty$, respectively. The thought-consideration for this evidence is the cohomological dimension count in families at a generic point. Indeed, for a particular choice $G_i^{-1}G_j = 1$ we get a trivial bundle which is semistable and hence minimizes the dimension of $H^0(\mathrm{End}(V))$ for V of degree 0. In this context, the trivial bundle is generic in the family of bundles for different G.

Despite the general argument here we propose a proof in spirit of the decomposition lemma in [56]. Let us introduce the notation

$$G_i = \begin{pmatrix} 1 & x_i \\ y_i & 1 \end{pmatrix}. \tag{4.39}$$

We can decompose the product

$$G(z) = G_i \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} G_i^{-1} G_j \begin{pmatrix} (z-1)^{-1} & 0 \\ 0 & 1 \end{pmatrix} G_j^{-1} \tag{4.40}$$

into the alternative product

$$G(z) = G_{ij}(z)G_\infty(z),$$

where $G_{ij}(z),\ G_\infty(z)$ are holomorphically invertible functions on $U_0,\ U_\infty$ respectively. The conventional calculations enable us to present the second factor in the form

$$G_\infty(z) = \begin{pmatrix} \frac{z(1-x_jy_i)(1-x_jy_j)-x_j(y_i-2y_j-x_jy_iy_j)}{(1-2x_jy_i+x_jy_j)(1-x_jy_i)(1-x_jy_j)(z-1)} & -\frac{x_j}{(1-x_jy_i)(1-x_jy_j)(z-1)} \\ \frac{y_i-2y_j+x_jy_iy_j}{(1-x_jy_i)(1-2x_jy_i+x_jy_j)} & \frac{1}{1-x_jy_i} \end{pmatrix}.$$

4.2.2 The action on connections

The action of Hecke transformations on classes of holomorphic bundles can be extended to the space of pairs: (bundle, connection) when certain conditions are satisfied. Let us describe in detail the induced action. A connection is a sheave map satisfying Leibniz rule with respect to the action of the structure sheaf:

$$\Delta : \mathcal{F} \to \mathcal{F} \otimes \Omega^1.$$

Hecke transformations can be defined on the space of connections preserving the space $Ann_l = \{v \in \mathcal{F}_x :< l, v >= 0\}$

$$\Delta_x : Ann_l \to Ann_l \otimes \Omega^1_x.$$

In our case we consider the composition of pairs of Hecke transformations localized at $z_i,\ z_j$, preserving the trivial rank 2 bundle.

As is mentioned above, the action can be defined by using the gluing functions language. Let us consider the trivial bundle specified by the gluing function 1. Hecke transformation change the bundle structure, the global section is defined by the pair S_0, S_∞, such that $S_0 = GS_\infty$, where $G = G_{ij}G_\infty$. One can define the action on connections as follows: let $\partial_z - A$ be a connection in the trivial bundle, determined by this expression on both opens, the transformed object is the pair of connection forms:

$$\begin{array}{rcl} \partial_z - A & \text{over} & U_\infty, \\ G(\partial_z - A)G^{-1} & \text{over} & U_0. \end{array}$$

After the basis change in U_∞ of the type $\widetilde{S_\infty} = G_\infty S_\infty$ we obtain the connection of the form

$$\partial_z - A \ \to \ G_\infty(\partial_z - A)G_\infty^{-1}. \tag{4.41}$$

The trivialization change in U_0 of the kind $\widetilde{S_0} = G_{ij}^{-1}S_0$ gives the following

$$G(\partial_z - A)G^{-1} \ \to \ G_{ij}^{-1}G(\partial_z - A)G^{-1}G_{ij} = G_\infty(\partial_z - A)G_\infty^{-1}. \tag{4.42}$$

Therefore, the transformed connection is of the same type as the initial one. The analytic properties at ∞ are preserved in virtue of the fact that G_∞ is holomorphically invertible in U_∞.

Using the results of the previous sections we calculate explicitly the Hecke action. To preserve the normalization condition $A(z)$ at ∞ it is necessary to consider transformations of the kind

$$\begin{aligned} \widetilde{G}(z) &= G_\infty^{-1}(\infty)G_\infty(z) \\ &= \frac{1}{z-1}\left(z - \begin{pmatrix} \frac{x_1(y_0-2y_1+x_1y_0y_1)}{(1-x_1y_0)(1-x_1y_1)} & \frac{x_1(1-2x_1y_0+x_1y_1)}{(1-x_1y_0)(1-x_1y_1)} \\ \frac{y_0-2y_1+x_1y_0y_1}{(1-x_1y_0)(1-x_1y_1)} & \frac{1-2x_1y_0+x_1y_1}{(1-x_1y_0)(1-x_1y_1)} \end{pmatrix}\right). \end{aligned} \tag{4.43}$$

Then one just needs to apply the gauge transformation $\tilde{G}(z)$ to the connection

$$A \mapsto \tilde{G}(z)A\tilde{G}^{-1}(z) + \partial_z\tilde{G}(z)\tilde{G}^{-1}(z).$$

The complete family of Hecke transformations in the case of 3 points associated with the analysis of the Painleve VI equation was described in [57].

Remark 4.4. The choice of the highest weights 1 in moving poles w_i is not obligatory, but in certain respect, the most general. One can consider a potential of the form

$$U = \sum_{j=1}^{m} \frac{-1/4(\eta_j + 2)\eta_j}{(z - w_j)^2} + \sum_{i=1}^{k} \frac{1/4 + detA_i}{(z - z_i)^2} + \sum_{j=1}^{k-2} \frac{H_{w_j}}{z - w_j} + \sum_{i=1}^{k} \frac{H_{z_i}}{z - z_i} \tag{4.44}$$

with the higher values of weights. It can be implemented if one requires that $a_{12}(z)$ have zeroes w_j with multiplicities η_j satisfying the condition $\sum_{j=1}^{m} \eta_j = k - 2$.

The local analysis at the poles shows that the eigenvalues of residues A_i transform according to the 4 following rules depending on the choice of the low and upper Hecke transformations subspaces:

$$\begin{array}{ccc}
(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) & \longmapsto & (\ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots), \\
(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) & \longmapsto & (\ldots, \lambda_i + 1, \ldots, \lambda_j + 1, \ldots), \\
(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) & \longmapsto & (\ldots, \lambda_i - 1, \ldots, \lambda_j - 1, \ldots), \\
(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) & \longmapsto & (\ldots, \lambda_i - 1, \ldots, \lambda_j + 1, \ldots).
\end{array}$$

The result obtained makes it possible to treat recurrent relations on the space of solutions for the Bethe equation system. The most interesting in the program of explicit solving of quantum systems is the set of transformations lowering the highest weight values at both points. The consecutive application of these transformations could reduce the highest weight to zero, which corresponds to the trivial representation of the quantum algebra and hence the trivial quantum problem.

4.3 Elliptic case

The elliptic $\mathfrak{sl}_2$ Gaudin model is provided by a similar technique of quantum model solution including the quantum spectral curve, quantum separated variables and Hecke symmetries on the spectrum.

4.3.1 Separated variables

Let us recall a conventional method of separation of variables for this system [53], [59]. As in the rational case we consider the $\mathfrak{sl}_2$ Gaudin model with fixed representation $V = V_1 \otimes \ldots \otimes V_k$ of the quantum algebra $U(\mathfrak{sl}_2)^{\otimes k}$, where V_i is the finite dimensional irreducible representation of the highest weight Λ_i. V_i can be realized as the quotient of the Verma module $\mathbb{C}[t_i]/t_i^{\Lambda_i+1}$, such that the generators of $\mathfrak{sl}_2$ act by differential operators:

$$h^{(s)} = -2t_s \frac{\partial}{\partial t_s} + \Lambda_s, \qquad e^{(s)} = -t_s \frac{\partial^2}{\partial t_s^2} + \Lambda_s \frac{\partial}{\partial t_s}, \qquad f^{(s)} = t_s.$$

Let us start with study of the quantum problem on the tensor product of Verma modules $W = \mathbb{C}[t_1, \ldots, t_k]$. We introduce the variables $C, \{y_j\}$ defined by:

$$\sum_{s=1}^{k} \frac{\theta(u-u_s-\lambda)}{\theta(u-u_s)\theta(-\lambda)} t_s = C \prod_{s=1}^{k} \frac{\theta(u-y_s)}{\theta(u-u_s)}.$$

Let us now represent the elliptic Gaudin model eigenvector as a function of introduced variables:

$$S_\lambda(u)\Omega(C, y_1, \ldots, y_k) = s_\lambda(u)\Omega(C, y_1, \ldots, y_k). \tag{4.45}$$

In this formula $s_\lambda(u)$ is a scalar-valued function on u of the form

$$s_\lambda(u) = \sum c_i \wp(u-u_i) + \sum d_i \frac{\theta'(u-u_i)}{\theta(u-u_i)}; \qquad c_i = \Lambda_i^2/4 + \Lambda_i/2. \tag{4.46}$$

Setting $u = y_j$ in (4.45) we obtain:

$$\Big(\frac{\partial}{\partial y_j} - \frac{1}{2}\sum_{s=1}^{k} \frac{\theta'(y_j-u_s)}{\theta(y_j-u_s)}\Lambda_s\Big)^2 \Omega(C, y_1, \ldots, y_k) = s_\lambda(y_j)\Omega(C, y_1, \ldots, y_k).$$

This equation induces a factorization of an eigenvector:

$$\Omega(C, y_1, \ldots, y_k) = C^a \prod_j \omega(y_j),$$

moreover it may be argued that the expression $w(u) = \prod_{s=1}^{k} \theta(u-u_s)^{-\Lambda_s/2}\omega(u)$ associated with the component of the eigenvector satisfies the equation

$$\big(\partial_u^2 - s_\lambda(u)\big)w(u) = 0 \tag{4.47}$$

Therefore, each equation (4.47) of the form (4.46) having solution $s_\lambda(u)$ with half-integer exponents at $\{u_1, \ldots, u_k\}$ and meromorphic outsides these points, corresponds to an eigenvector for the elliptic Gaudin Hamiltonians in representation V, obtained by projecting the vector Ω.

Conjecture 4.5. *There is a one-to-one correspondence between this kind of differential operators and the eigenvectors of the model in the representation V.*

Through the following sections we will consider only such eigenvectors for the Gaudin model that correspond to elliptic Sturm-Liouville operators with the described analytic properties.

4.3.2 Bethe ansatz

The traditional Bethe ansatz method in the elliptic case [59] can be obtained considering the following particular solution with simple zeroes

$$\psi(u)=\prod_i \theta^{-\Lambda_i/2}(u-u_i)\prod_j \theta(u-\gamma_j) \tag{4.48}$$

for the elliptic Sturm-Liouville equation

$$\left(\partial_u^2-\sum_i c_i\wp(u-u_i)-\sum_i d_i\frac{\theta'(u-u_i)}{\theta(u-u_i)}\right)\psi(u)=0. \tag{4.49}$$

This condition is equivalent to the following system of equations:

$$\begin{aligned} c_i &= \Lambda_i^2/4+\Lambda_i/2,\\ d_i &= \Lambda_i\left(\sum_j\frac{\theta'(u_i-\gamma_j)}{\theta(u_i-\gamma_j)}-\sum_{j\neq i}\frac{\Lambda_j\theta'(u_i-u_j)}{2\theta(u_i-u_j)}\right),\\ 0 &= \sum_i \Lambda_i/2\frac{\theta'(\gamma_j-u_i)}{\theta(\gamma_j-u_i)}-\sum_{i\neq j}\frac{\theta'(\gamma_j-\gamma_i)}{\theta(\gamma_j-\gamma_i)}, \end{aligned} \tag{4.50}$$

the latter is called the elliptic Bethe system.

4.3.3 Matrix form of the Bethe equations

In this section we find a matrix Fuchsian system equivalent to the elliptic Sturm-Liouville equation (4.49),

$$(\partial_u-A(u))\Psi(u)=0, \tag{4.51}$$

where

$$\Psi(u)=\begin{pmatrix}\psi_1(u)\\ \psi_2(u)\end{pmatrix},$$

$$A(u)=\begin{pmatrix}a_{11}(u) & a_{12}(u)\\ a_{21}(u) & a_{22}(u)\end{pmatrix}=\begin{pmatrix}\sum a_{11}^i\frac{\theta'(u-z_i)}{\theta(u-z_i)} & \sum a_{12}^i\frac{\theta(u-z_i-\lambda)}{\theta(u-z_i)\theta(-\lambda)}\\ \sum a_{21}^i\frac{\theta(u-z_i+\lambda)}{\theta(u-z_i)\theta(\lambda)} & \sum a_{11}^i\frac{\theta'(u-z_i)}{\theta(u-z_i)}.\end{pmatrix}$$

The equivalence relation of the matrix and scalar systems is the following: the function $w=\psi_1/\sqrt{a_{12}}$ solves the equation $w''-Uw=0$, of the same form as (4.47), with the potential whose highest term is given by the formula:

$$U(u)=-\sum(1/4+det(A_i))\wp(u-z_i)+\sum 3/4\wp(u-w_i)+...$$

Here the points w_j are defined by the condition

$$a_{12}(u) = c\frac{\prod \theta(u - w_i)}{\prod \theta(u - z_i)}.$$

In turn, A_i are defined as residues of $A(u)$ at z_i.

Remark 4.6. Note that the sets of poles of the Sturm-Liouville operator and of the matrix problem do not match, the first one is compiled from two subsets

$$\{u_1, \dots, u_{2l}\} = \{z_1, \dots, z_l, w_1, \dots, w_l\}.$$

It turns out that the method of construction of the solution to the matrix problem given a solution to the Sturm-Liouville equation is also explicit. Let us consider a scalar problem that corresponds to the set of marked points $\{z_1, \dots, z_l, w_1, \dots, w_l\}$, the set of highest weights $2s_1 - 1, \dots, 2s_k - 1, 1, \dots, 1\}$ and the set of Bethe roots $\{\gamma_1, \dots, \gamma_\rho\}$. Then the 2-vector function Ψ with components:

$$\begin{aligned} \psi_1 &= \prod_{i=1}^{k} \theta(u - u_i)^{s_i} \prod_{j=1}^{\rho} \theta(u - \gamma_j) \\ \psi_2 &= \sum_{j=1}^{\rho} \frac{\alpha_j \theta(u - \gamma_j + \lambda)}{\theta(u - \gamma_j)} \psi_1, \end{aligned} \tag{4.52}$$

where coefficients α_j are given by the formula

$$\alpha_j = \frac{\prod_i \theta(\gamma_j - w_i)}{\prod_i \theta(\gamma_j - u_i)},$$

satisfies the matrix equation (4.51).

An explicit calculation shows that the equation (4.51) for Ψ given by the expression (4.52) is equivalent to the following system of equations:

$$\begin{aligned} det \begin{pmatrix} a^i_{11} - s_i & a^i_{12} \\ a^i_{21} & a^i_{22} - s_i \end{pmatrix} &= 0, \\ -\sum_k (2s_k - 1)\frac{\theta'(\gamma_j - u_k)}{\theta(\gamma_j - u_k)} - \sum_k \frac{\theta'(\gamma_j - w_k)}{\theta(\gamma_j - w_k)} + 2\sum_{i \neq j} \frac{\theta'(\gamma_j - \gamma_i)}{\theta(\gamma_j - \gamma_i)} &= 0, \\ \frac{\prod_i \theta(\gamma_j - w_i)}{\prod_i \theta(\gamma_j - u_i)} &= \alpha_j. \end{aligned}$$

The system of equations means that exponents are eigenvalues of the residues of the connection and the set of γ_j satisfy the elliptic Bethe system (4.50) corresponding to the set of marked points

$$\{u_1, \dots, u_k, w_1, \dots, w_k\}$$

and the set of highest weights $\{2s_1 - 1, \dots, 2s_k - 1, 1, \dots, 1\}$.

4.3.4 Hecke transformations

Let us describe in more details how the Hecke transformations are calculated over an elliptic curve. The most suitable way of parameterization of holomorphic bundles for an elliptic curve Σ is the lift of a bundle to the universal covering $\mathbb{C}$ ([58] (2,6)). The monodromy group $\mathbb{Z}^2$ acts by homomorphisms on the sheaf of sections $\pi^*\mathcal{E}$ corresponding to the bundle E. In case of the degree 0 line bundle the only multiplier set, up to equivalence, is the set of quasiperiodic factors of the expression:

$$f(z) = \frac{\theta(z-\lambda)}{\theta(z)}$$

for $\lambda \in \Sigma$. Let us denote the corresponding line bundle by $\mathcal{O}_\lambda$. The Hecke transform at a point w supposes considering the subsheaf of $\mathcal{O}_\lambda$ taking values 0 at w. This sheaf is isomorphic to the sheaf of sections of some line bundle of degree 1

$$s(z) \mapsto \frac{s(z)}{\theta(z-w)}.$$

This map is an isomorphism due to the property that $\theta(z)$ has a unique zero at $z = 0$. The Hecke transformations on connections on the rank 2 bundle $\mathcal{O}_{\lambda/2} \oplus \mathcal{O}_{-\lambda/2}$ construct connections on a bundle $\mathcal{O}_{\mu/2} \oplus \mathcal{O}_{-\mu/2}$ as follows. Let the residues of the connection have the decomposition:

$$A_i = \begin{pmatrix} a_{11}^{(i)} & a_{12}^{(i)} \\ a_{21}^{(i)} & -a_{11}^{(i)} \end{pmatrix} = G_i \begin{pmatrix} d_i & 0 \\ 0 & -d_i \end{pmatrix} G_i^{-1}$$

where G_i are constant matrices. Then the connection is transformed by the gauge transformation with the group element

$$G_{ij}(z) = \tilde{G}_i \begin{pmatrix} 1 & 0 \\ 0 & \theta(z-z_i) \end{pmatrix} \tilde{G}_i^{-1} G_j \begin{pmatrix} \theta^{-1}(z-z_j) & 0 \\ 0 & 1 \end{pmatrix} G_j^{-1},$$

where

$$\tilde{G}_i = G_j \begin{pmatrix} \theta^{-1}(z_i - z_j) & 0 \\ 0 & 1 \end{pmatrix} G_j^{-1} G_i.$$

As well as in the rational case we consider a pair of Hecke transformations at different points $u_i,\ u_j$ with different signs $T_{ij} = T^{-1}_{(u_i,l_i)} T_{(u_j,l_j)}$ acting on rank 2 bundles with trivial determinant. Depending on the choice of subspaces of upper and lower transformations we get the following action of T_{ij} on the variety of highest weights of the Gaudin model

$$(\ldots,\lambda_i,\ldots,\lambda_j,\ldots) \ \longmapsto\ (\ldots,\lambda_i+1,\ldots,\lambda_j-1,\ldots),$$

$$(\ldots,\lambda_i,\ldots,\lambda_j,\ldots) \longmapsto (\ldots,\lambda_i+1,\ldots,\lambda_j+1,\ldots),$$
$$(\ldots,\lambda_i,\ldots,\lambda_j,\ldots) \longmapsto (\ldots,\lambda_i-1,\ldots,\lambda_j-1,\ldots),$$
$$(\ldots,\lambda_i,\ldots,\lambda_j,\ldots) \longmapsto (\ldots,\lambda_i-1,\ldots,\lambda_j+1,\ldots).$$

As in the rational case, the family of transformations that lower the weights of all representations, thus simplifying the diagonalization problem, is of particular interest.

5 Applications

This section is devoted to the two main applications of the quantum spectral curve method. The first application is related to the geometric Langlands correspondence, and mainly consists of an effective description of the center $U_{crit}(\widehat{\mathfrak{gl}_n})$ which in turn plays a key role in the Beilinson-Drinfeld quantization of the Hitchin system. Let us note that this problem is closely related to the representation theory of affine Lie algebras.

5.1 Geometric Langlands correspondence

5.1.1 The center of $U(\widehat{\mathfrak{gl}_n})$ on the critical level

We introduce the following notation $U_{crit}(\widehat{\mathfrak{gl}_n})$ for the local completion $U(\widehat{\mathfrak{gl}_n})/\{C-crit\}$, where C is a central element and $crit = -h^\vee = -n$ is the critical level inverse to the dual Coxeter number of the Lie algebra $\mathfrak{sl}_n$. It was proved in [61] that $U_{crit}(\widehat{\mathfrak{gl}_n})$ has a center isomorphic to the polynomial ring of the Cartan algebra as a linear space. Despite the geometric description of the center there was no explicit construction for the generators of this commutative algebra. For this purpose we use the Adler-Kostant-Symes scheme [60]. This approach plays an important role in the theory of integrable systems: it can be exploited to construct a wide family of commutative algebras, makes it possible to establish a relation of integrable systems to decomposition problems and provides an algebraic interpretation for the Lax representation, and r-matrix structures. The AKS scheme can be generalized to the quantum level and plays an important role in description, solution and classification of quantum integrable models. The most simple case is that of finite dimensional Lie algebra allowing a decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ into the sum of two Lie subalgebras. To each choice of normal ordering one can attach an isomorphism of linear spaces

$$\phi : U(\mathfrak{g}) \to U(\mathfrak{g}_+) \otimes U(\mathfrak{g}_-).$$

Let us introduce a notation $\mathfrak{g}_-^{op}$ for the inverse Lie algebra structure to the space $\mathfrak{g}_-$ defining by the formula $-\{\circ,\circ\}$. Let us denote the Lie algebra $\mathfrak{g}_+ \oplus \mathfrak{g}_-^{op}$ by the symbol $\mathfrak{g}_r$. The corresponding enveloping algebras can be identified as linear spaces with the help of the Poincare-Birkhoff-Witt basis:

$$U(\mathfrak{g}_-^{op}) \simeq U(\mathfrak{g}_-).$$

Lemma 5.1. *The center $\mathfrak{z}(U(\mathfrak{g}))$ is mapped by ϕ to a commutative subalgebra in $U(\mathfrak{g}_+) \otimes U(\mathfrak{g}_-^{op})$.*

Proof Let is denote the commutator in $U(\mathfrak{g}_+) \otimes U(\mathfrak{g}_-^{op})$ as follows $[*, *]_R$. Let c_1, c_2 be two central elements in $U(\mathfrak{g})$ represented as follows

$$c_i = \sum_j x_j^{(i)} y_j^{(i)} \qquad x_j^{(i)} \in U(\mathfrak{g}_+),\ y_j^{(i)} \in U(\mathfrak{g}_-).$$

The result of calculating the modified commutator is as follows

$$\begin{aligned} [\phi(c_1), \phi(c_2)]_R &= [\sum_j x_j^{(1)} y_j^{(1)}, \sum_k x_k^{(2)} y_k^{(2)}]_R \\ &= \sum_{j,k} [x_j^{(1)}, x_k^{(2)}]_R y_j^{(1)} y_k^{(2)} + x_j^{(1)} x_k^{(2)} [y_j^{(1)}, y_k^{(2)}]_R. \end{aligned}$$

In virtue of the definition above we have

$$[x_j^{(1)}, x_k^{(2)}]_R = [x_j^{(1)}, x_k^{(2)}] \qquad [y_j^{(1)}, y_k^{(2)}]_R = -[y_j^{(1)}, y_k^{(2)}]$$

$$[\phi(c_1), \phi(c_2)]_R = \sum_k [c_1, x_k^{(2)}] y_k^{(2)} - \sum_j x_j^{(1)} [y_j^{(1)}, c_2]$$

The last expression is equal to zero since c_1, c_2 are central elements. □

Remark 5.2. In what follows we will be interested in applying this scheme to $U_{crit}(\widehat{\mathfrak{gl}_n})$. To use the result of the AKS lemma in the infinite dimensional case one should choose an appropriate completion of the algebra. In our case we use the completion corresponding to the bigrading $deg(gt^k) = (k, 0),\ deg(gt^{-k}) = (0, k)$ for $k \geq 0$. One needs to prove that the considered central elements belong to this completion $\dot{U}_{crit}(\widehat{\mathfrak{gl}_n})$. This is a matter of fact due to the classical limit argument. In what follows we omit the completion in notation $U_{crit}(\widehat{\mathfrak{gl}_n})$, $U(\mathfrak{g}_r)$ and the tensor products for the sake of simplicity.

One considers also the linear space map

$$\varsigma : U(\mathfrak{g}) \to U(\mathfrak{g}_+) \oplus U(\mathfrak{g})\mathfrak{g}_-$$

defined by the direct sum decomposition for the Lie algebra. Let us denote by φ the projector to the first subspace $U(\mathfrak{g}_+)$.

Lemma 5.3. *The image of* $\mathfrak{z}(U(\mathfrak{g}))$ *with respect to* φ *is a commutative subalgebra of* $U(\mathfrak{g}_+)$.

Proof Let $c_1, c_2 \in Z$.

$$[c_1 - \varphi(c_1), c_2 - \varphi(c_2)] = [\varphi(c_1), \varphi(c_2)]$$

The r.h.s. belongs to $U(\mathfrak{g}_+)$; the l.h.s in an element of $U(\mathfrak{g})\mathfrak{g}_-$; this takes issue in vanishing of both sides □

We will identify $U_{crit}(\widehat{\mathfrak{gl}_n})$ with the loop algebra as linear spaces. Let us list several important facts about the loop algebra.

Proposition 5.4. *Let us consider* $\mathfrak{g} = \mathfrak{gl}_n[t, t^{-1}] = \mathfrak{gl}_n[t^{-1}] \oplus t\mathfrak{gl}_n[t]$ *whose generators* $e_{ij}^{(k)} = e_{ij}t^k$ *can be represented by the generating series*

$$L_{full}(z) = \sum_{s=-\infty,\infty} \Phi_s z^{-s-1} \tag{5.1}$$

where

$$\Phi_s = \sum_{ij} E_{ij} \otimes e_{ij}^{(s)}.$$

Here, as above, e_{ij} *are generators of the Lie algebra* $\mathfrak{gl}_n$, *and* E_{ij} *are matrix units. The Lie algebra structure on* $\mathfrak{g}_r$ *can be described by the following commutation relations*

$$\{L_{full}(z) \otimes L_{full}(u)\} = [\frac{P_{12}}{z-u}, L_{full}(z) \otimes 1 + 1 \otimes L_{full}(u)] \tag{5.2}$$

Let us remark that these relations are the same as for the Gaudin Lax operator (Section 2.42).

The center of $(U_{crit}(\widehat{\mathfrak{gl}_n}))$ **and a commutative subalgebra in** $U(t\mathfrak{gl}_n[t])$ Let us also introduce the "'positive"' Lax operator:

$$L(z) \;=\; \sum_{k>0} \Phi_k z^{-k-1},$$

which satisfies the following R-matrix relations:

$$\{L(z) \otimes L(u)\} = [\frac{P_{12}}{z-u}, L(z) \otimes 1 + 1 \otimes L(u)]. \tag{5.3}$$

Theorem 5.5. *The commutative subalgebra in* $U(t\mathfrak{gl}_n[t])$ *defined by the set of coefficients of the quantum characteristic polynomial* $det(L(z) - \partial_z)$ *coincides with the image of* $\mathfrak{z}(U_{crit}(\widehat{\mathfrak{gl}_n}))$ *by the projection* $\varphi : U_{crit}(\widehat{\mathfrak{gl}_n}) \to U(t\mathfrak{gl}_n[t])$.

Proof The proof is based on the results of [62] where it was proved that the centralizer of the set of quadratic Gaudin Hamiltonians H_2^i in $U(t\mathfrak{gl}_n[t])$ coincides with the projection of $U(\widehat{\mathfrak{gl}_n})$ on the critical level.

Remark 5.6. This particular property, namely the fact that the quadratic generators determine the complete commutative subalgebra is known also in the theory of Fomenko-Mishenko subalgebras [63] and in the theory of the Calogero-Moser system [64].

Following the proposed logic and using the fact that the subalgebra defined by the coefficients of $det(L(z) - \partial_z)$ commute with H_2^i, one can show that this subalgebra is a subalgebra of the algebra obtained from the center. In order to prove their coincidence it is sufficient to consider the classical limit □

Remark 5.7. The analogous strategy is applicable in the case of projection to $U(\mathfrak{gl}_n[t])$. One needs to take into account that both algebras are invariant with respect with the $GL(n)$ action.

5.1.2 Explicit description of the center of $U_{crit}(\widehat{\mathfrak{gl}_n}))$

Theorem 5.8. *The center of $U_{crit}(\widehat{\mathfrak{gl}_n})$ is isomorphic to a subalgebra in $U(\mathfrak{gl}_n[t^{-1}]\oplus t\mathfrak{gl}_n^{op}[t])$ defined by the coefficients of th quantum characteristic polynomial $det(L_{full}(z) - \partial_z)$. The isomorphism is induced by the mapping*

$$I : U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t\mathfrak{gl}_n^{op}[t]) \to U_{crit}(\widehat{\mathfrak{gl}_n}), \qquad I : h_1 \otimes h_2 \to h_1 h_2 \tag{5.4}$$

Proof follows the same lines as that in [62]. Let us firstly show that the algebra generated by the coefficients of the characteristic polynomial of the Lax operator $L_{full}(z)$ coincide with the centralizer of its quadratic elements. Further using the Sugawara formula for the quadratic center generators we prove that their image in $U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t\mathfrak{gl}_n[t])$ coincide with the quadratic elements of the quantum characteristic polynomial. For proving the first statement we consider a special limit of the commutative family.

Using the commutation relations 5.2, 5.3 and the traditional r-matrix calculations we show that $TrL_{full}^m(z)$ are central in the symmetric algebra $S(\mathfrak{gl}_n[t, t^{-1}])$ and moreover $TrL_{full}^m(z)$ generate the commutative Poisson subalgebra in $S(\mathfrak{gl}_n[t^{-1}]\oplus t\mathfrak{gl}_n^{op}[t])$.

Let us consider the family of automorphisms of the algebra $U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t\mathfrak{gl}_n^{op}[t])$ defined in terms of the Lax operator as follows: let K is a generic diagonal $n \times n$ matrix. The Lax operator

$$L_{full}^{\hbar}(z) = L_{full}(z) + \hbar K$$

also satisfies the r-matrix relations (5.2). This automorphism family is parameterized by $\hbar$.

Let us consider the family of commutative subalgebras

$$M^{\hbar} \subset U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t\mathfrak{gl}_n^{op}[t])$$

defined by the generating function $det(L^{\hbar}_{full}(z) - \partial_z)$. $M^{\hbar}$ centralizes the set of quadratic generators $QI_2(L^{\hbar}_{full}(z))$. $QI_k(z,\hbar)$ has the following leading term in expansion on $\hbar$

$$QI_k(z,\hbar) = \hbar^k Tr A_n K_1 K_2 \dots K_k + O(\hbar^{k-1}).$$

Changing the basis

$$QI_k(z,\hbar) \mapsto \widetilde{QI_k}(z,\hbar) = (QI_k(z,\hbar) - \hbar^k Tr A_n K_1 K_2 \dots K_k)\hbar^{-k+1}$$

and considering the limit $\hbar \to \infty$

$$\widetilde{QI_k}(z,\hbar) \to Tr(L_{full}(z)K^{k-1})$$

we obtain that these expressions generate the Cartan subalgebra

$$\mathfrak{H} = \mathfrak{H}_- \otimes \mathfrak{H}_+ = U(\mathfrak{h}[t^{-1}]) \otimes U(t\mathfrak{h}[t]).$$

Let us demonstrate that this subalgebra coincide with the centralizer of its quadratic generators

$$H_2^{\infty}(z) = lim_{\hbar\to\infty}\widetilde{QI}_2(z,\hbar) = \sum_{i=-\infty,\infty} Tr(\Phi_i K)z^{-i-1}.$$

Obviously $\mathfrak{H} \subset Z(H_2^{\infty}(z))$. Let us introduce the notations $(k_1,\dots,k_n)$ for the diagonal elements of K. Let us also denote by $h_i \in \mathfrak{H}$ the sum of the form

$$h_i = \sum_{s=1}^{n} (\Phi_i)_{ss} k_s,$$

then $H_2^{\infty}(z) = \sum_{i=-\infty,\infty} h_i z^{-i-1}$. The centralizer elements should commute with h_1 and h_{-1}. Let $\sum_{i=-\infty}^{\infty} x_i y_i$ be the infinite series such that $x_i \in U(\mathfrak{g}[t^{-1}]),\ y_i \in U(t\mathfrak{g}[t])$. We also suppose that this series is an element of the considered completion, i.e. such that it contains only finite number of elements of each bigrading. The operators $[h_1, *]$ and $[h_{-1}, *]$ are homogeneous of bigrading $(0,1)$ and $(1,0)$. Hence the centralizer description question is reduced to the analogous question in the polynomial algebra. The answer is given by the formulas

$$Z(h_1) = U(\mathfrak{gl}_n[t^{-1}]) \otimes \mathfrak{H}_+, \qquad Z(h_{-1}) = \mathfrak{H}_- \otimes U(t\mathfrak{gl}_n^{op}[t]).$$

An intersection of these subspaces in a completed sense coincides with the Cartan subalgebra $\mathfrak{H}$.

Summarizing we obtain that in the generic point $\hbar$ of the family the commutative subalgebra $M^{\hbar}$ belongs to the centralizer of the set QI_2 and in the limit $\hbar \to \infty$ generates the centralizer. From the arguments analogous to those of [62] at the generic point $M^{\hbar}$ should coincide with the centralizer of the quadratic generators. To finish the proof let us remind the Sugawara formula $U_{crit}(\widehat{\mathfrak{gl}_n})$

$$c_2(z) =: Tr(L^2_{full}(z)):$$

This uses the normal ordering symbol : : for currents in $\mathfrak{sl}_2$. These elements project to $QI_2(z)$ up to a central elements in $U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t\mathfrak{gl}_n[t])$ ■

5.1.3 The Beilinson-Drinfeld scheme

In [65] it was proposed a universal construction for the Hitchin system quantization. Let Σ be the connected smooth projective curve over $\mathbb{C}$ of genus $g > 1$, G - a semisimple Lie group, $\mathfrak{g}$ - the corresponding Lie algebra, Bun_G - the moduli stack of principal G-bundles on Σ. Let us also define the Langlands dual group ${}^LG$ as a group determined by the dual root data, namely such that its root lattice coincides with the dual lattice for G.
The main result of [65] can be reduced to the following:

- There exists a commutative ring of differential operators on $\mathfrak{z}(\Sigma, G)$, acting on sections of the canonical bundle K_{Bun_G} such that the symbol map produces the commutative subalgebra of classical Hitchin Hamiltonians on T^*Bun_G.

- The spectrum of the ring $\mathfrak{z}(\Sigma, G)$ is canonically isomorphic to the moduli space of ${}^L\mathfrak{g}$-opers (for the $G = SL_2$ case an ${}^L\mathfrak{g}$-oper is just the Sturm-Liouville operator on S; in general case this is a flat connection in a principal ${}^LG$ bundle with a parabolic structure).

- To each ${}^L\mathfrak{g}$-???per one can correspond a D-module on Bun_G by fixing eigenvalues of the Hitchin Hamiltonians. This D-module is an eigensheaf for the Hecke action defined naturally on the moduli stack of bundles. Moreover the eigenvalue in this case coincide with the corresponding ${}^L\mathfrak{g}$-oper.

The basement of this construction is the natural action of the center of $U_{crit}(\widehat{\mathfrak{g}})$ on the loop group of the corresponding Lie group. This action can induce an action by differential operators on $Bun_G(\Sigma)$ in virtue of one of the realizations of the moduli stack of principal bundles

$$Bun_G(\Sigma) \simeq G(F)\backslash G(\mathcal{A}_F)/G(\mathcal{O}_F) \simeq G_{in}\backslash G[[z, z^{-1}]/G_{out}$$

where G_{in} and G_{out} denote the subgroups of function converging in U_{in} and U_{out}, where U_{in} and U_{out} determine a covering of Σ of the type: U_{in} is an open disk centered in P with the local parameter z, $U_{out} = \Sigma\backslash P$. The middle part of the equality represents the so-called adelic realization of the moduli stack of principal G-bundles for an algebraic group G. The construction uses the adell group $G(\mathcal{A}_F)$ for the field F of rational functions on Σ, the group of entire adèles $G(\mathcal{O}_F)$ and the group of principal adèles $G(F)$. This realization is convenient for describing the complex geometry analogy between the arithmetic Langlands correspondence and the quantum Gaudin model.

5.1.4 Correspondence

Historically, the Langlands hypothesis generalizes the field-class theory [66, 67], one of whose principal results is the following statement in the case of a number field. Namely let F be a number field (this means a finite extension of $\mathbb{Q}$), $\bar{F}$ - its maximal algebraic extension, F^{ab} - its maximal abelian extension. The Galois group of an extension $F \subset F'$ is

$$Gal(F', F) \;=\; \{\sigma \in Aut(F') : \sigma(x) = x \;\; \forall x \in F\}.$$

The abelian reciprocity law
There exists a group isomorphism

$$Gal(F^{ab}, F) \simeq \text{The group of connectivity components of} \;\; F^{\times}\backslash\mathcal{A}_F^{\times}$$

where $\mathcal{A}_F^{\times}$ is the idèle group of the ring F, $F^{\times}$ is the group of invertible elements of F. The topology of of the completion product is considered.

The Langlands hypothesis is formulated as an n-dimensional (non-commutative) generalization of the abelian reciprocity law. Namely it is assumed the isomorphism between the category of the Galois group representations of the maximal algebraic extension of a ring and the category of automorphic representations for the corresponding idèle group. By an automorphic representation we mean a $GL_n(\mathcal{A}_F)$ - representation realized on the space of functions on

$$GL_n(F)\backslash GL_n(\mathcal{A}_F),$$

meet some additional conditions [68, 69]. The right part is traditionally called automorphic for the following reason. For $n = 2$ these representations are related with the theory of modular functions. It should be reminded that modular functions are functions on the upper-half Siegel plain matching the condition

$$f((az+b)/(cz+d)) = \chi(a)(cz+d)^k f(z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

In particular, the modular functions can be represented as functions on the following quotient space

$$SL_2(\mathbb{R})/SL_2(\mathbb{Z}) \simeq K\backslash GL_2(\mathcal{A}_{\mathbb{Q}})/GL_2(\mathbb{Q})$$

The Langlands program covers the following types of fields F:

- A number field.
- Field of functions on an algebraic curve over the finite field F_q (In this case, the hypothesis was proven in [70]).
- Field of functions on an algebraic curve over $\mathbb{C}$. This is called the geometrical case over $\mathbb{C}$. The following papers are on the subject [71].

The correspondence over $\mathbb{C}$:
In this case on the Galois side one considers classes of representations of the fundamental group or classes of flat connections in a holomorphic bundle of rank n. The automorphic side deals with the Hitchin D-module on

$$GL(F)\backslash GL(\mathcal{A}_F)/GL(\mathcal{O}_F) \simeq Bun_n(\Sigma).$$

The results of [65] and [61] ensures the correspondence between Hitchin D-modules and flat connections related to ${}^L\mathfrak{g}$-opers. Due to the construction of the quantum characteristic polynomial for the loop algebra, as well as an explicit construction for the center of $U_{crit}(\widehat{\mathfrak{gl}_n})$ in theorem 5.8 the correspondence for the Lie algebra $\mathfrak{gl}_n$ can be realized in a more effective way. The following scheme demonstrates the correspondence

$$\text{Hitchin } D\text{-module} \overset{FF,BD}{\Leftrightarrow} \text{Character } \chi \text{ on } \mathfrak{z}(U_{crit}(\widehat{\mathfrak{gl}_n})) \overset{CT}{\Leftrightarrow} \chi det(L_{full} - \partial_z).$$

Remark 5.9. The construction of a character on $\mathfrak{z}(U_{crit}(\widehat{\mathfrak{gl}_n}))$ by a Hitchin D-module is a corollary of the Feigin and Frenkel theorem on existence of the center and the Beilinson and Drinfeld quantization. To obtain the explicit description for the corresponding flat connection [26] one should exploit the identification of commutative algebras: the commutative subalgebra in $U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t\mathfrak{gl}_n[t])$ defined by the coefficients of the quantum characteristic polynomial on one side and the image of the center of $\mathfrak{z}(U_{crit}(\widehat{\mathfrak{g}}))$ by the AKS map on another side.

5.2 Non-commutative geometry

The main plot of these lectures is relevant to the emerging field of Noncommutative Geometry, substantive issues of which consist in geometric interpretation of algebraic structures in which the commutativity property is weakened. In this context the quantum characteristic polynomial is a natural generalization of classical one. Some properties of this object, in particular the role played by quantum characteristic polynomial in the program of effective solution of the quantum integrable models, suggest it to be a natural noncommutative generalization of an algebraic curve, the spectral curve of an integrable system. This section describes some linear algebraic properties of the quantum characteristic polynomial obtained in [27].

5.2.1 The Drinfeld-Sokolov form of the quantum Lax operator

Let $L(z) \in Mat_n \otimes U(\mathfrak{gl}_n)^{\otimes N} \otimes Fun(z)$ be the quantum Lax operator for the Gaudin model (2.14), here and further $Fun(z)$ means the space of rational functions on a parameter z. Let us denote by $L^{[i]}(z)$ *quantum powers* of the Lax operator defined by the formula:

$$\begin{aligned} L^{[0]} &= Id, \\ L^{[i]} &= L^{[i-1]}L + \partial_z L^{[i-1]}. \end{aligned}$$

Theorem 5.10. *The expression $C(z) \in Mat_n \otimes U(\mathfrak{gl}_n)^{\otimes N} \otimes Fun(z)$ defined by the formula*

$$C(z) = \begin{pmatrix} v \\ vL \\ \dots \\ vL^{[n-1]} \end{pmatrix}, \tag{5.5}$$

where $v \in \mathbb{C}^n$ is a generic vector defines a gauge transformation

$$C(z)(L(z) - \partial_z) = \left(\begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & 1 \\ QH_n & QH_{n-1} & ... & QH_2 & QH_1 \end{pmatrix} - \partial_z \right) C(z), \tag{5.6}$$

where the r.h.s. lower line coefficients are determined by the coefficients of the quantum characteristic polynomial

$$\begin{aligned} det(L(z) - \partial_z) &= TrA_n(L_1(z) - \partial_z) \dots (L_n(z) - \partial_z) \\ &= (-1)^n(\partial_z^n - \sum_i QH_{n-i}\partial_z^i). \end{aligned} \tag{5.7}$$

Knizhnik-Zamolodchikov equation Here and below we denote by V a finite-dimensional representation of $U(\mathfrak{gl}_n)^{\otimes N}$. It was shown in [72] that there exists a relation between solutions of the Knizhnik-Zamolodchikov (KZ) equation [73]

$$(L(z) - \partial_z)S(z) = 0,$$

where $S(z)$ is a function with values in $\mathbb{C}^n \otimes V$, solutions of the Baxter equation

$$det(L(z) - \partial_z)\Psi(z) = 0 \tag{5.8}$$

where $\Psi(z)$ is a function with values in V. To make this relation clear it is sufficient to take the antisymmetric projection of $U(z) = v_1 \otimes \dots \otimes v_{n-1} \otimes S(z)$ where v_i are some vectors in $\mathbb{C}^n$. In particular, for a special choice of such vectors one obtains that vector components of $S(z)$ solve the equation (5.8).

Proof of theorem 5.10 Let us consider both sides of (5.6) applied to a function $S(z) \in \mathbb{C}^n \otimes V \otimes Fun(z)$,

$$L.H.S = C(L-\partial_z)S = \begin{pmatrix} < v, LS - \partial_z S > \\ < v, L^{[1]}(LS - \partial_z S > \\ \dots \\ < v, L^{[n-1]}(LS - \partial_z S) > \end{pmatrix}, \tag{5.9}$$

$$R.H.S = (L_{DS} - \partial_z)CS = \begin{pmatrix} < v, LS - \partial_z S > \\ \dots \\ < v, L^{[n-1]}S - \partial_z(L^{[n-2]}S) > \\ < v, \sum_{i=0}^{n-1} QH_{n-i}L^{[i]}S - \partial_z(L^{[n-1]}S) > \end{pmatrix}. \tag{5.10}$$

Using the definition for quantum powers we obtain

$$L^{[k]}S - \partial_z(L^{[k-1]}S) = L^{[k-1]}(LS - \partial_z S).$$

The difference (5.10) - (5.9) takes the form

$$\begin{pmatrix} 0 \\ \dots \\ 0 \\ < v, \sum_{i=0}^{n-1} QH_{n-i}L^{[i]}S - L^{[n]}S > \end{pmatrix}. \tag{5.11}$$

Let us now consider this expression if $S(z)$ is a solution for the KZ equation

$$L(z)S(z) = \partial_z S(z).$$

Let $\Phi(z) = C(z)S(z)$, where $C(z)$ is given by the formula (5.5). Then

$$\begin{aligned} \Phi_1(z) &= < v, S(z) > \\ \Phi_2(z) &= < vL(z), S(z) >=< v, \partial_z S(z) > \\ \dots & \\ \Phi_k(z) &= < v(L^{[k-1]}L(z) + \partial_z L^{[k-1]}), S(z) > \\ &= < vL^{[k-1]}, \partial_z S(z) > + < v\partial_z L^{[k-1]}, S(z) >= \partial_z \Phi_{k-1}(z) \end{aligned}$$

One of the consequences of [72] is that $\Phi_1(z) =< v, S(z) >$ solves the Baxter equation

$$\sum_{i=0}^{n-1} QH_{n-i}\partial_z^i \Phi_1(z) - \partial_z^n \Phi_1(z) = 0 \tag{5.12}$$

for each solution $S(z)$ of the KZ equation and each vector $v \in \mathbb{C}^n$. The general position argument allows to claim that the n-th element of (5.11) vanishes identically on $S(z) \in \mathbb{C}^n \otimes V \otimes Fun(z)$. Theorem 2.5.7 [74] induces the equality of universal differential operators with values in the quantum algebra. ■

5.2.2 Caley-Hamilton identity

Corollary 5.11. *The quantum powers of the Lax operator satisfy the quantum version of the Caley-Hamilton identity*

$$L^{[n]}(z) = \sum_{i=1}^{n} QH_i(z) L^{[n-i]}(z). \tag{5.13}$$

Proof Let us consider the last line of the equation (5.6)

$$vL^{[n-1]}(z)(L(z) - \partial_z) = \sum_{i=1}^{n} vQH_i(z)L^{[n-i]}(z) - \partial_z vL^{[n-1]}(z).$$

The result follows from the general choice of the vector $v \in \mathbb{C}^n$. □

References

[1] Shiota T., *Characterization of Jacobian varieties in terms of soliton equations*, Invent. Math., **83**(2):333–382, 1986.

[2] Krichever I., *Integrable linear equations and the Riemann-Schottky problem*, in: Algebraic Geometry and Number Theory, Birkhäuser, Boston, 2006.

Krichever I., *Characterizing Jacobians via trisecants of the Kummer Variety*, arXiv:math/0605625.

[3] Atiyah M., *The geometry and physics of knots.* CUP, 1990

[4] Donaldson S., *An application of gauge theory to four- dimensional topology*, J. Differential Geometry **18** A983, 279-315.

[5] Kronheimer P.B., Mrowka T.S., *The genus of embedded surfaces in the protective plane*, Math. Research Letters **1** A994, 797-808.

[6] Nakayashiki, A., *Structure of Baker-Akhiezer modules of principally polarized abelian varieties, commuting partial differential operators and associated integrable systems*, Duke Math. J.62-2 (1991) 315–358,

Nakayashiki, A. and Smirnov, F., *Cohomologies of affine hyperelliptic Jacobi varieties and integrable systems*, Comm. Math. Phys. **217** (2001), 623-652.

[7] Thaddeus M. *Conformal field theory and the cohomology of the moduli space of stable bundles*, J. Differential Geom. 1992. V. **35**, ??? 1. P. 131-149.

[8] Feigin B., Finkelberg M., Negut A., Rybnikov L., *Yangians and cohomology rings of Laumon spaces*, arXiv:0812.4656.

Feigin B., Finkelberg M., Frenkel I., Rybnikov L., *Gelfand-Tsetlin algebras and cohomology rings of Laumon spaces*, arXiv:0806.0072.

[9] Gaudin M., *La Fonction d' Onde de Bethe*, Masson, Paris (1983).

[10] Dubrovin B.A., Matveev V.B., Novikov S.P., *Non-linear equations of the KdV type, finite zone linear operators and Abelian varieties.* Uspekhi Mat. Nauk, V. **31**, 1976, P. 55-136. English transl. in Russian Math.Surveys V. **31**, 1976.

Krichever I.M., *Methods of algebraic geometry in the theory of non-linear equations,* Russian Math. Surveys, **32** 6 (1977), 185–213

[11] Kowalevski S., *Sur le problème de la rotation d'un corps solide autour d'un point fixe.* Acta Mathematica **12** (1889),177-23.

[12] Jacobi C.G.J., *Vorlesungen uber Dynamik.* Georg Reimer, Berlin, 1866.

[13] Novikov S.P., Veselov A.P., *On Poisson brackets compatible with algebraic geometry and Korteweg-de Vries dynamics on the set of finite-zone potentials,* Sov. Math., Dokl. **26**, 357-362 (1982)

[14] Hitchin N., *Stable bundles and integrable systems.* Duke Math. Journal 1987 V **54** N**1** 91-114.

[15] Krichever I. M., *Rational solutions of the Kadomtsev???Petviashvili equation and integrable systems of N particles on a line*, Funkts. Anal. Prilozh., **12**, 1 (1978), 76–78

[16] Gardner C.S., Green J.M., Kruskal M.D., Miura R.M., *Method for solving the Korteweg-de Vries equation*, Phys.Rev.Lett. 1967. V. **19**. P. 1095-1097.

[17] Lax P., *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. 1968. V. **21**. P. 467-490.

[18] Dubrovin B.A., Krichever I.M., Novikov S.P., *Integrable systems. I, Dynamical systems*, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr., **4**, VINITI, Moscow, 1985, 179-277

[19] Perelomov A. M., *Integrable Systems of Classical Mechanics and Lie Algebras,* Birkhauser-Verlag, 1990.

[20] Demidov E.E., *The Kadomtsev-Petviashvili hierarchy and the Schottky problem*, Fundam. Prikl. Mat., **4**, 1 (1998), 367-460.

[21] Faddeev L.D., Sklyanin E.K., *Quantum mechanical approach to completely integrable field theory models*, Soviet Phys. Dokl., **23** (1978), 902-904.

Faddeev L.D., Sklyanin E.K., Takhtadzhyan L.A., *Quantum inverse problem method. I* Theoretical and Mathematical Physics, 1979, **40** 2, 688-706.

Kulish P.P., Sklyanin E.K. *Quantum inverse scattering method and the Heisenberg ferromagnet.* Phys. Lett. 1979 V. **70** A, P. 461-463.

[22] Bethe H., *Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette. (On the theory of metals. I. Eigenvalues and eigenfunctions of the linear atom chain)*, Zeitschrift fur Physik A, Vol. **71**, pp. 205-226 (1931).

[23] Drinfeld V.G., *Hopf algebras and the quantum Yang-Baxter equation,* Soviet Math. Dokl., 1985, V.**32**,

[24] Iwasaki K., Kimura H., Shimomura S., Yoshida M., *From Gauss to Painleve,* A modern theory of special functions, 1991.

[25] Chervov A., Falqui G., Rybnikov L., *Limits of Gaudin Systems: Classical and Quantum Cases*, arXiv:0903.1604v1 [math.QA].

[26] Chervov A., Talalaev D., *Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence.* hep-th/0604128.

[27] Chervov A.V., Talalaev D.V., *KZ equation, G-opers, quantum Drinfeld-Sokolov reduction, and quantum Cayley-Hamilton identity,* Zap. Nauchn. Sem. POMI, **360** (2008), 246–259.

[28] Babelon O., Bernard D., Talon M., *Introduction to classical integrable systems,* Cambridge University Press 2003.

[29] Reyman A.G., Semenov Tian-Shanski M.A., *Integrable Systems (group- theoretical approach),* Sovremennaya matematika, Izhevsk, 2003.

[30] Ramanan S., *The moduli space of vector bundles over an algebraic curve,* Math. Ann. **200** (1973) 69-84.

[31] Kodaira K., *Complex Manifolds and Deformation of Complex Structures* 1986 by Springer-Verlag.

[32] Nekrasov N., *Holomorphic bundles and many-body systems.* PUPT-1534, Comm. Math. Phys.,**180** (1996) 587-604; hep-th/9503157.

[33] Chervov A.V., Talalaev D.V., *Hitchin System on Singular Curves,* Theoretical and Mathematical Physics, 2004, **140**, (2), 1043–1072.

Chervov A., Talalaev D., *Hitchin systems on singular curves II. Gluing subschemes*, Int.J.Geom.Meth.Mod.Phys.4:751-787,2007.

[34] Krichever I., Babelon O., Billey E., Talon M., *Spin generalization of the Calogero-Moser system and the Matrix KP equation*, arXiv:hep-th/9411160.

[35] Kassel K., *Quantum groups,* Springer, 1995, Graduate Texts in Mathematics, **155**, Berlin, New York.

[36] Krichever I., Phong D., *Symplectic forms in the theory of solitons*, hep-th/9708170.

D'Hoker E., Krichever I., Phong D., *Seiberg-Witten Theory, Symplectic Forms, and Hamiltonian Theory of Solitons*, hep-th/0212313.

[37] Sklyanin E., *Separation of variables in the Gaudin model.* J.Sov.Math. **47**: 2473-2488, 1989, Zap.Nauchn.Semin. *164*: 151-169, 1987.

[38] Bayen F., Flato M., Fronsdal C., Lichnerowicz A., Sternheimer D., *Deformation theory and quantization. I. Deformations of symplectic structures,* Ann. Physics **111** (1978), 61–110.

Bayen F., Flato M., Fronsdal C., Lichnerowicz A., Sternheimer D., *Deformation theory and quantization. II. Physical applications,* Ann. Physics **111** (1978), 111–151.

[39] Gutt S., *An explicit $*$-product on the cotangent bundle of a Lie group,* Lett. Math. Phys. **7** (1983), 249–258.

[40] Kontsevich M., *Deformation quantization of Poisson manifolds,* Lett. Math. Phys. **66** (2003), 157–216.

[41] Fedosov B.V., *A simple geometrical construction of deformation quantization,* J. Differential Geom. **40** (1994), 213–238.

Fedosov B.V., *Deformation quantization and index theory,* Mathematical Topics, Vol. **9**, Akademie Verlag, Berlin, 1996.

[42] Talalaev D.V., *The Quantum Gaudin System,* Functional Analysis and Its Applications, 2006, **40**, 1, 73–77.

[43] Sklyanin E., *Separations of variables: new trends.* Progr. Theor. Phys. Suppl. **118** (1995), 35–60. solv-int/9504001.

[44] Kulish P., Sklyanin E., *Quantum spectral transform method. Recent developments*, Lect. Notes in Phys. **151** (1982), pp. 61-119.

[45] Kirillov A., Reshetikhin N., *The Yangians, Bethe Ansatz and combinatorics*, Lett. Math. Phys. **12**, 3 (1986), pp. 199-208.

[46] Molev A.I., *Yangians and their applications.* math.QA/0211288.

[47] Bogolubov N.M., Izergin A.G., Korepin V.E., *Quantum Inverse Scattering Method and Correlation Functions,* Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1993).

[48] Mukhin E., Tarasov V., Varchenko A., *Schubert calculus and representations of general linear group* arXiv:0711.4072.

[49] Rubtsov V., Silantiev A., Talalaev D., *Manin Matrices, Quantum Elliptic Commutative Families and Char-acteristic Polynomial of Elliptic Gaudin model*, SIGMA **5** (2009), 110-131.

[50] Felder G., *Conformal field theory and integrable systems associated to elliptic curves.* Proc. ICM Zürich 1994, 1247-55, Birkhäuser (1994); Elliptic quantum groups, Proc. ICMP Paris 1994, 211-8, International Press (1995).

[51] Tarasov V., Varchenko A., *Small Elliptic Quantum Group* $e_{\tau,\gamma}(\mathfrak{sl}_N)$, Mosc. Math. J., **1**, no. 2, (2001), 243–286, 303–304.

[52] Enriquez B., Feigin B., Rubtsov , *Separation of variables for Gaudin-Calogero systems*, Compositio Math. **110**, no. 1, (1998), 1–16.

[53] Felder G., Schorr A., *Separation of variables for quantum integrable systems on elliptic curves*, math.QA/9905072, J. Phys., **A 32**, (1999), no. 46, 8001–8022.

[54] Gould M. D., Zhang Y.-Z., Zhao S.-Y., *Elliptic Gaudin models and elliptic KZ Equations* nlin/0110038.

[55] Talalaev D.V., *Bethe ansatz and isomondromy deformations,* Theoretical and Mathematical Physics, **159**, 2, 627-639.

[56] Bolibrukh A.A., *Holomorphic vector bundles on the Riemann sphere and the 21 st Hilbert problem,* Journal of Mathematical Sciences (New York), **82** 6 (1996), 3759

[57] Muğan U., Sakka A., *Schlesinger transformations for the Painlevé VI equation*, J. Math. Phys. **36** (3) 1995.

[58] Griffiths Ph., Harris J., *Principles of Algebraic Geometry,* Wiley-Interscience, 1994.

[59] Schorr A., *Separation of variables for the 8-vertex SOS model with antiperiodic boundary conditions*, PhD Thesis, ETH, Zurich, 2000.

[60] Adler M., *On a trace functional for formal pseudodifferential operators and the symplectic structure for the KdV type equations*, Invent. Math. (1979) **50**, 219-248.

Kostant B., *Quantization and Representation Theory*, in: Representation Theory of Lie Groups, Proc. SRC/LMS Res. Symp., Oxford 1977. London Math. Soc. Lecture Notes Series, **34**, 287-316, 1979.

Symes W.W., *Systems of Toda type, inverse spectral problems, and representation theory.* Invent. Math. 59, 13-51 (1980).

[61] B. Feigin, E. Frenkel, Int. J. Mod. Phys. A7, Suppl. 1A 1992, 197-215.

[62] Rybnikov L., *Uniqueness of higher Gaudin hamiltonians*, archiv:math/0608588.

[63] Rybnikov L., *Centralizers of some quadratic elements in Poisson-Lie algebras and a method for the translation of invariants.* Russ. Math. Surv. 60, No.2, 367-369 (2005); translation from Usp. Mat. Nauk 60, No.2, 173-174 (2005) math.QA/0608586.

[64] Ochiai, H., Oshima, T., and Sekiguchi, H. *Commuting families of symmetric differential operators* , Proc. of the Japan Acad. 70, Ser. A 2 1994 62–68.

[65] Beilinson A., Drinfeld V., *Quantization of Hitchin's integrable system and Heche eigensheaves.* Preprint 1991.

[66] Serre J.-P., *Groupes algebriques et corps de classes.* Paris, Hermann, 1959.

[67] Iwasawa K., *Local Class Field Theory,* Oxford Univ. Press, 1986.

[68] Frenkel E., *Lectures on the Langlands Program and Conformal Field Theory*, hep-th/0512172.

[69] Vergne M., *All what I wanted to know about the Langlands program and was afraid to ask*, math.GR/0607479.

[70] Lafforgue L., *Chtoucas de Drinfeld et correspondance de Langlands*, Inv. Math. **147** 1-241 (2002).

[71] Hausel T., Thaddeus , *Mirror symmetry, Langlands duality, and the Hitchin system*, Invent. Math. **153** (2003) 197-229.

Frenkel E., Gaitsgory D., Vilonen K., *On the geometric Langlands conjecture*, Journal of AMS **15** (2001) 367–417.

Frenkel E., Gaitsgory D., *Local geometric Langlands correspondence and affine Kac-Moody algebras*, math.RT/0508382

Bezrukavnikov R., Braverman A., *Geometric Langlands correspondence for D-modules in prime characteristic: the GL(n) case*, math.AG/0602255.

[72] Talalaev D., Chervov A., *Universal G-oper and Gaudin eigenproblem* hep-th/0409007.

[73] Knizhnik V.G., Zamolodchikov A.B., Nucl.Phys.B **247,** (1984) 83-103.

[74] Dixmier J., *Enveloping algebras.* Providence: American Mathematical Soc., 1991.

[75] Frenkel E., *Affine Algebras, Langlands Duality and Bethe Ansatz*, q-alg/9506003.

Dmitry Talalaev
Lomonosov Moscow State University
Russia
dtalalaev@yandex.ru

Program of the School

- Jørgen Andersen (Aarhus University, CTQM, Denmark),
  Quantization of moduli spaces of bundles and the mapping class group
- Johannes Huebschmann (University of Lille 1, France),
  Stratified Kähler spaces
- Martin Schlichenmaier (University of Luxembourg, Luxembourg),
  An introduction to Berezin-Toeplitz quantization methods
- Armen Sergeev (Steklov Mathematical Institute, Moscow, Russia),
  Quantization of universal Teichmüller space
- Dmitry Talalaev (Institute for Theoretical and Experimental Physics, Moscow, Russia),
  Quantum integrable systems and Langlands program
- Tatsuya Tate (University of Nagoya, Japan),
  Problems on asymptotic analysis over convex polytopes
- Tilmann Wurzbacher (University Paul Verlaine, Metz, France),
  Geometry of loop spaces
- Weiping ZHANG (Nankai Institute of Mathematics, Tianjin, China),
  Analytic aspects of geometric quantization

List of Participants

1. Akbarov, Sergei, Russian Academy of Sciences (VINITI), Russia
2. Andersen, Jørgen, Aarhus Universitet, Denmark
3. Askaripour, Nadya, University of Western Ontario, Canada
4. Baier, Thomas, Instituto Superior Tecnico, Portugal
5. Blaga, Adara-Monica, West University of Timisoara, Romania
6. Charles, Laurent, Institut de Mathématiques de Jussieu, France
7. Chen, Zhengxiang, Otto-von-Guericke-Universität Magdeburg, Germany
8. Choudhary, Neelu, Barkhatulla University, India
9. Driver, Bruce, University of CA, San Diego, Canada
10. Efimov, Alexander, Moscow State University, Russia
11. Elchinger, Olivier, Université de Haute-Alsace, France
12. Engenhorst, Magnus, Universität Augsburg, Germany
13. Fonarev, Anton, Lomonosov Moscow State University, Russia
14. Frégier, Yaël, Université du Luxembourg, Luxembourg
15. Fuchssteiner, Martin, Technische Universität Darmstadt, Germany
16. Fujita, Hajime, Gakushuin University, Japan
17. Hajizadeh, Kobra, Azad University - South Tehran Branch, Iran
18. Hilgert, Joachim, Paderborn University, Germany
19. Himpel, Benjamin, University of Bonn, Bonn
20. Hofer, Laurent, Université du Luxembourg, Luxembourg
21. Huebschmann, Johannes, Université Lille 1, France
22. Jamshidi, Reza, Azad University-South Tehran Branch, Iran
23. Jellal, Ahmed, Chouaib Doukkali University, Morocco
24. Kaszewski, Kevin, Université du Luxembourg, Luxembourg
25. Kimouche, Louisa, Université du Luxembourg, Luxembourg
26. Knevel, Roland, Université du Luxembourg, Luxembourg
27. Kobayashi, Ryoichi, Nagoya University, Japan
28. Komlov, Alexander, Lomonosov Moscow State University, Russia
29. Kori, Tosiaki, Waseda University, Japan
30. Kotov, Alexei, Université du Luxembourg, Luxembourg
31. Kousidis, Stavros, Universität Köln, Germany
32. Kunze, Miriam, Universität Paderborn, Germany
33. La Fuente-Gravy, Laurent, Université Libre de Bruxelles, Belgium
34. Laubinger, Martin, Universitaet Paderborn, Germany
35. Lawn, Marie-Amé lie, Université du Luxembourg, Luxembourg
36. Leicht, Karl, Université Lille 1, France

37. Li, Hui, University of Bourgogne, France
38. Lim, Adrian, Université du Luxembourg, Luxembourg
39. Luo, Dejun, Université du Luxembourg, Luxembourg
40. Merigon, Stephane, Technische Universität Darmstadt, Germany
41. Michel, Jean-Philippe, Universite de la mediterranee, France
42. Mishra, Chayan Kumar, RML Avadh University, India
43. Mladenov, Ivailao, Bulgarian Academy of Sciences, Bulgaria
44. Olbrich, Martin, Université du Luxembourg, Luxembourg
45. Paleani, Christian, Technische Universität München, Germany
46. Palvelev, Roman, Lomonosov Moscow State University, Russia
47. Peter, Matthias, Université du Luxembourg, Luxembourg
48. Pysiak, Leszek, Warsaw University of Technology, Poland
49. Ronghe, Ashok, Barkatulla University, India
50. Sasin, Wieslaw, Warsaw University of Technology, Poland
51. Schlichenmaier, Martin, University of Luxembourg, Luxembourg
52. Schmidt, Viktoria, Universität Köln, Germany
53. Schottenloher, Martin, Ludwig-Maximilians-Universität München, Germany
54. Selinger, Christian, Université du Luxembourg, Luxembourg
55. Seppänen, Henrik, Technische Universität Darmstadt, Germany
56. Sergeev, Armen, Steklov Mathematical Institute, Russia
57. Sheinman, Oleg, Steklov Mathematical Institute, Russia
58. Shoikhet, Boris, Université du Luxembourg, Luxembourg
59. Stepnicki, Robert, The John Paul II Catholic University of Lublin, Poland
60. Stoica, George, University of New Brunswick, Canada
61. Talalaev, Dmitriy, Institut of Theoretical and Experimental Physics, Russia
62. Tate, Tatsuya, Nagoya University, Japan
63. Thalmaier, Anton, Université du Luxembourg, Luxembourg
64. Wagner, Stefan, TU Darmstadt, Germany
65. Wendland, Katrin, Universität Augsburg, Germany
66. Wurzbacher, Tilmann, Université Paul Verlaine - Metz, France
67. Yang, Xuan, Universität Osnabrück, Germany
68. Yoshida, Takahiko, Meiji University, Japan
69. Zaka, Malik, Blekinge Institute of Technology, Sweden
70. Zellner, Christoph, TU Darmstadt, Germany
71. Zhang, Weiping, Chern Institute of Mathematics Nankai University, China
72. Zheglov, Alexander, Moscow State University, Russia

Table of Content