HIGHER TRACE AND BEREZINIAN OF MATRICES OVER A CLIFFORD ALGEBRA

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Abstract. We define the notions of trace, determinant and, more generally, Berezinian of matrices over a $(\mathbb{Z}_2)^n$-graded commutative associative algebra A. The applications include a new approach to the classical theory of matrices with coefficients in a Clifford algebra, in particular of quaternionic matrices. In a special case, we recover the classical Dieudonné determinant of quaternionic matrices, but in general our quaternionic determinant is different. We show that the graded determinant of purely even $(\mathbb{Z}_2)^n$-graded matrices of degree 0 is polynomial in its entries. In the case of the algebra $A = \mathbb{H}$ of quaternions, we calculate the formula for the Berezinian in terms of a product of quasiminors in the sense of Gelfand, Retakh, and Wilson. The graded trace is related to the graded Berezinian (and determinant) by a $(\mathbb{Z}_2)^n$-graded version of Liouville's formula.

Contents

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1. Introduction

Linear algebra over quaternions is a classical subject. Initiated by Hamilton and Cayley, it was further developed by Study [1] and Dieudonné [2], see [3] for a survey. The best known version of quaternionic determinant is due to Dieudonné, it is far of being elementary and still attracts a considerable interest, see [4]. The Dieudonné determinant is not related to any notion of trace. To the best of our knowledge, the concept of trace is missing in the existing theories of quaternionic matrices.

The main difficulty of any theory of matrices over quaternions, and more generally over Clifford algebras, is related to the fact that these algebras are not commutative. It turns out however, that the classical algebra $\mathbb{H}$ of quaternions can be understood as a graded-commutative algebra. It was shown in [5], [6], [7] that $\mathbb{H}$ is a graded commutative algebra over the Abelian group $(\mathbb{Z}_2)^2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ (or over the even part of $(\mathbb{Z}_2)^3$, see [8]). Quite similarly, every Clifford algebra with n generators is $(\mathbb{Z}_2)^n$-graded commutative [7] (furthermore, a Clifford algebra is understood as even $(\mathbb{Z}_2)^{n+1}$-graded commutative algebra in [9]). This viewpoint suggests a natural approach to linear algebra over Clifford algebras as generalized Superalgebra.

Geometric motivations to consider $(\mathbb{Z}_2)^n$-gradings come from the study of higher vector bundles [10]. If E denotes a vector bundle with coordinates (x, ξ), a kind of universal Legendre transform

$$T^*E \ni (x, \xi, y, \eta) \leftrightarrow (x, \eta, y, -\xi) \in T^*E^*$$

provides a natural and rich $(\mathbb{Z}_2)^2$-degree $((0, 0), (1, 0), (1, 1), (0, 1))$ on $T^*[1]E[1]$. Multigraded vector bundles give prototypical examples of $(\mathbb{Z}_2)^n$-graded manifolds.

Quite a number of geometric structures can be encoded in supercommutative algebraic structures, see e.g. [11], [12], [13], [14]. On the other hand, supercommutative algebras define supercommutative geometric spaces. It turns out, however, that the classical $\mathbb{Z}_2$-graded commutative algebras $\mathrm{Sec}(\wedge E^*)$ of vector bundle forms are far from being sufficient. For instance, whereas Lie algebroids are in 1-to-1 correspondence with homological vector fields of split supermanifolds $\mathrm{Sec}(\wedge E^*)$, the supergeometric interpretation of Loday algebroids [15] requires a $\mathbb{Z}_2$-graded commutative algebra of non-Grassmannian type, namely the shuffle algebra $\mathcal{D}(E)$ of specific multidifferential operators. However, not only other types of algebras, but also more general grading groups must be considered.

Let us also mention that classical Supersymmetry and Supermathematics are not completely sufficient for modern physics (i.e., the description of anyons, paraparticles).

All the aforementioned problems are parts of our incentive to investigate the basic notions of linear algebra over a $(\mathbb{Z}_2)^n$-graded commutative unital associative algebra A. We consider the space $\mathsf{M}(\mathbf{r}; A)$ of matrices with coefficients in A and introduce the notions of graded trace and Berezinian (in the simplest case of purely even matrices we will talk of the determinant). We prove an analog of the Liouville formula that connects both concepts. Although most of the results are formulated and proved for arbitrary A, our main goal is to develop a new theory of matrices over Clifford algebras and, more particularly, over quaternions.

Our main results are as follows:

- There exists a unique homomorphism of graded A-modules and graded Lie algebras
$$\Gamma\mathrm{tr} : \mathsf{M}(\mathbf{r}; A) \to A \ ,$$
defined for arbitrary matrices with coefficients in A.
- There exists a unique map
$$\Gamma\mathrm{det} : \mathsf{M}^0(\mathbf{r}_0; A) \to A^0 \ ,$$
defined on purely even homogeneous matrices of degree 0 with values in the commutative subalgebra $A^0 \subset A$ consisting of elements of degree 0 and characterized by three properties: a) Γdet is multiplicative, b) for a block-diagonal matrix Γdet is the product of the determinants of the blocks, c) Γdet of a lower (upper) unitriangular matrix equals 1. In the case $A = \mathbb{H}$, the absolute value of Γdet coincides with the classical Dieudonné determinant.
- There exists a unique group homomorphism
$$\Gamma\mathrm{Ber} : \mathsf{GL}^0(\mathbf{r}; A) \to (A^0)^\times \ ,$$
defined on the group of invertible homogeneous matrices of degree 0 with values in the group of invertible elements of A^0, characterized by properties similar to a), b), c).
- The graded Berezinian is connected with the graded trace by a $(\mathbb{Z}_2)^n$-graded version of Liouville's formula
$$\Gamma\mathrm{Ber}(\exp(\varepsilon X)) = \exp(\Gamma\mathrm{tr}(\varepsilon X)) \ ,$$
where ε is a nilpotent formal parameter of degree 0 and X a graded matrix.
- For the matrices with coefficients in a Clifford algebra, there exists a unique way to extend the graded determinant to homogeneous matrices of degree different from zero, if and only if the total matrix dimension $|\,\mathbf{r}|$ satisfies the condition
$$|\,\mathbf{r}| = 0, 1 \pmod 4.$$
In the case of matrices over $\mathbb{H}$, this graded determinant differs from that of Dieudonné.

The reader who wishes to gain a quick and straightforward insight into some aspects of the preceding results, might envisage having a look at Section 8 at the end of this paper, which can be read independently.

Our main tools that provide most of the existence results and explicit formulæ of graded determinants and graded Berezinians, are the concepts of quasideterminants and quasiminors, see [16] and references therein.

Let us also mention that in the case of matrices over a Clifford algebra, the restriction for the dimension of the A-module, $|\,\mathbf{r}| = 0, 1 \pmod 4$, provides new insight into the old problem initiated by Arthur Cayley, who considered specifically two-dimensional linear algebra over quaternions. It follows that Cayley's problem has no solution, at least within the framework of graded algebra adopted in this paper. In particular, the notion of determinant of a quaternionic (2×2)-matrix related to a natural notion of trace does not exist.

Basic concepts of $(\mathbb{Z}_2)^n$-graded Geometry based on the linear algebra developed in the present paper are being studied in a separate work; applications to quaternionic functions and to Mathematical Physics are expected. We also hope to investigate the cohomological nature of $(\mathbb{Z}_2)^n$-graded Berezinians, as well as the properties of the characteristic polynomial, see [17] and references therein.

Another method to treat the problem of generalizing superalgebras and related notions, alternative to the one presented in this paper and which makes use of category theory, is being studied in a separate work. This approach follows from results by Scheunert in [18] (in the Lie algebra setting) and Nekludova (in the commutative algebra setting). An explicit description of the results of the latter first appeared in [19], and can also be found in [20].

2. $(\mathbb{Z}_2)^n$-Graded Algebra

In this section we fix terminology and notation used throughout this paper. Most of the definitions extend well-known definitions of usual superlagebra [21], see also [22].

2.1. **General Notions.**

Let $(\Gamma, +)$ be an Abelian group endowed with a symmetric bi-additive map

$$\langle\ ,\ \rangle : \Gamma \times \Gamma \to \mathbb{Z}_2\,.$$

That is,

$$\langle \gamma, \gamma' \rangle = \langle \gamma', \gamma \rangle \qquad \text{and} \qquad \langle \gamma + \gamma', \gamma'' \rangle = \langle \gamma, \gamma'' \rangle + \langle \gamma', \gamma'' \rangle\,.$$

The *even* subgroup Γ_0 consists of elements $\gamma \in \Gamma$ such that $\langle \gamma, \gamma \rangle = 0$. One then has a splitting

$$\Gamma = \Gamma_0 \cup \Gamma_1\,,$$

where Γ_1 consists of *odd* elements $\gamma \in \Gamma$ such that $\langle \gamma, \gamma \rangle = 1$. Of course, Γ_1 is not a subgroup of Γ.

A basic example is the additive group $(\mathbb{Z}_2)^n$, $n \in \mathbb{N}$, equipped with the standard scalar product $\langle\ ,\ \rangle$ of n-vectors, defined over $\mathbb{Z}_2$, Section 2.2.

A *graded vector space* is a direct sum

$$V = \bigoplus_{\gamma \in \Gamma} V^\gamma$$

of vector spaces V^γ over a commutative field $\mathbb{K}$ (that we will always assume of characteristic 0). A graded vector space is always a direct sum:

$$V = V_0 \oplus V_1$$

of its even subspace $V_0 = \bigoplus_{\gamma \in \Gamma_0} V^\gamma$ and its odd subspace $V_1 = \bigoplus_{\gamma \in \Gamma_1} V^\gamma$.

If V and W are graded vector spaces, then one has:

$$\mathsf{Hom}_{\mathbb{K}}(V, W) = \bigoplus_{\gamma \in \Gamma} \mathsf{Hom}^\gamma_{\mathbb{K}}(V, W)\,,$$

where $\mathsf{Hom}^\gamma_{\mathbb{K}}(V, W)$ (or simply $\mathsf{Hom}^\gamma(V, W)$) the vector space of $\mathbb{K}$-linear maps of weight γ

$$\ell : V \to W, \qquad \ell(V^\delta) \subset W^{\delta+\gamma}\,.$$

We also use the standard notation $\mathsf{End}_{\mathbb{K}}(V) := \mathsf{Hom}_{\mathbb{K}}(V, V)$.

A *graded algebra* is an algebra A which has a structure of a graded vector space A such that the operation of multiplication respects the grading:

$$A^\gamma A^\delta \subset A^{\gamma+\delta}\,.$$

If A is associative (resp., associative and unital), we call it a *graded associative algebra* (resp., *graded associative unital algebra*). In this case, the operation of multiplication is denoted by "$\cdot$". A graded associative algebra A is called *graded commutative* if, for any homogeneous elements $a, b \in A$, we have

$$b \cdot a = (-1)^{\langle \tilde{a}, \tilde{b} \rangle} a \cdot b\,. \tag{1}$$

Here and below $\tilde{a} \in \Gamma$ stands for the degree of a, that is $a \in A^{\tilde{a}}$. Note that graded commutative algebras are also known in the literature under the name of "color commutative" algebras.

Our main examples of graded commutative algebras are the classical Clifford algebras equipped with $(\mathbb{Z}_2)^n$-grading, see Section 2.2.

A graded algebra A is called a *graded Lie algebra* if it is graded anticommutative (or skewsymmetric) and satisfies the graded Jacobi identity. The operation of multiplication is then denoted by $[\ ,\ ]$. The identities read explicitly:

$$\begin{aligned} [a, b] &= -(-1)^{\langle \tilde{a}, \tilde{b} \rangle} [b, a]\,, \\ [a, [b, c]] &= [[a, b], c] + (-1)^{\langle \tilde{a}, \tilde{b} \rangle} [b, [a, c]]\,. \end{aligned}$$

Graded Lie algebras are often called "color Lie algebras", see [18]. The main example of a graded Lie algebra is the space $\mathsf{End}_{\mathbb{K}}(V)$ equipped with the commutator:

$$[X, Y] = X \circ Y - (-1)^{\langle \widetilde{X}, \widetilde{Y} \rangle}\, Y \circ X\,, \tag{2}$$

for homogeneous $X, Y \in \mathsf{End}_{\mathbb{K}}(V)$ and extended by linearity.

A graded vector space M is called a *graded (left) module* over a graded commutative algebra A if there is a $\mathbb{K}$-linear map $\lambda : A \to \mathsf{End}_{\mathbb{K}}(M)$ of weight $0 \in \Gamma$ that verifies

$$\lambda(a) \circ \lambda(b) = \lambda(a \cdot b) \quad \text{and} \quad \lambda(1_A) = \mathrm{id}_M\,,$$

where $a, b \in A$ and where 1_A denotes the unit of A; we often write am instead of $\lambda(a)(m)$, for $a \in A$, $m \in M$. The condition of weight 0 for the map λ reads:

$$\widetilde{am} = \widetilde{a} + \widetilde{m}\,.$$

As usual we inject $\mathbb{K}$ into A by means of $K \ni k \rightarrowtail k1_A \in A$, so that $\lambda(k)(m) = km$, $m \in M$. Graded right modules over A are defined similarly. Since A is graded commutative, any graded left A-module structure on M defines a right one,

$$ma := (-1)^{\langle \tilde{m}, \tilde{a} \rangle} am\,,$$

and vice versa. Hence, we identify both concepts and speak just about graded modules over a graded commutative algebra, as we do in the commutative and supercommutative contexts.

Let now M and N be two graded A-modules. Denote by $\mathsf{Hom}^\gamma_A(M, N)$ the subspace of $\mathsf{Hom}^\gamma_{\mathbb{K}}(V, W)$ consisting of A-linear maps $\ell : M \to N$ of weight γ that is

$$\ell(am) = (-1)^{\langle \gamma, \tilde{a} \rangle} a\, \ell(m) \quad \text{or, equivalently,} \quad \ell(ma) = \ell(m)a$$
$$\text{and} \qquad \ell(M^{\gamma'}) \subset N^{\gamma+\gamma'}\,.$$

The space

$$\mathsf{Hom}_A(M,N)=\bigoplus_{\gamma\in\Gamma}\mathsf{Hom}_A^{\gamma}(M,N)$$

carries itself an obviously defined graded A-module structure. The space

$$\mathsf{End}_A(M):=\mathsf{Hom}_A(M,M)$$

is a graded Lie algebra with respect to the commutator (2).

Graded A-modules and A-linear maps of weight 0 form a category $\mathtt{Gr}_\Gamma\mathtt{Mod}_A$. Hence, the categorical Hom is the vector space $\mathrm{Hom}(M,N)=\mathsf{Hom}_A^0(M,N)$.

As A is a graded module over itself, the internal (to $\mathtt{Gr}_\Gamma\mathtt{Mod}_A$) Hom provides the notion of *dual* module $M^*=\mathsf{Hom}_A(M,A)$ of a graded A-module M. Let us also mention that the categorical Hom sets corresponding to graded associative algebras (resp., graded associative unital algebras, graded Lie algebras) are defined naturally as the sets of those $\mathbb{K}$-linear maps of weight 0 that respect the multiplications (resp., multiplications and units, brackets).

A *free graded A-module* is a graded A-module M whose terms M^γ admit a basis

$$B^\gamma=(e_1^\gamma,\dots,e_r^\gamma)\,.$$

Assume that the Abelian group Γ is of finite order p, and fix a basis $\{\gamma_1,\dots,\gamma_p\}$. Assume also that M has a finite rank: $\mathbf{r}=(r_1,\dots,r_p)$, where $r_u\in\mathbb{N}$ is the cardinality of B^{γ_u}. If N is another free graded A-module of finite rank $\mathbf{s}=(s_1,\dots,s_p)$ and basis $({e'}_1^{\gamma_k},\dots,{e'}_{s_k}^{\gamma_k})_k$, then every homogeneous A-linear map $\ell\in\mathsf{Hom}_A(M,N)$ is represented by a matrix X defined by

$$\ell(e_j^{\gamma_u})=:\sum_{k=1}^{p}\sum_{i=1}^{s_k}{e'}_i^{\gamma_k}(X_{ku})_{ij}\,,$$

where $u\in\{1,\dots,p\}$ and $j\in\{1,\dots,r_u\}$.

Every homogeneous matrix can be written in the form:

$$X=\left(\begin{array}{c|c|c} X_{11} & \dots & X_{1p}\\ \hline \dots & \dots & \dots\\ \hline X_{p1} & \dots & X_{pp}\end{array}\right), \tag{3}$$

where each X_{ku} is a matrix of dimension $s_k\times r_u$ with entries in $A^{-\gamma_k+\gamma_u+x}$. We denote by $\mathsf{M}^x(\mathbf{s},\mathbf{r};A)$ the set of *homogeneous matrices* of degree $x\in\Gamma$ and

$$\mathsf{M}(\mathbf{s},\mathbf{r};A)=\bigoplus_{x\in\Gamma}\mathsf{M}^x(\mathbf{s},\mathbf{r};A)\,.$$

The set $\mathsf{M}^x(\mathbf{s},\mathbf{r};A)$ is in 1-to-1 correspondence with the space $\mathsf{Hom}_A^x(A^{\mathbf{r}},A^{\mathbf{s}})$ of all weight x A-linear maps between the free graded A-modules $A^{\mathbf{r}}$ and $A^{\mathbf{s}}$ of rank $\mathbf{r}$ and $\mathbf{s}$, respectively. This correspondence allows transferring the vector space structure of the latter space to weight x graded matrices. We thus obtain:

- the usual matrix sum of matrices;
- the usual multiplication of matrices.

The *multiplication of matrices by scalars* in A is less obvious. One has:

$$a\,X = \left(\begin{array}{c|c|c} (-1)^{\langle\tilde{a},\gamma_1\rangle} aX_{11} & \ldots & (-1)^{\langle\tilde{a},\gamma_1\rangle} aX_{1p} \\ \hline \ldots & \ldots & \ldots \\ \hline (-1)^{\langle\tilde{a},\gamma_p\rangle} aX_{p1} & \ldots & (-1)^{\langle\tilde{a},\gamma_p\rangle} aX_{pp} \end{array}\right) \tag{4}$$

so that the sign depends on the row of a matrix. Indeed, the graded A-module structure of $\mathsf{M}(\mathbf{s},\mathbf{r};A)$ is induced by the A-module structure on $\mathsf{Hom}_A(A^{\mathbf{r}},A^{\mathbf{s}})$.

The space

$$\mathsf{M}(\mathbf{r};A) := \mathsf{M}(\mathbf{r},\mathbf{r};A) \simeq \mathsf{End}_A(A^{\mathbf{r}})$$

is the most important example of the space of matrices. This space is a graded A-module and a graded associative unital algebra, hence a graded Lie algebra for the graded commutator (2). Invertible matrices form a group that we denote by $\mathsf{GL}(\mathbf{r};A)$.

2.2. **$(\mathbb{Z}_2)^n$- and $(\mathbb{Z}_2)^{n+1}$-Grading on Clifford Algebras.** From now on, we will consider the following Abelian group:

$$\Gamma = (\mathbb{Z}_2)^n$$

of order 2^n. Elements of $(\mathbb{Z}_2)^n$ are identified with n-vectors with coordinates 0 and 1, the element $0 := (0,\ldots,0)$ is the unit element of the group. We will need the following two simple additional definitions related to $(\mathbb{Z}_2)^n$.

- The group $(\mathbb{Z}_2)^n$ is equipped with the standard *scalar product* with values in $\mathbb{Z}_2$:

$$\langle\gamma,\gamma'\rangle = \sum_{i=1}^{n} \gamma_i\gamma'_i\,. \tag{5}$$

- An *ordering* of the elements of $(\mathbb{Z}_2)^n$, such that the first (resp., the last) 2^{n-1} elements are even (resp., odd). The order is termed *standard* if in addition, the subsets of even and odd elements are ordered lexicographically. For instance,

$$\begin{aligned} \mathbb{Z}_2 &= \{0,1\}\,, \\ (\mathbb{Z}_2)^2 &= \{(0,0),(1,1),(0,1),(1,0)\}\,, \\ (\mathbb{Z}_2)^3 &= \{(0,0,0),(0,1,1),(1,0,1),(1,1,0),(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}\,. \end{aligned}$$

The *real Clifford algebra* $\mathrm{Cl}_{p,q}(\mathbb{R})$ is the associative $\mathbb{R}$-algebra generated by e_i, where $1 \le i \le n$ and $n = p+q$, of $\mathbb{R}^n$, modulo the relations

$$\begin{aligned} e_ie_j &= -e_je_i\,, \quad i \neq j\,, \\ e_i^2 &= \begin{cases} +1\,, & i \le p \\ -1\,, & i > p\,. \end{cases} \end{aligned}$$

The pair of integers (p,q) is called the *signature*. Note that, as a vector space, $\mathrm{Cl}_{p,q}(\mathbb{R})$ is isomorphic to the Grassmann algebra $\bigwedge\langle e_1,\ldots,e_n\rangle$ on the chosen generators. Furthermore, $\mathrm{Cl}_{p,q}(\mathbb{R})$ is often understood as quantization of the Grassmann algebra.

Real Clifford algebras can be seen as graded commutative algebras essentially in two different ways.

A $(\mathbb{Z}_2)^n$-grading on $\mathrm{Cl}_{p,q}(\mathbb{R})$ was defined in [7] by setting for the generators

$$\widetilde{e_i} = (0, \ldots, 0, 1, 0, \ldots, 0) \ ,$$

where 1 occupies the i-th position. However, the graded commutativity condition (1) is not satisfied with respect to the standard scalar product (5), which has to be replaced by another binary function on $(\mathbb{Z}_2)^n$, see [7].

A $(\mathbb{Z}_2)^{n+1}_0$-grading on $\mathrm{Cl}_{p,q}(\mathbb{R})$ has been considered in [9]. This grading coincides with the preceding Albuquerque-Majid degree, if one identifies $(\mathbb{Z}_2)^n$ with the even subgroup $(\mathbb{Z}_2)^{n+1}_0$ of $(\mathbb{Z}_2)^{n+1}$. Indeed, the new degree is defined by

$$\widetilde{e_i} := (0, \ldots, 0, 1, 0, \ldots, 0, 1) \ . \tag{6}$$

An advantage of this "even" grading is that the condition (1) is now satisfied with respect to the standard scalar product. It was proven in [9] that the defined $(\mathbb{Z}_2)^{n+1}_0$-grading on $\mathrm{Cl}_{p,q}(\mathbb{R})$ is universal in the following sense: *every simple finite-dimensional associative graded-commutative algebra is isomorphic to a Clifford algebra equipped with the above $(\mathbb{Z}_2)^{n+1}_0$-grading.*

Example 2.1. The $(\mathbb{Z}_2)^3_0$-grading of the quaternions $\mathbb{H} = 1\mathbb{R} \oplus \mathrm{i}\,\mathbb{R} \oplus \mathrm{j}\,\mathbb{R} \oplus \mathrm{k}\,\mathbb{R}$ is defined by:

$$\begin{array}{rcl} \widetilde{1} & = & (0,0,0)\,, \\ \widetilde{\mathrm{i}} & = & (0,1,1)\,, \\ \widetilde{\mathrm{j}} & = & (1,0,1)\,, \\ \widetilde{\mathrm{k}} & = & (1,1,0)\,, \end{array} \tag{7}$$

see [8] for more details.

Remark 2.2. It is natural to understand Clifford algebras as *even algebras*. Moreover, sometimes it is useful to consider larger graded algebras that contain a given Clifford algebra as an even part, see [23]. It is therefore natural to use the even grading (6).

3. $(\mathbb{Z}_2)^n$-Graded Trace

In this section we introduce the notion of *graded trace* of a matrix over a $(\mathbb{Z}_2)^n$-graded commutative algebra A that extends the notion of supertrace. Although the proof of the main result is quite elementary, this is the first important ingredient of our theory. Let us also mention that the notion of trace is missing in the literature on quaternionic matrices (as well as on matrices with coefficients in Clifford algebras).

3.1. Fundamental Theorem and Explicit Formula.

The first main result of this paper is as follows.

Theorem 1. *There exists a unique (up to multiplication by a scalar of weight* 0*) A-linear graded Lie algebra homomorphism*

$$\Gamma\mathrm{tr} : \mathsf{M}(\mathbf{r}; A) \to A\,,$$

defined for a homogeneous matrix X of degree x by

$$\Gamma\mathrm{tr}(X) = \sum_k (-1)^{\langle \gamma_k + x,\, \gamma_k \rangle}\, \mathrm{tr}(X_{kk})\,, \tag{8}$$

where tr *is the usual trace and where X_{kk} is a graded block of X, see formula (3).*

Let us stress the fact that the term "homomorphism" must be understood in the categorical sense and means homomorphism of weight 0. For any $X \in \mathsf{M}(\mathbf{r}; A)$, we refer to $\Gamma\mathrm{tr}(X) \in A$ as the graded trace of X. Of course, if A is a usual supercommutative ($\mathbb{Z}_2$-graded) algebra, we recover the classical supertrace str.

Proof. It is straightforward to check that formula (8) is A-linear and indeed defines a graded Lie algebra morphism. Let us prove uniqueness.

Recall that a homogeneous matrix $X \in \mathsf{M}^x(\mathbf{r}; A)$ is a matrix that contains $p \times p$ blocks X_{ku} of dimension $r_k \times r_u$ with entries in $A^{\gamma_k + \gamma_u + x}$. We denoted the entry (i, j) of block X_{ku}, located on block row k and block column u, by $(X_{ku})_{ij}$. Let us emphasize that if we view X as an ordinary $\sum_k r_k \times \sum_k r_k$ matrix, we denote its entries by $x_{\alpha\beta}$.

Let $E_{\alpha\beta} \in \mathsf{M}(\mathbf{r}; A)$ be the matrix containing 1_A in entry (α, β) and zero elsewhere. As any row index α determines a unique block row index k and therefore a unique weight $w_\alpha := \gamma_k$, matrix $E_{\alpha\beta}$ is homogeneous of weight $w_\alpha + w_\beta$. It is easily seen that $E_{\alpha\beta}E_{\eta\varepsilon}$ equals $E_{\alpha\varepsilon}$, if $\beta = \eta$, and vanishes otherwise.

In view of Equation (4), any graded matrix $X \in \mathsf{M}^x(\mathbf{r}; A)$ reads

$$X = \sum_{\alpha,\beta} (-1)^{\langle w_\alpha + w_\beta + x, w_\alpha \rangle} x_{\alpha\beta} E_{\alpha\beta} \, .$$

It follows from the graded A-module morphism property of the graded trace that this functional is completely determined by its values on the matrices $E_{\alpha\beta}$. Moreover, the graded Lie algebra property entails

$$\Gamma\mathrm{tr}(E_{\alpha\beta}) = \Gamma\mathrm{tr}(E_{\alpha 1}E_{1\beta}) = (-1)^{\langle w_\alpha + w_1, w_1 + w_\beta \rangle}\, \Gamma\mathrm{tr}(E_{1\beta}E_{\alpha 1}) = (-1)^{\langle w_\alpha, w_\beta \rangle} \delta_{\alpha\beta}\, \Gamma\mathrm{tr}(E_{11}) \, ,$$

where $\delta_{\alpha\beta}$ is Kronecker's symbol. When combining the two last results, we get

$$\Gamma\mathrm{tr}(X) = \Gamma\mathrm{tr}(E_{11}) \sum_\alpha (-1)^{\langle w_\alpha + x, w_\alpha \rangle} x_{\alpha\alpha} = \sum_k (-1)^{\langle \gamma_k + x, \gamma_k \rangle}\, \mathrm{tr}(X_{kk}) \, .$$

Hence the uniqueness. □

Remark 3.1. Thanks to its linearity, the graded trace (8) is well-defined for an *arbitrary* (not necessarily homogeneous or even) matrix X. This will not be the case for the graded determinant or graded Berezinian. In this sense, the notion of trace is more universal. On the other hand, in algebra conditions of invariance can always be formulated infinitesimally so that the trace often suffices.

In Section 8, we will give a number of examples of traces of quaternionic matrices.

3.2. **Application: Lax Pairs.** Let us give here just one application of Theorem 1.

Corollary 3.2. *Given two families of even matrices $X(t), Y(t) \in \mathsf{M}(\mathbf{r}; A)$ (smooth or analytic, etc.) in one real or complex parameter t satisfying the equation*

$$\frac{d}{dt} X = [X, Y] \ ,$$

the functions $\Gamma\mathrm{tr}(X)$, $\Gamma\mathrm{tr}(X^2), \ldots$ are independent of t.

Proof. The function $\Gamma\mathrm{tr}$ obviously commutes with $\frac{d}{dt}$, therefore

$$\frac{d}{dt}\,\Gamma\mathrm{tr}(X) = \Gamma\mathrm{tr}\left([X,Y]\right) = 0\,.$$

Furthermore, let us show that

$$\frac{d}{dt}\,\Gamma\mathrm{tr}(X^2) = \Gamma\mathrm{tr}\left([X,Y]\,X + X\,[X,Y]\right) = 0\,.$$

Indeed, by definition of the commutator (2) one has:

$$[X,Y]\,X = XYX - (-1)^{\langle y,x\rangle}YXX = [X,\,YX]$$

and

$$X\,[X,Y] = XXY - (-1)^{\langle y,x\rangle}XYX = [X,\,XY]$$

due to the assumption that X is even, i.e., $\langle x,x\rangle = 0$.

This argument can be generalized to prove that $\frac{d}{dt}\,\Gamma\mathrm{tr}(X^k) = 0$ for higher k, since we find by induction that

$$\frac{d}{dt}X^k = \sum_{i=1}^{k}\left[X, X^{i-1}YX^{k-i}\right]\,.$$

□

The above statement is an analog of the Lax representation that plays a crucial role in the theory of integrable systems. The functions $\Gamma\mathrm{tr}(X)$, $\Gamma\mathrm{tr}(X^2), \ldots$ are first integrals of the dynamical system $\frac{d}{dt}X = [X,Y]$ that often suffice to prove its integrability. Note that integrability in the quaternionic and more generally Clifford case is not yet understood completely. We hope that interesting examples of integrable systems can be found within the framework of graded algebra.

4. $(\mathbb{Z}_2)^n$-Graded Determinant of Purely Even Matrices of Degree 0

Let A be a *purely even* $(\mathbb{Z}_2)^n$-graded commutative i.e. a $(\mathbb{Z}_2)^n_0$-graded commutative algebra. We also refer to matrices over A as *purely even* $(\mathbb{Z}_2)^n$-graded or $(\mathbb{Z}_2)^n_0$-graded matrices. Their space will be denoted by $\mathsf{M}(\mathbf{r}_0; A)$, where $\mathbf{r}_0 \in \mathbb{N}^q$, $q = 2^{n-1}$.

4.1. Statement of the Fundamental Theorem.

As in usual Superalgebra, the case of purely even matrices is special, in the sense that we obtain a concept of determinant which is polynomial (unlike the general Berezinian).

Theorem 2. *(i) There exists a unique map* $\Gamma\mathrm{det} : \mathsf{M}^0(\mathbf{r}_0; A) \to A^0$ *that verifies:*

(1) *For all* $X, Y \in \mathsf{M}^0(\mathbf{r}_0; A)$*,*

$$\Gamma\mathrm{det}(XY) = \Gamma\mathrm{det}(X)\cdot\Gamma\mathrm{det}(Y)\,.$$

(2) *If* X *is block-diagonal, then*

$$\Gamma\mathrm{det}(X) = \prod_{k=1}^{q}\det(X_{kk})\,.$$

(3) *If* X *is block-unitriangular then* $\Gamma\mathrm{det}(X) = 1\,.$

(ii) The map $\Gamma\mathrm{det}(X)$ *of any matrix* $X \in \mathsf{M}^0(\mathbf{r}_0; A)$ *is linear in the rows and columns of* X *and therefore it is polynomial.*

We refer to $\Gamma\mathrm{det}(X) \in A^0$, $X \in \mathsf{M}^0(\mathbf{r}_0; A)$, as the *graded determinant* of X.

To prove the theorem, we first work formally (see [24]), i.e. we assume existence of inverse matrices of all square matrices. Hence, in one sense, we begin by working on an open dense subset of $\mathsf{M}^0(\mathbf{r}_0; A)$. We show that the graded determinant $\Gamma\mathrm{det}$ formally exists and is unique, then we use this result to give evidence of the fact that $\Gamma\mathrm{det}$ is polynomial. The latter polynomial will be our final definition of $\Gamma\mathrm{det}$ and Theorem 2 will hold true not only formally but in whole generality.

4.2. **Preliminaries.** To find a (formal) explicit expression of the graded determinant, we use an UDL decomposition. For matrices over a not necessarily commutative ring the entries of the UDL factors are tightly related to quasideteminants. To ensure independent readability of the present text, we recall the concept of quasideterminants (see [25] and [16] for a more detailed and extensive survey on the subject).

4.2.1. *Quasideterminants.* Quasideterminants are an important tool in Noncommutative Algebra; known determinants with noncommutative entries are products of quasiminors. For matrices over a non-commutative ring the entries of the UDL factors are tightly related to quasideterminants. In general, a quasideterminant is a rational function in its entries; in the commutative situation, a quasideterminant is not a determinant but a quotient of two determinants.

Let R be a unital (not necessarily commutative) ring and let $X \in \mathrm{gl}(r; R)$, $r \in \mathbb{N}\backslash\{0\}$. Denote by $X^{i,j}$, $1 \leq i, j \leq r$, the matrix obtained from X by deletion of row i and column j. Moreover, let r_i^j (resp., c_j^i) be the row i (resp., the column j) without entry x_{ij}.

Definition 4.1. If $X \in \mathrm{gl}(r; R)$ and if the submatrix $X^{i,j}$ is invertible over R, the *quasideterminant* (i, j) of X is the element $|X|_{ij}$ of R defined by

$$|X|_{ij} := x_{ij} - r_i^j (X^{i,j})^{-1} c_j^i \,. \tag{9}$$

Any partition $r = r_1 + r_1 + \cdots + r_p$ determines a $p \times p$ block decomposition $X = (X_{ku})_{1 \leq k,u \leq p}$ with square diagonal blocks. According to common practice in the literature on quasideterminants, the entries $(X_{ku})_{ij}$ of the block matrices X_{ku} are in the following numbered consecutively, i.e. for any fixed k, u,

$$1 + \sum_{l<k} r_l \leq i \leq \sum_{l \leq k} r_l \quad \text{and} \quad 1 + \sum_{l<u} r_l \leq j \leq \sum_{l \leq u} r_l \,.$$

The most striking property of quasideterminants is the *heredity principle*, i.e. "a quasideterminant of a quasideterminant is a quasideterminant". The following statement is proved in [16].

Heredity principle. *Consider a decomposition of $X \in \mathrm{gl}(r; R)$ with square diagonal blocks, a fixed block index k, and two fixed indices*

$$1 + \sum_{l<k} r_l \leq i \,, \; j \leq \sum_{l \leq k} r_l \,.$$

If the quasideterminant $|X|_{kk}$ is defined, then the quasideterminant $||X|_{kk}|_{ij}$ exists if and only if the quasideterminant $|X|_{ij}$ does. Moreover, in this case

$$|\, |X|_{kk} \,|_{ij} = |X|_{ij} \;. \tag{10}$$

Observe that $|X|_{kk}$ is a quasideterminant, not over a unital ring, but over blocks of varying dimensions. Such blocks can only be multiplied, if they have the appropriate dimensions, the inverse can exist only for square blocks. However, under the usual invertibility condition, all operations involved in this quasideterminant make actually sense and $|X|_{kk} \in \mathrm{gl}(r_k; R)$.

In the following, we will need the next corollary of the heredity principle.

Corollary 4.2. *Consider a decomposition of $X \in \mathrm{gl}(r; R)$ with square diagonal blocks, a block index k, and two indices i, a (resp., j, b) in the range of block X_{kk}, such that $i \neq a$ (resp., $j \neq b$). If $|X|_{kk}$ exists, the LHS of Equation (11) exists if and only if the RHS does, and in this case we have*

$$|(|X|_{kk})^{i,j}|_{ab} = \left|X^{i,j}\right|_{ab} . \tag{11}$$

Proof. Assume for simplicity that $k = 1$ and set

$$X = \begin{pmatrix} X_{11} & B \\ C & D \end{pmatrix} .$$

By definition,

$$|X|_{11} = X_{11} - BD^{-1}C .$$

Clearly,

$$(|X|_{11})^{i,j} = (X_{11})^{i,j} - B^{i,0}D^{-1}C^{0,j} ,$$

where 0 means that no column or row has been deleted, coincides with $\left|X^{i,j}\right|_{11}$. The claim now follows from the heredity principle. □

Heredity shows that quasideterminants handle matrices over blocks (square diagonal blocks assumed) just the same as matrices over a ring. Quasideterminants over blocks (square diagonal blocks) that have themselves entries in a field were studied in [24]. In view of the preceding remark, it is not surprising that the latter theory coincides with that of [16]. Especially the heredity principle holds true for decompositions of block matrices. Moreover, the nature of the block entries is irrelevant, so that the results are valid for block entries in a ring as well.

Example 4.3. Let

$$X = \begin{pmatrix} x & a & b \\ c & y & d \\ e & f & z \end{pmatrix} \in \mathrm{gl}(3, \mathrm{R}) ,$$

where R is as above a unital ring. In this example, we work formally, i.e. without addressing the question of existence of inverses. A short computation allows to see that the formal inverse of a 2×2 matrix over R is given by

$$\begin{pmatrix} y & d \\ f & z \end{pmatrix}^{-1} = \begin{pmatrix} (y - dz^{-1}f)^{-1} & -(y - dz^{-1}f)^{-1}dz^{-1} \\ -z^{-1}f(y - dz^{-1}f)^{-1} & z^{-1} + z^{-1}f(y - dz^{-1}f)^{-1}dz^{-1} \end{pmatrix} . \tag{12}$$

It then follows from the definition of quasideterminants that

$$|X|_{11} = x - bz^{-1}e - (a - bz^{-1}f)(y - dz^{-1}f)^{-1}(c - dz^{-1}e) . \tag{13}$$

Note that, when viewing matrix X as block matrix

$$X = \left(\begin{array}{cc|c} x & a & b \\ c & y & d \\ \hline e & f & z \end{array}\right) ,$$

with square diagonal blocks, the quasideterminant

$$||X|_{11}|_{11} = \left| \begin{array}{cc} x - bz^{-1}e & a - bz^{-1}f \\ c - dz^{-1}e & y - dz^{-1}f \end{array} \right|_{11} ,$$

where the interior subscripts refer to the 2×2 block decomposition can be obtained without Inversion Formula (12) and coincides with the quasideterminant (13), as claimed by the heredity principle.

Remark 4.4. This example corroborates the already mentioned fact that a quasideterminant with respect to the ordinary row-column decomposition (resp., to a block decomposition) is a rational expression (resp., a block of rational expressions) in the matrix entries.

4.2.2. *UDL Decomposition of Block Matrices with Noncommutative Entries.* An *UDL decomposition* of a square matrix is a factorization into an upper unitriangular (i.e., triangular and all the entries of the diagonal are equal to 1) matrix U, a diagonal matrix D, and a lower unitriangular matrix L. In this section we study existence and uniqueness of a block UDL decomposition for invertible block matrices with square diagonal blocks that have entries in a not necessarily commutative ring R.

Definition 4.5. An invertible block matrix $X = (X_{ku})_{k,u}$ with square diagonal blocks X_{kk} and entries $x_{\alpha\beta}$ in R is called *regular* if and only if it admits a block UDL decomposition.

Lemma 4.6. *If X is regular, its UDL decomposition is unique.*

Proof. If $UDL = X = U'D'L'$ are two such decompositions, then $U'^{-1}UD = D'L'L^{-1}$ and $\mathcal{U} = U'^{-1}U$ (resp., $\mathcal{L} = L'L^{-1}$) is an upper (resp., lower) unitriangular matrix. Since $\mathcal{U}D$ (resp., $D'\mathcal{L}$) is an upper (resp., lower) triangular matrix with diagonal D (resp., D'), we have $D = D'$. The invertibility of X entails that D is invertible, so $\mathcal{U} = D\mathcal{L}D^{-1}$, where the LHS (resp., RHS) is upper (resp., lower) unitriangular. It follows that $U = U'$ and $L = L'$. □

Proposition 4.7. *An invertible $p \times p$ block matrix $X = (X_{ku})_{k,u}$ with square diagonal blocks X_{kk} and entries $x_{\alpha\beta}$ in R is regular if and only if its principal block submatrices*

$$X^{1,1}, X^{12,12}, \ldots, X^{12\ldots(p-1),12\ldots(p-1)} = X_{pp}$$

are all invertible over R. In this case, X factors as $X = \mathfrak{U}D^{-1}\mathfrak{L}$, where

$$\mathfrak{U} = \begin{pmatrix} |X|_{11} & |X^{2,1}|_{12} & |X^{23,12}|_{13} & \ldots & X_{1p} \\ & |X^{1,1}|_{22} & |X^{13,12}|_{23} & \ldots & X_{2p} \\ & & |X^{12,12}|_{33} & \ldots & X_{3p} \\ & & & \ddots & \vdots \\ & & & & X_{pp} \end{pmatrix} ,$$

$$D = \begin{pmatrix} |X|_{11} & & & & \\ & |X^{1,1}|_{22} & & & \\ & & |X^{12,12}|_{33} & & \\ & & & \ddots & \\ & & & & X_{pp} \end{pmatrix},$$

and

$$\mathfrak{L} = \begin{pmatrix} |X|_{11} & & & & \\ |X^{1,2}|_{21} & |X^{1,1}|_{22} & & & \\ |X^{12,23}|_{31} & |X^{12,13}|_{32} & |X^{12,12}|_{33} & & \\ \vdots & \vdots & \vdots & \ddots & \\ X_{p1} & X_{p2} & X_{p3} & \dots & X_{pp} \end{pmatrix}.$$

Observe that $\mathfrak{U}_{ku} = |X^{1\ldots\hat{k}\ldots u,1\ldots(u-1)}|_{ku}$ *and* $\mathfrak{L}_{uk} = |X^{1\ldots(u-1),1\ldots\hat{k}\ldots u}|_{uk}$ *(for* $k \leq u$*). Matrix* X *factors also as*

$$X = UDL\,,$$

where

$$U = \mathfrak{U}D^{-1} \quad (\textit{resp.},\ L = D^{-1}\mathfrak{L})$$

is an upper (resp., lower) unitriangular matrix.

Proof. The cases $p = 1$ and $p = 2$ are straightforward. Indeed, if X is an invertible 2×2 block matrix, if we write for simplicity A (resp., B, C, D) instead of X_{11} (resp., X_{12}, X_{21}, X_{22}), denote identity blocks by $\mathbb{I}$, and if submatrix D is invertible, we have

$$\begin{aligned} \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} \mathbb{I} & BD^{-1} \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ D^{-1}C & \mathbb{I} \end{pmatrix} \\ &= \begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix}^{-1} \begin{pmatrix} A - BD^{-1}C & 0 \\ C & D \end{pmatrix} \end{aligned}$$

(note that the first equality is valid even if X is not necessarily invertible). Conversely, if X is regular, its UDL decomposition is necessarily the preceding one, so that D is actually invertible.

For $p = 3$, i.e. for an invertible 3×3 block matrix X such that $X^{1,1}, X^{12,12} = X_{33}$ are invertible, the UDL part of Proposition 4.7 states that matrix X is given by

$$\begin{pmatrix} \mathbb{I} & |X^{2,1}|_{12}|X^{1,1}|_{22}^{-1} & X_{13}X_{33}^{-1} \\ & \mathbb{I} & X_{23}X_{33}^{-1} \\ & & \mathbb{I} \end{pmatrix} \begin{pmatrix} |X|_{11} & & \\ & |X^{1,1}|_{22} & \\ & & X_{33} \end{pmatrix} \begin{pmatrix} \mathbb{I} & & \\ |X^{1,1}|_{22}^{-1}|X^{1,2}|_{21} & \mathbb{I} & \\ X_{33}^{-1}X_{31} & X_{33}^{-1}X_{32} & \mathbb{I} \end{pmatrix}.$$

Observe first that the proven UDL decomposition for $p = 2$, applied to $X^{1,1}$, entails that $|X^{1,1}|_{22}$ is invertible. It is now easily checked that all the expressions involved in the preceding 3×3 matrix multiplication make sense (in particular the quasideterminants of the rectangular matrices $X^{2,1}$ and $X^{1,2}$ are well-defined). The fact that this product actually equals X is proved by means of a

$$\begin{pmatrix} 2\times 2 & 2\times 1 \\ 1\times 2 & 1\times 1 \end{pmatrix}$$

redivision. The result is then a consequence of two successive applications of the aforementioned 2×2 UDL decomposition and of the heredity principle.

The $\mathfrak{U}D^{-1}\mathfrak{L}$ decomposition of X follows immediately from its just proven UDL decomposition (again, invertibility of X is not needed for the proof of the UDL decomposition). Conversely, if matrix X is regular, we see that $|X^{1,1}|_{22}$ and X_{33} are invertible, then, from the case $p=2$, that $X^{1,1}$ is invertible. The passage from $p>2$ to $p+1$ is similar to the passage from $p=2$ to $p+1=3$. □

4.3. **Explicit Formula in Terms of Quasideterminants.** Recall that a degree 0 $(\mathbb{Z}_2)^n_0$-graded matrix $X\in \mathsf{M}^0(\mathbf{r}_0;A)$ is a $q\times q$ block matrix, where $q:=2^{n-1}$ is the order of $(\mathbb{Z}_2)^n_0$. The entries of a block X_{ku} of X are elements of $A^{\gamma_k+\gamma_u}$. Every such matrix X admits (formally) an UDL decomposition with respect to its block structure.

It follows from Proposition 4.7 that, if the graded determinant of any $X\in\mathsf{M}^0(\mathbf{r}_0;A)$ exists, then it is equal to

$$\Gamma\mathrm{det}(X)=\prod_{k=0}^{q-1}\det|X^{1\ldots k\,,\,1\ldots k}|_{k+1\;k+1}\,. \tag{14}$$

Hence, the graded determinant is (formally) unique. Observe that for $n=1$ and $n=2$ it coincides with the classical determinant (for $n=2$, see UDL decomposition). This observation is natural, since the entries of the considered matrices are in these cases elements of a commutative subalgebra of A. Note also that the graded determinant defined by Equation (14) verifies Conditions (2) and (3) of Theorem 2. To prove (formal) existence, it thus suffices to check Condition (1) (for $n>2$).

The proof of multiplicativity of Γdet given by (14) is based on an induction on n that relies on an equivalent inductive expression of Γdet.

This expression is best and completely understood, if detailed for low n, e.g. $n=3$. As recalled above, a purely even $(\mathbb{Z}_2)^3$-graded matrix X of degree 0 is a 4×4 block matrix. The degrees of the blocks of X are

$$\left(\begin{array}{cc|cc} 000 & 011 & 101 & 110 \\ 011 & 000 & 110 & 101 \\ \hline 101 & 110 & 000 & 011 \\ 110 & 101 & 011 & 000 \end{array}\right), \tag{15}$$

so that, if we consider the suggested 2×2 redecomposition

$$\mathfrak{X}=\left(\begin{array}{c|c} \mathfrak{X}_{11} & \mathfrak{X}_{12} \\ \hline \mathfrak{X}_{21} & \mathfrak{X}_{22} \end{array}\right) \tag{16}$$

of X, the quasideterminant $|\mathfrak{X}|_{11}$ and the block $\mathfrak{X}_{22}$ (as well as products of the type $\mathfrak{X}_{12}\mathfrak{X}_{22}^{-1}\mathfrak{X}_{21}$, $\mathfrak{X}_{12}\mathfrak{X}_{21}, \ldots$ – this observation will be used below) can be viewed as purely even $(\mathbb{Z}_2)^{n-1}$-graded matrices of degree 0. It follows that the inductive expression

$$\Gamma\mathrm{det}(X) = \Gamma\mathrm{det}(|\mathfrak{X}|_{11}) \cdot \Gamma\mathrm{det}(\mathfrak{X}_{22}) \tag{17}$$

actually makes sense. To check its validity, observe that the RHS of (17) reads, for $n = 3$,

$$\det ||\mathfrak{X}|_{11}|_{11} \cdot \det |(|\mathfrak{X}|_{11})^{1,1}|_{22} \cdot \det |\mathfrak{X}_{22}|_{11} \cdot \det |(\mathfrak{X}_{22})^{1,1}|_{22}\,,$$

where the indices in $|\mathfrak{X}|_{11}$ and $\mathfrak{X}_{22}$ (resp., the other indices) correspond to the 2×2 redecomposition $\mathfrak{X}$ of X (resp., the 2×2 decomposition of $|\mathfrak{X}|_{11}$ and $\mathfrak{X}_{22}$). When writing this result using the indices of the 4×4 decomposition of X, as well as the Heredity Principle, see Equations (10) and (11), we get

$$\det |X|_{11} \cdot \det |X^{1,1}|_{22} \cdot \det |X^{12,12}|_{33} \cdot \det |X^{123,123}|_{44} = \Gamma\mathrm{det}(X)\,.$$

Let us mention that the LHS and RHS Γdet-s in Equation (17) are slightly different. The determinant in the LHS is the graded determinant of a purely even $(\mathbb{Z}_2)^n$-graded matrix, whereas those in the RHS are graded determinants of purely even $(\mathbb{Z}_2)^{n-1}$-graded matrices.

To prove multiplicativity of the graded determinant Γdet defined by (14) and (17), we need the next lemma.

Lemma 4.8. *Let X and Y be two $(\mathbb{Z}_2)^n_0$-graded matrices of degree 0 of the same dimension. If $\mathfrak{X}_{12}$ or $\mathfrak{Y}_{21}$ is elementary, i.e. denotes a matrix that contains a unique nonzero element, then*

$$\Gamma\mathrm{det}\left(\mathbb{I} + \mathfrak{X}_{12}\mathfrak{Y}_{21}\right) = \Gamma\mathrm{det}\left(\mathbb{I} + \mathfrak{Y}_{21}\mathfrak{X}_{12}\right)\,. \tag{18}$$

Proof. In view of the above remarks, it is clear that both graded determinants are (formally) defined. Assume now that $\mathfrak{X}_{12}$ is elementary and has dimension $R \times S$, use numerations from 1 to R and 1 to S, and denote the position of the unique nonzero element x by (r, s).

One has:

$$\mathbb{I} + \mathfrak{X}_{12}\mathfrak{Y}_{21} = \begin{pmatrix} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ &&& \ddots &&&& \\ xY_{s1} & \cdots & \cdots & 1 + xY_{sr} & \cdots & \cdots & xY_{sR} \\ &&&& \ddots &&& \\ &&&& 1 &&& \\ &&&&& \ddots && \\ &&&&&& 1 \end{pmatrix},$$

where the element $1 + xY_{sr}$ is located at position (r, r), and Equation (14) entails that

$$\Gamma\mathrm{det}\left(\mathbb{I} + \mathfrak{X}_{12}\mathfrak{Y}_{21}\right) = 1 + xY_{sr}\,.$$

A similar computation shows that

$$\Gamma\mathrm{det}\left(\mathbb{I} + \mathfrak{Y}_{21}\mathfrak{X}_{12}\right) = 1 + Y_{sr}x\,.$$

Since the elements x at position (r,s) in $\mathfrak{X}_{12}$ and Y_{sr} at position (s,r) in $\mathfrak{Y}_{21}$ have the same even degree, they commute. $\square$

We are now prepared to prove multiplicativity of $\Gamma\mathrm{det}$. Let us begin by stressing that we will have to consider $2^n \times 2^n$ UDL decompositions $X_U X_D X_L$, as well as 2×2 UDL decompositions $\mathfrak{X}_U \mathfrak{X}_D \mathfrak{X}_L$. Whereas an upper unitriangular matrix $\mathfrak{X}_U$ is also an upper unitriangular matrix X_U (and similarly for lower unitriangular matrices), a diagonal matrix X_D is also of the type $\mathfrak{X}_D$ (but the converses are not valid). Such details must of course be carefully checked in the following proof, but, to increase its readability, we refrain from explicitly mentioning them.

Assume now that multiplicativity holds true up to n ($n \geq 2$) and consider the case $n+1$. If X, Y denote $(\mathbb{Z}_2)_0^{n+1}$-graded matrices, we need to show that $\Gamma\mathrm{det}(XY) = \Gamma\mathrm{det}(X) \cdot \Gamma\mathrm{det}(Y)$.

(i) Let first Y be lower unitriangular and set $X = X_U X_D X_L$. Since $X_L Y$ is again lower unitriangular, we have

$$\Gamma\mathrm{det}(XY) = \Gamma\mathrm{det}(X_D) = \Gamma\mathrm{det}(X) \cdot \Gamma\mathrm{det}(Y) \ .$$

(ii) Assume now that Y is diagonal,

$$Y = \begin{pmatrix} \mathfrak{Y}_1 & \\ & \mathfrak{Y}_2 \end{pmatrix} ,$$

where $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ are $(\mathbb{Z}_2)_0^n$-graded, and let $X = \mathfrak{X}_U \mathfrak{X}_D \mathfrak{X}_L$,

$$\mathfrak{X}_D = \begin{pmatrix} \mathfrak{X}_1 & \\ & \mathfrak{X}_2 \end{pmatrix}, \qquad \mathfrak{X}_L = \begin{pmatrix} \mathbb{I} & \\ \mathfrak{X}_3 & \mathbb{I} \end{pmatrix} .$$

Then, $XY = \mathfrak{X}_U \ (\mathfrak{X}_D \, \mathfrak{Y}) \ \mathfrak{Z}$, with

$$\mathfrak{Z} = \begin{pmatrix} \mathbb{I} & \\ (\mathfrak{Y}_2)^{-1} \, \mathfrak{X}_3 \, \mathfrak{Y}_1 & \mathbb{I} \end{pmatrix} .$$

Since $\mathfrak{X}_D \mathfrak{Y}$ is block-diagonal, we get

$$\begin{aligned} \Gamma\mathrm{det}(XY) = \Gamma\mathrm{det} \begin{pmatrix} \mathfrak{X}_1\mathfrak{Y}_1 & \\ & \mathfrak{X}_2\mathfrak{Y}_2 \end{pmatrix} &= \Gamma\mathrm{det}(\mathfrak{X}_1\mathfrak{Y}_1) \cdot \Gamma\mathrm{det}(\mathfrak{X}_2\mathfrak{Y}_2) \\ &= \Gamma\mathrm{det}(\mathfrak{X}_D) \cdot \Gamma\mathrm{det}(Y) = \Gamma\mathrm{det}(X) \cdot \Gamma\mathrm{det}(Y) \ , \end{aligned}$$

by induction.

(iii) Let finally Y be upper unitriangular. It is easily checked that Y can be written as a finite product of matrices of the form

$$\begin{pmatrix} \mathbb{I} & \mathfrak{E} \\ & \mathbb{I} \end{pmatrix}, \qquad \begin{pmatrix} \mathfrak{U} & \\ & \mathbb{I} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbb{I} & \\ & \mathfrak{U} \end{pmatrix} , \tag{19}$$

where $\mathfrak{U}$ is upper unitriangular and $\mathfrak{E}$ is elementary. It thus suffices to consider a matrix Y of each one of the preceding "elementary forms". Moreover, it also suffices to prove multiplicativity for a lower unitriangular X.

Set

$$X = \begin{pmatrix} \mathfrak{X}_1 & \\ \mathfrak{X}_3 & \mathfrak{X}_2 \end{pmatrix} ,$$

where $\mathfrak{X}_1, \mathfrak{X}_2$ are lower unitriangular.

(a) If Y is of the first above elementary form,

$$XY = \begin{pmatrix} \mathfrak{X}_1 & \mathfrak{X}_1\mathfrak{E} \\ \mathfrak{X}_3 & \mathfrak{X}_2 + \mathfrak{X}_3\mathfrak{E} \end{pmatrix} .$$

Hence, using the induction and Lemma 4.8, we get

$$\begin{aligned} \Gamma\mathrm{det}(XY) &= \Gamma\mathrm{det}(\mathfrak{X}_2 + \mathfrak{X}_3\mathfrak{E}) \cdot \Gamma\mathrm{det}(\mathfrak{X}_1 - \mathfrak{X}_1\mathfrak{E}(\mathfrak{X}_2 + \mathfrak{X}_3\mathfrak{E})^{-1}\mathfrak{X}_3) \\ &= \Gamma\mathrm{det}(\mathfrak{X}_1) \cdot \Gamma\mathrm{det}(\mathfrak{X}_2 + \mathfrak{X}_3\mathfrak{E}) \cdot \Gamma\mathrm{det}(\mathbb{I} - \mathfrak{E}(\mathfrak{X}_2 + \mathfrak{X}_3\mathfrak{E})^{-1}\mathfrak{X}_3) \\ &= \Gamma\mathrm{det}(\mathfrak{X}_1) \cdot \Gamma\mathrm{det}(\mathfrak{X}_2 + \mathfrak{X}_3\mathfrak{E}) \cdot \Gamma\mathrm{det}(\mathbb{I} - (\mathfrak{X}_2 + \mathfrak{X}_3\mathfrak{E})^{-1}\mathfrak{X}_3\mathfrak{E}) \\ &= \Gamma\mathrm{det}(X) \cdot \Gamma\mathrm{det}(Y) . \end{aligned}$$

(b) If Y is of the second elementary form,

$$XY = \begin{pmatrix} \mathfrak{X}_1\mathfrak{U} & \\ \mathfrak{X}_3\mathfrak{U} & \mathfrak{X}_2 \end{pmatrix} ,$$

then

$$\Gamma\mathrm{det}(XY) = \Gamma\mathrm{det}(\mathfrak{X}_2) \cdot \Gamma\mathrm{det}(\mathfrak{X}_1) \cdot \Gamma\mathrm{det}(\mathfrak{U}) = \Gamma\mathrm{det}(X) \cdot \Gamma\mathrm{det}(Y) .$$

(c) If Y is of the last form, the proof of multiplicativity is analogous to that in (b).

This completes the proof of multiplicativity and thus of the formal existence and uniqueness of the graded determinant.

4.4. **Polynomial Structure.**

4.4.1. *Quasideterminants and Homological Relations.* Let as above $X \in \mathsf{M}^0(\mathbf{r}_0; A)$, let all the components of $\mathbf{r}_0$ be 1 (or 0), and set $r_0 = |\,\mathbf{r}_0|$. We will need the following lemma.

Lemma 4.9. *For $r \neq i$ and $s \neq j$, we have*

$$\big|X\big|_{ij}\big|X^{i,j}\big|_{rl} = \pm\,\big|X\big|_{il}\big|X^{i,l}\big|_{rj} \quad \textit{and} \quad \big|X\big|_{ij}\big|X^{i,j}\big|_{ks} = \pm\,\big|X\big|_{kj}\big|X^{k,j}\big|_{is} . \tag{20}$$

Proof. The result is a consequence of an equivalent definition of quasideterminants and of the *homological relations* [16]. More precisely, the quasideterminant $|X|_{ij}$ can be defined by $|X|_{ij} = (X^{-1})_{ji}^{-1}$. It follows that Definition 4.1 of quasideterminants reads

$$|X|_{ij} = x_{ij} - \sum_{\substack{a\neq i\\ b\neq j}} x_{ib}\,\big|X^{i,j}\big|_{ab}^{-1}\,x_{aj} .$$

An induction on the matrix dimension then shows that any quasideterminant $|X|_{ij}$ is homogeneous. Hence, the mentioned homological relations

$$\big|X\big|_{ij}\big|X^{i,l}\big|_{rj}^{-1} = -\big|X\big|_{il}\big|X^{i,j}\big|_{rl}^{-1} \quad \text{and} \quad \big|X^{k,j}\big|_{is}^{-1}\big|X\big|_{ij} = -\big|X^{i,j}\big|_{ks}^{-1}\big|X\big|_{kj} , \tag{21}$$

valid for $r \neq i, s \neq j$, are equivalent to (20). $\square$

Proposition 4.10. *Set*

$$D(X) = |X|_{11}|X^{1,1}|_{22} \cdot \ldots \cdot x_{r_0 r_0}$$

and let $(i_1, \ldots, i_{r_0})$, $(j_1, \ldots, j_{r_0})$ *be two permutations of* $(1, \ldots, r_0)$. *Then,*

$$D(X) = \pm\, |X|_{i_1 j_1} |X^{i_1,j_1}|_{i_2 j_2} \cdot \ldots \cdot x_{i_{r_0} j_{r_0}} = \pm\, |X|_{i_1 j_1} D(X^{i_1,j_1})\ .$$

Proof. It suffices to use Lemma 4.9. □

Proposition 4.11. *The product* $D(X)$ *is linear with respect to the rows and columns of* X.

Proof. For $r_0 = 1$, the claim is obvious. For $r_0 = 2$, we obtain

$$D(X) = D\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \big(x_{11} - x_{12}x_{22}^{-1}x_{21}\big)\, x_{22} = x_{11}x_{22} \pm x_{12}x_{21}\ .$$

Assume now that the statement holds up to $r_0 = n$ ($n \geq 2$) and consider the case $r_0 = n + 1$. We have

$$D(X) = |X|_{11}\, D\big(X^{1,1}\big)$$

and

$$|X|_{11} = x_{11} - \sum_{\substack{a \neq 1 \\ b \neq 1}} x_{1b}\, \big|X^{1,1}\big|_{ab}^{-1}\, x_{a1}\ ,$$

with

$$\big|X^{1,1}\big|_{ab}^{-1} = \pm\, D\Big(X^{1a,1b}\Big)\, D^{-1}\big(X^{1,1}\big)\ ,$$

due to Proposition 4.10. Therefore,

$$D(X) = x_{11} D\big(X^{1,1}\big) \pm \sum_{\substack{a \neq 1 \\ b \neq 1}} x_{1b} D\Big(X^{1a,1b}\Big)\, x_{a1}\ .$$

By induction, the products $D\big(X^{1,1}\big)$ and $D\big(X^{1a,1b}\big)$ are linear with respect to the rows and columns of their arguments. Hence, the result. □

4.4.2. *Preliminary Remarks.* (i) Consider a block matrix

$$W = \begin{pmatrix} A & 0 & B \\ C & D & E \\ F & 0 & G \end{pmatrix}$$

with square diagonal blocks over a unital ring. A straightforward computation allows to check that the formal inverse of W is given by

$$W^{-1} = \begin{pmatrix} A' & 0 & B' \\ -D^{-1}(CA' + EF') & D^{-1} & -D^{-1}(CB' + EG') \\ F' & 0 & G' \end{pmatrix}, \tag{22}$$

where

$$\begin{pmatrix} A' & B' \\ F' & G' \end{pmatrix} = \begin{pmatrix} A & B \\ F & G \end{pmatrix}^{-1}.$$

(ii) Let $X \in \mathsf{M}^0(\mathbf{r}_0; A)$, denote by $E_{\alpha\beta}(\lambda) \in \mathsf{M}^0(\mathbf{r}_0; A)$ the elementary matrix whose unique nonzero element $\lambda \in A^{w_\alpha + w_\beta}$ is located at position (α, β), $\alpha \neq \beta$ (remember that any row index α defines a unique block row index k and therefore a unique degree $w_\alpha := \gamma_k$), and set

$$G_{\alpha\beta}(\lambda) := \mathbb{I} + E_{\alpha\beta}(\lambda) \in \mathsf{M}^0(\mathbf{r}_0; A)\,.$$

The rows of the product matrix $X_{\text{red}} := G_{\alpha\beta}(\lambda)X \in \mathsf{M}^0(\mathbf{r}_0; A)$ are the same than those of X, except that its α-th row is the sum of the α-th row of X and of the β-th row of X left-multiplied by λ. Since $\Gamma\mathrm{det}$ is formally multiplicative and as it immediately follows from its definition that $\Gamma\mathrm{det}\,(G_{\alpha\beta}(\lambda)) = 1$, we get

$$\Gamma\mathrm{det}(X) = \Gamma\mathrm{det}(X_{\text{red}})\,. \tag{23}$$

(iii) In the following we write $X^{i:j}$, if we consider the matrix obtained from X by deletion of its i-th row $(x_{i1}, x_{i2} \ldots)$ and j-th column, whereas in $X^{i,j}$ the superscripts refer, as elsewhere in this text, to a block row and column. Subscripts characterizing quasideterminants should be understood with respect to the block decomposition.

Lemma 4.12. *Let*

$$X = \left(\begin{array}{c|c} x_{11} & \star\cdots\star \\ \hline 0 & \\ \vdots & X^{1:1} \\ 0 & \end{array}\right) \in \mathsf{M}^0(\mathbf{r}_0 + \mathbf{e}_1; A)\,,$$

where $\mathbf{e}_1 = (1, 0, \ldots, 0)$. *Then,*

$$\Gamma\mathrm{det}(X) = x_{11}\,\Gamma\mathrm{det}(X^{1:1})\,.$$

Proof. If the first component of $\mathbf{r}_0 \in \mathbb{N}^q$ vanishes, the result is obvious. Otherwise, it suffices to remember that $(|X|_{11})^{1:1} = |X^{1:1}|_{11}$, so that

$$|X|_{11} = \left(\begin{array}{c|c} x_{11} & *\cdots* \\ \hline 0 & \\ \vdots & |X^{1:1}|_{11} \\ 0 & \end{array}\right)\,.$$

Therefore,

$$\Gamma\mathrm{det}(X) = \det|X|_{11}\,\Gamma\mathrm{det}\left(X^{1,1}\right) = x_{11}\det\left|X^{1:1}\right|_{11}\Gamma\mathrm{det}\left((X^{1:1})^{1,1}\right) = x_{11}\,\Gamma\mathrm{det}(X^{1:1})\,.$$

Hence the lemma. □

4.4.3. *Proof of the Polynomial Character.* We are now ready to give the proof of Theorem 2, Part (ii).

Fix n, so $q = 2^{n-1}$ is fixed as well. We first consider the case $\mathbf{r}_0 \in \{0,1\}^{\times q}$. Since the quasideterminants in Definition (14) of $\Gamma\mathrm{det}$ are then quasideterminants over A valued in A^0, we have $\Gamma\mathrm{det}(X) = D(X)$, we conclude that $\Gamma\mathrm{det}(X)$ is linear in the rows and columns of X, due to Proposition 4.11.

To prove that $\Gamma\mathrm{det}(X)$, $X \in \mathsf{M}^0(\mathbf{r}_0; A)$, is linear for any $\mathbf{r}_0 \in \mathbb{N}^q$, it suffices to show that, if $\Gamma\mathrm{det}$ is linear for $\mathbf{r}_0 = (r_1, \ldots, r_q) \in \{0, 1, \ldots, R\}^{\times q}$, $R \geq 1$, then it is linear as well for $\mathbf{r}_0 + \mathbf{e}_\ell = (r_1, \ldots, r_\ell + 1, \ldots, r_q)$, with $r_\ell \neq 0$.

(i) We just mentioned that linearity of $\Gamma\mathrm{det}$ with respect to the rows and columns of its argument holds true for $R = 1$.

(ii) Suppose now that it is valid for some $\mathbf{r}_0$ and some $R \geq 1$.

(ii1) We first prove that linearity then still holds for $\mathbf{r}_0 + \mathbf{e}_1$. More precisely, we set $|\mathbf{r}| = r_1 + \ldots + r_q$, consider a matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1,N+1} \\ \vdots & \vdots & & \vdots \\ x_{k1} & x_{k2} & \ldots & x_{k,N+1} \\ 0 & x_{k+1,2} & \ldots & x_{k+1,N+1} \\ \vdots & \vdots & & \vdots \\ 0 & x_{N+1,2} & \ldots & x_{N+1,N+1} \end{pmatrix} \in \mathsf{M}^0(\mathbf{r}_0 + \mathbf{e}_1; A)\,, \tag{24}$$

and prove linearity of $\Gamma\mathrm{det}(X)$ by induction on k. To differentiate the two mentioned inductions, we speak about the induction in $|\mathbf{r}|$ and in k.

(a) For $k = 1$, Lemma 4.12 yields $\Gamma\mathrm{det}(X) = x_{11}\,\Gamma\mathrm{det}(X^{1:1})$ and the $|\mathbf{r}|$-induction assumption allows to conclude that $\Gamma\mathrm{det}(X)$ is linear.

(b) If $k = 2$, consider $G_{21}(-x_{21}x_{11}^{-1})$, where $\lambda = -x_{21}x_{11}^{-1}$ has the same degree as x_{21}. Matrix $X_{\mathrm{red}} = G_{21}(-x_{21}x_{11}^{-1})X$ has the form (24) with $k = 1$. Indeed, its rows are those of X, except for the second one, which reads

$$0\ ,\ x_{22} - x_{21}x_{11}^{-1}x_{12}\ ,\ x_{23} - x_{21}x_{11}^{-1}x_{13}\ ,\ \ldots$$

Hence, by (23) and (a),

$$\Gamma\mathrm{det}(X) = \Gamma\mathrm{det}(X_{\mathrm{red}}) = x_{11}\,\Gamma\mathrm{det}(X_{\mathrm{red}}^{1:1})$$

is linear in the rows and columns of X_{red}. This means in fact that $\Gamma\mathrm{det}(X)$ is linear with respect to the rows and columns of X.

(c) Assume now that $\Gamma\mathrm{det}(X)$, X of the form (24), is linear up to $k = \kappa \geq 2$ and examine the case $k = \kappa + 1$. We use the same idea as in (b), but have now at least two possibilities. The matrix

$$X_{\mathrm{red}}^1 = G_{\kappa+1,1}(-x_{\kappa+1,1}x_{11}^{-1})\,X \quad (\text{ resp., } \ X_{\mathrm{red}}^2 = G_{\kappa+1,2}(-x_{\kappa+1,1}x_{21}^{-1})\,X\ )$$

has the form (24) with $k = \kappa$ and, in view of the k-induction assumption, $\Gamma\mathrm{det}(X) = \Gamma\mathrm{det}(X_{\mathrm{red}}^1)$ is linear in the rows and columns of X_{red}^1 and thus contains at the worst x_{11}^{-1} (resp., $\Gamma\mathrm{det}(X) = \Gamma\mathrm{det}(X_{\mathrm{red}}^2)$ is linear and contains at the worst x_{21}^{-1}). It follows e.g. that

$$x_{21}\,\Gamma\mathrm{det}(X_{\mathrm{red}}^1) = x_{21}\,\Gamma\mathrm{det}(X_{\mathrm{red}}^2)$$

is polynomial in the entries of X, so that $\Gamma\mathrm{det}(X_{\mathrm{red}}^1)$ cannot contain x_{11}^{-1}. Therefore, $\Gamma\mathrm{det}(X) = \Gamma\mathrm{det}(X_{\mathrm{red}}^1)$ is linear in the rows and columns of $X \in \mathsf{M}^0(\mathbf{r}_0 + \mathbf{e}_1; A)$.

(ii2) The case $X \in \mathsf{M}^0(\mathbf{r}_0 + \mathbf{e}_\ell; A)$, $\ell \neq 1$, can be studied in a quite similar way. Indeed, consider first a matrix of the form

$$X=\left(\begin{array}{ccc|cccc} x_{11} & \dots & x_{1,m-1} & 0 & x_{1,m+1} & \dots & x_{1,N+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{m-1,1} & \dots & x_{m-1,m-1} & 0 & x_{m-1,m+1} & \dots & x_{m-1,N+1} \\ \hline x_{m1} & \dots & x_{m,m-1} & x_{mm} & x_{m,m+1} & \dots & x_{m,N+1} \\ x_{m+1,1} & \dots & x_{m+1,m-1} & 0 & x_{m+1,m+1} & \dots & x_{m+1,N+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{N+1,1} & \dots & x_{N+1,m-1} & 0 & x_{N+1,m+1} & \dots & x_{N+1,N+1} \end{array}\right) \in \mathsf{M}^0(\mathbf{r}_0+\mathbf{e}_\ell;A)\,,$$

where the suggested redecomposition corresponds to the redecomposition

$$(r_1,\dots,r_{\ell-1}\mid r_\ell+1,\dots,r_q)\,.$$

Remember that the determinant of X is defined as

$$\Gamma\!\det(X)=\prod_{i=0}^{q-1}\det\left|X^{1\dots i,1\dots i}\right|_{i+1\;i+1}\,.$$

It follows from (an obvious extension of) Lemma 4.12 that

$$\prod_{i=\ell}^{q-1}\det\left|X^{1\dots i,1\dots i}\right|_{i+1\;i+1}=x_{mm}\prod_{i=\ell}^{q-1}\det\left|(X^{m:m})^{1\dots i,1\dots i}\right|_{i+1\;i+1}\,.$$

Let now $i<\ell$ and let

$$\left|X^{1\dots i,1\dots i}\right|_{i+1\;i+1}=X_{i+1\;i+1}-UW^{-1}V$$

be the corresponding quasideterminant. Since the m-th column of U vanishes (we maintain the numeration of X),

$$UW^{-1}=U^{0:m}(W^{-1})^{m:0}\,,$$

where 0 means that no row or column has been deleted. Using Equation (22), we similarly find that

$$UW^{-1}V=U^{0:m}(W^{-1})^{m:m}V^{m:0}\;=U^{0:m}(W^{m:m})^{-1}V^{m:0}\,.$$

Hence,

$$\left|X^{1\dots i,1\dots i}\right|_{i+1\;i+1}=\left|(X^{m:m})^{1\dots i,1\dots i}\right|_{i+1\;i+1}$$

and

$$\Gamma\!\det(X)=x_{mm}\,\Gamma\!\det(X^{m:m})\,.$$

Linearity now follows from the $|\mathbf{r}|$-induction hypothesis. To pass from an elementary m-th column, containing a unique nonzero element, to an arbitrary one, it suffices to “fill” the elementary column downwards and upwards using the arguments detailed in (b) and (c) of (ii1). This then completes the proof of the polynomial structure of the Γ-determinant, as well as that of Theorem 2.

4.5. **Example.** The graded determinant of a matrix

$$X = \left(\begin{array}{c|c|c|c} x & a & b & c \\ \hline d & y & e & f \\ \hline g & h & z & l \\ \hline m & n & p & w \end{array}\right) \in \mathsf{M}^0\left((1,1,1,1);A\right)$$

over a $(\mathbb{Z}_2)^3$-graded commutative algebra A, is given by

$$\Gamma\mathrm{det}(X) = |X|_{11}\left|X^{1,1}\right|_{22}\left|X^{12,12}\right|_{33}\left|X^{123,123}\right|_{44}.$$

Of course,

$$\left|X^{123,123}\right|_{44} = w \quad \text{ and } \quad \left|X^{12,12}\right|_{33} = z - l\alpha p\,,$$

where $\alpha = w^{-1}$. Using Inversion Formula (12) and the graded commutativity of the multiplication in A, we easily find

$$\left|X^{1,1}\right|_{22} = \alpha\beta\left(y(zx-lp) - ehw + fph + eln - fnz\right)\,,$$

with $\beta = (z - l\alpha p)^{-1}$, so that the product of the three last factors of $\Gamma\mathrm{det}(X)$ is equal to

$$v := y(zx-lp) - ehw + fph + eln - fnz\,.$$

As concerns the quasideterminant $|X|_{11}$, the inverse of the involved (3×3)-matrix can be computed for instance by means of the UDL-decomposition of this matrix. After simplifications based on graded commutativity, we obtain

$$\begin{pmatrix} y & e & f \\ h & z & l \\ n & p & w \end{pmatrix}^{-1} = \begin{pmatrix} v^{-1}(zw-lp) & v^{-1}(fp-ew) & v^{-1}(el-fz) \\ v^{-1}(ln-hw) & v^{-1}(yw-fn) & v^{-1}(hf-ly) \\ v^{-1}(ph-zn) & v^{-1}(ne-py) & v^{-1}(yz-eh) \end{pmatrix}.$$

Finally,

$$\begin{aligned} |X|_{11} \;=\; & v^{-1}\left[xv - (a(zw-lp) + b(ln-hw) + c(ph-zn))\,d\right. \\ & - (a(fp-ew) + b(yw-fn) + c(ne-py))\,g \\ & \left.- (a(el-fz) + b(hf-ly) + c(yz-eh))\,m\right] \end{aligned}$$

and

$$\begin{array}{rcrcrcrcrcrcr} \Gamma\mathrm{det}(X) & = & xyzw & - & xylp & - & xehw & - & xfhp & + & xeln & - & xfzn \\ & & -adzw & + & adlp & + & aegw & + & afgp & - & aelm & + & afzm \\ & & -bdhw & + & bdln & - & bygw & + & bfgn & + & bylm & + & bfhm \\ & & -cdhp & - & cdzn & - & cygp & + & cegn & - & cyzm & + & cehm\,. \end{array} \tag{25}$$

Further examples are given in Section 8.

Remark 4.13. (a) As claimed by Theorem 2, Part (ii), the determinant $\Gamma\mathrm{det}(X)$ is linear in the rows and columns of X. Result (25) is thus analogous to the Leibniz formula for the classical determinant. Of course, signs are quintessentially different. It is worth noticing that, if we use the LDU-decomposition of X, which can be obtained in the exact same manner as the UDL-decomposition, see Proposition 4.7, we find that

$$\Gamma\mathrm{det}(X) = \prod_{k=1}^{q} \det |X^{k+1\ldots q\,,\,k+1\ldots q}|_{kk}\,, \tag{26}$$

where $q = 2^{n-1}$. For the preceding example, we thus get

$$\Gamma\mathrm{det}(X) = |X^{234,234}|_{11}\,|X^{34,34}|_{22}\,|X^{4,4}|_{33}\,|X|_{44}$$

and computations along the same lines as above actually lead exactly to Expression (25).

(b) The reader might wish to check by direct inspection that the polynomials $\Gamma\mathrm{det}(XY)$ and $\Gamma\mathrm{det}(X)\cdot\Gamma\mathrm{det}(Y)$ coincide. However, even the simplest example in the $(\mathbb{Z}_2)^3$-graded case involves over a hundred of terms. Such (computer-based) tests preceded the elaboration of our above proofs. These computations can of course not be reproduced here.

5. $(\mathbb{Z}_2)^n$-Graded Berezinian of Invertible Graded Matrices of Degree 0

Let A be a $(\mathbb{Z}_2)^n$-graded commutative algebra. Its even part A_0 is clearly $(\mathbb{Z}_2)^n_0$-graded commutative. In the preceding section, we investigated the determinant $\Gamma\mathrm{det}(X)$ of degree zero $(\mathbb{Z}_2)^n_0$-graded matrices $X \in \mathsf{M}^0(\mathbf{r}_0; A_0)$, with $\mathbf{r}_0 \in \mathbb{N}^q$, $q := 2^{n-1}$. Below we now define the determinant $\Gamma\mathrm{Ber}(X)$ of invertible degree zero $(\mathbb{Z}_2)^n$-graded matrices $X \in \mathsf{GL}^0(\mathbf{r}; A)$, with $\mathbf{r} \in \mathbb{N}^p$, $p := 2^n$. These matrices, which contain $p \times p$ blocks X_{ku} of dimension $r_k \times r_u$ with entries in $A^{\gamma_k+\gamma_u}$, can also be viewed as 2×2 block matrices $\mathcal{X}$, whose decomposition is given by the even and odd subsets of $(\mathbb{Z}_2)^n$.

5.1. Statement of the Fundamental Theorem.

Recall that, $(A^0)^\times$ denotes the group of units of the unital algebra A^0. Clearly, $(A^0)^\times = \mathsf{M}^0((1,0,\ldots,0); A)$. Our result is as follows.

Theorem 3. *There is a unique group homomorphism*

$$\Gamma\mathrm{Ber} : \mathsf{GL}^0(\mathbf{r}; A) \to (A^0)^\times$$

such that:

1. *For every 2×2 block-diagonal matrix $\mathcal{X} \in \mathsf{GL}^0(\mathbf{r}; A)$,*
$$\Gamma\mathrm{Ber}(\mathcal{X}) = \Gamma\mathrm{det}(\mathcal{X}_{11}) \cdot \Gamma\mathrm{det}^{-1}(\mathcal{X}_{22}) \in (A^0)^\times\,.$$
2. *The image of any lower (resp., upper) 2×2 block-unitriangular matrix in $\mathsf{GL}^0(\mathbf{r}; A)$ equals $1 \in (A^0)^\times$.*

We call $\Gamma\mathrm{Ber}(X)$, where $X \in \mathsf{GL}^0(\mathbf{r}; A)$, the *graded Berezinian* of X.

To prove the theorem, we will need the following lemma.

Lemma 5.1. *A homogeneous degree 0 matrix $X \in \mathsf{M}^0(\mathbf{r}; A)$ is invertible, i.e. $X \in \mathsf{GL}^0(\mathbf{r}; A)$, if and only if its diagonal blocks $\mathcal{X}_{11}$ and $\mathcal{X}_{22}$ are invertible.*

Proof. Let J be the ideal of A generated by the odd elements, i.e. the elements of the subspace

$$A_1 := \bigoplus_{\gamma \in (\mathbb{Z}_2)^n_1} A^\gamma \,.$$

Note that any element in J reads as finite sum $\sum_{k=1}^{s} a_k o_k$, where $a_k \in A$ and $o_k \in A_1$ are homogeneous. Therefore, we have $j^{s+1} = 0$, so that $1 \notin J$ and J is proper. We denote by $\bar{A}$ the associative unital quotient algebra $A/_J \neq \{\bar{0}\}$ and by $\bar{\ }: A \to \bar{A}$ the canonical surjective algebra morphism. This map induces a map on matrices over A, which sends a matrix Y with entries $y_{ij} \in A$ to the matrix $\bar{Y}$ with entries $\bar{y}_{ij} \in \bar{A}$.

It suffices to prove the claim: *A matrix Y over A is invertible if and only if $\bar{Y}$ over $\bar{A}$ is invertible.* Indeed, if X is a degree 0 graded matrix over A, its 2×2 blocks $\mathcal{X}_{12}$ and $\mathcal{X}_{21}$ have exclusively odd entries. Thus, X is invertible if and only if

$$\bar{\mathcal{X}} = \begin{pmatrix} \bar{\mathcal{X}}_{11} & \bar{0} \\ \bar{0} & \bar{\mathcal{X}}_{22} \end{pmatrix}$$

is, i.e. if and only if $\bar{\mathcal{X}}_{11}$ and $\bar{\mathcal{X}}_{22}$ are invertible, or better still, if and only if $\mathcal{X}_{11}$ and $\mathcal{X}_{22}$ are invertible.

As for the mentioned claim, if Y is invertible, then, clearly, $\bar{Y}$ is invertible as well. Conversely, assume $\bar{Y}$ invertible and focus for instance on the right inverses (arguments are the same for the left ones). There then exists a matrix Z over A such that $YZ = \mathbb{I} + W$, for some matrix W over J. Hence, matrix Y has a right inverse, if $\mathbb{I} + W$ is invertible, which happens if W is nilpotent. Note that $W^{S+1} = 0$, then

$$(\mathbb{I} + W)^{-1} = \mathbb{I} + \sum_{k=1}^{S} (-W)^k \,.$$

To see that matrix W over J is actually nilpotent, remark that there is a finite number of homogeneous odd elements $o_1, \ldots, o_S$ such that each entry of W reads $\sum_{k=1}^{S} a_k o_k$, with homogeneous $a_k \in A$. Hence, $W^{S+1} = 0$. □

5.2. **Explicit Expression.** As for the graded determinant, we will prove uniqueness and existence of the graded Berezinian by giving a necessary explicit formula and then proving that a homomorphism defined by means of this formula fulfills all the conditions of Theorem 3.

Proposition 5.2. *Let A be a $(\mathbb{Z}_2)^n$-graded commutative algebra, and let $\mathbf{r} \in \mathbb{N}^{2^n}$. The $(\mathbb{Z}_2)^n$-graded Berezinian of a matrix $X \in \mathsf{GL}^0(\mathbf{r}; A)$ is given by*

$$\Gamma\mathrm{Ber}(X) = \Gamma\mathrm{det}(|\mathcal{X}|_{11}) \cdot \Gamma\mathrm{det}^{-1}(\mathcal{X}_{22}) \,, \tag{27}$$

where $\mathcal{X}_{k\ell}$ refers to the 2×2 redivision of X.

Of course, for $n = 1$, we recover the classical Berezinian Ber.

We will use the following lemma.

Lemma 5.3. *If $X, Y \in \mathsf{M}^0(\mathbf{r}; A)$ and $\mathcal{X}_{12}$ or $\mathcal{Y}_{21}$ is elementary, then*

$$\Gamma\mathrm{det}\left(\mathbb{I} - \mathcal{X}_{12}\mathcal{Y}_{21}\right) = \Gamma\mathrm{det}\left(\mathbb{I} + \mathcal{Y}_{21}\mathcal{X}_{12}\right) \,.$$

Proof. Note first that both sides of the result actually make sense, since $\mathcal{X}_{12}\mathcal{Y}_{21}$ and $\mathcal{Y}_{21}\mathcal{X}_{12}$ are $(\mathbb{Z}_2)^n_0$-graded matrices of degree 0. The proof is analogous to that of Lemma 4.8; the sign change is due to oddness of the entries of the off-diagonal blocks of $\mathcal{X}$ and $\mathcal{Y}$. □

Proof of Proposition 5.2 and Theorem 3. In view of Lemma 5.1, any matrix $X \in \mathsf{GL}^0(\mathbf{r}; A)$ admits a 2×2 UDL decomposition. The conditions of Theorem 3 then obviously imply that if $\Gamma\mathrm{Ber}$ exists it is necessarily given by Equation (27).

To prove existence of the graded Berezinian map, it suffices to check that the map $\Gamma\mathrm{Ber}$ defined by Equation (27) is multiplicative and satisfies the requirements (1) and (2). Properties (1) and (2) are obvious. As for multiplicativity, let $X, Y \in \mathsf{GL}^0(\mathbf{r}; A)$ and consider their 2×2 UDL decomposition

$$\mathcal{X} = \mathcal{X}_U\mathcal{X}_D\mathcal{X}_L = \begin{pmatrix} \mathbb{I} & \mathcal{X}_{12}\mathcal{X}_{22}^{-1} \\ 0 & \mathbb{I} \end{pmatrix}\begin{pmatrix} |\mathcal{X}|_{11} & 0 \\ 0 & \mathcal{X}_{22} \end{pmatrix}\begin{pmatrix} \mathbb{I} & 0 \\ \mathcal{X}_{22}^{-1}\mathcal{X}_{21} & \mathbb{I} \end{pmatrix}$$

and

$$\mathcal{Y} = \mathcal{Y}_U\mathcal{Y}_D\mathcal{Y}_L = \begin{pmatrix} \mathbb{I} & \mathcal{Y}_{12}\mathcal{Y}_{22}^{-1} \\ 0 & \mathbb{I} \end{pmatrix}\begin{pmatrix} |\mathcal{Y}|_{11} & 0 \\ 0 & \mathcal{Y}_{22} \end{pmatrix}\begin{pmatrix} \mathbb{I} & 0 \\ \mathcal{Y}_{22}^{-1}\mathcal{Y}_{21} & \mathbb{I} \end{pmatrix}.$$

The product XY then reads

$$\mathcal{X}\mathcal{Y} = \mathcal{X}_U \begin{pmatrix} |\mathcal{X}|_{11}\,|\mathcal{Y}|_{11} & |\mathcal{X}|_{11}\,\mathcal{Y}_{12} \\ \mathcal{X}_{21}\,|\mathcal{Y}|_{11} & \mathcal{X}_{21}\mathcal{Y}_{12} + \mathcal{X}_{22}\mathcal{Y}_{22} \end{pmatrix}\mathcal{Y}_L\,. \tag{28}$$

Comparing now the 2×2 UDL decomposition of $XY \in \mathsf{GL}^0(\mathbf{r}; A)$ given by the formula used above for $\mathcal{X}$ and $\mathcal{Y}$, to that obtained via the 2×2 UDL decomposition of the central factor of the RHS of Equation (28) (which exists e.g. as $\mathcal{X}_{21}\mathcal{Y}_{12} + \mathcal{X}_{22}\mathcal{Y}_{22} = (\mathcal{X}\mathcal{Y})_{22}$ is invertible), we find in particular that

$$|\mathcal{X}\mathcal{Y}|_{11} = |\mathcal{X}|_{11}\,|\mathcal{Y}|_{11} - |\mathcal{X}|_{11}\,\mathcal{Y}_{12}\,(\mathcal{X}_{21}\mathcal{Y}_{12} + \mathcal{X}_{22}\mathcal{Y}_{22})^{-1}\,\mathcal{X}_{21}\,|\mathcal{Y}|_{11}\,.$$

Consequently,

$$\Gamma\mathrm{Ber}(XY) = \tag{29}$$
$$\Gamma\mathrm{det}\Big(|\mathcal{X}|_{11}\,|\mathcal{Y}|_{11} - |\mathcal{X}|_{11}\,\mathcal{Y}_{12}\,(\mathcal{X}_{21}\mathcal{Y}_{12} + \mathcal{X}_{22}\mathcal{Y}_{22})^{-1}\,\mathcal{X}_{21}\,|\mathcal{Y}|_{11}\Big)\cdot \Gamma\mathrm{det}^{-1}\,(\mathcal{X}_{21}\mathcal{Y}_{12} + \mathcal{X}_{22}\mathcal{Y}_{22})\,.$$

Clearly, if X is 2×2 upper unitriangular or diagonal, or Y is lower unitriangular or diagonal (i.e. $X = \mathcal{X}_U$ or $X = \mathcal{X}_D$, $Y = \mathcal{Y}_L$ or $Y = \mathcal{Y}_D$), Equation (29) reduces to

$$\Gamma\mathrm{Ber}(XY) = \Gamma\mathrm{Ber}(X)\cdot\Gamma\mathrm{Ber}(Y)\,, \tag{30}$$

in view of the multiplicativity of $\Gamma\mathrm{det}$.

Due to the just established right multiplicativity of $\Gamma\mathrm{Ber}$ for matrices of the type $\mathcal{Y}_L$ and $\mathcal{Y}_D$, it suffices to still prove right multiplicativity for matrices of the type $\mathcal{Y}_U$. Since any matrix $\mathcal{Y}_U$ reads as a finite product of matrices of the form

$$\mathcal{E}_U := \begin{pmatrix} \mathbb{I} & E \\ 0 & \mathbb{I} \end{pmatrix} \in \mathsf{GL}^0(\mathbf{r}; A)\,,$$

where E is elementary, we only need prove this right multiplicativity for $\mathcal{E}_U$. However, since we know already that the graded Berezinian is left multiplicative for matrices of the form $\mathcal{X}_U$ and $\mathcal{X}_D$, we can even confine ourselves to showing that Equation (30) holds true for

$$X = \mathcal{X}_L = \begin{pmatrix} \mathbb{I} & 0 \\ C & \mathbb{I} \end{pmatrix}$$

and $Y = \mathcal{E}_U$, i.e. to showing that $\Gamma\mathrm{Ber}(\mathcal{X}_L\mathcal{E}_U) = 1$.

By definition,

$$\Gamma\mathrm{Ber}(\mathcal{X}_L\mathcal{E}_U) = \Gamma\mathrm{det}(\mathbb{I} - E\,(\mathbb{I} + CE)^{-1}\,C) \cdot \Gamma\mathrm{det}^{-1}(\mathbb{I} + CE)\,.$$

It is easily checked that any entry of CE and EC vanishes or is a multiple of the unique nonzero element of E. Since this element is odd, it squares to zero and $(CE)^2 = (EC)^2 = 0$. This implies in particular that $(\mathbb{I} + CE)^{-1} = \mathbb{I} - CE$, so that

$$\Gamma\mathrm{Ber}(\mathcal{X}_L\mathcal{E}_U) = \Gamma\mathrm{det}\,(\mathbb{I} - EC) \cdot \Gamma\mathrm{det}^{-1}\,(\mathbb{I} + CE)\ .$$

When combining the latter result with the consequence

$$\Gamma\mathrm{det}(\mathbb{I} - EC) = \Gamma\mathrm{det}(\mathbb{I} + CE)$$

of Lemma 5.3, we eventually get $\Gamma\mathrm{Ber}(\mathcal{X}_L\mathcal{E}_U) = 1$. This completes the proofs of Proposition 5.2 and Theorem 3. □

6. Liouville Formula

In this section we explain the relation between the graded trace and the graded Berezinian.

6.1. **Classical Liouville Formulas.**

The well-known Liouville formula

$$\det(\exp(X)) = \exp(\mathrm{tr}(X))\,, \tag{31}$$

$X \in \mathrm{gl}(r, \mathbb{C})$, expresses the fact that the determinant is the group analog of the trace. A similar statement

$$\mathrm{Ber}(\mathbb{I} + \varepsilon X) = 1 + \varepsilon\,\mathrm{str}(X)\,,$$

where ε is an even parameter such that $\varepsilon^2 = 0$ and X an even matrix, holds true in Superalgebra, see [26], [27], [21]. Moreover, if A denotes the Grassmann algebra generated over a commutative field $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, by a finite number of anticommuting parameters $\xi_1, \ldots, \xi_q$, i.e. if $A = \mathbb{K}[\xi_1, \ldots, \xi_q]$, we have

$$\frac{d}{dt}\,\mathrm{Ber}(X) = \mathrm{str}(M)\,\mathrm{Ber}(X)\ ,$$

if $\frac{d}{dt}X = MX$, where t is a real variable and $X = X(t)$ (resp., M) is an invertible even (resp., an even) matrix over A, see [28]. The super counterpart

$$\mathrm{Ber}(\exp(X)) = \exp(\mathrm{str}(X))\,,$$

is valid if X is nilpotent and in contexts where the exponential series converge, see [27]. For another variant of the correspondence between the Berezinian and the supertrace, using a formal parameter, see [29].

6.2. **Graded Liouville Formula.** In this section, we extend the preceding relationship to $\Gamma\mathrm{Ber}$ and $\Gamma\mathrm{tr}$. We introduce a formal nilpotent degree 0 parameter ε and work over the set $A[\varepsilon]$ of formal polynomials in ε with coefficients in a $(\mathbb{Z}_2)^n$-graded commutative algebra A over reals. It is easily seen that $A[\varepsilon]$ is itself a $(\mathbb{Z}_2)^n$-graded commutative algebra, so that the graded Berezinian and graded trace do exist over $A[\varepsilon]$.

For any $\mathbf{s} \in \mathbb{N}^p$ and any $M \in \mathsf{M}^0(\mathbf{s}; A)$, we have $\varepsilon M \in \mathsf{M}^0(\mathbf{s}; A[\varepsilon])$ and we define

$$\exp(\varepsilon M) := \sum_k \varepsilon^k \frac{M^k}{k!} \in \mathsf{M}^0(\mathbf{s}; A[\varepsilon]) \, .$$

Moreover, if t denotes a real variable, it is straightforward that

$$\frac{d}{dt} \exp(t\,\varepsilon M) = (\varepsilon M) \exp(t\,\varepsilon M) \; . \tag{32}$$

In particular, if $X \in \mathsf{GL}^0(\mathbf{r}; A)$, we get

$$\varepsilon\, \Gamma\mathrm{tr}(X) = \Gamma\mathrm{tr}(\varepsilon X) \in A^0[\varepsilon] = \mathsf{M}^0((1, 0, \ldots, 0); A[\varepsilon]) \, ,$$

so that

$$\exp(\Gamma\mathrm{tr}(\varepsilon X)) \in (A^0[\varepsilon])^\times \quad \text{and} \quad \Gamma\mathrm{Ber}(\exp(\varepsilon X)) \in (A^0[\varepsilon])^\times \, .$$

Theorem 4. *If ε denotes a formal nilpotent parameter of degree 0 and X a graded matrix of degree 0, we have*

$$\Gamma\mathrm{Ber}(\exp(\varepsilon X)) = \exp\left(\Gamma\mathrm{tr}(\varepsilon X)\right) \, .$$

Some preliminary results are needed to prove the preceding theorem. Let $X = X(t), M = M(t) \in \mathsf{M}^0(\mathbf{r}_0; A)$, where t runs through an open internal $I \subset \mathbb{R}$, let A be finite-dimensional, and assume that the dependence of $X = X(t)$ on t is differentiable.

Lemma 6.1. *If $\frac{d}{dt}X = MX$, then*

$$\frac{d}{dt} \Gamma\!\det(X) = \Gamma\mathrm{tr}(M)\, \Gamma\!\det(X) \, .$$

Proof. Denote by $|\mathbf{r}| = r_1 + \ldots + r_q$ the total matrix dimension and set $X = (x_{ij})$, $M = (m_{ij})$. As $\Gamma\!\det(X)$ is linear in the rows and columns of X, we have

$$\frac{d}{dt} \Gamma\!\det(X) = \sum_{i=1}^{|\mathbf{r}|} \Gamma\!\det(X_{i'}) \, ,$$

where $X_{i'}$ is the matrix X with i-th row derived with respect to t. For any fixed i and any $j \neq i$, the matrix $G_{ij}(-m_{ij}) \in \mathsf{M}^0(\mathbf{r}_0; A)$, see Proof of Theorem 2, Part (ii), has graded determinant 1. Hence, the graded determinant of $G_{ij}(-m_{ij})\, X_{i'}$, $j \neq i$, coincides with that of $X_{i'}$, although we subtract from the i-th row of $X_{i'}$ its j-th row left-multiplied by m_{ij}. In view of the assumption in Lemma 6.1,

$$\frac{d}{dt} x_{ia} = \sum_j m_{ij} x_{ja} \, ,$$

for all a. Consequently, the i-th row of $\prod_{j \neq i} G_{ij}(-m_{ij})\, X_{i'}$ contains the elements

$$\frac{d}{dt} x_{ia} - \sum_{j \neq i} m_{ij} x_{ja} = m_{ii} x_{ia} \, , \quad a \in \{1, \ldots, N\}$$

and
$$\prod_{j\neq i} G_{ij}(-m_{ij})\, X_{i'} = \mathrm{diag}(1,\ldots,1,m_{ii},1,\ldots,1)\, X\,,$$
with self-explaining notation. It follows that
$$\frac{d}{dt}\,\Gamma\mathrm{det}(X) = \sum_{i=1}^{|\mathbf{r}|} \Gamma\mathrm{det}(\prod_{j\neq i} G_{ij}(-m_{ij})\, X_{i'}) = \sum_{i=1}^{|\mathbf{r}|} m_{ii}\,\Gamma\mathrm{det}(X) = \Gamma\mathrm{tr}(M)\,\Gamma\mathrm{det}(X)\,.$$
□

Take now $X = X(t), M = M(t) \in \mathsf{GL}^0(\mathbf{r};A)$, $t \in I \subset \mathbb{R}$, where $X = X(t)$ depends again differentiably on t and is invertible for any t.

Lemma 6.2. *If* $\frac{d}{dt}X = MX$*, then*
$$\frac{d}{dt}\,\Gamma\mathrm{Ber}(X) = \Gamma\mathrm{tr}(M)\,\Gamma\mathrm{Ber}(X)\,.$$

The proof is exactly the same as for the classical Berezinian [28]. We reproduce it here to ensure independent readability of this text.

Proof. Set $Y := |\mathcal{X}|_{11}$ and $Z := \mathcal{X}_{22}^{-1}$. A short computation shows that
$$\frac{d}{dt}Y = (\mathcal{M}_{11} - \mathcal{X}_{12}\mathcal{X}_{22}^{-1}\mathcal{M}_{21})\,Y =: P\,Y \quad \text{and} \quad \frac{d}{dt}Z = -Z\,(\mathcal{M}_{21}\mathcal{X}_{12}\mathcal{X}_{22}^{-1} + \mathcal{M}_{22}) =: -Z\,Q\,.$$
Since, from Lemma 6.1, we now get
$$\frac{d}{dt}\,\Gamma\mathrm{det}(Y) = \Gamma\mathrm{tr}(P)\;\Gamma\mathrm{det}(Y) \quad \text{and} \quad \frac{d}{dt}\,\Gamma\mathrm{det}(Z) = -\,\Gamma\mathrm{tr}(Q)\;\Gamma\mathrm{det}(Z)\,,$$
we obtain
$$\begin{aligned}\frac{d}{dt}\,\Gamma\mathrm{Ber}(X) &= \left(\frac{d}{dt}\,\Gamma\mathrm{det}(Y)\right)\cdot\Gamma\mathrm{det}(Z) + \Gamma\mathrm{det}(Y)\cdot\left(\frac{d}{dt}\,\Gamma\mathrm{det}(Z)\right)\\ &= \Big(\Gamma\mathrm{tr}(P) - \Gamma\mathrm{tr}(Q)\Big)\;\Gamma\mathrm{det}(Y)\cdot\Gamma\mathrm{det}(Z)\,.\end{aligned}$$
It follows from the Lie algebra homomorphism property of $\Gamma\mathrm{tr}$ (see Theorem 1) that, for every $X \in \mathsf{M}^0(\mathbf{r};A)$, one has
$$\Gamma\mathrm{tr}(\mathcal{X}_{12}\mathcal{X}_{21}) = -\,\Gamma\mathrm{tr}(\mathcal{X}_{21}\mathcal{X}_{12})\,.$$
Using the preceding equation and recalling that above $\mathcal{M}_{22}$ is viewed as a purely even degree 0 matrix, we see that
$$\Gamma\mathrm{tr}(P) - \Gamma\mathrm{tr}(Q) = \Gamma\mathrm{tr}(\mathcal{M}_{11}) - \Gamma\mathrm{tr}(\mathcal{M}_{22}) = \Gamma\mathrm{tr}(M)\,.$$
Hence Lemma 6.2. □

Proof of Theorem 4. It follows from Equation (32) that
$$\frac{d}{dt}\exp(t\,\varepsilon X) = (\varepsilon X)\exp(t\,\varepsilon X) \quad \text{and} \quad \frac{d}{dt}\exp(t\,\Gamma\mathrm{tr}(\varepsilon X)) = \Gamma\mathrm{tr}(\varepsilon X)\exp(t\,\Gamma\mathrm{tr}(\varepsilon X))\,.$$
However,
$$\frac{d}{dt}\,\Gamma\mathrm{Ber}(\exp(t\,\varepsilon X)) = \Gamma\mathrm{tr}(\varepsilon X)\,\Gamma\mathrm{Ber}(\exp(t\,\varepsilon X))\,,$$
due to Lemma 6.2. It now suffices to observe that both solutions $\exp(t\,\Gamma\mathrm{tr}(\varepsilon X))$ and $\Gamma\mathrm{Ber}(\exp(t\,\varepsilon X))$ of the equation $\frac{d}{dt}y = \Gamma\mathrm{tr}(\varepsilon X)\,y$ coincide at 0. □

7. $(\mathbb{Z}_2)^n$-Graded Determinant over Quaternions and Clifford Algebras

It this section we obtain the results specific for the Clifford Algebras and, in particular, for the algebra of quaternions. We show that, in the quaternionic case, the graded determinant is related to the classical Dieudonné determinant. We then examine whether the graded determinant can be extended to (purely even) homogeneous matrices of degree $\gamma \neq 0$. It turns out that this is possible under the condition that the global dimension is equal to 0 or 1 modulo 4.

7.1. **Relation to the Dieudonné Determinant.**

In this section the algebra A is the classical algebra $\mathbb{H}$ of quaternions equipped with the $(\mathbb{Z}_2)^3_0$-grading (see Example 2.1). It turns out that the graded determinant of a purely even homogeneous quaternionic matrix of degree 0 coincides (up to a sign) with the classical Dieudonné determinant.

Proposition 7.1. *For any matrix $X \in \mathsf{M}^0(\mathbf{r}_0;\mathbb{H})$ of degree 0, the absolute value of the graded determinant coincides with the Dieudonné determinant:*

$$|\Gamma\mathrm{det}(X)| = \mathrm{Ddet}(X)\,.$$

Proof. We will first show that the graded determinant can be written as a product of quasiminors

$$\Gamma\mathrm{det}(X) = |X|_{i_1j_1}|X^{i_1:j_1}|_{i_2j_2}\dots x_{i_Nj_N}\,,$$

for appropriate permutations $I = (i_1,\dots,i_N), J = (j_1,\dots,j_N)$ of $(1,\dots,N)$,

$$|\mathbf{r}| = |\mathbf{r}_0| = r_1 + \dots + r_4\,.$$

and then compare this formula with the classical Dieudonné determinant.

Following [4], [16], define the *predeterminants* of X by

$$D_{IJ}(X) := |X|_{i_1j_1}|X^{i_1:j_1}|_{i_2j_2}\ \dots\ x_{i_Ni_N}\ \in \mathbb{H}$$

where $I = (i_1,\dots,i_N)$ and $J = (j_1,\dots,j_N)$ are some are permutations of $(1,\dots,N)$. It is shown in the above references, that the $D_{IJ}(X)$ are polynomial expressions with real coefficients in the entries x_{ij} and their conjugates $\overline{x}_{ij}$. Moreover, for any of these permutations I, J, the *Dieudonné determinant* $\mathrm{Ddet}(X)$ of X is given by

$$\mathrm{Ddet}(X) = ||D_{IJ}(X)||\,, \tag{33}$$

where $||-||$ denotes the quaternionic norm.

Observe that in our case, $X \in \mathsf{GL}^0(\mathbf{r}_0;\mathbb{H})$, (i.e., X is an invertible 4×4 block matrix with square diagonal blocks, such that the entries of block X_{ku} are elements of $\mathbb{H}^{\gamma_k+\gamma_u}$), the entries x_{ij} of X and their conjugates coincide (up to sign). Hence, every $D_{IJ}(X)$ is polynomial in the entries x_{ij}. Moreover, these polynomials are clearly valued in $\mathbb{R}$. Therefore,

$$\mathrm{Ddet}(X) = |D_{IJ}(X)|\,, \tag{34}$$

where $|-|$ is the absolute value of real numbers.

For any matrix $X \in \mathsf{M}^0(\mathbf{r}_0;\mathbb{H})$, we obtain the graded determinant of X by writing the rational expression

$$\det|X|_{11}\det\left|X^{1,1}\right|_{22}\det\left|X^{12,12}\right|_{33}\det X_{44} \tag{35}$$

which is, indeed, a polynomial (see Theorem 2, Part (ii)). Let us recall that, for a matrix C with commutative entries, a quasideterminant is a ratio of classical determinants [16]:

$$\det C = (-1)^{a+b}|C|_{ab}\det(C^{a:b})\,.$$

When applying this result iteratively to the determinants in (35), we get

$$\det|X^{1\dots k,1\dots k}|_{k+1\;k+1} = \pm \prod_{i=0}^{r_{k+1}-1} \left| \left(|X^{1\dots k,1\dots k}|_{k+1\;k+1} \right)^{1\dots i:1\dots i} \right|_{i+1\;i+1} .$$

Corollary 4.2 now entails that the rational expression (35) coincides with

$$\pm|X|_{11}|X^{1:1}|_{22}\dots x_{NN} = \pm\, D(X) = \pm|X|_{i_1j_1}|X^{i_1:j_1}|_{i_2j_2}\dots x_{i_Nj_N} = \pm\, D_{IJ}(X)\,,$$

see Proposition 4.10, for any permutations $I = (i_1, \dots, i_N), J = (j_1, \dots, j_N)$ of $(1, \dots, N)$. However, see Equation (34), for appropriate permutations I, J, the product $\pm\, D_{IJ}(X) \in \mathbb{R}$ is polynomial and thus coincides with $\Gamma\mathrm{det}(X)$. □

7.2. **Graded Determinant of Even Homogeneous Matrices of Arbitrary Degree.** In this section, A denotes a $(\mathbb{Z}_2)^n_0$-graded commutative associative unital algebra, such that each subspace A^γ contains at least one invertible element. Every Clifford algebra satisfies the required property since it is graded division algebra, see Section 2.2.

Consider a homogeneous matrix $X \in \mathsf{M}^\gamma(\mathbf{r}_0; A)$, where $\gamma \in (\mathbb{Z}_2)^n_0$ is not necessarily equal to 0. Every such matrix can be written (in many different ways) in the form $X = q\,X_0$, where X_0 is homogeneous of degree 0 and $q \in A$ is invertible. We define the *graded determinant* of X by

$$\Gamma\mathrm{det}(X) := q^{|\mathbf{r}|}\,\Gamma\mathrm{det}(X_0) \tag{36}$$

with values in $A^{|\mathbf{r}|\,\gamma}$.

Let us first check that the graded determinant is *well-defined.* Given two invertible elements $q, q' \in A^\gamma$, one has two different expressions: $X = (q\mathbb{I})X_0 = (q'\mathbb{I})X'_0$. Since

$$X_0 = (q^{-1}\mathbb{I})(q'\mathbb{I})X'_0 = \left(q^{-1}q'\mathbb{I}\right)X'_0\,,$$

where both factors of the RHS are of degree 0, we obtain

$$q^{|\mathbf{r}|}\,\Gamma\mathrm{det}(X_0) = q^{|\mathbf{r}|}(q^{-1}q')^{|\mathbf{r}|}\,\Gamma\mathrm{det}(X'_0) = q'^{|\mathbf{r}|}\,\Gamma\mathrm{det}(X'_0)\,.$$

Therefore, Formula (36) is independent of the choice elements q.

Proposition 7.2. *The graded determinant (36) is multiplicative:*

$$\Gamma\mathrm{det}(XY) = \Gamma\mathrm{det}(X)\cdot\Gamma\mathrm{det}(Y)\,.$$

for any purely even homogeneous $(|\mathbf{r}|\times|\mathbf{r}|)$*-matrices* X, Y*, if and only if* $|\mathbf{r}| = 0, 1 \pmod 4$.

Proof. Recall that the A-module structure (4) of the space $\mathsf{M}(\mathbf{r}; A)$ is compatible with the associative algebra structure in the sense that, for any $a, b \in A$, and matrices $X \in \mathsf{M}^x(\mathbf{r}; A)$, and $Y \in \mathsf{M}(\mathbf{r}; A)$, we have

$$(aX)(bY) = (-1)^{\langle\tilde{b},x\rangle}(ab)(XY)\,.$$

Let X and Y be two purely even graded matrices of even degree γ_ℓ and γ_m, respectively. We then have

$$\Gamma\mathrm{det}(X)\,\Gamma\mathrm{det}(Y) = q_\ell^{|\mathbf{r}|}q_m^{|\mathbf{r}|}\,\Gamma\mathrm{det}(X_0)\,\Gamma\mathrm{det}(Y_0)$$

and, since $XY = (q_\ell q_m\mathbb{I})(X_0Y_0)$, we get

$$\Gamma\mathrm{det}(XY) = (q_\ell q_m)^{|\mathbf{r}|}\,\Gamma\mathrm{det}(X_0)\,\Gamma\mathrm{det}(Y_0) = (-1)^{\frac{|\mathbf{r}|(|\mathbf{r}|-1)}{2}}\,q_\ell^{|\mathbf{r}|}q_m^{|\mathbf{r}|}\;\Gamma\mathrm{det}(X_0)\,\Gamma\mathrm{det}(Y_0)\,.$$

Therefore, multiplicativity is equivalent to the condition $(-1)^{\frac{|\mathbf{r}|(|\mathbf{r}|-1)}{2}} = 1$, that holds if and only if $|\mathbf{r}| = 0, 1 \pmod 4$. □

Remark 7.3. It is well-known that the classical super determinant can be extended to odd matrices, only if the numbers p of even and q of odd dimensions coincide, hence only if the total dimension $|\mathbf{r}| = p + q = 0 \pmod 2$. Although our situation is not completely analogous, this can explain that a condition on the total dimension shows up in our situation.

8. Examples of Quaternionic $(\mathbb{Z}_2)^n$-Graded Determinants

In this last section we present several examples of matrices, their traces and determinants, in the $(\mathbb{Z}_2)^3$-graded case. A natural source of such matrices is provided by endomorphisms of modules over the classical algebra $\mathbb{H}$ of quaternions equipped with the $(\mathbb{Z}_2)^3_0$-grading (7). Although this section is based on the general theory developed in the present work, it can be read independently and might provide some insight into the more abstract aspects of this text.

8.1. **Quaternionic Matrices of Degree Zero.**

The examples given in this section are obtained by straightforward computations that we omit.

8.1.1. *Matrix Dimension* $|\mathbf{r}| = 4 = 1+1+1+1$. The first interesting case of $(\mathbb{Z}_2)^3$-graded matrices is that of dimension $|\mathbf{r}| = 4$. More precisely, let V be a real 4-dimensional vector space, graded by the even elements of $(\mathbb{Z}_2)^3$:

$$V = V_{(0,0,0)} \oplus V_{(0,1,1)} \oplus V_{(1,0,1)} \oplus V_{(1,1,0)}\,. \tag{37}$$

Each of the preceding subspaces is 1-dimensional. We then define a $(\mathbb{Z}_2)^3$-graded $\mathbb{H}$-module $M = V \otimes_{\mathbb{R}} \mathbb{H}$. A homogeneous degree $(0,0,0)$ endomorphism of M is then represented by a matrix of the form

$$X = \left(\begin{array}{c|c|c|c} x & a\,\mathrm{i} & b\,\mathrm{j} & c\,\mathrm{k} \\ \hline d\,\mathrm{i} & y & e\,\mathrm{k} & f\,\mathrm{j} \\ \hline g\,\mathrm{j} & h\,\mathrm{k} & z & \ell\,\mathrm{i} \\ \hline m\,\mathrm{k} & n\,\mathrm{j} & p\,\mathrm{i} & w \end{array}\right),$$

where the coefficients $x, a, \ldots, w$ are real numbers and where $\mathrm{i}, \mathrm{j}, \mathrm{k} \in \mathbb{H}$ stand for the standard basic quaternions.

The graded trace of X is, in the considered situation of a purely even grading and a degree (0,0,0) matrix, just the usual trace $\Gamma\mathrm{tr}(X) = x+y+z+w$. The graded determinant is given by

$$\begin{array}{rcrcrcrcrcrcr} \Gamma\mathrm{det}\,(X) & = & xyzw & + & xy\ell p & + & xehw & + & xfhp & - & xe\ell n & + & xfzn \\ & & adzw & + & ad\ell p & + & aegw & + & afgp & + & ae\ell m & - & afzm \\ & & -bdhw & + & bd\ell n & + & bygw & + & bfgn & + & by\ell m & + & bfhm \\ & & cdhp & + & cdzn & - & cygp & + & cegn & + & cyzm & + & cehm\,. \end{array} \tag{38}$$

The signs look at first sight quite surprising. However, in this quaternionic degree 0 case,

$$|\,\Gamma\mathrm{det}\,(X)| = \mathrm{Ddet}\,(X)\ ,$$

where Ddet denotes the Dieudonné determinant. Note also that (38) is a particular case of formula (25).

8.1.2. *Matrix Dimension* $|\mathbf{r}| = 4 = 0+2+1+1$. When choosing other dimensions for the homogeneous subspaces of the 4-dimensional real vector space V, see (37), namely

$$V_{(0,0,0)} = 0\,, \qquad \dim V_{(0,1,1)} = 2\,, \qquad \dim V_{(1,0,1)} = \dim V_{(1,1,0)} = 1\,,$$

we obtain a different type of matrix representation of degree $(0,0,0)$ endomorphisms of the $\mathbb{H}$-module $M = V \otimes_{\mathbb{R}} \mathbb{H}$:

$$X = \left(\begin{array}{cc|c|c} x & a & b\,\mathrm{k} & c\,\mathrm{j} \\ d & y & e\,\mathrm{k} & f\,\mathrm{j} \\ \hline g\,\mathrm{k} & h\,\mathrm{k} & z & \ell\,\mathrm{i} \\ \hline m\,\mathrm{j} & n\,\mathrm{j} & p\,\mathrm{i} & w \end{array}\right).$$

The graded determinant of X is then given by

$$\begin{array}{rcrcrcrcrcrcr} \Gamma\mathrm{det}\,(X) & = & xyzw & + & xy\ell p & + & xehw & + & xfhp & - & xe\ell n & + & xfzn \\ & & -adzw & - & ad\ell p & - & aegw & - & afgp & + & ae\ell m & - & afzm \\ & & -bdhw & + & bd\ell n & + & bygw & + & bfgn & - & by\ell m & - & bfhm \\ & & -cdhp & - & cdzn & + & cygp & - & cegn & + & cyzm & + & cehm\,. \end{array}$$

The signs are of course different from those in (38). The graded determinant is multiplicative, i.e.

$$\Gamma\mathrm{det}(XY) = \Gamma\mathrm{det}\,(X) \cdot \Gamma\mathrm{det}\,(Y)\ ,$$

(this property can be checked by direct computation) and it verifies the Liouville formula

$$\Gamma\mathrm{det}\,(\exp(\varepsilon X)) = \exp\left(\Gamma\mathrm{tr}(\varepsilon X)\right)\ ,$$

where ε denotes a degree zero nilpotent parameter.

8.1.3. *Matrix Dimension* $|\mathbf{r}| = d+d+d+d$. In this example, the graded components of the space (37) are of equal dimension d. Then, there exists an embedding of the quaternion algebra $\mathbb{H}$ into the algebra of quaternionic matrices of homogeneous degree $(0,0,0)$. Indeed, consider $q = x + a\,\mathrm{i} + b\,\mathrm{j} + c\,\mathrm{k}$ and set

$$X_q = \left(\begin{array}{c|c|c|c} x & a\,\mathrm{i} & b\,\mathrm{j} & c\,\mathrm{k} \\ \hline a\,\mathrm{i} & x & c\,\mathrm{k} & b\,\mathrm{j} \\ \hline b\,\mathrm{j} & c\,\mathrm{k} & x & a\,\mathrm{i} \\ \hline c\,\mathrm{k} & b\,\mathrm{j} & a\,\mathrm{i} & x \end{array}\right),$$

where the blocks are $(d \times d)$-matrices proportional to the identity.

In this case, the graded determinant is $\Gamma\mathrm{det}(X_q) = ||q||^{2d}$.

8.2. **Homogeneous Quaternionic Matrices of Nonzero Degrees.** In this last subsection, the $(\mathbb{Z}_2)^3$-graded space (37) is of dimension

$$|\mathbf{r}| = r_1 + r_2 + r_3 + r_4\,, \qquad |\mathbf{r}| = 0, 1 \pmod 4\,,$$

where the r_i are the dimensions of the four homogeneous subspaces. Let us emphasize that the condition $|\mathbf{r}| = 0, 1 \pmod 4$ is necessary and sufficient for consistency.

8.2.1. *Multiplying by a Scalar.* If q denotes a nonzero homogeneous quaternion (i.e. it is a nonzero multiple of an element of the standard basis of $\mathbb{H}$) and if X is a quaternionic matrix of degree $(0, 0, 0)$, then

$$\Gamma\!\det(q\,X) = q^{|\mathbf{r}|}\ \Gamma\!\det(X)\,.$$

Since every homogeneous quaternionic matrix, of any even degree, is of the form $q\,X$, this definition allows to calculate the determinants from the results of Subsection 8.1.

Let us emphasize that the multiplication of a graded matrix X by a homogeneous scalar q obeys a nontrivial sign rule.

(a) For instance, in the case of the decomposition $(1, 1, 1, 1)$, one has

$$\mathrm{i}\left(\begin{array}{c|c|c|c} 1 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \end{array}\right) = \left(\begin{array}{c|c|c|c} \mathrm{i} & & & \\ \hline & \mathrm{i} & & \\ \hline & & -\mathrm{i} & \\ \hline & & & -\mathrm{i} \end{array}\right), \qquad \mathrm{j}\left(\begin{array}{c|c|c|c} 1 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \end{array}\right) = \left(\begin{array}{c|c|c|c} \mathrm{j} & & & \\ \hline & -\mathrm{j} & & \\ \hline & & \mathrm{j} & \\ \hline & & & -\mathrm{j} \end{array}\right),$$

and similarly for k, with $-$ signs at the second and the third blocks.

(b) For the decomposition $(0, 2, 1, 1)$,

$$\mathrm{j}\left(\begin{array}{cc|c|c} 1 & & & \\ & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \end{array}\right) = \left(\begin{array}{cc|c|c} -\mathrm{j} & & & \\ & -\mathrm{j} & & \\ \hline & & \mathrm{j} & \\ \hline & & & -\mathrm{j} \end{array}\right), \quad \mathrm{k}\left(\begin{array}{cc|c|c} 1 & & & \\ & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \end{array}\right) = \left(\begin{array}{cc|c|c} -\mathrm{k} & & & \\ & -\mathrm{k} & & \\ \hline & & -\mathrm{k} & \\ \hline & & & \mathrm{k} \end{array}\right),$$

and similarly for i, with $-$ signs at the third and the fourth blocks.

8.2.2. *An Example in Dimension* $|\mathbf{r}| = 1 + 1 + 2 + 1$. Consider the example

$$\mathrm{i}\,\mathbb{I} = \mathrm{i}\left(\begin{array}{c|c|cc|c} 1 & & & & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ & & & 1 & \\ \hline & & & & 1 \end{array}\right) = \left(\begin{array}{c|c|cc|c} \mathrm{i} & & & & \\ \hline & \mathrm{i} & & & \\ \hline & & -\mathrm{i} & & \\ & & & -\mathrm{i} & \\ \hline & & & & -\mathrm{i} \end{array}\right) \in \mathsf{M}^{(011)}((1, 1, 2, 1); \mathbb{H})\,.$$

According to the definition (36), one has: $\Gamma\!\det(\mathrm{i}\,\mathbb{I}) = \mathrm{i}^5 = \mathrm{i}$. Applying (heuristically) Liouville's formula, we find

$$\Gamma\!\det(q\,\mathbb{I}) = \Gamma\!\det(q\,\mathbb{I}_N) = \exp\left(\Gamma\mathrm{tr}\left(\mathrm{i}\,\frac{\pi}{2}\,\mathbb{I}_N\right)\right) = \exp\left(\mathrm{i}\,\frac{5\pi}{2}\right) = \mathrm{i}\,,$$

in full accordance with the definition.

8.2.3. *The Diagonal Subalgebra* $\mathbb{H}$. The diagonal $(|\mathbf{r}| \times |\mathbf{r}|)$-matrices

$$I = \begin{pmatrix} \mathrm{i} & & \\ & \ddots & \\ & & \mathrm{i} \end{pmatrix}, \qquad J = \begin{pmatrix} \mathrm{j} & & \\ & \ddots & \\ & & \mathrm{j} \end{pmatrix}, \qquad K = \begin{pmatrix} \mathrm{k} & & \\ & \ddots & \\ & & \mathrm{k} \end{pmatrix} \tag{39}$$

are homogeneous of degree $(0,1,1)$, $(1,0,1)$, $(1,1,0)$, respectively. These matrices I, J, K, together with the identity matrix, span a subalgebra of the algebra of quaternionic graded matrices, which is isomorphic to $\mathbb{H}$.

(a) For the matrices (39), the graded trace is

$$\begin{aligned} \Gamma\mathrm{tr}(I) &= (r_1 + r_2 - r_3 - r_4)\,\mathrm{i}\,, \\ \Gamma\mathrm{tr}(J) &= (r_1 - r_2 + r_3 - r_4)\,\mathrm{j}\,, \\ \Gamma\mathrm{tr}(K) &= (r_1 - r_2 - r_3 + r_4)\,\mathrm{k}\,. \end{aligned}$$

In particular, for the decomposition $|\mathbf{r}| = 1+1+1+1$, one obtains

$$\Gamma\mathrm{tr}\,(I) = \Gamma\mathrm{tr}\,(J) = \Gamma\mathrm{tr}\,(K) = 0\,,$$

while for $|\mathbf{r}| = 0+2+1+1$,

$$\Gamma\mathrm{tr}\,(I) = 0\,, \qquad \Gamma\mathrm{tr}\,(J) = -2\,\mathrm{j}\,, \qquad \Gamma\mathrm{tr}\,(K) = -2\,\mathrm{k}\,.$$

(b) The graded determinant of the matrices (39) is as follows:

$$\Gamma\!\det(I) = \mathrm{i}^{(r_1+r_2-r_3-r_4)}\,, \qquad \Gamma\!\det(J) = \mathrm{j}^{(r_1-r_2+r_3-r_4)}\,, \qquad \Gamma\!\det(K) = \mathrm{k}^{(r_1-r_2-r_3+r_4)}\,.$$

For example, if $|\mathbf{r}| = 1+1+1+1$, one has

$$\Gamma\!\det(I) = \Gamma\!\det(J) = \Gamma\!\det(K) = 1\,.$$

If $|\mathbf{r}| = 0+2+1+1$, then

$$\Gamma\!\det(I) = 1\,, \qquad \Gamma\!\det(J) = \Gamma\!\det(K) = -1\,.$$

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