Approximations of the Lovász Extension of Pseudo-Boolean Functions;

Applications to Multicriteria Decision Making

by

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Approximations of pseudo-Boolean functions

Preliminary result (Hammer and Rudeanu, 1968)

Any pseudo-Boolean function $f : \{0, 1\}^n \rightarrow \mathbb{R}$ has a unique expression as a multilinear polynomial in $n$ variables:

$$f(x) = \sum_{T \subseteq N} a_T \prod_{i \in T} x_i, \quad x \in \{0, 1\}^n,$$

where $N = \{1, \ldots, n\}$ and $a_T \in \mathbb{R}$.

Definition

Let $f : \{0, 1\}^n \rightarrow \mathbb{R}$ and $k \in \{0, \ldots, n\}$. The best $k$-th approximation of $f$ is the multilinear polynomial $f^{(k)} : \{0, 1\}^n \rightarrow \mathbb{R}$ of degree $\leq k$ defined by

$$f^{(k)}(x) = \sum_{\substack{T \subseteq N \mid |T| \leq k}} a_T^{(k)} \prod_{i \in T} x_i$$

which minimizes

$$\sum_{x \in \{0, 1\}^n} [f(x) - f^{(k)}(x)]^2$$

among all multilinear polynomials of degree $\leq k$. 
Let $S \subseteq N$. The $S$-derivative of $f$ at $x \in \{0,1\}^n$, denoted $\Delta_S f(x)$, is defined inductively as

$$
\Delta_i f(x) := f(x|x_i = 1) - f(x|x_i = 0),
$$

$$
\Delta_{ij} f(x) := \Delta_i(\Delta_j f)(x) = \Delta_j(\Delta_i f)(x),
$$

$$
\Delta_S f(x) := \Delta_i(\Delta_{S\setminus i} f)(x)
$$

**Theorem** (Hammer and Holzman, 1992)

The best $k$-th approximation $f^{(k)}$ is given by the unique solution of the triangular linear system

$$
\frac{1}{2^n} \sum_{x \in \{0,1\}^n} \Delta_S f^{(k)}(x) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \Delta_S f(x), \quad \forall S \subseteq N, \ |S| \leq k.
$$

**Theorem** (Grabisch, Marichal, and Roubens, 1998)

The coefficients $a_S^{(k)}$ of the best $k$-th approximation $f^{(k)}$ are given from those of $f$ by

$$
a_S^{(k)} = a_S + (-1)^{k-|S|} \sum_{\substack{T \supseteq S \nu T \neq S \nu |T| > k}} \binom{|T\setminus S|-1}{k-|S|-1} \frac{1}{2^{|T\setminus S|}} a_T, \quad S \subseteq N, \ |S| \leq k.
$$
Approximations of Lovász extensions

The Lovász extension (Lovász, 1983; Singer, 1985)

Let $\Pi_n$ denote the family of all permutations $\pi$ of $N$. The Lovász extension $\hat{f} : [0, 1]^n \to \mathbb{R}$ of any pseudo-Boolean function $f$ is defined on each $n$-simplex

$$\mathcal{B}_\pi = \{ x \in [0, 1]^n \mid x_{\pi(1)} \leq \cdots \leq x_{\pi(n)} \}, \quad \pi \in \Pi_n,$$

as the unique affine function which interpolates $f$ at the $n + 1$ vertices of $\mathcal{B}_\pi$:

$$\hat{f}(x) = \sum_{T \subseteq N} a_T \bigwedge_{i \in T} x_i, \quad x \in [0, 1]^n.$$

Definition

Let $\hat{f}$ be the Lovász extension of a pseudo-Boolean function $f$, and let $k \in \{0, \ldots, n\}$. The best $k$-th approximation of $\hat{f}$ is the min-polynomial $\hat{f}^{(k)} : [0, 1]^n \to \mathbb{R}$ of degree $\leq k$ defined by

$$\hat{f}^{(k)}(x) = \sum_{\substack{T \subseteq N \mid |T| \leq k}} a_T^{(k)} \bigwedge_{i \in T} x_i,$$

which minimizes

$$\int_{[0,1]^n} [\hat{f}(x) - \hat{f}^{(k)}(x)]^2 \, dx$$

among all min-polynomials of degree $\leq k$.

We write $\hat{f}^{(k)} = A^{(k)}(\hat{f})$. 
We set
\[ V^{(n)} := \{ \hat{f} \mid \hat{f}(x) = \sum_{T \subseteq N} a_T \wedge x_i, ~ a_T \in \mathbb{R} \} \]

\( V^{(n)} \) is a vector space isomorphic to \( \mathbb{R}^{2^n} \):
\[ \hat{f} \quad \longleftrightarrow \quad (a_T)_{T \subseteq N} \]

In particular,
\[ \dim( V^{(n)} ) = 2^n \]

In \( V^{(n)} \), we define
- a scalar product: \( \langle \hat{f}_1, \hat{f}_2 \rangle := \int_{[0,1]^n} \hat{f}_1(x) \hat{f}_2(x) \, dx \)
- a norm: \( \| \hat{f} \| := (\langle \hat{f}, \hat{f} \rangle)^{1/2} \)
- a distance: \( d(\hat{f}_1, \hat{f}_2) := \| \hat{f}_1 - \hat{f}_2 \| \) in \( V^{(n)} \)

For any \( k \in \{0, \ldots, n\} \), we set
\[ V^{(k)} := \{ \hat{f}^{(k)} \mid \hat{f}^{(k)}(x) = \sum_{T \subseteq N \atop |T| \leq k} a_T^{(k)} \wedge x_i, ~ a_T^{(k)} \in \mathbb{R} \} \]

(vector subspace in \( V^{(n)} \))

A basis for \( V^{(k)} \) is given by
\[ B^{(k)} = \{ \wedge x_j \mid S \subseteq N, |S| \leq k \} \]

In particular,
\[ \dim( V^{(k)} ) = \sum_{s=0}^{k} \binom{n}{s} \]
The best $k$-th approximation $\hat{f}^{(k)} = A^{(k)}(\hat{f})$ is given by

minimize: $\|\hat{f} - \hat{f}^{(k)}\|

subject to: $\hat{f}^{(k)} \in V^{(k)}$

(orthogonal projection of $\hat{f}$ onto $V^{(k)}$)

\[
\begin{align*}
\int_{[0,1]^n} [\hat{f}(x) - \hat{f}^{(k)}(x)] \wedge x_j \, dx = 0, & \quad \forall S \subseteq N, |S| \leq k \\
\sum_{T \subseteq N} \sum_{|T| \leq k} a_T \int_{[0,1]^n} x_i \wedge x_j \, dx = \sum_{|T| \leq k} a_T \int_{[0,1]^n} x_i \wedge x_j \, dx \\
\forall S \subseteq N, |S| \leq k
\end{align*}
\]

with $\int_{[0,1]^n} x_i \wedge x_j \, dx = \frac{|T| + |S| + 2}{(|T \cup S| + 2)(|T| + 1)(|S| + 1)}$

Theorem

The coefficients $a_S^{(k)}$ of $A^{(k)}(\hat{f})$ are given from those of $\hat{f}$ by

$\quad a_S^{(k)} = a_S + (-1)^{k-|S|} \sum_{T \supseteq S, |T| > k} \binom{|S|}{|S|+1} \binom{|T\setminus S|-1}{k-|S|} \binom{k+|S|+1}{k+1} a_T, \quad S \subseteq N, |S| \leq k.$

For $k = 1$ we obtain:

$\quad a_\emptyset^{(1)} = \sum_{T \subseteq N} \frac{-2(|T| - 1)}{(|T| + 1)(|T| + 2)} a_T$

$\quad a_i^{(1)} = \sum_{T \ni i} \frac{6}{(|T| + 1)(|T| + 2)} a_T, \quad i \in N$
Approximations having two fixed values

We set
\[ V^{(k,0,1)} := \{ \hat{f}^{(k,0,1)} \mid \hat{f}^{(k,0,1)} \in V^{(k)}, \hat{f}^{(k,0,1)}(0) = 0, \hat{f}^{(k,0,1)}(1) = 1 \} \]
where \(0 = (0, \ldots, 0)\) and \(1 = (1, \ldots, 1)\).

Any element in \(V^{(k,0,1)}\) is of the form
\[ \hat{f}^{(k,0,1)}(x) = \sum_{\substack{T \subseteq N \\mid |T| \leq k}} a_T^{(k,0,1)} \land_{i \in T} x_i \]
with
\[ \begin{cases} a_{\emptyset}^{(k,0,1)} = 0 \\ \sum_{\substack{T \subseteq N \\mid |T| \leq k \\forall T \neq j \\land \mid T \mid \leq k \\forall i \in T}} a_T^{(k,0,1)} = 1 \end{cases} \]

For a fixed \(j \in N\), these functions can also be written as
\[ \hat{f}^{(k,0,1)}(x) = x_j + \sum_{\substack{T \subseteq N \mid |T| \leq k \\forall T \neq j \\land \mid T \mid \leq k}} a_T^{(k,0,1)} (\land_{i \in T} x_i - x_j) \]

\(V^{(k,0,1)}\) is an affine subspace in \(V^{(k)}\). A basis for \(V^{(k,0,1)}\) is given by
\[ B_j^{(k,0,1)} = \{ \land_{i \in S} x_i - x_j \mid S \subseteq N, S \neq \{ j \}, 1 \leq |S| \leq k \} \]
for one \(j \in N\). In particular
\[ \dim(V^{(k,0,1)}) = \dim(V^{(k)}) - 2. \]
Problem

Given $\hat{f} \in V^{(n)}$, we search for the solution $\hat{f}^{(k,0,1)} = A^{(k,0,1)}(\hat{f})$ of

minimize: $\| \hat{f} - \hat{f}^{(k,0,1)} \|$  
subject to: $\hat{f}^{(k,0,1)} \in V^{(k,0,1)}$

(orthogonal projection of $\hat{f}$ onto $V^{(k,0,1)}$)

Since $V^{(k,0,1)} \subset V^{(k)}$, we have $A^{(k,0,1)}(\hat{f}) = A^{(k,0,1)}(A^{(k)}(\hat{f}))$ and the problem becomes

minimize: $\| A^{(k)}(\hat{f}) - \hat{f}^{(k,0,1)} \|$  
subject to: $\hat{f}^{(k,0,1)} \in V^{(k,0,1)}$

(orthogonal projection of $A^{(k)}(\hat{f})$ onto $V^{(k,0,1)}$)

$\Downarrow$

$\int_{[0,1]^n} [(A^{(k)}(\hat{f})(x) - \hat{f}^{(k,0,1)}(x))] (\wedge_{i \in S} x_i - x_j) \, dx = 0$

$\forall S \subseteq N, S \neq \{j\}, 1 \leq |S| \leq k$

For $k = 1$ we obtain:

$a_{\emptyset}^{(1,0,1)} = 0$

$a_i^{(1,0,1)} = a_i^{(1)} + \frac{1}{n} (1 - \sum_{j \in N} a_j^{(1)}), \quad i \in N$
Increasing approximations having two fixed values

We set

\[ V^{[k,0,1]} := \{ \hat{f}^{[k,0,1]} \mid \hat{f}^{[k,0,1]} \in V^{(k,0,1)}, \text{\hat{f}^{[k,0,1]} is increasing} \} \]

Any element in \( V^{[k,0,1]} \) is of the form

\[ \hat{f}^{[k,0,1]}(x) = \sum_{T \subseteq N, |T| \leq k} a_T^{[k,0,1]} \land x_i \]

with

\[
\begin{align*}
    a_0^{[k,0,1]} &= 0 \\
    \sum_{T \subseteq N, |T| \leq k} a_T^{[k,0,1]} &= 1 \\
    \sum_{T : i \in T \subseteq S, |T| \leq k} a_T^{[k,0,1]} &\geq 0, \quad S \subseteq N, i \in S.
\end{align*}
\]

\( V^{[k,0,1]} \) is a non-empty closed convex polyhedron in \( V^{(k,0,1)} \).

**Problem**

Given \( \hat{f} \in V^{(n)} \), we search for the solution \( \hat{f}^{[k,0,1]} = A^{[k,0,1]}(\hat{f}) \) of

minimize: \( \| \hat{f} - \hat{f}^{[k,0,1]} \| \)

subject to: \( \hat{f}^{[k,0,1]} \in V^{[k,0,1]} \)

(projection of \( \hat{f} \) onto the polyhedron \( V^{[k,0,1]} \))

Since \( V^{[k,0,1]} \subset V^{(k,0,1)} \), we can replace \( \hat{f} \) by \( A^{(k,0,1)}(\hat{f}) \).
Case of $k = 1$: the closest weighted arithmetic mean to a Lovász extension

Problem

Find the solution $(a_1^{[1,0,1]}, \ldots, a_n^{[1,0,1]}) \in \mathbb{R}^n$ of:

$$\min \int_{[0,1]^n}[\sum_{i=1}^{n} a_i^{[1,0,1]} x_i - \sum_{i=1}^{n} a_i^{[1,0,1]} x_i]^2 dx$$

subject to:

$$\begin{cases} 
\sum_{i=1}^{n} a_i^{[1,0,1]} = 1 \\
 a_i^{[1,0,1]} \geq 0, \quad i \in \mathbb{N}.
\end{cases}$$

Recall that

$$V^{(1,0,1)} = \left\{ \sum_{i=1}^{n} \omega_i x_i \bigg| \sum_{i=1}^{n} \omega_i = 1, \omega_i \in \mathbb{R} \right\}$$

$$V^{[1,0,1]} = \left\{ \sum_{i=1}^{n} \omega_i x_i \bigg| \sum_{i=1}^{n} \omega_i = 1, \omega_i \geq 0 \right\}$$

$V^{(1,0,1)}$ is the affine hull of $x_1, \ldots, x_n$

$V^{[1,0,1]}$ is the convex hull of $x_1, \ldots, x_n$

$$\dim(V^{(1,0,1)}) = \dim(V^{[1,0,1]}) = n - 1$$

$$||x_i - x_j|| = \frac{1}{\sqrt{6}}, \quad \forall i, j \in \mathbb{N}, i \neq j$$

$P := V^{[1,0,1]}$ is a regular simplex in $V^{(1,0,1)}$
Assume \( \hat{g} := A^{(1,0,1)}(\hat{f}) \in V^{(1,0,1)} \setminus P \).
Then \( A^{[1,0,1]}(\hat{f}) \) can be obtained by projecting \( \hat{g} \) onto \( P \).

There exists a facet \( F_\hat{g} \) of \( P \) such that the affine hull \( \text{aff}(F_\hat{g}) \) of its vertices contains \( \hat{g} \) or separates \( \hat{g} \) from \( P \).

**Theorem**

Let \( \hat{g} \in V^{(1,0,1)} \setminus P \). Then the projection of \( \hat{g} \) onto \( P \) is in \( F_\hat{g} \).

The projection of \( \hat{g} \) onto \( P \) can be obtained by first projecting \( \hat{g} \) onto \( \text{aff}(F_\hat{g}) \) and then projecting, if necessary, the obtained projection onto \( F_\hat{g} \).
If more than one affine hull contain $\hat{g}$ or separate it from $P$ then the projection onto $P$ is clearly in the intersection of the corresponding facets.

Prestep. $P := V^{[1,0,1]}$, $\hat{g} := A^{(1,0,1)}(\hat{f})$.

Step 1. $F_{\hat{g}}^{\cap}$ := intersection of all the facets of $P$ whose affine hull contains $\hat{g}$ or separates it from $P$.

Step 2. $\hat{h} :=$ projection of $\hat{g}$ onto $\text{aff}(F_{\hat{g}}^{\cap})$.

Step 3. If $\hat{h} \in F_{\hat{g}}^{\cap}$ then $\hat{h}$ is the projection of $A^{(1,0,1)}(\hat{f})$ onto $P$, --- stop, else $P \leftarrow F_{\hat{g}}^{\cap}$, $\hat{g} \leftarrow \hat{h}$, return to Step 1.
Input:
\[ \hat{f}(x) = \sum_{T \subseteq N} \alpha_T \land_{i \in T} x_i \]

\[ a_i^{(1)} := \sum_{T \ni i} (|T| + 1)(|T| + 2) \alpha_T, \quad i \in N, \]
\[ a_i^* := a_i^{(1)} + \frac{1}{n} (1 - \sum_{j \in N} a_j^{(1)}), \quad i \in N. \]

\[ a_i^* \geq 0 \quad \forall i \in N \]

\[ R := \{ i \in N \mid a_i^* \leq 0 \}, \]
\[ a_i^* \left\{ \begin{array}{ll}
0 & \text{if } i \in R, \\
\frac{1}{n-|R|} \sum_{j \in R} a_j^* & \text{if } i \notin R.
\end{array} \right. \]

Output:
\[ (A^{[1,0,1]} \hat{f})(x) = \sum_{i \in N} a_i^* x_i. \]
Example:

Let \( \hat{f}: [0, 1]^4 \rightarrow \mathbb{R} \) be given by

\[
\hat{f}(x) = \frac{3}{10} \left[ x_1 + x_2 + x_3 + (x_1 \land x_2) + (x_1 \land x_3) + (x_2 \land x_3) \right] \\
- \frac{21}{25} (x_1 \land x_2 \land x_3) + \frac{1}{25} (x_1 \land x_2 \land x_3 \land x_4).
\]

The best linear approximation is given by

\[
(A^{(1)} \hat{f})(x) = \frac{1}{100} + \frac{89}{250} (x_1 + x_2 + x_3) + \frac{1}{125} x_4
\]

and the best min-quadratic approximation by

\[
(A^{(2)} \hat{f})(x) = -\frac{27}{700} + \frac{803}{1750} (x_1 + x_2 + x_3) - \frac{8}{875} x_4 \\
- \frac{19}{175} [(x_1 \land x_2) + (x_1 \land x_3) + (x_2 \land x_3)] \\
+ \frac{2}{175} [(x_1 \land x_4) + (x_2 \land x_4) + (x_3 \land x_4)].
\]

We also have

\[
(A^{(1,0,1)} \hat{f})(x) = \frac{337}{1000} (x_1 + x_2 + x_3) - \frac{11}{1000} x_4
\]

\[
(A^{(2,0,1)} \hat{f})(x) = \frac{29419}{67000} (x_1 + x_2 + x_3) - \frac{1937}{67000} x_4 \\
- \frac{181}{1675} [(x_1 \land x_2) + (x_1 \land x_3) + (x_2 \land x_3)] \\
+ \frac{4}{335} [(x_1 \land x_4) + (x_2 \land x_4) + (x_3 \land x_4)].
\]

and

\[
(A^{[1,0,1]} \hat{f})(x) = \frac{1}{3} (x_1 + x_2 + x_3).
\]
Applications to Multicriteria Decision Making

Example (Grabisch, 1995)

3 students: a, b, c
3 criteria: mathematics (M), physics (P), and literature (L)
Aggregation operator: weighted arithmetic mean
Weights: 3, 3, 2.

<table>
<thead>
<tr>
<th>student</th>
<th>M</th>
<th>P</th>
<th>L</th>
<th>global evaluation</th>
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<tr>
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<td>c</td>
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<td>15</td>
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No weight vector \((\omega_M, \omega_P, \omega_L)\) satisfying \(\omega_M = \omega_P > \omega_L\) is able to favor student c:

\[ c \succ a \iff \omega_L > \omega_M \]

We substitute a non-additive measure to the weight vector (additive measure)

Definition (Choquet, 1953)

A Choquet capacity on \(N\) is a set function \(\mu : 2^N \rightarrow [0, 1]\) satisfying

\[
\begin{align*}
  i) & \quad \mu_\emptyset = 0, \mu_N = 1 \\
  ii) & \quad S \subseteq T \Rightarrow \mu_S \leq \mu_T
\end{align*}
\]
For example, we can define

\[ \mu_0 = 0 \quad \mu_M = 0.45 \quad \mu_{MP} = 0.50 \quad \mu_{MPL} = 1 \]
\[ \mu_P = 0.45 \quad \mu_{ML} = 0.90 \]
\[ \mu_L = 0.30 \quad \mu_{PL} = 0.90 \]

Any real valued set function can be assimilated unambiguously with a pseudo-Boolean function. Particularly, to any Choquet capacity \( \mu \) corresponds a unique increasing pseudo-Boolean function \( f : \{0, 1\}^n \rightarrow \mathbb{R} \) such that \( f(0) = 0 \) and \( f(1) = 1 \):

\[
\mu \quad \longleftrightarrow \quad f_\mu(x) = \sum_{T \subseteq N} a_T \prod_{i \in T} x_i
\]

with

\[
a_\emptyset = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \mu_T, \quad S \subseteq N.
\]

\[
a_0 = 0 \quad a_M = 0.45 \quad a_{MP} = -0.40 \quad a_{MPL} = -0.10
\]
\[
a_P = 0.45 \quad a_{ML} = 0.15
\]
\[
a_L = 0.30 \quad a_{PL} = 0.15
\]

**Definition** (Choquet, 1953)

Let \( \mu \) be a Choquet capacity on \( N \). The (discrete) Choquet integral of a function \( x : N \rightarrow [0, 1] \) w.r.t. \( \mu \) is defined by

\[
C_\mu(x) := \sum_{i=1}^{n} x(i) [\mu_{\{(i),\ldots,(n)\}} - \mu_{\{(i+1),\ldots,(n)\}}],
\]

with the convention that \( x(1) \leq \cdots \leq x(n) \).

We also have

\[
C_\mu(x) = \hat{f}_\mu(x) = \sum_{T \subseteq N} a_T \land x_i
\]
Theorem

Let $M_\mu : [0, 1]^n \to \mathbb{R}$ be an aggregation operator depending on a Choquet capacity $\mu$ on $N$. Then $M_\mu$ is

- **linear w.r.t. the Choquet capacity**:
  
  there exist functions $g_T(x) : [0, 1]^n \to \mathbb{R}$, $T \subseteq N$, such that
  
  $$M_\mu(x) = \sum_{T \subseteq N} a_T g_T(x), \quad \forall \mu.$$

- **increasing in each variable**

- **stable for the positive linear transformations**

  $$M_\mu(r x_1 + s, \ldots, r x_n + s) = r M_\mu(x_1, \ldots, x_n) + s$$

  for all $x \in [0, 1]^n$ and all $r > 0, s \in \mathbb{R}$.

- **an extension of $\mu$**:

  $$M_\mu(e_T) = \mu_T, \quad T \subseteq N.$$ 

if and only if $M_\mu = C_\mu$.

Back to the example:

<table>
<thead>
<tr>
<th>student</th>
<th>M</th>
<th>P</th>
<th>L</th>
<th>WAM$_\omega$</th>
<th>$C_\mu$</th>
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$C_\mu : c > a > b$
Linear approximation:

\[ A^{[1,0,1]}(C_\mu) = 0.29 x_M + 0.29 x_P + 0.42 x_L \]

Proposition

When \( n \leq 3 \), the weights of the linear approximation identify with the Shapley value:

\[ a_i^{[1,0,1]} = \phi_\mu(i) = \sum_{T \ni i} \frac{1}{|T|} a_T, \quad i \in N. \]

However, for

\[
\hat{f}(x) = \frac{3}{10} [x_1 + x_2 + x_3 + (x_1 \wedge x_2) + (x_1 \wedge x_3) + (x_2 \wedge x_3)] - \frac{21}{25} (x_1 \wedge x_2 \wedge x_3) + \frac{1}{25} (x_1 \wedge x_2 \wedge x_3 \wedge x_4),
\]

we have

\[ a_1^{[1,0,1]} = a_2^{[1,0,1]} = a_3^{[1,0,1]} = \frac{1}{3} \text{ and } a_4^{[1,0,1]} = 0 \]

\[ \phi(1) = \phi(2) = \phi(3) = \frac{33}{100} \text{ and } \phi(4) = \frac{1}{100} \]

Min-quadratic approximation:

\[ A^{[2,0,1]}(C_\mu) = 0.47 x_M + 0.47 x_P + 0.31 x_L \]

\[ -0.45 (x_M \wedge x_P) + 0.10 [(x_M \wedge x_L) + (x_P \wedge x_L)] \]

<table>
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<th>student</th>
<th>M</th>
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<th>WAM_\omega</th>
<th>C_\mu</th>
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