

The Weil–Petersson metric and the renormalized volume of hyperbolic 3-manifolds

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1 Introduction

We survey the renormalized volume of hyperbolic 3-manifolds, as a tool for Teichmüller theory, using simple differential geometry arguments to recover results sometimes first achieved by other means. One such application is McMullen’s quasifuchsian (or more generally Kleinian) reciprocity, for which different arguments are proposed. Another is the fact that the renormalized volume of quasifuchsian (or more generally geometrically finite) hyperbolic 3-manifolds provides a Kähler potential for the Weil–Petersson metric on Teichmüller space. Yet another is the fact that the grafting map is symplectic, which is proved using a variant of the renormalized volume defined for hyperbolic ends.

1.1 Liouville theory

One of the early approaches to the problem of uniformization of Riemann surfaces was based on the so-called Liouville equation. Consider a closed Riemann surface S of genus g and fix an arbitrary reference metric h_0 in the conformal class of S . Then, consider a conformally equivalent metric $h = e^{2\phi}h_0$. The condition that h has constant curvature minus one reads

$$\Delta_0\phi - K_0 = e^{2\phi}, \quad (1.1)$$

where Δ_0 and K_0 are the Laplacian and Gauss curvature of h_0 respectively. One then tries to solve this equation for ϕ and thus find a hyperbolic metric on S .

Historically, this approach to the uniformization turned out to be too difficult, and was abandoned in favor of the one based on Fuchsian groups. More recently, the set of ideas related to the Liouville equation came to the spotlight due to the central role it plays in Polyakov's approach to string theory. In the so-called non-critical string theory a very important role is played by the *Liouville functional*, which (in one of its versions) can be written as

$$S[h_0, \phi] = \frac{1}{8\pi} \int d\text{vol}_0 (|\nabla\phi|^2 + e^{2\phi} - 2\phi K_0). \quad (1.2)$$

When varied with respect to ϕ this functional gives the Liouville equation (1.1). As is implied by the uniformization theorem, there is indeed a unique solution to (1.1) on any given Riemann surface S . Let us denote this solution by ϕ_{hyp} , where hyp stands for hyperbolic. One can evaluate the functional (1.2) on ϕ_{hyp} and obtain a functional $S[h_0] = S[h_0, \phi_{\text{hyp}}]$. When h_0 is taken to be the hyperbolic metric, $\phi_{\text{hyp}} = 0$ and the value of the above functional is just $1/4\pi$ times the area of S evaluated using the hyperbolic metric, i.e. minus half the Euler characteristics of X . The values of $S[h_0]$ at other points are not easy to find and one gets a rather non-trivial functional on the space of metrics h_0 on S , which, in a certain sense, measures how far the metric h_0 is from the hyperbolic one. This functional is not so interesting by itself, but serves as a prototype for the construction of a whole class of Liouville-type functionals that will play the central role in this chapter. The reason why $S[h_0]$ is not so interesting is that one would rather have a functional on the space of conformal equivalence classes of metrics on S , i.e. the *moduli space* of Riemann surfaces, or at least on the Teichmüller space \mathcal{T}_g , hoping that such a functional could be used to characterize the geometry of the moduli space in an interesting way. Such functionals are not easy to construct. Indeed, one could have obtained a functional on the moduli from $S[h_0]$ by taking h_0 to be some canonical metric in the given conformal class. However, as we have already seen, taking h_0 to be the canonical metric of constant negative curvature gives a functional on the moduli space whose value at all points is a constant – the Euler characteristic.

The reader can find another, discrete approach to Liouville theory in the survey by Kashaev in this volume [22].

1.2 Liouville theory and projective structures

It turned out to be possible to use Liouville theory for a construction of interesting functions on the moduli space by first considering a certain function on the space of complex projective structures on Riemann surfaces. Then choosing the projective structure to be some canonical one, e.g. one related to the Schottky or quasifuchsian uniformization, one can get a functional on the (appropriate) moduli space itself. Since projective structures on Riemann surfaces are intimately connected to hyperbolic

3-manifolds, such Liouville functionals of projective structure turn out to be related to the geometry of the corresponding 3-manifolds, and this is where the notion of the renormalized volume enters into the story.

However, the first examples of non-trivial Liouville functionals on the Teichmüller space of Riemann surfaces had nothing to do with 3-manifolds, and worked entirely in the 2-dimensional setting. Thus, the first construction for the moduli space of surfaces of genus g was proposed in [39]. The idea was to base the construction on the so-called Schottky uniformization of the surface S . The latter is unique given a *marking* of S , which is a choice of g generators of $\pi_1(S)$ corresponding to disjoint simple closed curves on S . The surface is then cut along the generators chosen, which results in a $2g$ -holed sphere. Thus, in the Schottky uniformization, the Riemann surface S is represented as a quotient of the complex plane under the action of the Schottky group Γ , which is a group freely generated by g (loxodromic, i.e. non-elliptic) elements of $\mathrm{PSL}(2, \mathbb{C})$. A fundamental region for the action of the Schottky group on \mathbb{C} is the exterior of a set of $2g$ Jordan curves that get mapped into each other pairwise by the generators of Γ . The flat metric $|dz|^2$ of \mathbb{C} can now be used as the reference structure for constructing a non-trivial Liouville functional, in place of h_0 above. This functional is given by essentially the same quantity as in (1.2), with the flat metric Laplacian and $K_0 = 0$. However, the Liouville field ϕ now has rather non-trivial transformation properties under the action of Γ (so that the metric $e^\phi|dz|^2$ can be pulled back to the quotient Riemann surface). As a result, the integral of the term $|\nabla\phi|^2$ depends on a choice of the fundamental region. Thus, in order to define the functional in an unambiguous way one has to correct the “bulk” term (1.2) with a rather non-trivial set of boundary terms, see [39] for details. After this is done, one gets a well-defined functional of the conformal structure of S , of the marking of S that was used to get the Schottky uniformization as well as the Liouville field ϕ . One of the main properties of this functional is that the variation with respect to ϕ gives rise precisely to the Liouville equation $\Delta\phi = e^\phi$, which can then be shown to have a unique solution with the required transformation properties. Evaluating the Liouville functional on this solution gives a non-trivial functional on the Schottky space, i.e. the moduli space of marked (by g curves) Riemann surfaces. One of the key properties of this functional is that its first variation (when the moduli are varied) gives rise to a certain holomorphic quadratic differential on the complex plane, which measures the difference between the Schottky and Fuchsian (used as reference) projective structures. From this one finds that the extremal Liouville functional is the Kähler potential for the so-called Weil–Petersson metric on the Schottky space, see [39] for details (and [46] for many key properties of the Weil–Petersson metrics).

Even though this point of view is not at all developed in [39], it is convenient to think of the functional constructed as a special case of a more general Liouville functional on a surface equipped with a projective structure, specialized to the case where the projective structure in question is the Schottky one. This point of view suggests that there are other Liouville functionals out there, namely when one chooses the projective structure to be different. Indeed, a Liouville functional of the same

type, but this time for the so-called quasifuchsian (or even more generally arbitrary Kleinian) projective structure was constructed in [41]. Since quasifuchsian projective structures are parametrized by the product of two copies of Teichmüller space \mathcal{T}_g , one gets a functional on $\mathcal{T}_g \times \mathcal{T}_g$. Similar to the story in the Schottky case, this functional turns out to be a Kähler potential for the Weil–Petersson metric on \mathcal{T}_g , as its $(1, 1)$ derivative with respect to moduli of the first copy of \mathcal{T}_g turns out to be independent of the point in the second copy, see below for more details.

In [40], a Liouville functional of the same type was constructed for the moduli space $\mathcal{M}_{(0,n)}^{(\alpha_1, \dots, \alpha_n)}$ of Riemann surfaces with n conical singularities of given angle deficits $2\pi\alpha_i$, $i = 1, \dots, n$. Moreover, a set of metrics on the space $\mathcal{M}_{(0,n)}^{(\alpha_1, \dots, \alpha_n)}$ was introduced, which for all $\alpha_i = 1$ (angle deficits 2π) coincides with the usual Weil–Petersson metric on $\mathcal{M}_{(0,n)}$. It was moreover shown that all these metrics are Kähler with the Kähler potential given by the Liouville functional. A choice of the projective structure is implicit in this case, as the natural projective structure on the complex plane with n marked points is used.

Below we shall see how these Liouville functionals are intimately related to the geometry of the hyperbolic 3-manifolds which the corresponding projective structures define. We note however that, although hyperbolic surfaces with cone singularities fit nicely in the 3D geometric context considered here (see Section 8), the question of a geometrical (3-dimensional) interpretation of our last example – the functional on $\mathcal{M}_{(0,n)}^{(\alpha_1, \dots, \alpha_n)}$ constructed in [40] – remains open.

1.3 The AdS-CFT correspondence and holography

An initially unrelated development occurred in the context of the so-called AdS/CFT correspondence of string theory, see [44], in which asymptotically-hyperbolic manifolds play the key role. These are non-compact, infinite volume (typically Einstein, or Einstein with non-trivial “stress-energy tensor” on the right hand side of Einstein equations) Riemannian manifolds that have a conformal boundary, near which the manifold looks like the hyperbolic space of the corresponding dimension. Most interesting for physics is the case of 5-dimensional asymptotically-hyperbolic manifolds, for in this case the 5-dimensional gravity theory induces (or, as physicists say, is dual to) a certain non-trivial gauge theory on the boundary, see [44]. However, the simplest situation is that in three dimensions, where the Einstein condition implies constant curvature and one is led to consider simply a hyperbolic manifold M .

Given such an asymptotically-hyperbolic manifold, physicists are interested in computing the Einstein–Hilbert functional (given by an integral of the scalar curvature plus a multiple of the volume form) on the metric of M . For the hyperbolic metric, the integrand reduces to a multiple of the volume form, so one is computing the volume of the hyperbolic manifold M , which is infinite. However, physicists are masters of extracting a finite answer from a divergent expression. And so it was found that in many situations there is a “canonical” way to extract a finite answer by

“regularizing” the divergent volume, then subtracting the divergent contribution, and finally taking a limit to “remove the regulator”. This procedure, somewhat mysterious for a mathematician, was applied in the context of Schottky 3-dimensional hyperbolic manifolds (quotients of the hyperbolic space by the action of a Schottky group) in [24], with a somewhat unexpected result. Namely, it was shown that the “renormalized volume” of a Schottky manifold, as a function of the conformal factor for a metric in the conformal class of the hyperbolic boundary, is equal to the Schottky Liouville functional as defined in [39]. When one takes the conformal factor corresponding to the canonical metric of curvature -1 one gets the Liouville functional on the Schottky moduli space defined and studied in [39]. In view of the results of this reference, one finds that the renormalized volume, which is a purely 3-dimensional quantity in its definition, equals the Kähler potential for the Weil–Petersson metric on the Schottky space, which is a purely 2-dimensional quantity. One thus gets an example of what physicists like to refer to as a “holographic” correspondence (a relation between one quantity (or even theory) in $n + 1$ dimensions and another in n dimensions).

The methods of [24], originally applicable only in the context of (classical) Schottky manifolds, were generalized and applied to arbitrary Schottky, quasifuchsian and even Kleinian manifolds in [41], with the end result being always the same: the renormalized volume turns out to be equal to the (appropriately defined) Liouville functional, and the latter is shown to be the Kähler potential on the corresponding moduli space. Even prior to the work [41], the set of ideas building upon the Kähler property of the Weil–Petersson metric on Teichmüller space (and closely related to the renormalized volume ideas, as was later shown in [41]) was used in [30] for a proof of the Kähler hyperbolicity property of the moduli space.

The above story makes it clear that there is a deep relation between the geometry of the Teichmüller space of a Riemann surface S and the geometry of hyperbolic three-manifolds that realize S as its conformal boundary. This relation was recently demonstrated from a more geometrical perspective in [27], where it was shown that the key property of the Kähler potential, namely, that its first variation is given by a certain canonical quadratic differential on S , is a simple consequence of the Schläfli-like formula of [35]. This geometrical perspective on the renormalized volume (and the Liouville functional) also made it clear that such a quantity may be defined not only for the Schottky, quasifuchsian and Kleinian projective structures considered in the literature, but, in fact, for an arbitrary projective structure. This idea leads to the notion of relative volume, defined and studied in [28]. It is on this more geometrical and, we believe, very simple, perspective on the renormalized volume that we would like to emphasize in this review.

1.4 The definitions of the renormalized volume

Let M be a convex co-compact hyperbolic manifold, for instance a quasifuchsian manifold. The hyperbolic volume of M is infinite, but an interesting finite quantity can be extracted by a procedure that physicists refer to as *renormalization*. This

proceeds by introducing a “regulator” that makes the quantity of interest finite, then removing the divergent contribution and then removing the regulator. For the case at hand an appropriate regulator is given by equidistant surfaces. Thus, the first step in the definition of the renormalized volume of M is to define a volume depending on an equidistant foliation F of M in the complement of a compact convex subset N . By an equidistant foliation we mean a foliation of $M \setminus N$ by closed, smooth, convex surfaces, so that, in each connected component of $M \setminus N$, the surfaces are pairwise at constant distance. Since the foliation is equidistant, it is uniquely determined by N .

Section 2 gives two different but equivalent definitions of a volume associated to N in M . The first definition is in terms of the asymptotic behavior of the volume of the set of points at distance at most ρ from N as $\rho \rightarrow \infty$. This definition can be surprising but it extends to higher dimensions in the setting of conformally compact, Poincaré–Einstein metrics, see [16] for an original reference and [4] for a review.

The other definition is simpler, but limited to 3 dimensions. It is in terms of the volume of N , corrected by a term involving the integral of the mean curvature of the boundary of N , as

$$W(M, N) = V(N) - \frac{1}{4} \int_{\partial N} H da.$$

In 3 dimensions a convex co-compact hyperbolic manifold is completely specified by the conformal structure of its boundary ∂M . It then turns out that the dependence of $W(N, M)$ on N is just the dependence on a metric in the conformal class of the boundary. Moreover, when this metric is varied the functional $W(N, M)$ reaches an extremum on metrics of constant (negative) curvature. Thus, $W(N, M)$ is nothing but the Liouville functional whose 2-dimensional realizations have been discussed above. Note that the hyperbolic 3-manifold M in which the volume is computed comes equipped with a projective structure on ∂M , and explains why the Liouville action in all cases needed a projective structure to be defined.

1.5 The first and second fundamental forms at infinity

There is a natural description of the connected components of the complement of N in terms of the induced metric and second fundamental forms I, II of the corresponding boundary component of N . Section 3 contains an alternative description, in terms of naturally defined “induced metric” and “second fundamental form” I^*, II^* at infinity in the same end. There are simple transformation formulas from I, II to I^*, II^* and conversely. The conformal class of I^* is the conformal class at infinity of M . Moreover, I^* and II^* satisfy the Codazzi equation and an analog of the Gauss equation, which involves the trace of II^* instead of the determinant of II as in the usual Gauss equation for surfaces in H^3 .

A subtle point, explained at more length in Section 3, is that, as we have already mentioned, not only N determines the metric at infinity I^* , but I^* also determines uniquely N . In general a metric I^* in the conformal class at infinity does not come

from a choice of some convex subset N of M . However, for any such metric I^* , there is associated an equidistant foliation of a neighborhood of infinity in M . This is sufficient to define the renormalized volume W , and this makes W a function of M and I^* , rather than of M and N as defined above.

The modified Gauss and Codazzi equations for I^* , II^* are described in Section 3. They have an interesting consequence. When K^* is constant, the trace H^* of II^* is also constant, so that the traceless part II_0^* of II^* also satisfies the Codazzi equation $d^{\nabla^*} II_0^* = 0$, where ∇^* is the Levi-Civita connection of I^* . It then follows (by an argument going back to Hopf [21]) that II_0^* is the real part of a holomorphic quadratic differential. In other words, II_0^* is canonically identified with a cotangent vector in $T_{[I^*]}^* \mathcal{T}_{\partial M}$, where $[I^*]$ is the conformal class of I^* , a property that is going to play an important role below.

1.6 Variation formulas

The function $W(M, N)$ has a simple first-order variation formula in terms of the data on the boundary of N , recalled in Section 4. For any first-order deformation of M or of N in M ,

$$\delta W = \frac{1}{4} \int_{\partial N} \delta H + \langle \delta I, II_0 \rangle da,$$

where $H = \text{tr}_I II$ is the mean curvature of the boundary and II_0 is the traceless part of the second fundamental form.

A lengthy but direct computation then shows that the same formula (except for the sign) holds (when this function is interpreted as $W(M, I^*)$) in terms of the data at infinity:

$$\delta W = -\frac{1}{4} \int_{\partial N} \delta H^* + \langle \delta I^*, II_0^* \rangle da, \quad (1.3)$$

where $H^* = \text{tr}_{I^*} II^*$ and $II_0^* = II^* - (H^*/2)I^*$. A version of this formula that is valid in any dimension has been first obtained in [11] by a direct computation of the variation of the regularized volume, see also [4] for a more mathematical exposition. A geometrical viewpoint adopted in this review (that originates in [27]) interprets this variational formula as a version of the Schläfli formula.

1.7 Maximization

A first consequence of Equation (1.3), together with a simple integration by parts argument which is explained in Subsection 4.3, is that the only critical point of W over the space of metrics of fixed area in a conformal class on ∂M is attained for the unique metric of constant curvature. Conversely, metrics of constant curvature are critical points of the restriction of W to metrics of fixed area in a conformal class, and those critical points happen to be always non-degenerate minima.

This leads to the definition of the renormalized volume of M (no choice of I^* involved) as the maximum of $W(M, I^*)$, obtained precisely when the metric at infinity, I^* , has constant curvature $K^* = -1$. We call $W(M)$ this function. By the Ahlfors–Bers theorem [1], M is uniquely determined by its conformal class at infinity, so W can also be considered as a function from the Teichmüller space $\mathcal{T}_{\partial M}$ of the boundary to \mathbb{R} .

The second step is to vary the conformal class on ∂M while considering only constant curvature metrics. Then it follows directly from (1.3) that

$$dW(I^{*'}) = \frac{-1}{4} \int_{\partial M} \langle I^{*'}, II_0^* \rangle da^*,$$

which means that dW is identified (up to the factor $-1/4$) with II_0^* , considered as a 1-form on $\mathcal{T}_{\partial M}$.

1.8 The renormalized volume and the Schwarzian derivative

There is another way to give a geometric meaning to II_0^* . Given a convex co-compact hyperbolic metric on M , it defines on the boundary at infinity $\partial_\infty M$ a complex projective structure σ , see [12]. Let c be the underlying complex structure, so that c is the complex structure at infinity of M . There is another special complex projective structure on ∂M , obtained by applying the Riemann uniformization theorem to $(\partial M, c)$, we call it σ_0 . The image by the developing map of σ_0 of each connected component of ∂M is a disk, so σ_0 is called the Fuchsian complex projective structure associated to c .

Let ϕ be the map between $(\partial M, \sigma_0)$ and $(\partial M, \sigma)$. By construction ϕ is holomorphic, so that we can consider its *Schwarzian derivative* $\mathcal{S}(\phi)$, which is a holomorphic quadratic differential on $(\partial M, c)$. This holomorphic quadratic differential can also be considered as the difference between the projective structures σ and σ_0 . We have:

Proposition 1.1. $II_0^* = -\operatorname{Re}(\mathcal{S}(\phi))$.

1.9 The renormalized volume and Kleinian reciprocity

Suppose now that M is a quasifuchsian manifold, that is, it is convex co-compact and homeomorphic to the product of a closed oriented surface S of genus at least 2 with an interval. Let \mathcal{T} be the Teichmüller space of S , and let $\tilde{\mathcal{T}}$ be the Teichmüller space of S with the opposite orientation. Given two complex structures $c_+ \in \mathcal{T}$, $c_- \in \tilde{\mathcal{T}}$, there is by Bers' Double Uniformization Theorem a unique hyperbolic metric on M such that the complex structure at infinity on the upper boundary $\partial_+ M$ of M is c_+ , while the complex structure at infinity on the lower boundary $\partial_- M$ of M is c_- .

Given a quasifuchsian manifold M , we can also consider the corresponding complex projective structure σ_+ on $\partial_+ M$, and the Fuchsian complex projective structure $\sigma_{0,+}$ on $\partial_+ M$ obtained by applying the Riemann uniformization theorem to the

complex structure c_+ . This defines, through the Schwarzian derivative construction recalled above, a holomorphic quadratic differential q_+ on $(\partial_+ M, c_+)$, and the real part of q_+ defines a cotangent vector $\beta_+(c_-, c_+) \in T_{c_+}^* \mathcal{T}$. The same construction for $\partial_- M$ yields a cotangent vector $\beta_-(c_-, c_+) \in T_{c_-}^* \bar{\mathcal{T}}$.

McMullen's quasifuchsian reciprocity [30] gives a subtle relation between the ways the complex projective structures on the two boundary components vary when the complex structure at infinity changes. In the following statement we consider a fixed conformal structure c_+ and vary c_- (resp. fix c_- and vary c_+), β_+ is then a map from $\bar{\mathcal{T}}$ to $T_{c_+}^* \mathcal{T}$ (resp. β_- is a map from \mathcal{T} to $T_{c_-}^* \bar{\mathcal{T}}$). We call $D\beta_+$ (resp. $D\beta_-$) the differential of this map.

Theorem 1.2 (McMullen [30]). *The tangent maps*

$$D\beta_+(\cdot, c_+): T_{c_-} \bar{\mathcal{T}} \rightarrow T_{c_+}^* \mathcal{T}$$

and

$$D\beta_-(c_-, \cdot): T_{c_+} \mathcal{T} \rightarrow T_{c_-}^* \bar{\mathcal{T}}$$

are adjoint.

This can be stated in different terms using the standard cotangent bundle symplectic structure on $T^* \mathcal{T}_{\partial M}$, which we will call ω_* here. Given $(c_-, c_+) \in \bar{\mathcal{T}} \times \mathcal{T}$, we call $\beta(c_-, c_+) = (\beta_-(c_-, c_+), \beta_+(c_-, c_+)) \in T_{c_-}^* \bar{\mathcal{T}} \times T_{c_+}^* \mathcal{T}$, so that $\beta(c_-, c_+) \in T_{(c_-, c_+)}^* \mathcal{T}_{\partial M}$. Thus β defines a section of $T^* \mathcal{T}_{\partial M}$. Theorem 1.2 is a direct consequence of the following simpler statement.

Theorem 1.3. *The image $\beta(\mathcal{T}_{\partial M})$ is Lagrangian in $(T^* \mathcal{T}_{\partial M}, \omega_*)$.*

In this form, the statement extends as it is to a much more general setting of convex co-compact (or geometrically finite) hyperbolic 3-manifolds.

The proof of Theorem 1.2 from Theorem 1.3 is straightforward, as is the proof of Theorem 1.3 from the first-order variation of the renormalized volume as described above. Both are done in Subsection 5.2. It is proved there that Theorem 1.3 is actually equivalent to Theorem 1.2 along with the following proposition, which uses the same notation as Theorem 1.2 and is another direct consequence of Theorem 1.3.

Corollary 1.4. *For fixed c_- , $\beta_+(c_-, \cdot)$ is a closed 1-form on \mathcal{T} .*

Part 5.3 describes a second proof of Theorem 1.3, based on the geometry of the convex core and on the fact that the grafting map is symplectic (Theorem 1.5 below). The idea here is to prove first that the data on the boundary of the convex core defines, as the representation varies, a Lagrangian submanifold of the cotangent bundle of Teichmüller space, understood here in terms of hyperbolic metrics and measured laminations. But the natural map sending this data on the boundary of the convex core to the corresponding data at infinity is symplectic, and Theorem 1.3 follows.

In Part 5.4 we describe a third argument, due to Kerckhoff, which works in the setting of deformations of the holonomy representation of the fundamental group of M . This uses topological arguments, more precisely a long exact sequence and Poincaré duality between cohomology spaces with value in an $sl(2, \mathbb{R})$ -bundle over M . The equivalence with Theorem 1.3 is made through a result of Kawai [23] connecting the Goldman symplectic form on the space of complex projective structures on a surface to the cotangent symplectic structure on the cotangent bundle of Teichmüller space.

1.10 The renormalized volume as a Kähler potential

A rather direct consequence of McMullen’s quasifuchsian reciprocity, explained in Section 6, is that although $\beta_+(c_-, c_+)$ clearly depends on c_- , its exterior (anti-holomorphic, because $\beta_+(c_-, c_+)$ is a holomorphic one-form on \mathcal{T}) differential does not depend on c_- . So this exterior differential can be computed explicitly in the simplest possible case – when $c_- = c_+$, that is, in the neighborhood of a “Fuchsian” hyperbolic manifold. This leads to a key result on the renormalized volume, namely that it is a Kähler potential for the Weil–Petersson metric on Teichmüller space.

1.11 The relative volume of hyperbolic ends and the grafting map

So far we have considered the renormalized volume in the context of hyperbolic convex co-compact 3-manifolds. Such manifolds come equipped with a complex projective structure on each boundary component, and the renormalized volume we have discussed can be said to be the Liouville functional for the corresponding projective structure, as discussed in the beginning of this section. Section 7 is centered on a notion of renormalized volume, or Liouville functional, that is defined for an arbitrary projective structure on a Riemann surface S . This has been developed in [28]. The main idea is to use a variant of the renormalized volume, called the *relative volume* of a hyperbolic end, and then use an analog of Theorem 1.3 to obtain results on the grafting map.

Thus, consider a closed surface S of genus at least 2. We denote by \mathcal{ML}_S the space of measured geodesic laminations on S , see e.g. [10], and by \mathcal{CP}_S the space of complex projective structures (or $\mathbb{C}P^1$ -structures) on S , see e.g. [12].

The grafting map $\text{Gr}: \mathcal{T}_S \times \mathcal{ML}_S \rightarrow \mathcal{CP}_S$ was defined by Thurston, see e.g. [13], [12]. It can be briefly described as follows. Let $m \in \mathcal{T}_S$ be a hyperbolic metric, and let $l \in \mathcal{ML}_S$ be a measured lamination with support a disjoint union of closed curves c_1, \dots, c_n , each with a positive weight w_1, \dots, w_n . The “grafted metric” on S is obtained by cutting open (S, m) along each of the c_i and gluing in a flat strip of width equal to w_i . Then $\text{Gr}(m, l)$ is the complex projective structure naturally associated to this metric. The map Gr extends by continuity to a map defined on all measured laminations (not only those supported by multicurves).

Thurston showed that the grafting map is a homeomorphism. His proof, although quite subtle, relies on a simple geometric idea, which is simpler to explain for a complex projective structure $\sigma \in \mathcal{CP}_S$ whose developing map $\text{dev}: \tilde{S} \rightarrow \mathbb{C}P^1$ is injective. In this case, the boundary of the convex hull in H^3 of $\mathbb{C}P^1 \setminus \text{dev}(\tilde{S})$ – here $\mathbb{C}P^1$ is identified with the boundary at infinity of H^3 – is a convex pleated surface, on which $\pi_1(S)$ acts properly discontinuously. The quotient surface is endowed with a hyperbolic metric (its induced metric) and a measured lamination (its pleating lamination). This gives an element $(m, l) \in \mathcal{T}_S \times \mathcal{ML}_S$, which is the inverse image of σ under the grafting map.

In this picture – which can be extended to the case where the developing map of σ is not injective – $\pi_1(S)$ acts properly discontinuously on the connected component bounded at infinity by $\text{dev}(\tilde{S})$ of the complement in H^3 of the convex hull of $\mathbb{C}P^1 \setminus \text{dev}(\tilde{S})$, and the quotient is a *hyperbolic end*. In Section 7 we explain how to define the *relative volume* of such a hyperbolic end, and show that it satisfies a simple first-order variation formula, involving both a term “at infinity” similar to the one which we already mentioned for the renormalized volume, and a term on the “compact” boundary, involving the hyperbolic metric and the measured pleating lamination, which is very close to a Schläfli-type formula for the convex core of a quasifuchsian manifold, discovered by Bonahon [9].

There is a natural identification between $\mathcal{T}_S \times \mathcal{ML}_S$ and the cotangent space $T^*\mathcal{T}_S$, obtained by considering the differential of the length of a measured lamination as a cotangent vector. Using this map, the grafting map can be considered as a map $\overline{\text{Gr}}: T^*\mathcal{T}_S \rightarrow \mathcal{CP}_S$, and both sides are naturally symplectic manifolds (\mathcal{CP}_S is actually a complex symplectic manifold, but we consider here only the real part of its complex symplectic structure). It is proved in [28] – using mostly tools from Bonahon’s work [8], [9] – that this map is C^1 smooth (it is however not C^2).

Theorem 1.5. $(1/2)\overline{\text{Gr}}$ is symplectic.

The proof is a direct consequence of the first-order variation formula for the relative volume of hyperbolic ends, similar to the proof of Theorem 1.2 that follows from the first-order variations of the renormalized volume.

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2 Two definitions of the renormalized volume

2.1 The renormalized volume

As mentioned in the introduction, the renormalized volume was motivated by physical considerations. This notion was first envisaged and used in the more general context of conformally compact (in particular Einstein) manifolds in an arbitrary number of dimensions, and only later specialized to the 3-dimensional setting considered here. In the more general case its definition uses a foliation of a particular type close to infinity (associated to a “defining function” of the boundary). The “canonical” renormalized volume independent of which foliation is used is then either the constant term (in even dimensions) or the logarithmic term (in odd dimensions) in the asymptotic expansion of the volume in terms of the parameter of the foliation. In this review, however, we are interested in the constant term in this asymptotic expansion for the odd-dimensional (3d) manifolds. This quantity depends on the choice of a metric in the conformal class at infinity. The higher-dimensional applications can be found e.g. in [16], [19], [4], see also [18], [11], while this review concentrates on a simple(r), but still extremely rich case of 3-dimensional manifolds.

2.2 The renormalized volume via equidistant foliations

The limiting procedure via which the volume is defined can be somewhat de-mystified by considering for regularization a family of surfaces equidistant to a given one, following an idea already used by C. Epstein [33] (and more recently put to use in [26]). Thus, the main idea is to obtain the renormalized volume by taking a convex domain $N \subset M$, and compute the renormalized volume of M *with respect to* N as

$$V_R(M, N) = V(N) + \lim_{\rho \rightarrow \infty} \left(V(\partial N, \partial N_\rho) - (1/2)A(\partial N_\rho) - \sum_i 2\pi\rho(g_i - 1) \right) \quad (2.1)$$

where $V(\partial N, \partial N_\rho)$ is the volume between the boundary ∂N of the domain N and the surface ∂N_ρ located a distance ρ from ∂N . The quantity $A(\partial N_\rho)$ is the area of the surface ∂N_ρ , the sum in the last term is taken over all boundary components of M and g_i are the genera of these boundary components. The convexity of the domain N ensures that the equidistant surfaces ∂N_ρ exist all the way to infinity. This ensures that the limit $\rho \rightarrow \infty$ can be taken. This limit exists and can be computed in terms of the volume of N , see below for the corresponding expressions. The limiting procedure used in [24], [41] is an example of the limiting procedure described above, for the Epstein surfaces [14] used in these references are equidistant.

2.3 Renormalized volume as the W -volume

The following formula for the renormalized volume (2.1) can be shown by an explicit (simple) computation:

$$V_R(M, N) = W(N) - \sum_i \pi(g_i - 1),$$

where the sum in the last term is taken over all the boundary components. Here the W -volume is defined as

$$W(N) := V(N) - \frac{1}{4} \int_{\partial N} H da. \quad (2.2)$$

This formula for V_R is a special case of a formula found by C. Epstein [33] for the renormalized volume of hyperbolic manifolds in any dimension.

Thus, the renormalized volume of M with respect to N is, apart from an uninteresting term given by a multiple of the Euler characteristic of the boundary, just the W -volume of the domain N . It then makes sense to study this geometrical W -volume instead. Note already that $W(N)$ is not equal to the Einstein–Hilbert functional of N with its usual boundary term, it differs from it in the coefficient of the boundary term.

2.4 Self-duality

One of the interesting properties of the W -volume is that it is self-dual. Thus, we recall that the Einstein–Hilbert functional

$$I_{\text{EH}}(N) := V(N) - \frac{1}{2} \int_{\partial N} H da \quad (2.3)$$

for a compact domain $N \subset H^3$ of hyperbolic space (note a different numerical factor in front of the second term) is nothing but the dual volume. Indeed, we recall that there is a duality between objects in H^3 and objects in dS_3 , the $(2 + 1)$ -dimensional de Sitter space. Under this duality geodesic planes in H^3 are dual to points in dS_3 , etc. This duality between domains in the two spaces is easiest to visualize for convex polyhedra (see [34]), but the duality works for general domains as well. The fact that (2.3) is the volume of the dual domain is a simple consequence of the Schläfli formula, see below.

Thus, we can write

$${}^*V(N) = V({}^*N) = V(N) - \frac{1}{2} \int_{\partial N} H da$$

for the volume of the dual domain. This immediately shows that

$$W(N) = V(N) - \frac{1}{4} \int_{\partial N} H da = \frac{V(N) + {}^*V(N)}{2}.$$

Thus, the W -volume is self-dual in that this quantity for N is equal to this quantity for the dual domain $*N$: $W(N) = W(*N)$.

2.5 The W -volume and the Chern–Simons formulation

An interesting remark is that there is a very simple expression for the W -volume in terms of the so-called Chern–Simons formulation of 2+1 gravity [45]. Let us briefly review this formulation. In the so-called first order formalism for gravity the independent variables are not components of the spacetime metric but instead the triads and the spin connection. Thus, for Riemannian signature gravity in 3 spacetime dimensions let us introduce a collection of 3 one-forms θ^i , $i = 1, 2, 3$, such that the spacetime interval can be written as $ds^2 = \sum_i \theta^i \otimes \theta^i$. We can now construct from θ^i an $\mathfrak{su}(2)$ Lie algebra-valued one-form by taking $\theta = \sum_i i\theta^i \sigma^i$, where the σ^i are the 2×2 anti-symmetric Pauli matrices $\sigma^i \sigma^j = \delta^{ij} \text{Id} + i\epsilon^{ijk} \sigma^k$. Using the Lie algebra-valued form θ one can write the metric as $ds^2 = -(1/2)\text{Tr}(\theta \otimes \theta)$.

The Einstein–Hilbert action as a functional of the metric g is given by

$$I_{\text{EH}}[g] = -\frac{1}{4} \int_M dv (R + 2) - \frac{1}{2} \int_{\partial M} da H, \quad (2.4)$$

where R is the Ricci scalar of g , H is the mean curvature of the boundary, and dv, da are the volume and area forms on M and ∂M respectively. When evaluated on a constant curvature metric with $R = -6$ the Einstein–Hilbert action reduces to (2.3). The functional (2.4) can be re-written in a very simple form in terms of θ by introducing a spin connection ω , which is locally an $\mathfrak{su}(2)$ -valued one-form. The action is then

$$I_{\text{EH}}[\theta, \omega] = \frac{1}{2} \int_M \text{Tr} \left(\theta \wedge f(\omega) - \frac{1}{12} \theta \wedge \theta \wedge \theta \right) + \frac{1}{2} \int_{\partial M} \text{Tr}(\theta \wedge \omega). \quad (2.5)$$

Here $f(\omega) = d\omega + \omega \wedge \omega$ is the curvature of the spin connection ω . When one varies this action with respect to ω one obtains the equation $d_\omega \theta = 0$, where d_ω is the covariant derivative with respect to the connection ω . This equation can be solved for ω in terms of the derivatives of θ . Once one substitutes the solution back into the action one gets (2.4).

In contrast, the linear combination of the two terms on the right-hand side of (2.2) that plays the role of the renormalized volume is obtained by evaluating on the constant curvature metric the following action:

$$I_W[g] = -\frac{1}{4} \int_M dv (R + 2) - \frac{1}{4} \int_{\partial M} da H. \quad (2.6)$$

This can be written in terms of the tetrad and the spin connection forms as follows:

$$I_W[\theta, \omega] = \frac{1}{2} \int_M \text{Tr} \left(\theta \wedge f(\omega) - \frac{1}{12} \theta \wedge \theta \wedge \theta \right) + \frac{1}{4} \int_{\partial M} \text{Tr}(\theta \wedge \omega). \quad (2.7)$$

One then notes that this is precisely the combination that appears in the Chern–Simons formulation. Indeed, let us introduce the Chern–Simons action of an $\mathfrak{su}_{\mathbb{C}}(2)$ connection A via

$$I_{\text{CS}}[A] = \frac{1}{4i} \int_M \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (2.8)$$

Now, defining the two $\mathfrak{su}_{\mathbb{C}}(2)$ connections

$$A = \omega + \frac{i}{2}\theta, \quad \bar{A} = \omega - \frac{i}{2}\theta \quad (2.9)$$

it is not hard to see that (2.7) is given by

$$I_W = I_{\text{CS}}[A] - I_{\text{CS}}[\bar{A}], \quad (2.10)$$

with precisely the right boundary term that comes from having to integrate by parts. In contrast, to obtain via the Chern–Simons formulation the combination (2.3) one has to add a separate boundary term that is constructed from both A, \bar{A} . For more details on the argument presented the reader is referred to [25], see formula (3.7) of this reference as well as the related discussion. It would be of interest to understand the relation, if any, between the self-duality of the W -volume and the fact that it has such a simple expression in the Chern–Simons formulation.

3 Description “from infinity”

3.1 The metric at infinity

In this section we switch from a description of the renormalized volume from the boundary of a convex subset to the boundary at infinity of M . This description from infinity is remarkably similar to the previous one from the boundary of a convex subset.

Lemma 3.1. *Let M be a convex co-compact hyperbolic 3-manifold, and let $N \subset M$ be compact and “strongly” convex with smooth boundary. Let S_ρ be the equidistant surfaces from ∂N . The induced metric on S_ρ is asymptotic, as $\rho \rightarrow \infty$, to $(1/2)e^{2\rho} I^*$, where $I^* = (1/2)(I + 2II + III)$ is defined on ∂N .*

Proof. Follows from the following lemma. □

Lemma 3.2. *Let S be a surface in H^3 , with bounded principal curvatures, and let I, B be the first fundamental form and the shape operator of S correspondingly. Let S_ρ be the surface at distance ρ from S . Then, for sufficiently small ρ the induced metric on S_ρ is*

$$I_\rho(x, y) = I \left((\cosh(\rho)E + \sinh(\rho)B) x, (\cosh(\rho)E + \sinh(\rho)B) y \right). \quad (3.1)$$

Here E is the identity operator.

Note that this lemma also holds for a surface S in any hyperbolic 3-manifold M , not necessarily H^3 . We also note that when the surface S is convex, then the expression (3.1) gives the induced metric on any surface $\rho > 0$, where ρ increases in the convex direction. A proof of this lemma can be found in, e.g., [26].

It is the metric I^* that will play such a central role in what follows, so we would like to state some of its properties.

Lemma 3.3. *The curvature of I^* is*

$$K^* := \frac{2K}{1 + H + K_e}. \quad (3.2)$$

Proof. The Levi-Civita connection of I^* is given, in terms of the Levi-Civita connection ∇ of I , by

$$\nabla_x^* y = (E + B)^{-1} \nabla_x ((E + B)y).$$

This follows from checking the three points in the definition of the Levi-Civita connection of a metric:

- ∇^* is a connection,
- ∇^* is compatible with I^* ,
- it is torsion-free (this follows from the fact that $E + B$ verifies the Codazzi equation: $(\nabla_x(E + B))y = (\nabla_y(E + B))x$).

Let (e_1, e_2) be an orthonormal moving frame on S for I , and let β be its connection 1-form, i.e.,

$$\nabla_x e_1 = \beta(x)e_2, \quad \nabla_x e_2 = -\beta(x)e_1.$$

Then the curvature of I is defined as $d\beta = -Kda$.

Now let $(e_1^*, e_2^*) := \sqrt{2}((E + B)^{-1}e_1, (E + B)^{-1}e_2)$; clearly it is an orthonormal moving frame for I^* . Moreover the above expression of ∇^* shows that its connection 1-form is also β . It follows that $Kda = -d\beta = K^*da^*$, so that

$$K^* = K \frac{da}{da^*} = \frac{K}{(1/2) \det(E + B)} = \frac{2K}{1 + H + K_e}. \quad \square$$

We note that the metric I^* is defined for any surface $S \subset M$. However, it might have singularities (even when the surface S is smooth) unless S is strictly *horospherically convex*, i.e., its principal curvatures are less than 1 (which implies that it remains on the concave side of the tangent horosphere at each point). If S is a strictly horospherically convex surface S embedded in a hyperbolic end of M then the metric I^* is guaranteed to be in the conformal class of the (conformal) boundary at infinity of M . For a general surface S the ‘‘asymptotic’’ metric is not directly related to the conformal infinity, and in particular, it does not have to be in the conformal class of the boundary.

3.2 Second fundamental form at infinity

We have already defined the metric “at infinity”. Let us now add to this a definition of what can be called the second fundamental form at infinity.

Definition 3.1. Given a surface S with first, second and third fundamental forms I , II and III , we define the first and second fundamental forms “at infinity” as follows:

$$\begin{aligned} I^* &= \frac{1}{2}(I + 2II + III) = \frac{1}{2}(I + II)I^{-1}(I + II) \\ &= \frac{1}{2}I((E + B)\cdot, (E + B)\cdot), \\ II^* &= \frac{1}{2}(I - III) = \frac{1}{2}(I + II)I^{-1}(I - II) = \frac{1}{2}I((E + B)\cdot, (E - B)\cdot). \end{aligned} \quad (3.3)$$

It is then natural to define

$$B^* := (I^*)^{-1}II^* = (E + B)^{-1}(E - B), \quad (3.4)$$

and

$$III^* := I^*(B^*\cdot, B^*\cdot) = I((E - B)\cdot, (E - B)\cdot).$$

An interesting point is that I^* , II^* and III^* determine the full asymptotic development of the metric close to infinity: the induced metrics I_ρ on the surfaces S_ρ are given by Equation (3.6) below. This extends Lemma 3.1, and can be considered as an analog of Equation (3.1).

Note that, for a surface which has principal curvatures strictly bounded between -1 and 1 , III^* is also a smooth metric and its conformal class corresponds to that on the other component of the boundary at infinity. This is a simple consequence of Lemma 3.2 and the fact that when the principal curvatures are strictly bounded between $-1, 1$ the foliation by surfaces equidistant to S extends all the way through the manifold M . Such manifolds were called *almost Fuchsian* in our work [26].

As before, these definitions make sense for any surface, but it is only for a convex surface (or more generally for a horospherically convex surface) that the fundamental forms so introduced are guaranteed to have something to do with the actual conformal infinity of the space.

3.3 The Gauss and Codazzi equations at infinity

We also define $H^* := \text{tr}(B^*)$. The Gauss equation for “usual” surfaces in H^3 is replaced by a slightly twisted version.

Lemma 3.4. $H^* = -K^*$: *the mean curvature at infinity is equal to minus the curvature of I^* .*

Proof. By definition, $H^* = \text{tr}((E + B)^{-1}(E - B))$. An elementary computation (for instance based on the eigenvalues of B) shows that

$$H^* = \frac{2 - 2 \det(B)}{1 + \text{tr}(B) + \det(B)}.$$

But we have seen (as Equation (3.2)) that $K^* = 2K/(1 + H + K_e)$. The result follows because, by the Gauss equation, $K = -1 + \det(B)$. \square

However, the ‘‘usual’’ Codazzi equation holds at infinity.

Lemma 3.5. $d^{\nabla^*} B^* = 0$.

Proof. Let u, v be vector fields on $\partial_\infty M$. Then it follows from the expression of ∇^* found above that

$$\begin{aligned} (d^{\nabla^*} B^*)(x, y) &= \nabla_x^*(B^*y) - \nabla_y^*(B^*x) - B^*[x, y] \\ &= (E + B)^{-1} \nabla_x((E + B)B^*y) \\ &\quad - (E + B)^{-1} \nabla_y((E + B)B^*x) - B^*[x, y] \\ &= (E + B)^{-1} \nabla_x((E - B)y) \\ &\quad - (E + B)^{-1} \nabla_y((E - B)x) - (E + B)^{-1}(E - B)[x, y] \\ &= (E + B)^{-1}(d^\nabla(E - B))(x, y) \\ &= 0. \end{aligned} \quad \square$$

3.4 Inverse transformations

The transformation $I, II \rightarrow I^*, II^*$ is invertible. The inverse is given explicitly, and the inversion formula exhibits a remarkable symmetry.

Lemma 3.6. *Given I^*, II^* , the fundamental forms I, II such that (3.3) holds are obtained as*

$$\begin{aligned} I &= \frac{1}{2}(I^* + II^*)(I^*)^{-1}(I^* + II^*) = \frac{1}{2}I^*((E + B^*)\cdot, (E + B^*)\cdot), \\ II &= \frac{1}{2}(I^* + II^*)(I^*)^{-1}(I^* - II^*) = \frac{1}{2}I^*((E + B^*)\cdot, (E - B^*)\cdot). \end{aligned} \quad (3.5)$$

Moreover,

$$B = (E + B^*)^{-1}(E - B^*).$$

Having an expression for the fundamental forms of a surface in terms of the ones at infinity, one can re-write the metric of Lemma 3.2 induced on surfaces equidistant to S in terms of I^*, II^* .

Lemma 3.7. *The metric (3.1) induced on the surfaces equidistant to S can be re-written in terms of the fundamental forms “at infinity” as*

$$I_\rho = \frac{1}{2}e^{2\rho}I^* + II^* + \frac{1}{2}e^{-2\rho}III^*. \quad (3.6)$$

This lemma shows the significance of II^* as being the constant term of the metric. This lemma also shows clearly that when the equidistant foliation extends all the way through M (i.e. when the principal curvatures on S are in $(-1, 1)$), the conformal structure at the second boundary component of M is that of $III^* = II^*(I^*)^{-1}II^*$. Thus, in this particular case of almost Fuchsian manifolds, the knowledge of I^* on both boundary components of M is equivalent to the knowledge of I^* , II^* near either component. In other words, II^* is determined by I^* . This statement is, of course, more general and works for manifolds other than almost Fuchsian ones.

3.5 Fundamental Theorem of surface theory “from infinity”

Let us now recall that the Fundamental Theorem of surface theory states that given I, II on S satisfying the Gauss and Codazzi equations, there is a unique embedding of S into the hyperbolic space with induced metric and second fundamental form equal to I and to II . Then (3.1) gives an expression for the metric on equidistant surfaces to S , and thus describes a hyperbolic manifold M in which S is embedded, in some neighborhood of S . It would be possible to state a similar result for hyperbolic ends, uniquely determined by I^* and II^* at infinity. But there is also an analogous theorem, based on a classical result of Bers [5], in which the first (and only the first) form at infinity is used. This can be compared with arguments used in [36], where a corresponding second fundamental form and the Gauss and Codazzi equations “at infinity” were introduced.

Theorem 3.8. *Given a convex co-compact 3-manifold M , and a metric I^* (on all the boundary components of M) in the conformal class of the boundary, there is a unique foliation of each end of M by convex equidistant surfaces $S_\rho \subset M$ such that $(1/2)(I_\rho + 2II_\rho + III_\rho) = e^{2\rho}I^*$, where $I_\rho, II_\rho, III_\rho$ are the fundamental forms of S_ρ .*

Remark 3.2. Note that one does not need to specify II^* . The first fundamental form I^* (but on all the boundary components) is sufficient.

Proof. The surfaces in question can be given explicitly as an embedding of the universal cover \tilde{S} of S into the hyperbolic space. Thus, let $(\xi, y), \xi > 0, y \in \mathbb{C}$ be the usual upper half-space model coordinates of H^3 . Let us write the metric at infinity as

$$I^* = e^\phi |dz|^2, \quad (3.7)$$

where ϕ is the Liouville field covariant under the action of the Kleinian group giving M on S^2 . The surfaces are given by the following set of maps: $\text{Eps}_\rho: S^2 \rightarrow H^3, z \mapsto (\xi, y)$ (here Eps stands for Epstein, who described these surfaces in [14]):

$$\begin{aligned}\xi &= \frac{\sqrt{2}e^{-\rho}e^{-\phi/2}}{1 + (1/2)e^{-2\rho}e^{-\phi}|\phi_z|^2}, \\ y &= z + \phi_{\bar{z}} \frac{e^{-2\rho}e^{-\phi}}{1 + (1/2)e^{-2\rho}e^{-\phi}|\phi_z|^2}.\end{aligned}\tag{3.8}$$

As is shown by an explicit computation, the metric induced on the surfaces S_ρ is given by (3.6) with

$$II^* = \frac{1}{2}(\theta dz^2 + \bar{\theta} d\bar{z}^2) + \phi_{z\bar{z}} dz d\bar{z},\tag{3.9}$$

$$\theta = \phi_{zz} - \frac{1}{2}(\phi_z)^2.\tag{3.10}$$

Thus, we see that II^* is determined by the conformal factor in (3.7). \square

Remark 3.3. This theorem implies that the renormalized volume only depends on I^* . Indeed, the foliation (S_ρ) of the ends does depend only on I^* , and this foliation can be used for regularization and subtraction procedure. Then the fact that the W -volume is essentially the renormalized volume implies that the W -volume is a functional of I^* only. In the next section we will find a formula for the first variation of this functional.

Corollary 3.9. *If the principal curvatures at infinity (eigenvalues of B^*) are positive the map Eps_ρ is a homeomorphism onto its image for any ρ .*

Proof. We first note that the map Eps_ρ is not always a homeomorphism, and the surfaces S_ρ are not necessarily convex, but for sufficiently large ρ both things are true. A condition that guarantees that Eps_ρ is a homeomorphism for any ρ is stated above. This condition can be obtained from the requirement that the principal curvatures of surfaces S_ρ are in $[-1, 1]$. Let us consider the surface $S := S_{\rho=0}$ the first and second fundamental forms of which are given by (3.5) (this immediately follows from (3.6)). The shape operator of this surface is then given by $B = (E + B^*)^{-1}(E - B^*)$. It is then clear that the principal curvatures of S are given by $k_i = (1 - k_i^*)/(1 + k_i^*)$, where the k_i^* are the ‘‘principal curvatures’’ (eigenvalues) of B^* . The latter are easily shown to be given by

$$k_{1,2}^* = e^{-\phi}(\phi_{z\bar{z}} \pm \sqrt{\theta\bar{\theta}}).\tag{3.11}$$

It is now easy to see that the condition $k_{1,2} \in (-1, 1)$ is equivalent to the condition $k_{1,2}^* > 0$. This is a necessary and sufficient condition for the foliation by surfaces S_ρ to extend throughout M . If this condition is satisfied the map Eps_ρ is a homeomorphism for any ρ . \square

Interestingly, this condition makes sense not only in the quasifuchsian situation but is more general. Thus, for example, it applies to Schottky manifolds. But for Schottky manifolds, with their single boundary component, the foliation by equidistant surfaces S_ρ cannot be smooth for arbitrary ρ . It is clear that surfaces must develop singularities for some value of ρ . We therefore get an interesting corollary:

Corollary 3.10. *There is no Liouville field ϕ on \mathbb{C} invariant under a Schottky group such that $\phi_{z\bar{z}}$ is greater than $|\phi_{zz} - (1/2)\phi_z^2|$ everywhere on \mathbb{C} .*

Proof. Indeed, if such a Liouville field existed, we could have used it to construct a smooth equidistant foliation for arbitrary values of ρ , but this is impossible. \square

A similar statement holds for a Kleinian group with more than two components of the domain of discontinuity.

4 The Schläfli formula “from infinity”

In this section we obtain a formula for the variation of the renormalized volume.

4.1 The Schläfli formula for hyperbolic polyhedra

Recall first the classical Schläfli formula (see e.g. [31]), which is a good motivation for what follows. Consider a hyperbolic polyhedron P . Under a first-order deformation of P , the first-order variation of the volume of P is given by

$$dV = \frac{1}{2} \sum_e L_e d\theta_e. \quad (4.1)$$

Here the sum is over the edges of P , L_e is the length of the edge e , and θ_e is its exterior dihedral angle.

There is also an interesting “dual” Schläfli formula. Let

$$V^* = V - \frac{1}{2} \sum_e L_e \theta_e$$

be the *dual volume* of P ; then, still under a first-order deformation of P ,

$$dV^* = -\frac{1}{2} \sum_e \theta_e dL_e. \quad (4.2)$$

This follows from the Schläfli formula (4.1) by an elementary computation.

4.2 The Schläfli formula for hyperbolic manifolds with boundary

As we have seen in the previous sections, the renormalized volume of a convex co-compact hyperbolic 3-manifold M can be expressed as the W -volume of any convex domain $N \subset M$. The W -volume is equal to the volume of N minus the quarter of the integral of the mean curvature over the boundary of N . Let us consider what happens if one changes the metric in M . As was shown in [35], the following formula for the variation of the volume holds

$$2\delta V(N) = \int_{\partial N} \left(\delta H + \frac{1}{2} \langle \delta I, II \rangle \right) da. \quad (4.3)$$

Here H is the trace of the shape operator $B = I^{-1}II$, and the expression $\langle A, B \rangle$ stands for $\text{tr}(I^{-1}AI^{-1}B)$. We can use this to get the following expression for the variation of the W -volume:

$$\delta W(N) = \frac{1}{2} \int_{\partial N} \left(\delta H + \frac{1}{2} \langle \delta I, II \rangle \right) da - \frac{1}{4} \int_{\partial N} \delta H da - \frac{1}{4} \int_{\partial N} H \delta(da),$$

so that

$$\delta W(N) = \frac{1}{4} \int_{\partial N} \left(\delta H + \left\langle \delta I, II - \frac{H}{2}I \right\rangle \right) da. \quad (4.4)$$

To get the last equality we have used the obvious equality

$$\delta da = \frac{1}{2} \text{tr}(I^{-1}\delta I) = \frac{1}{2} \langle \delta I, I \rangle da. \quad (4.5)$$

The formula (4.4) can be further modified using

$$\begin{aligned} \delta H &= \delta(\text{tr}(I^{-1}II)) = -\text{tr}(I^{-1}(\delta I)I^{-1}II) + \text{tr}(I^{-1}\delta II) \\ &= -\langle \delta I, II \rangle + \langle I, \delta II \rangle. \end{aligned} \quad (4.6)$$

We get

$$\delta W(N) = \frac{1}{4} \int_{\partial N} \left\langle \delta II - \frac{H}{2}\delta I, I \right\rangle da. \quad (4.7)$$

It is this formula that will be our starting point for transformations to express the variation in terms of the data at infinity.

4.3 Parametrization by the data at infinity

Let us now recall that given the data I, II on the boundary of N one can introduce the first and second fundamental forms “at infinity” via (3.3). Conversely, knowing the fundamental forms I^*, II^* “at infinity” one can recover the fundamental forms on ∂N via (3.5). Our aim is to rewrite the variation (4.7) of the W -volume in terms of the variations of the forms I^*, II^* .

Lemma 4.1. *The first-order variation of W can be expressed as*

$$\delta W(N) = -\frac{1}{4} \int_{\partial N} \left\langle \delta II^* - \frac{H^*}{2} \delta I^*, I^* \right\rangle da^*. \quad (4.8)$$

A proof can be found in [27]; it follows from a direct computation based on the formulas expressing I , II in terms of I^* , II^* .

This formula could be compared to a more general formula applicable in any dimension given in [11] and reviewed in [4]. A similar formula for the case of 4-dimensional manifolds is given by Anderson in [3], and more recently by Albin [2] in higher dimensions. Our derivation here is different from that in the cited references, for we interpret the variational formula as a version of the Schläfli formula. We also note that the situation is simpler for even-dimensional manifolds, since the renormalized volume is then canonically defined, while in odd dimension it depends on the choice of a metric in the conformal class at infinity. In odd dimension there is another, canonically defined “renormalized volume”, namely the logarithmic term in the asymptotic expansion of the volume as a function of the parameter of an equidistant foliation. This quantity is different from the one used here (defined in (2.1)) which is the constant term in the same asymptotic expansion.

Formula (4.8) looks very much like the original formula (4.7), except for the minus sign and the fact that the quantities at infinity are used. The fact that we have got the same variational formula as in terms of the data on ∂N is not too surprising. Indeed, the variational formula (4.8) was obtained from (4.7) by applying the transformation (3.5). As it is clear from (3.3), this transformation applied twice gives the identity map. In view of this, it is hard to think of any other possibility for the variational formula in terms of δI^* , δII^* except being given by the same expression (4.7), maybe with a different sign. This is exactly what we see in (4.8).

There is another expression of the first-order variation of W , dual to (4.4), which will be useful below.

Corollary 4.2. *The first-order variation of W can also be expressed as*

$$\delta W = -\frac{1}{4} \int_{\partial N} \delta H^* + \langle \delta I^*, II_0^* \rangle da^*,$$

where II_0^* is the traceless part (for I^*) of II^* .

4.4 Conformal variations of the metric at infinity

We can now use Corollary 4.2 to show that, when varying the W -volume with the area of the boundary defined by the I^* metric kept fixed, the variational principle forces the metric I^* to have constant negative curvature. The variations we consider here do not change the conformal structure of the metric I^* , and thus do not change the manifold M . Geometrically they correspond to small movements of the surface ∂N inside the fixed manifold M . Thus, let us consider a conformal deformation of the

metric I^* of the form $\delta I^* = 2uI^*$, where u is some function on ∂N . Clearly for such variations $\langle \delta I^*, II_0^* \rangle = 0$, precisely because II_0^* is traceless.

Let us consider the following functional

$$F(N) = W(N) - \frac{\lambda}{4} \int_{\partial N} da^* \quad (4.9)$$

appropriate for finding an extremum of the W -volume with the area computed using the metric I^* kept fixed. The first variation of this functional gives, using Corollary 4.2:

$$\delta F = -\frac{1}{4} \int_{\partial N} (\delta H^*) da^* - \frac{\lambda}{4} \int_{\partial N} 4u da^* = \frac{1}{4} \int_{\partial N} (\delta K^*) da^* - \frac{\lambda}{4} \int_{\partial N} 4u da^*.$$

But

$$\delta \int_{\partial N} K^* da^* = \int_{\partial N} (\delta K^*) + 4uK^* da^* = 0$$

by the Gauss-Bonnet formula, so that

$$\delta F = \int_{\partial N} (-uK^* - u\lambda) da^*.$$

It follows that critical points of F are characterized by the fact that $K^* = -\lambda$. It is not hard to compute the second variation and show that the critical points of F are local maxima, see [27] for details.

Remark 4.1. As we have already discussed, the renormalized volume $W(N)$ is actually a functional of metrics I^* on all the boundary components of M . We have just established that this functional has an extremum, for variations that keep the total area of the boundary components fixed, at the constant curvature metric I^* . However, this is precisely the defining property of the Liouville functional we have discussed in the Introduction. This establishes the renormalized volume – Liouville action functional relation. Moreover, one can turn the argument around and use the renormalized volume $W(N)$ (as a functional of the metrics I^* on all the boundary components) as a *definition* of the Liouville functional. This point of view explains why there is not one, but a whole set of Liouville functionals – depending on which hyperbolic three-manifold is used – and it also explains why it is so hard to define the Liouville functional in intrinsically 2-dimensional terms – because it is in fact a 3-dimensional quantity.

4.5 The renormalized volume as a function on Teichmüller space

Let us now consider the renormalized volume as a function over the Teichmüller space of ∂N ; in other terms, for each conformal class on ∂N , we consider the extremum of W over metrics of given area within this conformal class. We have just seen that this extremum is obtained at the (unique) constant curvature metric. The main goal here is to recover by simple differential geometric methods important results of Takhtajan and

Zograf [39], Takhtajan and Teo [41] – showing that the renormalized volume provides a Kähler potential for the Weil–Petersson metric. So the “volume” that we consider here is now defined as follows.

Definition 4.2. Let g be a convex co-compact hyperbolic metric on M , and let $c \in \mathcal{T}_{\partial M}$ be the conformal structure induced on $\partial_\infty M$. We call $W_M(c)$ the *value of W* on the equidistant foliation of M near infinity for which I^* has constant curvature -1 .

In other terms, by the results obtained above, $W_M(c)$ is the maximum of W over the metrics at infinity which have the same area as a hyperbolic metric, for each boundary component of M . Throughout this section the metric at infinity I^* that we consider is the hyperbolic metric, while the second fundamental form at infinity, II^* , is uniquely determined by the choice that I^* is hyperbolic. Its traceless part is denoted by II_0^* .

4.6 The second fundamental form at infinity as the real part of a holomorphic quadratic differential

It is interesting to remark that, in the context considered here – when I^* has constant curvature – the second fundamental form at infinity has a complex interpretation. This can be compared with the same phenomenon, discovered by Hopf [21], for the second fundamental form of constant mean curvature surfaces in 3-dimensional constant curvature spaces.

Lemma 4.3. *When K^* is constant, II_0^* is the real part of a holomorphic quadratic differential (for the complex structure associated to I^*) on $\partial_\infty M$. This holomorphic quadratic differential is given explicitly by (3.10).*

Proof. By definition II_0^* is traceless, which means that it is at each point the real part of a quadratic differential: $II_0^* = \operatorname{Re}(h)$. Moreover, we have seen in Remark 3.5 that B^* satisfies the Codazzi equation, $d^{\nabla^*} B^* = 0$. It follows as for constant mean curvature surfaces (see e.g. [26]) that h is holomorphic relative to the complex structure of I^* . \square

4.7 The second fundamental form as a Schwarzian derivative

The next step is to show that, for each boundary component $\partial_i M$ of M , II_{0i}^* is just the real part of the Schwarzian derivative of a natural equivariant map from the hyperbolic plane (with its canonical complex projective structure) to $\partial_i M$ with its complex projective structure induced by the hyperbolic metric on M . In the terminology used by McMullen [30], II_{0i}^* is the difference between the complex projective structure at infinity on $\partial_i M$ and the Fuchsian projective structure on $\partial_i M$.

To state this result, let us call σ_F the “Fuchsian” complex projective structure on $\partial_i M$, obtained by applying the Poincaré uniformization theorem to the conformal

metric at infinity on $\partial_i M$. The universal cover of $\partial_i M$, with the complex projective structure lifted from σ_F , is projectively equivalent to a disk in $\mathbb{C}P^1$. We also call σ_{QF} the projective structure induced on $\partial_i M$ by the hyperbolic metric on M . Here “ QF ” stands for quasifuchsian (while M is only supposed to be convex co-compact). This notation is used to keep close to the notation in [30]. The map $\phi: (\partial_i M, \sigma_F) \rightarrow (\partial_i M, \sigma_{QF})$ is conformal but not projective between $(\partial_i M, \sigma_F)$ and $(\partial_i M, \sigma_{QF})$, so we can consider its Schwarzian derivative $\mathcal{S}(\phi)$.

Lemma 4.4. $II_0^* = -\operatorname{Re}(\mathcal{S}(\phi))$.

A simple way to prove this assertion is to use the formula (3.10) for the holomorphic quadratic differential θ whose real part gives the traceless part of II^* . The Liouville field ϕ that enters into this formula can be simply expressed in terms of the conformal map from $\partial_i M$ to the hyperbolic plane. It is then a standard and simple computation to verify that θ is equal to the Schwarzian derivative of this map, see e.g. [39].

It is possible to reformulate the statement (4.4) slightly, setting θ_i to be the restriction to $\partial_i M$ of $\mathcal{S}(\phi)$.

This notation is analogous to the notation used in [30], where the index i is useful to recall that this quantity is related to $\partial_i M$.

Then θ_i is a quadratic holomorphic differential (QHD) on $\partial_i M$, and, still using the notation in [30], the definition of θ_i can be rephrased as $\theta_i = \sigma_{QF} - \sigma_F$. The lemma can then be written as $II_{0i}^* = \operatorname{Re}(\theta_i)$. A geometric proof of this lemma is given in the appendix of [27].

Remark 4.3. Note that θ_i can also be considered as a complex-valued 1-form on the Teichmüller space of $\partial_i M$. Indeed, it is well known that the cotangent vectors to \mathcal{T}_S , where S is a Riemann surface, can be described as holomorphic quadratic differentials q on S . The pairing with a tangent vector (Beltrami differential μ) is given by the integral of $q\mu$ over S . The complex structure on \mathcal{T}_S can then be described as follows: the image of the cotangent vector q under the action of the complex structure J is simply $J(q) = iq$. Another, more geometric way to state the action of J is to note that it exchanges the horizontal and vertical trajectories of q . Thus, holomorphic quadratic differentials q on S are actually holomorphic 1-forms on \mathcal{T}_S .

4.8 The second fundamental form as the differential of W_M

There is another simple interpretation of the traceless part of the second fundamental form at infinity.

Lemma 4.5. *The differential dW_M of the renormalized volume W_M , as a 1-form over the Teichmüller space of ∂M , is equal to $(-1/4)II_0^*$.*

Proof. This is another direct consequence of Corollary 4.2 because, as one varies I^* among hyperbolic metrics, H^* (which is equal to K^*) remains equal to -1 , so that $\delta H^* = 0$. \square

Corollary 4.6. $\theta_i = -4\partial W_M$.

Proof. This follows directly from the lemma, since we already know that θ_i is a holomorphic differential. \square

Remark 4.4. We would like to emphasize how much simpler is the proof given above than that given in [39], [41]. Unlike in these references, which obtain the above result on the gradient of W_M via an involved computation using a reasonably complicated cohomology machinery, Corollary 4.2 implies this result in one line. This demonstrates the strength of the geometric method used here. Our proof can be immediately extended even to situations where the methods of [41] are inapplicable, such as manifolds with cone singularities. See more remarks on this case below.

Lemma 4.5 and, in particular, its corollary above, is the key fact needed to demonstrate that the renormalized volume plays the role of the Kähler potential on Teichmüller space. The remainder of the proof is in part 6.3 below, after some considerations on quasifuchsian reciprocity, which are partly based on the results of this section and are needed in the proof.

5 Kleinian reciprocity

5.1 Statement

Kleinian reciprocity, as defined by McMullen [30], is the extension of Theorem 1.3 to the more general setting of a geometrically finite hyperbolic 3-manifold M . Let $\mathcal{GF}(M)$ be the space of complete geometrically finite hyperbolic metrics on M . Each metric $g \in \mathcal{GF}$ induces a complex projective structure $\sigma(g)$ on ∂M .

Theorem 5.1. $\sigma(\mathcal{GF})$ is Lagrangian in $\mathcal{CP}_{\partial M}$.

As explained in the introduction and proved in Subsection 5.2.1 below, Theorem 1.2 is a direct corollary of this statement. The proof of Theorem 1.2 given in [30] can be described as analytic, as it takes place in the universal cover of M and uses the symmetry of a certain kernel related to the Beltrami problem. By contrast, the arguments considered here are mostly geometric and take place in M .

We will describe here three (other) proofs of Theorem 5.1, corresponding to different ways to consider the space of complex projective structures \mathcal{CP} .

- When \mathcal{CP} is considered in complex terms, and identified with $T^*\mathcal{T}_C$, the cotangent bundle of the space of complex structures on ∂M , Theorem 5.1 follows from the first variation of the renormalized volume, as explained in the introduction. This is the argument described in the introduction.
- When \mathcal{CP} is considered in hyperbolic terms, and identified with $T^*\mathcal{T}_H$, Theorem 5.1 follows from the dual Bonahon–Schläfli formula for the first variation of the dual volume of the convex core. The equivalence with the previous viewpoint is clear through Theorem 1.5.
- When \mathcal{CP} is considered as (a connected component of) a space of equivalence classes of representations of $\pi_1(\partial M)$ into $\mathrm{PSL}(2, \mathbb{C})$, Theorem 5.1 can be proved by a completely different argument, based on exact sequences and Poincaré duality, which was discovered (previously) by S. Kerckhoff. The equivalence with the complex or the hyperbolic viewpoint follows from the fact that the Goldman form on the space of representations of $\pi_1(\partial M)$ into $\mathrm{PSL}(2, \mathbb{C})$ is equal (up to multiplication by a constant) to the symplectic form obtained from the cotangent symplectic form on $T^*\mathcal{T}$, as proved by Kawai [23].

We briefly describe these three arguments in the next subsections. We consider here for simplicity the case of quasifuchsian manifolds; however all arguments can be extended without difficulty to the more general context of geometrically finite hyperbolic 3-manifolds.

5.2 The first variation of the renormalized volume

We give here the proofs announced in the introduction of quasifuchsian reciprocity from the first-order variation formula for the renormalized volume.

5.2.1 From Theorem 1.3 to Theorem 1.2. Suppose that Theorem 1.3 holds, and consider a pair $(c'_-, 0), (0, c'_+)$ of tangent vectors in $T_{c_-}\bar{\mathcal{T}} \times T_{c_+}\mathcal{T}$. In addition to the notation from Section 1.8, we introduce the notation $B: \mathcal{T}_{\partial M} \rightarrow T^*\mathcal{T}_{\partial M}$ for the map to the total space of the bundle $T^*\mathcal{T}_{\partial M}$, corresponding to the section β of $T^*\mathcal{T}_{\partial M}$. We use the notation D for the differential as in the introduction, and call ∇ the Levi-Civita connection of the Weil–Petersson metric on the cotangent bundle of both \mathcal{T} and $\bar{\mathcal{T}}$ (we could use another connection). Thus ∇ determines a connection on $\mathcal{T}_{\partial M} = \mathcal{T} \times \bar{\mathcal{T}}$, and this connection defines an identification $T(T^*\mathcal{T}_{\partial M}) \simeq T\mathcal{T}_{\partial M} \times T^*\mathcal{T}_{\partial M}$. With this notation we have

$$\begin{aligned}
& \langle D\beta_+(c'_-, 0), c'_+ \rangle - \langle D\beta_-(0, c'_+), c'_- \rangle \\
&= \langle (\nabla_{c'_-}\beta_-, D\beta_+(c'_-, 0)), (0, c'_+) \rangle \\
&\quad - \langle (D\beta_-(0, c'_+), \nabla_{c'_+}\beta_+), (c'_-, 0) \rangle \\
&= \omega_*(B_*(c'_-, 0), B_*(0, c'_+)) = 0,
\end{aligned}$$

where the last equality follows from Theorem 1.3. This proves Theorem 1.2.

Corollary 1.4 is also a direct consequence of Theorem 1.3. Indeed, for fixed c_- , $\beta_+(c_-, \mathcal{T})$ is a Lagrangian submanifold in $T^*\mathcal{T}$ by Theorem 1.3. It is also the graph of a 1-form by definition, and it is well known (and can be checked by a direct computation) that a 1-form is closed if and only if its graph is Lagrangian.

Conversely, Theorem 1.3 follows quite directly from Theorem 1.2 together with Corollary 1.4. To see this, let $c_- \in \widetilde{\mathcal{T}}$, $c_+ \in \mathcal{T}$, $v_- \in T_{c_-}\widetilde{\mathcal{T}}$, $v_+, v'_+ \in T_{c_+}\mathcal{T}$. Then the computation above, done backwards, shows that

$$\omega_*(B_*(v_-, 0), B_*(0, v_+)) = 0.$$

In addition,

$$\begin{aligned} \omega_*(B_*(0, v_+), B_*(0, v'_+)) &= \langle (D_{v_+}\beta_-, \nabla_{v_+}\beta_+), (0, v'_+) \rangle \\ &\quad - \langle (D_{v'_+}\beta_-, \nabla_{v'_+}\beta_+), (0, v_+) \rangle \\ &= \langle \nabla_{v_+}\beta_+, v'_+ \rangle - \langle \nabla_{v'_+}\beta_+, v_+ \rangle \\ &= d\beta_+(v_+, v'_+) \\ &= 0, \end{aligned}$$

where the last equality comes from Corollary 1.4. Theorem 1.3 follows by linearity.

5.2.2 Proof of Theorem 1.3. Theorem 1.3 follows very directly from the basic properties of the renormalized volume W_M as they are described above. Indeed we have seen that

$$\beta = -H_0^* = \frac{1}{4}dW_M.$$

So β is closed. This argument extends as it is to the more general setting of Theorem 5.1.

5.3 The boundary of the convex core and the grafting map

The second argument leading to quasifuchsian reciprocity is also based on hyperbolic geometry, and more specifically on the geometry of the convex core of quasifuchsian 3-manifolds. It rests on an extension of the classical Schläfli identity for convex cores of quasifuchsian manifolds, found by Bonahon [9], which leads to an analog of Theorem 5.1 where the renormalized volume is replaced by the volume of the convex core, and the cotangent bundle of Teichmüller space is considered in “hyperbolic” terms.

5.3.1 The convex core of quasifuchsian manifolds. Let M be a quasifuchsian hyperbolic 3-manifold. M contains a smallest non-empty geodesically convex subset, its *convex core* $C(M)$, which is compact and homeomorphic to M . The boundary of $C(M)$ is therefore the disjoint union of two copies of a surface S of genus at least 2, which we call the “upper” and “lower” boundary components of $C(M)$. Since

$C(M)$ is a minimal convex subset, it has no extreme points, so $\partial C(M)$ is locally convex and ruled (there is a geodesic segment of the ambient manifold, contained in $C(M)$, containing each point). It follows that the induced metric m on $\partial C(M)$ is hyperbolic (it has constant curvature -1), but $\partial C(M)$ is pleated along a measured geodesic lamination l . More details can be found in [42].

5.3.2 Measured laminations as cotangent vectors. When thinking of Teichmüller space in terms of hyperbolic metrics on surfaces, it is natural to associate its cotangent bundle with measured laminations, rather than with holomorphic quadratic differentials. The identification goes as follows. Let $l \in \mathcal{ML}$ be a measured lamination, and let $m \in \mathcal{T}$ be a hyperbolic metric, both on S . It is then possible (see [10]) to define the length of l for m , $L_m(l)$. This defines a smooth function

$$L_m(l): \mathcal{T} \rightarrow \mathbb{R}_{\geq 0}.$$

The differential $dL_m(l)$ at m is an element of $T_m^*\mathcal{T}$, and the map $\mathcal{ML} \rightarrow T_m^*\mathcal{T}$ is a homeomorphism (see e.g. [28]).

This construction defines an identification δ between $\mathcal{T} \times \mathcal{ML}$ and $T^*\mathcal{T}$. But $T^*\mathcal{T}$ has a cotangent symplectic structure, which we call ω_H here. It can be pulled back to $\mathcal{T} \times \mathcal{ML}$, where we still call it ω_H . It can be defined quite explicitly in terms of the intersection form on \mathcal{ML} , see [38].

5.3.3 A Lagrangian submanifold. Given a quasifuchsian metric $g \in \mathcal{QF}$, we can consider the induced metrics on the upper and lower boundary components of the convex core, $m_+, m_- \in \mathcal{T}$, and the corresponding measured bending laminations, $l_+, l_- \in \mathcal{ML}$. So we have two points $(m_+, l_+), (m_-, l_-) \in \mathcal{T} \times \mathcal{ML}$. This defines a map $H: \mathcal{QF} \rightarrow T^*\mathcal{T}_{\partial M}$.

Theorem 5.2. $H(\mathcal{QF})$ is a Lagrangian submanifold of $(T^*\mathcal{T}_{\partial M}, \omega_H)$.

The main ideas of the proof are explained in the next subsection. Theorem 1.3 directly follows from this result and from Theorem 1.5, according to which the grafting map is symplectic (up to a constant).

5.3.4 The Bonahon–Schläfli formula for the volume of the convex core. The convex core of a quasifuchsian manifold is, in some ways, reminiscent of a convex polyhedron. The main differences are: it has no vertices and edges are replaced by a measured lamination. This gives, in a sense, a much richer structure.

Bonahon [8] has extended the Schläfli formula recalled in Subsection 4.1 to this setting as follows. Let M be a convex co-compact hyperbolic manifold (for instance, a quasifuchsian manifold), let μ be the induced metric on the boundary of the convex core, and let λ be its measured bending lamination. By a “first-order variation” of M we mean a first-order variation of the representation of the fundamental group of M . Bonahon shows that the first-order variation of λ under a first-order variation of M is

described by a transverse Hölder distribution λ' , and there is a well-defined notion of length of such transverse Hölder distributions. This leads to a version of the Schläfli formula.

Lemma 5.3 (The Bonahon–Schläfli formula [8]). *The first-order variation of the volume V_C of the convex core of M , under a first-order variation of M , is given by*

$$dV_C = \frac{1}{2}L_\mu(\lambda').$$

Here λ' is the first-order variation of the measured bending lamination, which is a Hölder cocycle so that its length for μ can be defined, see [6], [7], [8], [9].

5.3.5 The dual volume. Just as for polyhedra above, we define the dual volume of the convex core of M as

$$V_C^* = V_C - \frac{1}{2}L_\mu(\lambda).$$

Lemma 5.4 (The dual Bonahon–Schläfli formula). *The first-order variation of V^* under a first-order variation of M is given by*

$$dV_C^* = -\frac{1}{2}L'_\mu(\lambda).$$

This formula has a very simple interpretation in terms of the geometry of Teichmüller space: up to the factor $-1/2$, dV^* is equal to the pull-back by δ of the Liouville form of the cotangent bundle $T^*\mathcal{T}$. Note also that this formula can be understood in an elementary way, without reference to a transverse Hölder distribution: the measured lamination λ is fixed, and only the hyperbolic metric μ varies. The proof can be found in [28], it is based on Lemma 5.3.

Theorem 5.2 is a direct consequence of Lemma 5.4: since dV_C^* coincides with the Liouville form of $T^*\mathcal{T}_{\partial M}$ on $H(\mathcal{QF})$, it follows immediately that $H(\mathcal{QF})$ is Lagrangian for the symplectic form ω_H on $T^*\mathcal{T}_{\partial M}$.

5.4 Deformations of representations and Poincaré duality

Steve Kerckhoff found another (unpublished) proof of Theorem 5.1 based on topological ideas and in particular on his earlier work with Craig Hodgson [20].

This proof works in the context of deformations of $\mathrm{PSL}(2, \mathbb{C})$ representations, the symplectic form on \mathcal{CP} is here the Goldman symplectic form ω_G on \mathcal{CP} . Recall (from [15]) that given a complex projective structure σ on ∂M , its holonomy representation is a morphism ρ from $\pi_1(\partial M)$ to $\mathrm{PSL}(2, \mathbb{C})$. The tangent space to \mathcal{CP} at σ is then naturally identified with the cohomology space $H^1(\partial M; E)$, where E is an $\mathfrak{sl}(2, \mathbb{C})$ bundle over ∂M naturally associated to ρ – it is the bundle of local projective vector fields for σ on ∂M . Given two cohomology classes $u, v \in H^1(\partial M; E)$, one can

consider their cup product $u \cup v \in H^2(\partial M, \mathbb{C})$, and integrate it over ∂M . This defines the Goldman symplectic form

$$\omega_G : H^1(\partial M; E) \times H^1(\partial M; E) \rightarrow \mathbb{C}.$$

Kawai [23] proved that this symplectic form is equal, up to a constant, to the canonical symplectic form on $\mathcal{CP}_{\partial M}$, obtained for instance by identification of $\mathcal{CP}_{\partial M}$ with $T^*\widetilde{\mathcal{J}}_{\partial M}$ through the Schwarzian derivative, see [12].

Kerckhoff's proof is based on the long exact sequence in cohomology for the pair $(M, \partial M)$:

$$\cdots \rightarrow H^1(M, \partial M; E) \rightarrow H^1(M; E) \xrightarrow{\alpha} H^1(\partial M; E) \xrightarrow{\beta} H^2(M, \partial M; E) \rightarrow \cdots.$$

Here α is restriction of the deformation from M to ∂M . Note that the map

$$H^1(M, \partial M; E) \rightarrow H^1(M; E)$$

is zero, since any non-trivial deformation of the hyperbolic structure on M induces a non-zero deformation of the complex projective structure on the boundary (this follows for instance from the Ahlfors–Bers theorem). As a consequence, α is injective.

Part of the long exact sequence above can be extended as the commutative diagram below, taken from [20], p. 42:

$$\begin{array}{ccccc} H^1(M; E) & \xrightarrow{\alpha} & H^1(\partial M; E) & \xrightarrow{\beta} & H^2(M, \partial M; E) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(M, \partial M; E)^* & \xrightarrow{\beta^*} & H^1(\partial M; E)^* & \xrightarrow{\alpha^*} & H^1(M; E)^*. \end{array}$$

Here the vertical arrows are the Poincaré duality maps. Recall that Poincaré duality can be defined using the cup product as above. In particular, the Poincaré dual u^* of a form u , for instance in $H^1(\partial M; E)$, is such that, for all $v \in H^1(\partial M; E)$, $\omega_G(u, v) = \langle u^*, v \rangle$.

Let $u, v \in H^1(M; E)$. It follows from the above commutative diagram that

$$\begin{aligned} \omega_G(\alpha(u), \alpha(v)) &= \langle \alpha(u), \alpha(v)^* \rangle \\ &= \langle \alpha(u), \beta^*(v^*) \rangle \\ &= \langle \beta \circ \alpha(u), v^* \rangle \\ &= 0. \end{aligned}$$

It also follows from the above exact sequence (or from the upper part of the commutative diagram and the fact that α is injective) that $\dim H^1(\partial M; E) = 2 \dim H^1(M; E)$ (see [20] for the details). So $\alpha(H^1(M; E))$ is Lagrangian in $H^1(\partial M; E)$, and this, along with Kawai's result [23], provides yet another proof of Theorem 5.1.

6 The renormalized volume as a Kähler potential

In this section we again consider the setting of quasifuchsian manifolds and recover the result of Takhtajan and Teo [41]: the renormalized volume W_M with c_- fixed is a Kähler potential for the Weil–Petersson metric on $\mathcal{T}_{\partial_+ M}$.

6.1 Notation

To simplify notation a little, we set $\theta_{c_-} := \theta(c_-, \cdot)$, so that the real part of θ_{c_-} is $\beta_+(c_-, \cdot)$. Since we already know that $\theta_{c_-} = 4\partial W_M$, we only need to prove that $\partial(i\theta_{c_-}) = -2\omega_{WP}$, where ω_{WP} is the Kähler form of the Weil–Petersson metric on $\mathcal{T}_{\partial_+ M}$.

An important part of the argument is that $d\theta_{c_-}$, as a 2-form on $\mathcal{T}_{\partial_+ M}$, does not depend on c_- . This appears as Theorem 7.2 in McMullen’s paper [30]. We include a proof for completeness, following the proof given in [30].

Proposition 6.1. *The differential $d\theta_{c_-}$, considered as a complex-valued 2-form on $\mathcal{T}_{\partial_+ M}$, does not depend on c_- .*

Proof. Let $v_- \in T_{c_-} \mathcal{T}_{\partial_+ M}$, we want to show that the corresponding first-order variation $D_{v_-}(d\theta_{c_-})$ of $d\theta_{c_-}$ vanishes. This will follow from the fact that the first-order variation of θ_{c_-} corresponding to v_- , $D_{v_-}\theta_{c_-}$, is the differential of a function defined on $\mathcal{T}_{\partial_+ M}$, namely the function f_{v_-} defined by

$$f_{v_-}(c_+) = \langle \theta_{c_-}(c_+), v_- \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing.

The fact that $D_{v_-}\theta_{c_-} = df_{v_-}$ can be proved by evaluating both sides on a vector $v_+ \in T_{c_+} \mathcal{T}_{\partial_+ M}$ and using the quasifuchsian reciprocity. Since θ_{c_-} is a complex 1-form with real part equal to $\beta_+(c_-, \cdot)$, Theorem 1.2 indicates that

$$\begin{aligned} \langle D_{v_-}\theta_{c_-}, v_+ \rangle &= \langle D\beta_+(c_-, c_+)(v_-, 0), v_+, \rangle \\ &= \langle D\beta_-(c_-, c_+)(0, v_+), v_- \rangle = df_{v_-}(v_+). \end{aligned}$$

It clearly follows that $d\theta_{c_-}$, as a 2-form on $\mathcal{T}_{\partial_+ M}$, does not depend on c_- . \square

6.2 Local deformations near the Fuchsian locus

That W_M is a Kähler potential is now reduced to a simple computation in the Fuchsian situation.

Proposition 6.2. *Suppose that M is a Fuchsian manifold, with $c_+ = c_-$. Let I^* be the hyperbolic metric in the conformal class c_+ . Under a first-order deformation which does not change c_- , the variation of I^* and II_0^* on $\partial_+ M$ are related by*

$$\delta II_0^* = -\delta I^*.$$

The proof is quite elementary, it is based on the fact that, for a quasifuchsian manifold which is “close to Fuchsian”, the metrics at infinity on the upper and lower components of the boundary are I^* and III^* respectively (where III^* is the third fundamental form at infinity on the upper boundary component). Moreover, if I^* has constant curvature, then III^* also has constant curvature (see [27]).

6.3 Kähler potential

We can reformulate this statement by setting $\theta_R := \operatorname{Re}(\theta_{c_-})$, so that, by Lemma 4.4, $\theta_R(X) = -\langle X, II_0^* \rangle$. Using the previous proposition, this can then be stated as

$$(D_X \theta_R)(Y) = \langle X, Y \rangle_{WP},$$

where D is the Levi-Civita connection of the Weil–Petersson metric on $\tilde{\mathcal{T}}_{\partial_+ M}$.

We can now compute explicitly an expression of $\bar{\partial}\theta_{c_-}$, denoting by J the complex structure on $\tilde{\mathcal{T}}_{\partial_- M}$.

$$\begin{aligned} \bar{\partial}\theta_{c_-}(X, Y) &= (D_X \theta_{c_-})(Y) + i(D_{JX} \theta_{c_-})(Y) \\ &= (D_X \theta_R)(Y) - i(D_X \theta_R)(JY) \\ &\quad + i((D_{JX} \theta_R)(Y) - i(D_{JX} \theta_R)(JY)) \\ &= \langle X, Y \rangle - i\langle X, JY \rangle + i\langle JX, Y \rangle + \langle JX, JY \rangle \\ &= 2(\langle X, Y \rangle - i\langle X, JY \rangle). \end{aligned}$$

This means precisely that $\bar{\partial}\theta_{c_-}(X, JX) = 2i\|X\|_{WP}^2$, and we recover the result of Takhtajan and Teo [41] that W_M is a Kähler potential for the Weil–Petersson metric.

Note that this statement could be compared to the fact (proved recently in [17]) that, on the Fuchsian locus, the Hessian of the area of the (unique) closed minimal surface, as a function on $\tilde{\mathcal{T}} \times \mathcal{T}$, is compatible with the hypothesis that this area is also a Kähler potential for the Weil–Petersson metric.

7 The relative volume of hyperbolic ends

So far we have considered a version of the renormalized volume defined for a hyperbolic 3-manifold M as a whole. This means that only certain very special projective structures can arise at boundary components ∂M . It is interesting, however, to extend the notion of renormalized volume (and thus of Liouville action) to arbitrary projective structures at the boundary. This is achieved by the notion of the relative volume that we consider in this section. When the projective structure in question is such that a non-singular hyperbolic 3-manifold M realizing it exists, then the renormalized volume of M is just the sum of relative volumes of its hyperbolic ends and the (dual) volume of the convex core $C(M)$.

We first recall some results due to Bonahon concerning the first variation of the volume of the convex core of a quasifuchsian manifold. We then introduce the relative volume of a hyperbolic end, and give a first variation formula for it. This establishes an analog of Kleinian reciprocity in the relative volume context, and proves that the grafting map is symplectic.

7.1 Definition

The relative volume is defined for (geometrically finite) hyperbolic ends rather than for hyperbolic manifolds. Thus, consider a hyperbolic end M . The procedure used in the definition of the renormalized volume can be used in this setting, leading to the relative volume of the end. We will say that a geodesically convex subset $K \subset M$ is a *collar* if it is relatively compact and contains the metric boundary $\partial_0 M$ of M (possibly all geodesically convex relatively compact subsets of M are collars, but it is not necessary to consider this question here). Then $\partial K \cap M$ is a locally convex surface in M .

The relative volume of M is related both to the (dual) volume of the convex core and to the renormalized volume; it is defined as the renormalized volume, but starting from the metric boundary of the hyperbolic end. We follow the same path as for the renormalized volume and start from a collar $K \subset M$. We set

$$W(K) = V(K) - \frac{1}{4} \int_{\partial K} H da + \frac{1}{2} L_\mu(\lambda),$$

where H is the mean curvature of the boundary of K , μ is the induced metric on the metric boundary $\partial_0 M$ of M , and λ is its measured bending lamination.

As for the renormalized volume we define the metric at infinity as

$$I^* := \lim_{\rho \rightarrow \infty} 2e^{-2\rho} I_\rho,$$

where I_ρ is the set of points at distance ρ from K . The conformal structure of I^* is equal to the canonical conformal structure c_∞ at infinity of M .

Here again, W only depends on I^* (and on M). Not all metrics in c_∞ can be obtained from a compact subset of M , however all metrics do define an equidistant foliation close to infinity in M , and it is still possible to define $W(I^*)$ even when I^* is not obtained from a convex subset of M . So W defines a function, still called W , from metrics in the conformal class c_∞ to \mathbb{R} .

Lemma 7.1. *For fixed area of I^* , W is maximal exactly when I^* has constant curvature.*

The proof follows directly from the arguments used in [27] (and reviewed in Section 7) so we do not repeat it here. It takes place entirely on the boundary at infinity so it makes no difference whether one considers a hyperbolic end or a geometrically finite hyperbolic manifold.

Definition 7.1. The relative volume V_R of M is defined to be $W(I^*)$ when I^* is the hyperbolic metric in the conformal class at infinity on M .

7.2 The first variation of the relative volume

Proposition 7.2. *Under a first-order variation of the hyperbolic end, the first-order variation of the relative volume is given by*

$$V'_R = \frac{1}{2}L'_\mu(\lambda) - \frac{1}{4} \int_{\partial_\infty E} \langle I^{*'}, II_0^* \rangle da^*. \quad (7.1)$$

The proof is based on the arguments described in the previous sections, both for the first variation of the renormalized volume and for the first variation of the volume of the convex core.

7.3 The grafting map is symplectic

Since hyperbolic ends are in one-to-one correspondence with $\mathbb{C}P^1$ -structures, we can consider the relative volume V_R as a function on the space of projective structures \mathcal{CP} . This space is canonically identified with the (complex) cotangent bundle $T^*\mathcal{T}_C$, where the subscript C indicates that one is talking about the complex Teichmüller space. Let β_C be the Liouville form on $T^*\mathcal{T}_C$. Consider now the space $\mathcal{T} \times \mathcal{ML}$ associated with the metric boundary of our hyperbolic end. In Section 5.3.2 we have discussed how this space can be naturally identified by the map δ with $T^*\mathcal{T}_H$. Let λ_H be the Liouville form on $T^*\mathcal{T}_H$.

We can now consider the grafting map $\text{Gr}: \mathcal{T} \times \mathcal{ML} \rightarrow \mathbb{C}P^1$, and the composition $\delta \circ \text{Gr}^{-1}: \mathcal{CP} \rightarrow T^*\mathcal{T}_H$. The latter map turns out to be C^1 (see [28]), a fact which is somewhat surprising since there is no C^1 structure on \mathcal{ML} . It pulls back λ_H as

$$(\delta \circ \text{Gr}^{-1})^* \lambda_H = L'_\mu(\lambda).$$

Under the identification of \mathcal{CP} with $T^*\mathcal{T}_C$ through the Schwarzian derivative, the expression of λ_C is

$$\lambda_C = \int_{\partial_\infty M} \langle I^{*'}, II_0^* \rangle da^*.$$

So Proposition 7.2 can be formulated as

$$dV_R = \frac{1}{2}(\delta \circ \text{Gr}^{-1})^* \lambda_H - \frac{1}{4} \lambda_C,$$

and it follows that $2(\delta \circ \text{Gr}^{-1})^* \omega_H = \omega_C$. This means that the grafting map preserves (up to a constant) the symplectic form and is thus symplectic. This statement can also be rephrased in a way analogous to (5.1) by saying that the subspace of the space $(\mathcal{T} \times \mathcal{ML}) \times \mathcal{CP}$ that can be realized on the two boundaries of a hyperbolic end is a Lagrangian submanifold in $(\mathcal{T} \times \mathcal{ML}) \times \mathcal{CP}$.

8 Manifolds with particles and the Teichmüller theory of surfaces with cone singularities

One key feature of the arguments presented in this work is that they are always local, in the sense that they depend on local quantities defined on the boundaries of compact subsets of quasifuchsian manifolds. Thus, we make only a very limited use of the fact that the quasifuchsian manifolds are actually quotients of hyperbolic 3-space by a group of isometries. One place where this is used is in the proof of the fact that II^* is determined by I^* (actually a direct consequence of the Bers double uniformization theorem). We expect that all the results should extend from quasifuchsian (more generally geometrically finite) manifolds to the “quasifuchsian manifolds with particles” which were studied e.g. in [26], [32]. Those are actually cone-manifolds, with cone singularities along infinite lines running from one connected component of the boundary at infinity to the other, along which the cone angle is less than π .

In the (non-singular) quasifuchsian setting the Bers double uniformization theorem shows that everything is determined by the conformal structure at infinity. The corresponding statement holds for “quasifuchsian manifolds with particles”; a first step towards it is made in [32], while the second step is made in [29]. The corresponding statement for more general, convex co-compact manifolds, remains however elusive.

Those results can be used to obtain results on the Teichmüller-type space of hyperbolic metrics with n cone singularities of prescribed angles on a closed surface of genus g . Note that this space, which can be denoted by $\mathcal{T}_{g,n,\theta}$ (with $\theta = (\theta_1, \dots, \theta_n) \in (0, \pi)^n$) is topologically the same as the “usual” Teichmüller space $\mathcal{T}_{g,n}$ of hyperbolic metrics with n cusps (with a one-to-one correspondence from [43]) but it has a natural “Weil–Petersson” metric which is different. It follows from the considerations made here, extended to quasifuchsian manifolds with particles, that this “Weil–Petersson” metric is still Kähler, with the renormalized volume playing the role of a Kähler potential – a result also obtained by different arguments by Schumacher and Trapani [37]. We leave the detailed investigation of this extension to quasifuchsian cone manifolds for future work.

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