A SYMPLECTIC MAP BETWEEN HYPERBOLIC AND COMPLEX TEICHMÜLLER THEORY

KIRILL KRASNOV and JEAN-MARC SCHLENKER

Abstract

Let S be a closed, orientable surface of genus at least 2. The space $\mathcal{T}_H \times \mathcal{ML}$, where $\mathcal{T}_H$ is the "hyperbolic" Teichmüller space of S and $\mathcal{ML}$ is the space of measured geodesic laminations on S, is naturally a real symplectic manifold. The space $\mathcal{CP}$ of complex projective structures on S is a complex symplectic manifold. A relation between these spaces is provided by Thurston's grafting map Gr. *We prove that this map, although not smooth, is symplectic. The proof uses a variant of the renormalized volume defined for hyperbolic ends.*

1. Introduction and main results

1.1. Historical background

This article addresses an old question—what relation exists between "real" and "complex" descriptions of the Teichmüller space of Riemann surfaces. The milestones in a century of work on this question are presented in the following paragraphs.

The early approach to the uniformization problem, due to Poincaré, was based on the so-called Liouville equation, which, if solved, allows one to find a hyperbolic (constant negative curvature) metric conformal to a given metric. This approach proved too difficult and was later abandoned in favor of an approach based on conformal mappings into the unit circle and Fuchsian groups. Thus, according to the classical uniformization theorem [P], any Riemann surface can be conformally mapped into the interior of the unit disc and can thus be realized as the quotient H^2/Γ, where $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ is a Fuchsian group.

A deformation theory for Riemann surfaces was developed much later in several key works by Teichmüller (see, in particular, [Te]). The key technology used was that of quasi-conformal mappings. Such mappings naturally give rise to holomorphic quadratic differentials (as their Schwartzian derivatives) and can in turn be

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reconstructed from quadratic differentials via a simple second-order differential equation. Quasi-conformal mappings also naturally lead to the concept of Beltrami differentials, and the later can be seen to serve as tangent vectors to the space of Riemann surfaces. The Teichmüller space can then be defined as the space of all Fuchsian groups that are produced from a given one by quasi-conformal mappings. It can be shown to be an open ball in the Banach space of Beltrami differentials (see [A] for details). The described complex viewpoint on the Teichmüller space also allows us to define a natural complex structure on it and then shows that it is a Kähler manifold, where the Kähler metric is that of Weil-Petersson.

At the same time, the complex viewpoint originating in works of Teichmüller makes the geometrical aspects of the question quite hidden. An alternative description that makes the geometry manifest was developed by Fenchel and Nielsen (see, e.g., [FN]). This describes a Riemann surface as being glued from the so-called pairs of pants. The arising coordinates on the Teichmüller space are those of the length and twist parameters. Despite its intuitive attractiveness, the Fenchel-Nielsen description is often much harder to deal with in practice, as it turns out to be very hard to characterize the Fuchsian group thats results from gluing in sufficient generality. It also proved remarkably difficult to relate this "real" description to the "complex" one at the root of Teichmüller's theory.

A much closer connection between the real and complex pictures was provided by Thurston in [Th] with his deformation theory of hyperbolic metrics on a surface via measured geodesic laminations, earthquakes (not considered in this article), and grafting. In grafting, one (heuristically) cuts the surface in question along leaves of a lamination and glues in flat strips of the width prescribed by the measure, thus obtaining a surface with a new projective structure on it; one can then read off the underlying complex structure to get a new point in Teichmüller space. One thus gets a version of deformation theory in which all geometric aspects are manifest. At the same time, there is a link to the "complex" description via projective structures.

This article goes one step further in the direction opened by Thurston. Thus, we study a map very closely related to the grafting, but this time not from the Teichmüller space $\mathscr{T}$ to itself, as would be relevant for the deformation theory. Instead, we consider the grafting as a map from the twice bigger space $\mathscr{T}$ times the space of measured geodesic laminations to the space of complex projective structures. Both spaces are naturally symplectic manifolds, but the first is real and the second is complex. We then show that the corresponding grafting map is symplectic. Or, using physics terminology, we show that the grafting map is a canonical transformation from real to holomorphic coordinates on the phase space—the cotangent bundle over the Teichmüller space of Riemann surfaces. This space can be shown to be the phase space of $(2+1)$-dimensional gravity (see, e.g., [KS1] for a description emphasizing this aspect). Thus, our results can be seen as providing an analog of the $q \pm ip$

coordinates of classical mechanics for the dynamical system of $(2+1)$-gravity. The proof we give is geometrical and, in fact 3-dimensional in nature.

Let us now describe the results of this work in more technical terms. Throughout this article, S is a closed, orientable surface of genus g at least 2, $\mathscr{T}$ is the Teichmüller space of S, $\mathscr{CP}$ the space of complex projective structures on S, and $\mathscr{ML}$ is the space of measured laminations on S.

1.2. The "hyperbolic" Teichmüller space

There are several quite distinct ways to define the Teichmüller space of S, for example, as the space of complex structures on S, or as a space of (particular) representations of $\pi_1(S)$ in $\mathrm{PSL}(2,\mathbb{R})$ (modulo conjugation). In this section, we consider what can be called the "hyperbolic" Teichmüller space, defined as the space of hyperbolic metrics on S, considered up to isotopy. In this guise it is sometimes called the Fricke space of S. Here we denote this space by $\mathscr{T}_H$ to highlight its "hyperbolic" nature. This description emphasizes geometric properties of $\mathscr{T}$, while some other properties, notably the complex structure on $\mathscr{T}$, are not expressed.

There is a natural identification between $\mathscr{T}_H \times \mathscr{ML}$ and the cotangent bundle of $\mathscr{T}_H$, which can be defined as follows. Let $l \in \mathscr{ML}$ be a measured lamination on S. For each hyperbolic metric $m \in \mathscr{T}_H$ on S, let $L_m(l)$ be the geodesic length of l for m (as studied, e.g., in [Ke1], [Wo3]). Thus $m \mapsto L_m(l)$ is a function on $\mathscr{T}$, which is differentiable. For $m_0 \in \mathscr{T}_H$, the differential of $m \mapsto L_m(l)$ at m_0 is a vector in $T^*_{m_0}\mathscr{T}_H$, which we call $\delta(m,l)$. This defines a function $\delta : \mathscr{T}_H \times \mathscr{ML} \to T^*\mathscr{T}_H$, which is the identification we wish to use here. It is proved in Section 2 (see Lemma 2.3) that δ is indeed one-to-one (this fact should be quite obvious to specialists; a proof is included here for completeness). Bonahon proved (see [B1]) that δ is tangentiable; that is, the image by δ of a tangent vector is well defined. Moreover, the tangent map is invertible at each point.

Let ω_H denote the cotangent symplectic structure on $T^*\mathscr{T}_H$. The map δ can be used to pull back to ω_H to $\mathscr{T}_H \times \mathscr{ML}$, making $\mathscr{T}_H \times \mathscr{ML}$ into a symplectic manifold. Somewhat abusing notation, we call ω_H again the symplectic form on $\mathscr{T}_H \times \mathscr{ML}$ obtained in this manner. A more involved but explicit description of ω_H on $\mathscr{T}_H \times \mathscr{ML}$ is recalled below in Section 1.9.

Note that the identification δ between $\mathscr{T}_H \times \mathscr{ML}$ and $T^*\mathscr{T}_H$ is *not* identical with the better-known identification, which goes through measured foliations and quadratic differentials (see, e.g., [FLP]).

1.3. The "complex" Teichmüller space and complex projective structures

We now consider the "complex" Teichmüller space of S, denoted here by $\mathscr{T}_C$, which is the space of complex structures on S. Of course, there is a canonical identification between $\mathscr{T}_H$ and $\mathscr{T}_C$ — there is a unique hyperbolic metric in each conformal class on S. As this map is not explicit, however, it appears helpful to keep in mind the distinction between the two viewpoints. Note that the term *complex* could be used here in two different, albeit related, senses. One is the above definition of $\mathscr{T}_C$ as the space of complex structures on S. The other is related to the well-known deformation theory of $\mathscr{T}_C$ in terms of Beltrami differentials. Considered this way, the complex structure on $\mathscr{T}_C$ becomes manifest. So it is useful to keep in mind that the word *complex* refers both to the complex structures on S and on $\mathscr{T}_C$ itself.

Now let $\mathscr{CP}$ denote the space of (equivalence classes of) $\mathbb{C}\mathrm{P}^1$-structures (or complex projective structures) on S. Recall that a (complex) projective structure on S is a maximal atlas of charts from S into $\mathbb{C}\mathrm{P}^1$ such that all transition maps are Möbius transformations. Such a structure naturally yields a holonomy representation $\mathrm{hol} : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$, as well as an $\mathrm{hol}(\pi_1(S))$-equivariant developing map $\mathrm{dev} : \tilde{S} \to \mathbb{C}\mathrm{P}^1$.

There is a natural relation between complex projective structures on S and complex structures, along with a holomorphic quadratic differential on S. Thus, let σ be a complex projective structure on S, and let σ_0 be the Fuchsian $\mathbb{C}\mathrm{P}^1$-structure on S obtained by the Fuchsian uniformization of the complex structure underlying σ. Then the Schwarzian derivative of the complex map from (S, σ_0) to (S, σ) is a quadratic differential q on S, holomorphic with respect to the conformal structure c of both σ and σ_0, and the map sending σ to (c, q) is a homeomorphism (see, e.g., [D], [Mc]).

Recall also that the space of couples (c, q), where c is a complex structure on S and q is a quadratic holomorphic differential on (S, c), is naturally identified with the complexified cotangent bundle of $\mathscr{T}_C$ (see, e.g., [A]). Thus, $\mathscr{CP}$ is naturally a complex symplectic manifold. We denote the associated (complex) symplectic form by $\omega_{\mathbb{C}}$, and its real part by ω_C.

An equivalent way to describe the complex symplectic structure on $\mathscr{CP}$ is via the holonomy representation. This viewpoint naturally leads to a complex symplectic structure on $\mathscr{CP}$ (see [G]), defined in terms of the cup-product of two 1-cohomology classes on S with values in the appropriate Lie algebra bundle over S. We call this complex symplectic structure ω_G here. The fact that this complex symplectic structure is the same (up to a constant) as $\omega_{\mathbb{C}}$ was established by Kawai in [K].

Note that [K] uses another way to associate a holomorphic quadratic differential to a complex projective structure on S, using as a reference point a complex projective structure given by the simultaneous uniformization (Bers slice) instead of the Fuchsian structure σ_0. This identification is not as canonical as the one above since it depends on a chosen reference conformal structure needed for the simultaneous uniformization. It turns out that the symplectic structure obtained in this way on $\mathscr{CP}$ is independent of the reference point and is the same as the one coming from the above construction using the Fuchsian projective structure σ_0 (see Lemma 4.8).

1.4. The grafting map

The "hyperbolic" and the "complex" descriptions of Teichmüller space behave differently in some key aspects, and it is interesting to understand the relation between them. This relation is given by the well-known grafting map $\mathrm{Gr} : \mathcal{T}_H \times \mathcal{ML} \mapsto \mathcal{CP}$.

The grafting map is defined as follows. When $m \in \mathcal{T}_H$ is a hyperbolic metric and $l \in \mathcal{ML}$ is a weighted multicurve, $\mathrm{Gr}_l(m)$ can be obtained by cutting (S, m) open along the leaves of l, gluing in each cut a flat cylinder of width equal to the weight of the curve in l, and considering the complex projective structure underlying this metric. This map extends by continuity from weighted multicurves to measured laminations, a fact discovered by Thurston (see, e.g., [D]). Out of Gr one can obtain a map from the Teichmüller space to itself by fixing a measured lamination $l \in \mathcal{ML}$ and reading the conformal structure underlying $\mathrm{Gr}_l(m)$; this map is known to be a homeomorphism (see [SW]). Grafting on a fixed hyperbolic surface also defines a homeomorphism between $\mathcal{ML}$ and $\mathcal{T}$ (see [DW]).

It is possible to compose $\delta^{-1} : T^*\mathcal{T}_H \to \mathcal{T}_H \times \mathcal{ML}$ with the grafting map $\mathrm{Gr} : \mathcal{T}_H \times \mathcal{ML} \to \mathcal{CP}$. The resulting map between smooth manifolds is tangentiable by the results mentioned above, but it turns out to be smoother.

LEMMA 1.1
The map $\mathrm{Gr} \circ \delta^{-1} : T^*\mathcal{T}_H \to \mathcal{CP}$ *is* C^1.

The proof of Lemma 1.1 is in Section 2.

Our main result in this article is to prove that this composed map is symplectic. This can be stated as follows, using the symplectic structure induced on $\mathcal{T} \times \mathcal{ML}$ by ω_H.

THEOREM 1.2
The pullback of the symplectic form ω_C on $\mathcal{CP}$ by the grafting map is the form ω_H on $\mathcal{T}_H \times \mathcal{ML}$, up to a factor of 2*:* $\mathrm{Gr}^*\omega_C = 2\omega_H$.

The meaning of the theorem is easier to appreciate when considering the composed map $\mathrm{Gr} \circ \delta^{-1} : T^*\mathcal{T}_H \to \mathcal{CP}$. Since this map is C^1 by Lemma 1.1 we can use it to pull back the symplectic form ω_C.

We work with symplectic forms ω given in terms of Liouville forms β as $d\beta = \omega$. The proof shows that the image by Gr of the Liouville form of $2\omega_H$ is the Liouville form of ω_C plus the differential of a function. In Theorem 1.6 we give an alternative statement of Theorem 1.2 in terms of Lagrangian submanifolds.

Our proof of Theorem 1.2 is based on geometrically finite 3-dimensional hyperbolic ends. We recall this notion in Section 1.5.

1.5. Hyperbolic ends

Definition 1.3

A *hyperbolic end* is a 3-manifold M, homeomorphic to $S \times \mathbb{R}_{>0}$, where S is a closed surface of genus at least 2, endowed with a (noncomplete) hyperbolic metric such that

- the metric completion corresponds to $S \times \mathbb{R}_{\geq 0}$;
- the metric g extends to a hyperbolic metric in a neighborhood of the boundary in such a way that $S \times \{0\}$ corresponds to a pleated surface;
- $S \times \mathbb{R}_{>0}$ is concave in the neighborhood of this boundary.

Given such a hyperbolic end, we call $\partial_0 M$ the *metric boundary corresponding to* $S \times \{0\}$, and we call $\partial_\infty M$ the *boundary at infinity*. We call $\mathcal{G}_S$ the *space* of those hyperbolic ends.

It is simpler to consider a quasi-Fuchsian hyperbolic manifold N. The complement of its convex core is the disjoint union of two hyperbolic ends. A hyperbolic end, however, as defined above, does not always extend to a quasi-Fuchsian manifold. Note also that the hyperbolic ends as defined here are always convex cocompact, so our definition is more restrictive than others found elsewhere, and the longer name "convex cocompact hyperbolic end" would perhaps be more precise. We do not consider here degenerate hyperbolic ends, with an end invariant which is a lamination rather than a conformal structure; the fact that $S \times \{0\}$ is a convex pleated surface in our definition prevents the other end from being degenerate at infinity.

There are two natural ways to describe a hyperbolic end, either from the metric boundary or from the boundary at infinity, both of which are well known. On the metric boundary side, $\partial_0 M$ has an induced metric m which is hyperbolic and pleated along a measured lamination l. It is well known that m and l uniquely determine M (see, e.g., [D]).

In addition, $\partial_\infty M$ carries naturally a complex projective structure, σ, because it is locally modeled on the boundary at infinity of H^3 and hyperbolic isometries act at infinity by Möbius transformations. This complex projective structure has an underlying conformal structure, c. Moreover, the construction described above assigns to $\partial_\infty M$ a quadratic holomorphic differential q, which is none other than the Schwartzian derivative of the complex map from (S, σ_0) to (S, σ). It follows from Thurston's original construction of the grafting map that $\sigma = \mathrm{Gr}_l(m)$.

1.6. Convex cores

Before we describe how the above hyperbolic ends can be of any use for the questions considered in this article, let us consider what is perhaps a more familiar situation. Thus, consider a hyperbolic 3-manifold with boundary N, which admits a convex cocompact hyperbolic metric. We call $\mathcal{G}(N)$ the space of such convex cocompact hyperbolic metrics on N. Let $g \in \mathcal{G}$; then (N, g) contains a smallest nonempty subset K which is geodesically convex (any geodesic segment with endpoints in K

is contained in K) (its convex core is denoted here by $CC(N)$). So $CC(N)$ is then homeomorphic to N, its boundary is the disjoint union of closed pleated surfaces, each of which has an induced metric which is hyperbolic, and each is pleated along a measured geodesic lamination (see, e.g., [EM]). So we obtain a map

$$i' : \mathscr{G}(N) \to \mathscr{T}_H(\partial N) \times \mathscr{ML}(\partial N).$$

Composing i' with the identification δ between $\mathscr{T}_H \times \mathscr{ML}$ and $T^*\mathscr{T}_H$, we obtain an injective map

$$i : \mathscr{G}(N) \to T^*\mathscr{T}_H(\partial N).$$

THEOREM 1.4
The image $i(\mathscr{G}(N))$ is a Lagrangian submanifold of $(T^*\mathscr{T}_H(\partial N), \omega_H)$.

The map i is not smooth, but as in the case of the map δ defined above, it is tangentiable (see [B3], [B4]). The natural map from $\mathscr{G}(N)$ to the space of complex projective structure on each connected component of the boundary at infinity is smooth, and it follows from Theorem 2.4 that i is C^1.

The proof given below shows that the restriction to $i(\mathscr{G}(N))$ of the Liouville form of $T^*\mathscr{T}_H(\partial N)$ is the differential of a function.

The reason for considering convex cores in our context becomes clear in Sections 1.7 and 1.8.

1.7. Kleinian reciprocity
There is a direct relationship between Theorems 1.4 and 1.2 in that Theorem 1.4 can be considered a corollary of Theorem 1.2. This follows the so-called "Kleinian reciprocity" of McMullen. Thus, consider a Kleinian manifold M, and let $\mathscr{G}(M)$ be the space of complete convex cocompact hyperbolic metrics on M. Each $g \in \mathscr{G}(M)$ gives rise to a projective structure on the boundary at infinity $\partial_\infty M$. This gives an injective map $j : \mathscr{G}(M) \to T^*\mathscr{T}_C(\partial_\infty M)$.

THEOREM 1.5 (see McMullen [Mc])
The image $j(\mathscr{G}(M))$ is a Lagrangian submanifold of $(T^*\mathscr{T}_C(\partial_\infty M), \omega_C)$.

This statement is quite analogous to Theorem 1.4, with the only difference being that the "hyperbolic" cotangent bundle at boundaries of the convex core is replaced by the "complex" one. This statement (under a different formulation) is proved in [Mc, Appendix] under the name of "Kleinian reciprocity", and it is an important technical statement allowing the author to prove the Kähler hyperbolicity of Teichmüller space.

Let us note that Theorem 1.4 is a direct consequence of Theorems 1.2 and 1.5. This becomes clearer later when we present another statement of Theorem 1.2. In Section 1.8, we give a direct proof of Theorem 1.4, thus also giving a more direct proof of the Kleinian reciprocity result.

Using the result of Kawai [K], Theorem 1.5 is equivalent to the fact that the subspace of complex projective structures on ∂N obtained from hyperbolic metrics on N is a Lagrangian submanifold of $(\mathscr{CP}(\partial N), \mathrm{Re}(\omega_G))$, a fact previously known to Kerckhoff [Ke2] through a different, topological argument involving Poincaré duality.

1.8. A Lagrangian translation of Theorem 1.2

In a similar vein to what we have done above, let us consider the space $\mathscr{G}_S$ of hyperbolic ends. Each such space gives a point in $\mathscr{T}_H \times \mathscr{ML}$ for its pleated surface boundary and a point in $T^*\mathscr{T}_C$ for its boundary at infinity. Thus, composing this with the map δ, we get an injective map

$$k : \mathscr{G} \to T^*\mathscr{T}_H \times T^*\mathscr{T}_C.$$

Our main Theorem 1.2 can then be restated as follows.

THEOREM 1.6

The image $k(\mathscr{G})$ is a Lagrangian submanifold of $T^*\mathscr{T}_H \times T^*\mathscr{T}_C$.

We prove our main result in this version, which is clearly equivalent to Theorem 1.2. Let us stress again that $k(\mathscr{G})$ is not smooth but only the graph of a C^1 map.

1.9. The intersection form and ω_H

An efficient combinatorial description of $\mathscr{T}_H$ was given in Thurston's article on earthquakes. Later, a powerful analytical realization of the same ideas was developed in a series of articles by Bonahon [B1], [B2]. The earthquake description of $\mathscr{T}_H$ is somewhat related to a much earlier parametrization of the same space in terms of the Fenchel-Nielsen coordinates, whose idea is to glue a Riemann surface from pairs of pants, the pants being characterized by the length of their boundary components and the gluing being characterized by twists parameters. Thurston's earthquake description of $\mathscr{T}_H$ describes a hyperbolic metric $m \in \mathscr{T}_H$ as obtained by a left earthquake on a measured lamination from another base hyperbolic metric m_0. It is remarkable that this measured lamination completely determines the earthquake and is in turn completely determined by the two metrics $m, m_0 \in \mathscr{T}_H$.

In Thurston's description, the hyperbolic Teichmüller space is parametrized by the space of measured geodesic laminations. However, the space $\mathscr{ML}$ does not possess a natural differentiable structure, which makes the analysis on this space hardly possible. One of the key achievements of Bonahon's work in [B1] and [B2] was to develop the calculus on $\mathscr{ML}$ using $\mathbb{R}$-valued transverse cocycles or, equivalently, transverse Hölder distributions for geodesic laminations. Essentially, Bonahon gave a very elegant description of the tangent space to $\mathscr{ML}$. This allowed him to provide a characterization of the space $\mathscr{T}_H$ itself as homeomorphic to an open cone in a vector space $\mathscr{H}(\lambda, \mathbb{R})$ of $\mathbb{R}$-valued transverse cocycles for a lamination λ (see [B1, Theorem A]) and also prove that the vector fields tangent to $\mathscr{ML}$ are Hamiltonian vector fields with respect to Thurston's symplectic structure on $\mathscr{H}(\lambda, \mathbb{R})$, with the Hamiltonian function being essentially the hyperbolic length (see [B1]).

In a later work, Sözen and Bonahon [SB] established that Thurston's symplectic form on $\mathscr{H}(\lambda, \mathbb{R})$ is (up to a constant) an image of the Weil-Petersson symplectic form on $\mathscr{T}_H$ under the homeomorphism of this space into $\mathscr{H}(\lambda, \mathbb{R})$. The proof goes through Goldman's characterization of the Weil-Petersson symplectic form in terms of a cup product in a twisted cohomology group [G]. Thus, Bonahon's description of $\mathscr{T}_H$ in terms of shearing coordinates can be said to provide a symplectic map from $\mathscr{T}_H$ with its usual Weil-Petersson symplectic form to the vector space $\mathscr{H}(\lambda, \mathbb{R})$ with its Thurston's symplectic form. Theorem 1.2 can be construed as an analog concerning related but twice bigger spaces: on one hand, $\mathscr{T}_H \times \mathscr{ML}$, while on the other, the space $\mathscr{CP}$ of complex projective structures on S.

The space $\mathscr{T}_H \times \mathscr{ML}$ is naturally a real symplectic manifold. The length function has an extension $L_m : \mathscr{H}(\lambda, \mathbb{R}) \to \mathbb{R}, \dot{l} \in \mathscr{H}(\lambda, \mathbb{R}) \to L_m(\dot{l}) \in \mathbb{R}$ to geodesic laminations with transverse cocycles, where it can be interpreted as a differential of the corresponding function on $\mathscr{ML}$ (see [B1]). Now, given a vector field $\dot{m}$ tangent to $\mathscr{T}_H$, we obtain a pairing $L_{\dot{m}}(\dot{l})$ as the derivative of $L_m(\dot{l})$ in the direction of $\dot{m}$. Consider the following two-form on $\mathscr{T}_H \times \mathscr{ML}$:

$$\omega'_H\big((\dot{m}_1, \dot{l}_1), (\dot{m}_2, \dot{l}_2)\big) = L_{\dot{m}_1}(\dot{l}_2) - L_{\dot{m}_2}(\dot{l}_1). \tag{1}$$

LEMMA 1.7
This two-form is equal to the symplectic form $\omega'_H = \omega_H$.

Proof
Let $(m, l) \in \mathscr{T} \times \mathscr{ML}$, then $\delta(m, l) = (m, dL_{\cdot}(l)) \in T^*\mathscr{T}_H$. Denote by β the Liouville form of $(T^*\mathscr{T}_H, \omega_H)$, so that, if $(m, u) \in T^*\mathscr{T}_H$ and $(\dot{m}, \dot{u}) \in T_{(m,u)}T^*\mathscr{T}_H$, then

$$\beta(\dot{m}, \dot{u}) = u(\dot{m}).$$

If $(\dot{m}, \dot{l}) \in T(\mathscr{T} \times \mathscr{ML})$ then $\beta(d\delta(\dot{m}, \dot{l})) = L_{\dot{m}}(l)$. This corresponds precisely to the Liouville form of ω'_H, and the result follows. $\square$

1.10. Another possible proof ?

Let us note that an alternative proof that only uses 2-dimensional quantities may be possible, based essentially on the result of [SB]. For this, one would need to extend the ideas developed in this work to shear-bend coordinates on $\mathcal{T}_H \times \mathcal{ML}$ and show that the symplectic form ω_H on $\mathcal{H}(\lambda, \mathbb{R})$ extends to a complex symplectic form on the vector space of complex-valued cocycles for λ. Presumably this complex symplectic form would then coincide with the complexified Thurston symplectic form on $\mathcal{H}(\lambda, \mathbb{C})$.

According to [SB], Thurston's symplectic form on $\mathcal{H}(\lambda, \mathbb{R})$ is equal to the Weil-Petersson symplectic form on the real character variety. This equality should extend to the complexified Thurston symplectic form and the Weil-Petersson symplectic form on the complex character variety, because both are holomorphic.

It was proved by Kawai [K], however, that the Weil-Petersson form on the complex character variety corresponds to the complex symplectic cotangent symplectic form on $T^*\mathcal{T}_C$, and so Theorem 1.2 should follow.

Note that this line of reasoning is quite different from the proof considered below, which uses mostly the Bonahon-Schläfli formula in Lemma 2.1 (or more precisely, the dual formula in Lemma 2.2). The arguments outlined in this section could therefore be combined with those developed in this article, for instance to obtain a new proof of Kawai's result in [K] from the results of [SB], holomorphic continuation, and the use of the renormalized volume.

1.11. Cone singularities

One interesting feature of the arguments used here is that they appear likely to extend to the setting of hyperbolic surfaces with cone singularities of angle less than π. One should then use hyperbolic ends with particles (i.e., cone singularities of angle less than π going from the interior boundary to the boundary at infinity, as already done in [KS1] and to some extent in [KS2]).

2. The Schläfli formula and the dual volume

In this section we recall the Schläfli formula, first in the simple case of hyperbolic polyhedra, then in the more involved setting of convex cores of hyperbolic 3-manifolds (as extended by Bonahon). We then deduce from Bonahon's Schläfli formula a dual formula for the first-order variation of the dual volume of the convex core. Finally, we give the proof of Lemma 1.1.

2.1. The Schläfli formula for hyperbolic polyhedra

Let $P \subset H^3$ be a convex polyhedron. The Schläfli formula (see, e.g., [Mi]) describes the first-order variation of the volume of P, under a first-order deformation, in terms of the lengths and the first-order variations of the angles, and so

$$dV = \frac{1}{2} \sum_e L_e d\theta_e, \tag{2}$$

where the sum is over the edges of P, L_e is the length of the edge e, and θ_e is its exterior dihedral angle.

There is also an interesting dual Schläfli formula. Let

$$V^* = V - \frac{1}{2} \sum_e L_e \theta_e$$

be the *dual volume* of P; then, still under a first-order deformation of P,

$$dV^* = -\frac{1}{2} \sum_e \theta_e dL_e. \tag{3}$$

This follows from the Schläfli formula (2) by an elementary computation.

2.2. First-order variations of the volume of the convex core

In many ways, the convex core of a quasi-Fuchsian manifold is reminiscent of a polyhedron, with the edges and their exterior dihedral angles being replaced by a measured lamination describing the pleating of the boundary (see, e.g., [Th], [EM]).

Bonahon [B3] has extended the Schläfli formula to this setting as follows. Let M be a convex cocompact hyperbolic manifold (e.g., a quasi-Fuchsian manifold), let μ be the induced metric on the boundary of the convex core, and let λ be its measured bending lamination. By a first-order variation of M we mean a first-order variation of the representation of the fundamental group of M. Bonahon shows that the first-order variation of λ under a first-order variation of M is described by a transverse Hölder distribution λ', and there is a well-defined notion of length of such transverse Hölder distributions. This leads to a version of the Schläfli formula.

LEMMA 2.1 (Bonahon-Schläfli formula [B3])

The first-order variation of the volume V_C of the convex core of M, under a first-order variation of M, is given by

$$dV_C = \frac{1}{2} L_\mu(\lambda').$$

Here λ' is the first-order variation of the measured bending lamination, which is a Hölder cocycle, so that its length for μ can be defined (see [B1], [B2], [B3], [B4]).

2.3. The dual volume

Just as for polyhedra above, we define the dual volume of the convex core of M as

$$V_C^* = V_C - \frac{1}{2} L_\mu(\lambda).$$

LEMMA 2.2 (The dual Bonahon-Schläfli formula)

The first-order variation of V^* under a first-order variation of M is given by

$$dV_C^* = -\frac{1}{2} L'_\mu(\lambda).$$

This formula has a very simple interpretation in terms of the geometry of Teichmüller space: up to the factor $-1/2$, dV^* is equal to the pullback by δ of the Liouville form of the cotangent bundle $T^*\mathcal{T}_H$. Note also that this formula can be understood in an elementary way, without reference to a transverse Hölder distribution: the measured lamination λ is fixed, and only the hyperbolic metric μ varies. The proof we give here, however, is based on Lemma 2.1 and thus on the whole machinery developed in [B3].

Theorem 1.4 is a direct consequence of Lemma 2.2; since dV_C^* coincides with the Liouville form of $T^*\mathcal{T}_H(\partial N)$ on $i(\mathcal{G}(N))$, it follows immediately that $i(\mathcal{G}(N))$ is Lagrangian for the symplectic form ω_H on $T^*\mathcal{T}_H(\partial N)$.

Proof of Lemma 2.2

Thanks to Lemma 2.1, we only have to show a purely 2-dimensional statement, valid for any closed surface S of genus at least 2; that is, the function

$$\begin{aligned} L : \mathcal{T} \times \mathcal{ML} &\to \quad \mathbb{R} \\ (\mu, \lambda) \quad &\mapsto L_\mu(\lambda) \end{aligned}$$

admits directional derivatives, and its derivative with respect to a tangent vector $(\mu', \lambda',)$ is equal to

$$L_\mu(\lambda)' = L'_\mu(\lambda) + L_\mu(\lambda'). \tag{4}$$

Two special cases of this formula were proved by Bonahon: the case when μ is kept constant [B2], and the case when λ is kept constant [B1].

To prove equation (4), suppose that μ_t, λ_t depend on a real parameter t chosen so that the derivatives μ'_t, λ'_t exist for $t = 0$, with

$$\frac{d\mu_t}{dt}_{|t=0} = \mu', \qquad \frac{d\lambda_t}{dt}_{|t=0} = \lambda'.$$

We can also suppose that (m_t) is a smooth curve for the differentiable structure of Teichmüller space. We can then decompose as follows.

$$\frac{L_{\mu_t}(\lambda_t) - L_{\mu_0}(\lambda_0)}{t} = \frac{L_{\mu_t}(\lambda_t) - L_{\mu_0}(\lambda_t)}{t} + \frac{L_{\mu_0}(\lambda_t) - L_{\mu_0}(\lambda_0)}{t}.$$

The second term on the right-hand side converges to $L_\mu(\lambda')$ by [B2], so we now concentrate on the first term.

To prove that the first term converges to $L'_\mu(\lambda)$, it is sufficient to prove that $L'_\mu(\lambda)$ depends continuously on μ, μ' and on λ. This can be proved by a nice and simple argument suggested to us by Bonahon. Here μ can be replaced by a representation of the fundamental group of S in $\mathrm{PSL}_2(\mathbb{C})$, as in [B1]. For fixed λ, the function $\mu \to L_\mu(\lambda)$ is then holomorphic in μ, and continuous in λ. Since it is holomorphic, it is continuous with respect to μ and to μ', and the result follows. □

2.4. A cotangent space interpretation

Here we sketch for completeness the argument showing that the map $\delta : \mathscr{T}_H \times \mathscr{ML} \to T^*\mathscr{T}_H$ defined in the introduction is a homeomorphism. This is equivalent to the following statement.

LEMMA 2.3

Let $m_0 \in \mathscr{T}_H$ be a hyperbolic metric on S. For each cotangent vector $u \in T^*_{m_0}\mathscr{T}_H$, there exists a unique $l \in \mathscr{ML}$ such that the differential of the function $m \mapsto dL_m(l)$ is equal to u at m_0.

Proof

Wolpert [Wo1] discovered that the Weil-Petersson symplectic form on $\mathscr{T}_H$ has a remarkably simple form in Fenchel-Nielsen coordinates,

$$\omega_{WP} = \sum_i dL_i \wedge d\theta_i,$$

where the sum is over the simple closed curves in the complement of a pants decomposition of S. A direct consequence is that, given a weighted multicurve w on S, the dual for ω_{WP} of the differential of the length L_w of w is equal to the infinitesimal fractional Dehn twist along w.

This extends when w is replaced by a measured lamination λ, with the infinitesimal fractional Dehn twist replaced by the earthquake vector along λ (see [Wo2], [SB]). So the Weil-Petersson symplectic form provides a duality between the differential of the lengths of measured laminations and the earthquake vectors.

Moreover, the earthquake vectors associated to the elements of $\mathscr{ML}$ cover $T_m\mathscr{T}_H$ for all $m \in \mathscr{T}_H$ (see [Ke1]), so it follows that the differentials of the lengths of the measured laminations cover $T^*_m\mathscr{T}_H$. □

Note that this argument extends directly to hyperbolic surfaces with cone singularities, when the cone angles are less than π. In that case, the fact that earthquake vectors still span the tangent to Teichmüller space follows from [BS].

2.5. Proof of Lemma 1.1

Lemma 1.1 is mostly a consequence of the tools developed by Bonahon in [B1] and [B2]. We first recall some of his results. Given a lamination λ on S, he defined the space $\mathscr{H}(\lambda, \mathbb{R})$ of real-valued transverse cocycles for λ and proved that it is related to measured laminations in interesting ways.

- If $l \in \mathscr{ML}$, and if λ is a lamination which contains the support of l, then l defines a real-valued transverse cocycle on λ (see [B1]).
- Transverse cocycles can be used to define a polyhedral tangent cone to $\mathscr{ML}$ at a point l. Given a lamination λ containing the support of l, the transverse cocycles on λ satisfying a positivity condition (essentially, that the transverse measure remains positive) can be interpreted as tangent vectors to $\mathscr{ML}$ at l (i.e., velocities at 0 of curves in $\mathscr{ML}$ starting from l). The laminations containing the support of l therefore correspond to the faces of the tangent cone to $\mathscr{ML}$ at l.
- There is a well-defined notion of length of a transverse cocycle h for a hyperbolic metric m on S, extending the length of a measured lamination. If $l \in \mathscr{ML}$, then L_m is tangentiable at l; if λ is a lamination containing the support of l, and if $h \in \mathscr{H}(\lambda, \mathbb{R})$, then $L_m(h)$ is equal to the first-order variation of $L_m(l)$ under the deformation of l given by h.

Transverse cocycles are also related to pleated surfaces.

- Transverse cocycle provide shear coordinates on Teichmüller space. Given a reference hyperbolic metric $m_0 \in \mathscr{T}_H$ and another hyperbolic metric $m \in \mathscr{T}_H$, there is a unique element $h \in \mathscr{H}(\lambda, \mathbb{R})$ such that shearing m_0 along h yields m. The elements of $\mathscr{H}(\lambda, \mathbb{R})$ which can be obtained in this way have a simple characterization in terms of a positivity condition.
- Transverse cocycles also describe the bending of a pleated surface: $\mathscr{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z})$ is in one-to-one correspondence with the space of equivariant pleated surfaces of given induced metrics for which the support of the pleating locus is contained in λ.
- Pleated surfaces with a pleating locus contained in λ are associated to a complex-valued transverse cocycle $h \in \mathscr{H}(\lambda, \mathbb{C}/2\pi i\mathbb{Z})$, with real part

describing the induced metric (in terms of its shear coordinates with respect to a given reference metric) and imaginary part describing the bending measure.

Each pleated equivariant surface in H^3 defines a representation of its fundamental group in PSL(2, $\mathbb{C}$). In the neighborhood of a convex pleated surface, this representation is the holonomy representation of a complex projective structure. If the induced metric and measured bending lamination of the convex pleated surface are $m \in \mathcal{T}_H$ and $l \in \mathcal{ML}$, respectively, if λ contains the support of l, and if $\bar{l}$ is the projection of l in $\mathcal{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z})$, then there is a well-defined map from $\mathcal{H}(\lambda, \mathbb{C}/2\pi i\mathbb{Z})$ to $\mathcal{CP}$ defined in the neighborhood of $i\bar{l}$ sending a complex-valued transverse cocycle h to the complex projective structure σ of the pleated surface obtained from h. This map is differentiable; taking its tangent at $i\bar{l}$ yields a map

$$\phi_\lambda : \mathcal{H}(\lambda, \mathbb{C}) \to T_\sigma \mathcal{CP}.$$

THEOREM 2.4 ([B4])
The map ϕ_λ is complex-linear (with respect to the complex structure on $\mathcal{CP}$).

A pleated surface is also described, however, by its induced metric and measured bending lamination and thus by an element of $\mathcal{T}_H \times \mathcal{ML}$. Using the map $\delta : \mathcal{T}_H \times \mathcal{ML} \to T^*\mathcal{T}_H$ defined above, we obtain a map, defined in the neighborhood of $i\bar{l}$, from $\mathcal{H}(\lambda, \mathbb{C}/2\pi i\mathbb{Z})$ to $T^*\mathcal{T}_H$ which by definition is also differentiable. Taking the differential of this map yields another linear map

$$\psi_\lambda : \mathcal{H}(\lambda, \mathbb{C}) \to T^*\mathcal{T}_H.$$

The definitions (and the arguments of [B1], [B4]) then show that $\phi_\lambda \circ \psi_\lambda^{-1}$ is partially equal to the tangent map of $\mathrm{Gr} \circ \delta^{-1}$, in the following sense. Let $m \in \mathcal{T}_H$, and let $u \in T_m^*\mathcal{T}_H$, and let $(m, l) = \delta^{-1}(m, u) \in \mathcal{T}_H \times \mathcal{ML}$. Let then $(\dot{m}, \dot{u}) \in T_{(m,u)}(T^*\mathcal{T}_H)$, and let $\dot{l}$ be the tangent vector to $\mathcal{ML}$ at l corresponding to $\dot{u}$. There is then a lamination λ containing the support of both l and $\dot{l}$, and $\phi_\lambda \circ \psi_\lambda^{-1}(\dot{m}, \dot{u}) = T(\mathrm{Gr} \circ \delta^{-1})(\dot{m}, \dot{u})$. Its definition shows that $\phi_\lambda \circ \psi_\lambda^{-1}$ is linear. So, to prove that $\mathrm{Gr} \circ \delta^{-1}$ is differentiable, it is sufficient to show that $\phi_\lambda \circ \psi_\lambda^{-1}$ does not depend on λ.

We now consider a fixed lamination λ containing the support of l and a variation $(\dot{m}, \dot{u})$ inducing a variation $\dot{l}$ of l with support contained in λ. To $(\dot{m}, \dot{l})$ is associated, through Bonahon's shear-bend coordinates, a complex transverse cocycle $h \in \mathcal{H}(\lambda, \mathbb{C})$, with real part h_0 corresponding to $\dot{m}$ and imaginary part h_1 corresponding to $\dot{l}$. The first-order variation of σ corresponding to $\dot{m}$, $\phi_\lambda \circ \psi_\lambda^{-1}(\dot{m}, 0)$ is shown in [B4] to be independent of λ. The following is clearly equivalent to [B4, Lemma 13].

LEMMA 2.5 (Bonahon)
For fixed $l \in \mathcal{ML}$, the restriction map $\mathrm{Gr}_l : \mathcal{T}_H \to \mathcal{CP}$ is C^1.

We now focus on $\dot{l}$ and on the corresponding imaginary part h_1 of the transverse cocycle h. We have already recalled that $L_m(h_1) = L_m(l)'$, where the prime denotes the first-order variation under the tangent vector to $\mathscr{ML}$ at l corresponding to h_1. It follows that for any $m' \in T_m\mathscr{T}_H$, we have $\dot{u}(m') = dL_{\cdot}(h_1)(m')$. However, it was proved in [B1] that $dL_{\cdot}(h_1)(m') = \omega_{WP}(m', e_{h_1})$, where ω_{WP} is the Weil-Petersson symplectic form on $\mathscr{T}_H$ and e_{h_1} is the tangent vector to $\mathscr{T}_H$ at m corresponding to the infinitesimal shear along h_1. So e_{h_1} is the dual of $\dot{u}$ for the symplectic form ω_H on $T^*\mathscr{T}_H$. We write this as $e_{h_1} = \dot{u}^*$ (the star stands for the Weil-Petersson symplectic duality).

We can now apply Theorem 2.4, and we conclude that

$$\begin{aligned}(\phi_\lambda \circ \psi_\lambda^{-1})(0, \dot{u}) &= \phi_\lambda(ih_1) = i\phi_\lambda(h_1) = i(\phi_\lambda \circ \psi_\lambda^{-1})(e_{h_1}, 0) \\ &= i(\phi_\lambda \circ \psi_\lambda^{-1})(\dot{u}^*, 0) = \mathrm{id}\,\mathrm{Gr}_l(\dot{u}^*). \end{aligned} \tag{5}$$

In particular, $(\phi_\lambda \circ \psi_\lambda^{-1})(0, \dot{u})$ is independent of λ by Lemma 1.1, so that $\phi_\lambda \circ \psi_\lambda^{-1}$ is linear, making $\mathrm{Gr} \circ \delta^{-1}$ differentiable.

The fact that $\mathrm{Gr} \circ \delta^{-1}$ is actually C^1 then follows from Theorem 2.4 applied twice, once for the first-order variations of the metric, and another time (through the composition (5)) for the first-order variation of $\dot{u}$ (resp., $\dot{l}$).

Note that this map $\mathrm{Gr} \circ \delta^{-1}$ is probably not C^2. This is indicated by the fact, shown by Bonahon in [B4], that the composition of the inverse grafting map $\mathrm{Gr}^{-1} : \mathscr{CP} \to \mathscr{T}_H \times \mathscr{ML}$ with the projection on the first factor is C^1 but not C^2.

3. The renormalized volume

3.1. Definition

We recall in this section, very briefly, the definition and one key property of the renormalized volume of a quasi-Fuchsian—or, more generally, a geometrically finite—hyperbolic 3-manifold (more details can be found in [KS2]). The definition can be made as follows. Let M be a quasi-Fuchsian manifold, and let K be a compact subset which is *geodesically convex* (any geodesic segment with endpoints in K is contained in K), with smooth boundary.

Definition 3.1
We call

$$W(K) = V(K) - \frac{1}{4}\int_{\partial K} H da,$$

where H is the mean curvature of the boundary of K.

K defines a metric I^* on the boundary of M. For $\rho > 0$, let S_ρ be the set of points at distance ρ from K; then $(S_\rho)_{\rho>0}$ is an equidistant foliation of $M \setminus K$. It is then possible to define a metric on ∂M as

$$I^* := \lim_{\rho\to\infty} 2e^{-2\rho} I_\rho, \tag{6}$$

where I_ρ is the induced metric on S_ρ. Then I^* is in the conformal class at infinity of M, which we call c_∞.

Defined in this way, both I^* and W are functions of the convex subset K. However, K is itself uniquely determined by I^*, and it is possible to consider W as a function of I^*, considered as a metric in ∂M in the conformal class at infinity c_∞, although such a metric in c_∞ is not necessarily associated to a convex subset of M. The reason for this is that each metric $I^* \in c_\infty$ is associated to a unique foliation of a neighborhood of infinity in M by equidistant convex surfaces $(S_\rho)_{\rho\geq\rho_0}$ (see [E2], [E1], [S] or Theorem 5.8 in [KS2]). This foliation does not always extend to $\rho \to 0$, which would mean that it is the equidistant foliation from a convex subset with boundary S_0.

To understand the construction of W in this setting, we need to revert to another definition of the renormalized volume as it is defined for higher-dimensional conformally compact Einstein manifolds. If V_ρ is the volume of the set of points of M at distance at most ρ from K, then V_ρ behaves as $\rho \to \infty$ as

$$V_\rho = V_2 e^{2\rho} + V_1\rho + V_0 + \epsilon(\rho),$$

where $\lim_0 \epsilon = 0$. Epstein proved in [PP, Appendix] (see also [KS1, Lemma 4.5]) that $V_0 = W$ (as defined above) is equal to V_0, while V_1 depends only on the topology of M (it is equal to $-\pi\chi(\partial M)$). Suppose now that K is replaced by

$$K_r = \{x \in M \mid d(K,x) \leq r\}.$$

Let $\overline{V}_\rho$ be the volume of the set of points at distance at most ρ from K_r; then clearly,

$$\overline{V}_\rho = V_{\rho+r} = V_2 e^{2(\rho+r)} + V_1(\rho+r) + V_0 = (V_2 e^{2r})e^{2\rho} + V_1\rho + (V_0 + V_1 r) + \epsilon(\rho),$$

so that V_0 is replaced by $\overline{V}_0 = V_0 + V_1 r$. This means that W can be read off from any of the surfaces V_ρ, since, for any $\rho > 0$, we have

$$W = V_\rho - \frac{1}{4}\int_{S_\rho} H da + \pi\chi(\partial M)\rho.$$

Starting from a metric I^* in the conformal class at infinity c_∞, there is an associated equidistant foliation by convex surfaces $(S_\rho)_{\rho\geq\rho_0}$ of a neighborhood of infinity in M, and the previous formula can be used to define W even if the foliation does not extend to $\rho \to 0$. As a consequence, W defines a function, still called W, which, to any metric $I^* \in c_\infty$, associates a real number $W(I^*)$.

LEMMA 3.2 (see [Kr], [TT], [ZT])
Over the space of metrics $I^* \in c_\infty$ of fixed area, W has a unique maximum, which is obtained when I^* has constant curvature.

This, along with the Bers simultaneous uniformization theorem, defines a function $V_R : \mathscr{T}(\partial M) \to \mathbb{R}$, sending a conformal structure on the boundary of M to the maximum value of $W(I^*)$ when I^* is in the fixed conformal class of metrics and is restricted to have area equal to $-2\pi\chi(\partial M)$. This number V_R is called the renormalized volume of M.

3.2. The first variation of the renormalized volume
The first variation of the renormalized volume involves a kind of Schläfli formula, in which some terms appear that need to be defined. One such term is the second fundamental form at infinity $I\!I^*$ associated to an equidistant foliation in a neighborhood of infinity, as in Section 3.1. The definition comes from the following lemma, which is taken from [KS2].

LEMMA 3.3
Consider an equidistant foliation (S_ρ) as above, recalling that I_ρ is the induced metric on S_ρ and that I^* is the metric on the boundary at infinity defined by $I^* = \lim_{\rho\to\infty} e^{-\rho} I_\rho$. There is a unique bundle morphism $B^* : T\partial M \to T\partial M$, self-adjoint for I^*, such that

$$I_\rho = \frac{1}{2}(e^{2\rho} I^* + 2I\!I^* + e^{-2\rho} I\!I\!I^*),$$

where $I\!I^* = I^*(B^*\cdot, \cdot)$ and $I\!I\!I^* = I^*(B^*\cdot, B^*\cdot)$.

The first variation of W under a deformation of M or of the equidistant foliation is given by another lemma from [KS2], which can be seen as a version at infinity of the Schläfli formula for hyperbolic manifolds with boundary found in [RS1], [RS2].

LEMMA 3.4
Under a first-order deformation of the hyperbolic metric on M or of the equidistant foliation close to infinity, the first-order variation of W is given by

$$dW = -\frac{1}{4}\int_{\partial M} \left\langle dI\!I^* - \frac{H^*}{2} dI^*, I^* \right\rangle da^*,$$

where $H^* := tr(B^*)$ and da^* is the area form of I^*.

The second fundamental form at infinity, $I\!I^*$, is actually quite similar to the usual second fundamental form of a surface. It satisfies the Codazzi equation

$$d^{\nabla^*} I\!I^* = 0,$$

where ∇^* is the Levi-Civita connection of I^*, and it satisfies a modified form of the Gauss equation,

$$\mathrm{tr}_{I^*}(I\!I^*) = -K^*,$$

where K^* is the curvature of I^*. The proof can again be found in [KS2, Section 5]. A direct consequence is that, if I^* has constant curvature -1, the traceless part $I\!I_0^*$ of $I\!I^*$ is the real part of a holomorphic quadratic differential on ∂M for the complex structure of I^*. In addition, the first-order variation of V_R follows from Lemma 3.4.

LEMMA 3.5
In a first-order deformation of M, we have

$$dV_R = -\frac{1}{4}\int_{\partial M} \langle dI^*, I\!I_0^* \rangle da^*.$$

This statement is very close in spirit to Lemma 2.2, with the dual volume of the convex core replaced by the renormalized volume. The right-hand term is, up to the factor $-1/4$, the Liouville form on the cotangent bundle $T^*\mathcal{T}_C(\partial M)$.

Proof A simple proof of Theorem 1.5
We have just seen that dV_R coincides (up to the constant $-1/4$) with the Liouville form of $T^*\mathcal{T}_C(\partial M)$ on $j(\mathcal{G})$. It follows that the symplectic form of $T^*\mathcal{T}_C(\partial M)$ vanishes on $j(\mathcal{G}(\partial M))$, which is precisely the statement of the theorem. □

4. The relative volume of hyperbolic ends

4.1. Definition

We consider in this part yet another notion of volume, defined for (geometrically finite) hyperbolic ends rather than for hyperbolic manifolds. Here we consider a hyperbolic end M. The definition of the renormalized volume can be used in this setting, leading to the relative volume of the end. We call a geodesically convex subset $K \subset M$ a *collar* if it is relatively compact and contains the metric boundary $\partial_0 M$ of M (possibly all geodesically convex relatively compact subsets of M are collars, but it is not necessary to consider this question here). Then $\partial K \cap M$ is a locally convex surface in M.

The relative volume of M is related both to the (dual) volume of the convex core and to the renormalized volume; it is defined as the renormalized volume, but starting from the metric boundary of the hyperbolic end. We follow the same path as for the renormalized volume, and we start from a collar $K \subset M$. We set

$$W(K) = V(K) - \frac{1}{4}\int_{\partial K} H da + \frac{1}{2}L_\mu(\lambda),$$

where H is the mean curvature of the boundary of K, μ is the induced metric on the metric boundary of M, and λ is its measured bending lamination.

As for the renormalized volume, we define the metric at infinity as

$$I^* := \lim_{\rho\to\infty} 2e^{-2\rho} I_\rho,$$

where I_ρ is the induced metric on the set S_ρ of points at distance ρ from K and the (implicit) identification between $S_{\rho'}$ and S_ρ, for $\rho' > \rho$, is by normal projection on S_ρ. The conformal structure of I^* is equal to the canonical conformal structure at infinity c_∞ of M.

Here again, W only depends on I^*. Not all metrics in c_∞ can be obtained from a compact subset of M; however, all metrics do define an equidistant foliation close to infinity in M, and it remains possible to define $W(I^*)$ even when I^* is not obtained from a convex subset of M. So W defines a function, still called W, from the conformal class c_∞ to $\mathbb{R}$.

LEMMA 4.1
For a fixed area of I^*, W is maximal exactly when I^* has constant curvature.

The proof follows directly from the arguments used in [KS2, Section 7], so we do not repeat the proof here. This proof takes place entirely on the boundary at infinity, so considering a hyperbolic end or a geometrically finite hyperbolic manifold has no impact.

Definition 4.2
The relative volume V_R of M is $W(I^*)$ when I^* is the hyperbolic metric in the conformal class at infinity on M.

4.2. The first variation of the relative volume

PROPOSITION 4.3
Under a first-order variation of the hyperbolic end, the first-order variation of the relative volume is given by

$$V_R' = \frac{1}{2}L_\mu'(\lambda) - \frac{1}{4}\int_{\partial_\infty M} \langle I^{*\prime}, I\!I_0^*\rangle da^*. \qquad (7)$$

The proof is based on the arguments described above, both for the first variation of the renormalized volume and for the first variation of the volume of the convex core. Some preliminary definitions are required.

Definition 4.4
A *polyhedral collar* in a hyperbolic end M is a collar $K \subset M$ such that $\partial K \cap M$ is a polyhedral surface.

LEMMA 4.5
Let K be a polyhedral collar in M, and let L_e, θ_e be the length and the exterior dihedral angle of edge e in $\partial K \cap M$. In any deformation of M, the first-order variation of the measured bending lamination on the metric boundary of M is given by a transverse Hölder distribution λ'. The first-order variation of the volume of K is given by

$$2V' = \sum_e L_e d\theta_e - L_\mu(\lambda').$$

Proof
This is very close in spirit to the main result of [B3], with the difference that here we consider a compact domain bounded on one side by a pleated surface, on the other by a polyhedral surface. The argument of [B3] can be followed line by line, keeping one surface polyhedral (of fixed combinatorics, say), while on the other boundary component the approximation arguments of [B3] can be used. □

COROLLARY 4.6
Let $V^*(K) := V(K) + (1/2)L_\mu(\lambda)$; then, in any deformation of K

$$2V^{*\prime} = \sum_e L_e d\theta_e + L'_\mu(\lambda).$$

Proof
We have seen in the proof of Lemma 2.1 that $L_\mu(\lambda)' = L'_\mu(\lambda) + L_\mu(\lambda')$. So the corollary follows from Lemma 4.5 exactly as Lemma 2.2 follows from Lemma 2.1. □

It is possible to define the renormalized volume of the complement of a polyhedral collar in a hyperbolic end, in the same way as for quasi-Fuchsian manifolds above. Let C be a closed polyhedral collar in the hyperbolic end M, and let D be its complement. Let K' be a compact geodesically convex subset of M containing C in its interior, and let $K := K' \cap D$. We define

$$W(K) = V(K) - \frac{1}{4}\int_{D\cap\partial K} H da.$$

In addition, K defines a metric at infinity, I^*, according to (6), and the arguments explained after Lemma 3.1 show that K is uniquely determined by I^*, so that W can be considered as a function of I^*, a metric in the conformal class at infinity of M. (In general, as explained in Section 3.1, I^* only defines an equidistant foliation near infinity which might not extend all the way to K.) The first-variation of W with respect to I^* shows (as in [KS2]) that $W(I^*)$ is maximal (under the constraint that I^* has fixed area) if and only if I^* has constant curvature. We then define the renormalized volume $V_R(D)$ as the value of this maximum.

LEMMA 4.7
Under a first-order deformation of D, the first-order variation of its renormalized volume is given by

$$V_R(D)' = -\frac{1}{4}\int_{\partial_\infty D} \left\langle I\!I_0^*, I^{*\prime}\right\rangle da^* + \frac{1}{2}\sum_e L_e\theta_e'.$$

Here L_e and θ_e are the length and exterior dihedral angle of edge e of the (polyhedral) boundary of D.

Proof
The proof can be obtained by following the argument used in [KS2]; the fact that D is not complete and has a polyhedral boundary just adds some terms relative to this polyhedral boundary in the variations formulae. □

Proof of Proposition 4.3
The statement follows directly from Corollary 4.6 applied to a polyhedral collar and from Lemma 4.7 applied to its complement, since the terms corresponding to the polyhedral boundary between the two cancel. □

4.3. Proof of Theorem 1.2
Since hyperbolic ends are in one-to-one correspondence with $\mathbb{C}\mathrm{P}^1$-structures, we can consider the relative volume V_R as a function on $\mathcal{CP}$. Let β_H (resp., β_C) be the Liouville form on $T^*\mathcal{T}_H$ (resp., $T^*\mathcal{T}_C$). We can consider the composition $\delta\circ\mathrm{Gr}^{-1}$: $\mathcal{CP}\to T^*\mathcal{T}_H$, it is C^1, and it pulls back β_H as

$$(\delta\circ\mathrm{Gr}^{-1})^*\beta_H = L'_\mu(\lambda).$$

Under the identification of $\mathscr{CP}$ with $T^*\mathscr{T}_C$ through the Schwarzian derivative, the expression of β_C is

$$\beta_C = \int_{\partial_\infty M} \langle I^{*\prime}, I\!I_0^* \rangle da^*.$$

So Proposition 4.3 can be formulated as

$$dV_R = \frac{1}{2}(\delta \circ \mathrm{Gr}^{-1})^* \beta_H - \frac{1}{4}\beta_C,$$

and it follows that $2(\delta \circ \mathrm{Gr}^{-1})^* \omega_H = \omega_C$. □

4.4. The Fuchsian slice versus Bers slices

Here we prove for the reader's convenience that the identification considered here between $\mathscr{CP}$ and $T^*\mathscr{T}_C$, based on the Fuchsian slice, determines the same symplectic structure on $\mathscr{CP}$ as the identification based on a Bers slice (as used, e.g., in [K]). This can be compared to [Mc, Theorem 9.2], where a related result is proved (by different arguments).

We consider a fixed conformal structure $c_- \in \mathscr{T}$. Then, for each $c \in \mathscr{T}$, we define $\sigma_{c_-}(c)$ as the complex projective structure on the upper boundary at infinity of the (unique) quasi-Fuchsian manifold for which the lower conformal metric at infinity is c_- and the upper conformal metric at infinity is c. The Schwarzian derivative of the identity map from (S, σ_{c_-}) to (S, σ) is a holomorphic quadratic differential on S, which can be considered as a point of the (complexified) cotangent space $T_c^*\mathscr{T}_C$. Taking its real part defines a map from $\mathscr{CP}$ to $T^*\mathscr{T}_C$, which we can use to pull back the cotangent symplectic map on $T^*\mathscr{T}_C$ to a symplectic form ω_{c_-} on $\mathscr{CP}$. Recall that the symplectic form ω_C considered in this article is obtained in the same manner, but using the Fuchsian complex projective structure σ_0 rather than the complex projective structure of the Bers slice σ_{c_-}.

LEMMA 4.8

The two symplectic forms are equal: $\omega_{c_-} = \omega_C$.

Proof

Consider $\sigma \in \mathscr{CP}$, let c be its underlying complex structure, let $\alpha_0(\sigma) = \mathscr{S}(Id : (S, \sigma_0(c)) \to (S, \sigma))$, and let $\alpha_{c_-}(\sigma) = \mathscr{S}\big(Id : (S, \sigma_{c_-}(c)) \to (S, \sigma)\big)$. Both $\alpha_0(\sigma)$ and $\alpha_{c_-}(\sigma)$ can be considered as vectors in the (complexified) cotangent space $T_c^*\mathscr{T}_C$. The properties of the Schwarzian derivative under composition show that $\alpha_0(\sigma) - \alpha_{c_-}(\sigma) = \mathscr{S}\big(\mathrm{Id} : (S, \sigma_0(c)) \to (S, \sigma_{c_-}(c))\big)$. So $\alpha_0(\sigma) - \alpha_{c_-}(\sigma)$ depends only on the underlying complex structure c of σ (and on c_-), and it defines a section of the complexified cotangent bundle $T^*\mathscr{T}_C$. By definition this is precisely the section called θ_{c_-} in [KS2, text following Theorem 8.8].

Still by construction, $\omega_{c_-} - \omega_C = \mathrm{Re}(d\alpha_{c_-} - d\alpha_0) = \mathrm{Re}(d\theta_{c_-})$. According to [KS2, Proposition 8.9], $d\theta_{c_-}$ does not depend on c_-. So $d\theta_{c_-}$ can be computed by choosing $c_- = c$ (fixed). An explicit computation is possible (see [KS2, Proposition 8.10]), which shows that, for any two tangent vectors $X, Y \in T_c\mathscr{T}_C$, we have

$$\big(D_X Re(\theta_{c_-})\big)(Y) = \langle X, Y\rangle_{WP},$$

and it follows that $d(\mathrm{Re}(\theta_{c_-})) = 0$. So $\omega_{c_-} = \omega_0$ as claimed. □

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Krasnov
School of Mathematical Sciences, University of Nottingham, Nottingham, NG7 2RD, UK; krasnov@maths.nottingham.ac.uk

Schlenker
Institut de Mathématiques de Toulouse, UMR CNRS 5219, Université de Toulouse III, 31062 Toulouse CEDEX 9, France; schlenker@math.ups-tlse.fr