Reliability analysis and lattice polynomial system representation

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Definition. A system consists of several interconnected units.

Assumptions:
1. The system and the units are of the crisply on/off kind.
2. A serially connected segment of units is functioning if and only if every single unit is functioning.
3. A system of parallel units is functioning if and only if at least one unit is functioning.
Example. Home video system

1. Blu-ray player
2. DVD player
3. LCD monitor
4. Amplifier
5. Speaker A
6. Speaker B
Structure function

Definition.
The state of a component \( i \in [n] = \{1, \ldots, n\} \) can be represented by a Boolean variable

\[
x_i = \begin{cases} 
1 & \text{if component } i \text{ is functioning} \\
0 & \text{if component } i \text{ is in a failed state}
\end{cases}
\]

The state of the system is described from the component states through a Boolean function \( \phi : \{0, 1\}^n \rightarrow \{0, 1\} \)

\[
\phi(x_1, \ldots, x_n) = \begin{cases} 
1 & \text{if the system is functioning} \\
0 & \text{if the system is in a failed state}
\end{cases}
\]

This function is called the structure function of the system.
Structure function

Series structure

\[ \phi(x) = x_1 \cdot x_2 \cdot x_3 = \prod_{i=1}^{3} x_i \]

Parallel structure

\[ \phi(x) = 1 - (1 - x_1)(1 - x_2)(1 - x_3) = \prod_{i=1}^{3} x_i \]
Home video system

\[ \phi(x) = (x_1 \text{ II } x_2) \cdot x_3 \cdot x_4 \cdot (x_5 \text{ II } x_6) \]
Coherent and semicoherent systems

Definition.
Let $\phi : \{0, 1\}^n \rightarrow \{0, 1\}$ be a structure function on $[n] = \{1, \ldots, n\}$.

The system is said to be semicoherent if
- $\phi$ is nondecreasing: $x \leq x' \Rightarrow \phi(x) \leq \phi(x')$
- $\phi(0) = 0$, $\phi(1) = 1$

The system is said to be coherent if, in addition
- every component is relevant to $\phi$: 
  $$\exists x \in \{0, 1\}^n : \phi(1_i, x) \neq \phi(0_i, x)$$

where
- $(1_i, x) = (x_1, \ldots, \text{i}^\text{th} \text{value} \in \{1, \ldots, x_n\})$
- $(0_i, x) = (x_1, \ldots, \text{i}^\text{th} \text{value} \in \{0, \ldots, x_n\})$
Representations of Boolean functions

Boolean function $\phi : \{0, 1\}^n \rightarrow \{0, 1\}$  
set function $\nu : 2^{[n]} \rightarrow \{0, 1\}$

$$\nu(A) = \phi(e_A) \quad A \subseteq [n]$$

→ We write $\phi_\nu$ instead of $\phi$

Representations of a Boolean function

$$\phi_\nu(x) = \sum_{A \subseteq [n]} \nu(A) \prod_{i \in A} x_i \prod_{i \in [n] \setminus A} (1 - x_i)$$
Representations of Boolean functions

Alternative representations

\[ \phi_v(x) = \sum_{A \subseteq [n]} m_v(A) \prod_{i \in A} x_i \]

where

\[ m_v(A) = \sum_{B \subseteq A} (-1)^{|A| - |B|} v(B) \]

If \( \phi_v \) is nondecreasing and nonconstant:

\[ \phi_v(x) = \prod_{A \subseteq [n]} \prod_{i \in A} x_i \]

(Hammer and Rudeanu 1968)
Any component $i \in [n]$ has a random lifetime: $T_i$

The system has a random lifetime: $T_S$

The structure function induces a functional relationship between the variables $T_1, \ldots, T_n$ and the variable $T_S$

Example:

$$\phi(x) = x_1 \cdot x_2 \cdot x_3 = \prod_{i=1}^{3} x_i$$

$$T_S = T_1 \land T_2 \land T_3 = \bigwedge_{i=1}^{3} T_i$$
System and component lifetimes

In general,

\[ T_S = p(T_1, \ldots, T_n) \]

where \( p : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+ \) is an \( n \)-ary lattice polynomial function

⇒ Formal parallelism between two representations of systems: structure functions and lattice polynomial functions
Lattice polynomial functions

Let $L \subseteq [-\infty, \infty]$ a totally ordered bounded lattice
\[ \Rightarrow \land = \min \text{ and } \lor = \max \]

The class of $n$-ary lattice polynomial (l.p.) functions is defined as follows:

(i) For any $k \in [n]$, the projection $(t_1, \ldots, t_n) \mapsto t_k$ is an $n$-ary l.p. function

(ii) If $p$ and $q$ are $n$-ary l.p. functions then $p \land q$ and $p \lor q$ are $n$-ary l.p. functions

(iii) Every $n$-ary l.p. function is constructed by finitely many applications of the rules (i) and (ii).

Example:

\[ p(t_1, t_2, t_3) = (t_1 \land t_2) \lor t_3 \]
Lattice polynomial functions

Let $a = \inf L$ and $b = \sup L$

l.p. function $\leftrightarrow$ set function

$p : L^n \to L \quad w : 2^{[n]} \to \{a, b\}$

$$w(A) = p(e^{a,b}_A) \quad A \subseteq [n]$$

Example: $e^{a,b}_{\{1,2\}} = (b, b, a, \ldots, a)$

→ We write $p_w$ instead of $p$

Representations of an l.p. function (Birkhoff 1967)

$$p_w(t) = \bigvee_{A \subseteq [n]} \bigwedge_{i \in A} t_i$$
Formal parallelism between the two representations

\[ T_i = \text{random lifetime of component } i \in [n] \]
\[ X_i(t) = \text{Ind}(T_i > t) = \text{random state of } i \text{ at time } t \geq 0 \]

\[ X_i(t) = \begin{cases} 
1 & \text{if } i \text{ is functioning at time } t \\
0 & \text{if } i \text{ is in a failed state at time } t
\end{cases} \]

For the system:
\[ T_S = \text{system lifetime} \]
\[ X_S(t) = \text{Ind}(T_S > t) = \text{random state of the system at time } t \geq 0 \]
Formal parallelism between the two representations

Home video system

\[ p_w(T) = (T_1 \lor T_2) \land T_3 \land T_4 \land (T_5 \lor T_6) \]
\[ \phi_v(X(t)) = (X_1(t) \sqcup X_2(t)) \sqcap X_3(t) \sqcap X_4(t) \sqcap (X_5(t) \sqcup X_6(t)) \]

\[ \phi_v \] is also an l.p. function that has just the same max-min form as \[ p_w \] but applied to binary arguments

\[ \phi_v \leftrightarrow p_w \]
\[ w = \gamma \circ v \]
\[ \gamma: \{0, 1\} \rightarrow \{a, b\}, \quad \gamma(0) = a, \quad \gamma(1) = b \]

As the lifetimes are \([0, \infty]\)-valued, we now assume that \(a = 0\) and \(b = \infty\)
Formal parallelism between the two representations

**Theorem.** (Dukhovny and M. 2008)

Consider a system whose structure function \( \phi_v : \{0, 1\}^n \rightarrow \{0, 1\} \) is nondecreasing and nonconstant. Then we have

\[
T_S = p_w(T_1, \ldots, T_n)
\]

(1)

where \( w = \gamma \circ v \). Conversely, any system fulfilling (1) for some l.p. function \( p_w : L^n \rightarrow L \) has the nondecreasing and nonconstant structure function \( \phi_v \), where \( v = \gamma^{-1} \circ w \)

The proof mainly lies on the immediate identities

\[
\begin{align*}
\text{Ind}(E \land E') &= \text{Ind}(E) \land \text{Ind}(E') \\
\text{Ind}(E \lor E') &= \text{Ind}(E) \lor \text{Ind}(E')
\end{align*}
\]

valid for all events \( E \) and \( E' \)
Formal parallelism between the two representations

**Proof.** For every $t \geq 0$ we have

$$
\phi_v(X(t)) = \prod_{A \subseteq [n]} \prod_{i \in A} X_i(t)
$$

$$
= \bigvee_{A \subseteq [n]} \bigwedge_{i \in A} \text{Ind}(T_i > t) = \text{Ind}\left( \bigvee_{A \subseteq [n]} \bigwedge_{i \in A} T_i > t \right)
$$

$$
= \text{Ind}(p_w(T) > t)
$$

Hence, we have

$$
X_S(t) = \phi_v(X(t)) \quad \forall t \geq 0
$$

$$
\Leftrightarrow \text{Ind}(T_S > t) = \text{Ind}(p_w(T) > t) \quad \forall t \geq 0
$$

$$
\Leftrightarrow T_S = p_w(T)
$$
Properties of l.p. functions reveal properties of structure functions

Example. Any l.p. function $p : L^n \rightarrow L$ satisfies trivially the functional equations

\[
\begin{align*}
p(u \land t_1, \ldots, u \land t_n) &= u \land p(t_1, \ldots, t_n) \quad \forall u \in L \\
p(u \lor t_1, \ldots, u \lor t_n) &= u \lor p(t_1, \ldots, t_n) \quad \forall u \in L
\end{align*}
\]

The corresponding equations for the structure functions are

\[
\begin{align*}
\phi(y x_1, \ldots, y x_n) &= y \phi(x_1, \ldots, x_n) \quad \forall y \in \{0, 1\} \\
\phi(y \lor x_1, \ldots, y \lor x_n) &= y \lor \phi(x_1, \ldots, x_n) \quad \forall y \in \{0, 1\}
\end{align*}
\]
Advantages of the lattice polynomial language

Properties of structure functions reveal properties of l.p. functions

**Example.** Pivotal decomposition of the structure function

\[ \phi(x) = x_i \phi(1_i, x) + (1 - x_i)\phi(0_i, x) \]

Bridge structure

\[ \phi(x) = x_3 \phi(1_3, x) + (1 - x_3)\phi(0_3, x) \]

\[ \phi(1_3, x) = (x_1 \uplus x_2)(x_4 \uplus x_5) \]

\[ \phi(0_3, x) = (x_1 x_4) \uplus (x_2 x_5) \]
Advantages of the lattice polynomial language

Corresponding property of the l.p. functions?

\[ p(t) = \text{median}(p(a_i, t), t_i, p(b_i, t)) \]

where

\[ \text{median}(x_1, x_2, x_3) = (x_1 \land x_2) \lor (x_2 \land x_3) \lor (x_3 \land x_1) \]

Proof.

\[
\phi(x) = \text{median}(\phi(0_i, x), x_i, \phi(1_i, x)) \\
= \left( \phi(0_i, x) \land x_i \right) \lor (x_i \lor \phi(1_i, x)) \lor \left( \phi(1_i, x) \land \phi(0_i, x) \right) \\
= x_i \phi(1_i, x) \lor \phi(0_i, x) \\
= x_i \phi(1_i, x) + \phi(0_i, x) - x_i \phi(1_i, x) \phi(0_i, x) \\
= x_i \phi(1_i, x) + (1 - x_i) \phi(0_i, x)
\]
Reliability analysis

Reliability function of component $i \in [n]$

$$R_i(t) = \Pr(T_i > t) = \Pr(X_i(t) = 1) = \mathbb{E}[X_i(t)]$$

= probability that component $i$ does not fail in the interval $[0, t]$

System reliability function

$$R_S(t) = \Pr(T_S > t) = \Pr(X_S(t) = 1) = \mathbb{E}[X_S(t)]$$

= probability that the system does not fail in the interval $[0, t]$
Reliability analysis

**Theorem.** (Dukhovny 2007)

\[
R_S(t) = \sum_{A \subseteq [n]} v(A) \Pr(X(t) = e_A)
\]

**Remarks.**

(i) All the needed information is the distribution of \(X(t)\) (the knowledge of the joint distribution of \(T\) is not necessary)

(ii) The distribution of \(X(t)\) can be easily expressed in terms of the *joint probability generating function* of \(X(t)\)

\[
G(z, t) = E\left[ \prod_{i=1}^{n} z_i^{X_i(t)} \right] \quad (|z_i| \leq 1).
\]

We have

\[
\Pr(X(t) = e_A) = \sum_{B \subseteq A} (-1)^{|A| - |B|} G(e_B, t)
\]
When $T_1, \ldots, T_n$ are independent, we have

$$RS(t) = \sum_{A \subseteq [n]} v(A) \prod_{i \in A} R_i(t) \prod_{i \in [n] \setminus A} (1 - R_i(t))$$

Alternative expression (Dukhovny and M. 2008)

$$RS(t) = \sum_{A \subseteq [n]} m_v(A) \Pr(T_i > t \ \forall i \in A)$$

In case of independence

$$RS(t) = \sum_{A \subseteq [n]} m_v(A) \prod_{i \in A} R_i(t)$$
Mean time-to-failure of the system

The *mean time-to-failure* of the system is defined as

\[ \text{MTTF}_S = E[T_S] \]

It is easy to show that

\[ \text{MTTF}_S = \int_0^\infty R_S(t) \, dt \]

(Rausand and Høyland 2004)

In case of independence

\[
\begin{align*}
\text{MTTF}_S &= \sum_{A \subseteq [n]} v(A) \int_0^\infty \prod_{i \in A} R_i(t) \prod_{i \in [n] \setminus A} (1 - R_i(t)) \, dt \\
\text{MTTF}_S &= \sum_{A \subseteq [n]} m_v(A) \int_0^\infty \prod_{i \in A} R_i(t) \, dt
\end{align*}
\]
Mean time-to-failure of the system

**Example.** Assume \( R_i(t) = e^{-\lambda_i t}, \ i = 1, \ldots, n \)

\[
\begin{align*}
\text{MTTF}_S &= \sum_{A \subseteq [n]} m_v(A) \int_0^\infty \prod_{i \in A} e^{-\lambda_i t} \, dt \\
&= \sum_{A \subseteq [n]} m_v(A) \int_0^\infty e^{-\lambda_A t} \, dt \quad \left( \lambda_A = \sum_{i \in A} \lambda_i \right) \\
&= \sum_{A \subseteq [n]} m_v(A) \frac{1}{\lambda_A}
\end{align*}
\]

**Series structure:** \( \text{MTTF}_S = \frac{1}{\lambda_{[n]}} \)

**Parallel structure:** \( \text{MTTF}_S = \sum_{A \subseteq [n]} \left( -1 \right)^{|A|-1} \frac{1}{\lambda_A} \)
Generalization to weighted lattice polynomial functions

Suppose there are

(i) collective upper bounds on lifetimes of certain subsets of units (imposed by the physical properties of the assembly)

\[
\text{subset lifetime } = T \land c
\]

(ii) collective lower bounds (imposed by back-up blocks with constant lifetimes)

\[
\text{subset lifetime } = T \lor c
\]
Another advantage of the lattice polynomial language

The lifetime of a general system with upper and/or lower bounds can be described through a weighted lattice polynomial function

\[ T_S = p(T_1, \ldots, T_n) \]

Example.

Suppose that the lifetime of component \#2 must lies in the time interval \([c, d]\)

\[
T_S = T_1 \land \text{median}(c, T_2, d) \\
= T_1 \land (c \lor (T_2 \land d)) \\
= (c \land T_1) \lor (d \land T_1 \land T_2)
\]
The class of $n$-ary weighted lattice polynomial (w.l.p.) functions is defined as follows:

(i) For any $k \in [n]$ and any $c \in L$, the projection $(t_1, \ldots, t_n) \mapsto t_k$ and the constant function $(t_1, \ldots, t_n) \mapsto c$ are $n$-ary w.l.p. function

(ii) If $p$ and $q$ are $n$-ary w.l.p. functions then $p \land q$ and $p \lor q$ are $n$-ary w.l.p. functions

(iii) Every $n$-ary w.l.p. function is constructed by finitely many applications of the rules (i) and (ii).
Weighted lattice polynomial functions

\[ w \text{.l.p. function} \iff \text{set function} \]
\[ p : L^n \to L \quad w : 2^n \to L \]

\[ w(A) = p(e_A^{a,b}) \quad A \subseteq [n] \]

\[ \rightarrow \text{ We write } p_w \text{ instead of } p \]

Representations of a \textbf{w.l.p. function} (Goodstein 1967)

\[ p_w(t) = \bigvee_{A \subseteq [n]} (w(A) \land \bigwedge_{i \in A} t_i) \]
Example (cont’d)

\[ T_S = (c \land T_1) \lor (d \land T_1 \land T_2) \]

\[ p_w(t_1, t_2) = (c \land t_1) \lor (d \land t_1 \land t_2) \]

<table>
<thead>
<tr>
<th>( A )</th>
<th>( w(A) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \emptyset )</td>
<td>0</td>
</tr>
<tr>
<td>{1}</td>
<td>c</td>
</tr>
<tr>
<td>{2}</td>
<td>0</td>
</tr>
<tr>
<td>{1, 2}</td>
<td>d</td>
</tr>
</tbody>
</table>

We can show that

\[ X_S(t) = (\text{Ind}(c > t) \ X_1(t)) \ \Pi \ (\text{Ind}(d > t) \ X_1(t) \ X_2(t)) \]
Representation of w.l.p. functions (DNF)

\[
p_w(t) = \bigvee_{A \subseteq [n]} w(A) \land \bigwedge_{i \in A} t_i
\]

**Theorem.** (Dukhovny and M. 2008)

If \( T_S = p_w(T_1, \ldots, T_n) \) then

\[
X_S(t) = \prod_{A \subseteq [n]} v_t(A) \prod_{i \in A} X_i(t) \quad (t \geq 0)
\]

where \( v_t(A) = \text{Ind}(w(A) > t) \)
Exact reliability formulas (Dukhovny and M. 2008)

\[
R_S(t) = \sum_{A \subseteq [n]} v_t(A) \Pr(X(t) = e_A)
\]

\[
R_S(t) = \sum_{A \subseteq [n]} m_{v_t}(A) \Pr(T_i > t \, \forall i \in A)
\]

In case of independence

\[
R_S(t) = \sum_{A \subseteq [n]} v_t(A) \prod_{i \in A} R_i(t) \prod_{i \in [n] \setminus A} (1 - R_i(t))
\]

\[
R_S(t) = \sum_{A \subseteq [n]} m_{v_t}(A) \prod_{i \in A} R_i(t)
\]
Mean time-to-failure of the system

\[ \text{MTTF}_S = \int_0^\infty R_S(t) \, dt \]

\[ = \sum_{A \subseteq [n]} \int_0^\infty m_{v_t}(A) \prod_{i \in A} R_i(t) \, dt \]

\[ = \sum_{A \subseteq [n]} \int_0^\infty \left( \sum_{B \subseteq A} (-1)^{|A| - |B|} v_t(B) \right) \prod_{i \in A} R_i(t) \, dt \]

\[ = \sum_{A \subseteq [n]} \sum_{B \subseteq A} (-1)^{|A| - |B|} \int_0^\infty \text{Ind}(w(B) > t) \prod_{i \in A} R_i(t) \, dt \]

\[ = \sum_{A \subseteq [n]} \sum_{B \subseteq A} (-1)^{|A| - |B|} \int_0^{w(B)} \prod_{i \in A} R_i(t) \, dt \]
Mean time-to-failure of the system

**Example.** Assume $R_i(t) = e^{-\lambda_i t}$, $i = 1, \ldots, n$

$$
\text{MTTF}_S = \sum_{A \subseteq [n]} \sum_{B \subseteq A} (-1)^{|A| - |B|} \int_0^{w(B)} \prod_{i \in A} e^{-\lambda_i t} \, dt
$$

$$
= \sum_{A \subseteq [n]} \sum_{B \subseteq A} (-1)^{|A| - |B|} \int_0^{w(B)} e^{-\lambda_A t} \, dt \quad (\lambda_A = \sum_{i \in A} \lambda_i)
$$

$$
= w(\emptyset) + \sum_{A \subseteq [n]} \sum_{B \subseteq A \neq \emptyset} (-1)^{|A| - |B|} \frac{1 - e^{-\lambda_A w(B)}}{\lambda_A}
$$
Conclusion

We have discussed the formal parallelism between two representations of systems

- Structure functions
- Lattice polynomial functions

→ Their languages are equivalent in many ways
Advantages

- Generalization to w.l.p. functions + exact reliability formulas
- Exact formulas for the distribution functions of w.l.p. functions of random variables

\[ Y = p_w(X_1, \ldots, X_n) \]

- Several special cases can be investigated
  - Symmetric w.l.p. functions: \( w(A) = f(|A|) \)
  - The reliability of any subsystem depends only on the number of units in the subsystem \( \Pr(\mathbf{X}(t) = e_A) = g_t(|A|) \)
  - \( \ldots \)
Thank you for your attention!

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