Cumulative Distribution Functions and Moments of Weighted Lattice Polynomials

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Part I: Weighted lattice polynomials
- Definitions
- Representation and characterization

Part II: Cumulative distribution functions of aggregation operators
- Weighted lattice polynomials
- Applications
Sketch of the Presentation

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Part I: Weighted lattice polynomials
Let $L$ be a lattice with lattice operations $\land$ and $\lor$

We assume that $L$ is

- bounded (with bottom 0 and top 1)
- distributive

**Definition (Birkhoff 1967)**

An $n$-ary *lattice polynomial* is a well-formed expression involving $n$ variables $x_1, \ldots, x_n \in L$ linked by the lattice operations $\land$ and $\lor$ in an arbitrary combination of parentheses

**Example.**

$$p(x_1, x_2, x_3) = (x_1 \land x_2) \lor x_3$$
Lattice polynomials

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Lattice polynomial functions

Any lattice polynomial naturally defines a *lattice polynomial function* (l.p.f.) \( p : L^n \to L \).

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\[
p(x_1, x_2, x_3) = (x_1 \land x_2) \lor x_3
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If \( p \) and \( q \) represent the same function, we say that \( p \) and \( q \) are equivalent and we write \( p = q \).

Example.

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x_1 \lor (x_1 \land x_2) = x_1
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**Notation.** \([n] := \{1, \ldots, n\} \).

**Proposition (Birkhoff 1967)**

Let \( p : L^n \to L \) be any l.p.f.

Then there are nonconstant set functions \( v, w : 2^n \to \{0, 1\} \), with \( v(\emptyset) = 0 \) and \( w(\emptyset) = 1 \), such that

\[
p(x) = \bigvee_{S \subseteq [n], \; v(S) = 1} \bigwedge_{i \in S} x_i = \bigwedge_{S \subseteq [n], \; w(S) = 0} \bigvee_{i \in S} x_i.
\]

**Example.**

\[
(x_1 \land x_2) \lor x_3 = (x_1 \lor x_3) \land (x_2 \lor x_3)
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v(\{3\}) = v(\{1, 2\}) = 1
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The set functions \( v \) and \( w \), which generate \( p \), are not unique:

\[
x_1 \lor (x_1 \land x_2) = x_1 = x_1 \land (x_1 \lor x_2)
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**Notation.** \( 1_S \) := characteristic vector of \( S \subseteq [n] \) in \( \{0, 1\}^n \).

**Proposition (Marichal 2002)**

From among all the set functions \( v \) that disjunctively generate the l.p.f. \( p \), only one is isotone:

\[
v(S) = p(1_S)
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From among all the set functions \( w \) that conjunctively generate the l.p.f. \( p \), only one is antitone:

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$\begin{array}{c|c|c}
S & p(1_S) & p(1_{[n]\setminus S}) \\
\emptyset & 0 & 1 \\
\{1\} & 0 & 1 \\
\{2\} & 0 & 1 \\
\{3\} & 1 & 1 \\
\{1, 2\} & 1 & 1 \\
\{1, 3\} & 1 & 0 \\
\{2, 3\} & 1 & 0 \\
\{1, 2, 3\} & 1 & 0 \\
\end{array}$

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$p(\emptyset) = 1$
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$p(\{2\}) = 1$
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**Example.** \( p(x) = (x_1 \land x_2) \lor x_3 \)

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p(x) = x_3 \lor (x_1 \land x_2) \lor (x_1 \land x_3) \lor (x_2 \land x_3) \lor (x_1 \land x_2 \land x_3)
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Particular cases: order statistics

Denote by $x_{(1)}, \ldots, x_{(n)}$ the order statistics resulting from reordering $x_1, \ldots, x_n$ in the nondecreasing order: $x_{(1)} \leq \cdots \leq x_{(n)}$.

**Proposition (Ovchinnikov 1996, Marichal 2002)**

$p$ is a symmetric l.p.f. $\iff$ $p$ is an order statistic

**Notation.** Denote by $os_k : L^n \to L$ the $k$th order statistic function.

$$os_k(x) := x_{(k)}$$

Then we have

$$os_k(1_S) = 1 \iff |S| \geq n - k + 1$$

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We can generalize the concept of l.p.f. by regarding some variables as parameters.

**Example.** For \( c \in L \), we consider

\[
p(x_1, x_2) = (c \lor x_1) \land x_2
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**Definition**

\( p : L^n \rightarrow L \) is an \( n \)-ary **weighted lattice polynomial** function (w.l.p.f.) if there exist parameters \( c_1, \ldots, c_m \in L \) and a l.p.f. \( q : L^{n+m} \rightarrow L \) such that

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- \( p \) is a l.p.f. if \( v \) and \( w \) range in \( \{0, 1\} \), with \( v(\emptyset) = 0 \) and \( w(\emptyset) = 1 \).

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Proposition (Lausch & Nöbauer 1973)

Let \( p : L^n \to L \) be any w.l.p.f.
Then there are set functions \( v, w : 2^{[n]} \to L \) such that

\[
p(x) = \bigvee_{S \subseteq [n]} \left[ v(S) \land \bigwedge_{i \in S} x_i \right] = \bigwedge_{S \subseteq [n]} \left[ w(S) \lor \bigvee_{i \in S} x_i \right].
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Remarks.

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From among all the set functions $v$ that disjunctively generate the w.l.p.f. $p$, only one is isotone:

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Particular case: the Sugeno integral

Let us generalize the concept of discrete Sugeno integral in the framework of bounded distributive lattices.

**Definition (Sugeno 1974)**

An $L$-valued *fuzzy measure* on $[n]$ is an isotone set function $\mu : 2^{[n]} \to L$ such that $\mu(\emptyset) = 0$ and $\mu([n]) = 1$.

The *Sugeno integral* of a function $x : [n] \to L$ with respect to $\mu$ is defined by

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**Remark.** A function $f : L^n \to L$ is an $n$-ary Sugeno integral if and only if $f$ is a w.l.p.f. fulfilling $f(1_{\emptyset}) = 0$ and $f(1_{[n]}) = 1$. 
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**Notation.** The median function is the function \( os_2 : L^3 \rightarrow L \).

**Proposition (Marichal 2006)**

For any w.l.p.f. \( p : L^n \rightarrow L \), there is a fuzzy measure \( \mu : 2^n \rightarrow L \) such that

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p(x) = \text{median} \left[ p(\emptyset), S_\mu(x), p(1_{[n]}) \right]
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**Corollary (Marichal 2006)**

Consider a function \( f : L^n \rightarrow L \).

The following assertions are equivalent:

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Inclusion properties

Weighted lattice polynomials

Sugeno integrals

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Order statistics
The median based decomposition formula

Let $f : L^n \to L$ and $k \in [n]$ and define $f_k^0, f_k^1 : L^n \to L$ as

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    f_k^0(x) &:= f(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) \\
    f_k^1(x) &:= f(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_n)
\end{align*}
$$

**Remark.** If $f$ is a w.l.p.f., so are $f_k^0$ and $f_k^1$

Consider the following system of $n$ functional equations, called the *median based decomposition formula*

$$f(x) = \text{median}[f_k^0(x), x_k, f_k^1(x)] \quad (k = 1, \ldots, n)$$
The median based decomposition formula

Any solution of the median based decomposition formula

\[ f(x) = \text{median}[f^0_k(x), x_k, f^1_k(x)] \quad (k = 1, \ldots, n) \]

is an \( n \)-ary w.l.p.f.

**Example.** For \( n = 2 \) we have

\[ f(x_1, x_2) = \text{median}[f(x_1, 0), x_2, f(x_1, 1)] \]

with

\[ f(x_1, 0) = \text{median}[f(0, 0), x_1, f(1, 0)] \quad (\text{w.l.p.f.}) \]

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Theorem (Marichal 2006)
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Part II: Cumulative distribution functions of aggregation operators
Consider

- an aggregation operator $A : \mathbb{R}^n \rightarrow \mathbb{R}$
- $n$ independent random variables $X_1, \ldots, X_n$, with cumulative distribution functions $F_1(x), \ldots, F_n(x)$

\[
\begin{align*}
X_1 & \\
\vdots \\
X_n & \\
\longrightarrow & \\
Y_A = A(X_1, \ldots, X_n)
\end{align*}
\]

Problem. We are searching for the cumulative distribution function (c.d.f.) of $Y_A$:

\[
F_A(y) := \Pr[Y_A \leq y]
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Cumulative distribution functions of aggregation operators

From the c.d.f. of $Y_A$, we can calculate the expectation

$$E[g(Y_A)] = \int_{-\infty}^{\infty} g(y) \, dF_A(y)$$

for any measurable function $g$.

Some useful examples:

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Example: the arithmetic mean

\[ AM(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i \]

\( F_{AM}(y) \) is given by the convolution product of \( F_1, \ldots, F_n \)

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For uniform random variables \( X_1, \ldots, X_n \) on \([0, 1]\), we have

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Graph of $F_{AM}(y)$

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Example: Łukasiewicz $t$-norm

\[
T_L(x_1, \ldots, x_n) = \max \left[ 0, \sum_{i=1}^{n} x_i - (n - 1) \right]
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F_{T_L}(y) = \Pr \left[ \max \left[ 0, \sum_i X_i - (n - 1) \right] \leq y \right]
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= \Pr \left[ 0 \leq y \quad \text{and} \quad \sum_i X_i - (n - 1) \leq y \right]
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\]

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where $H_c(y)$ is the Heaviside function

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Case of $n = 3$ uniform random variables $X_1, X_2, X_3$ on $[0, 1]$

Graph of $F_{T_L}(y)$

Remark.

$F_{T_L}(y)$ is discontinuous

$\Rightarrow$ The p.d.f. does not exist
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$F_{T_L}(y)$ is discontinuous  
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Graph of $F_{T_L}(y)$
Example: order statistics on $\mathbb{R}$

$$
OS_k(x_1, \ldots, x_n) = x(k)
$$

$$
F_{os_k}(y) = \sum_{S \subseteq [n]} \prod_{i \in S} F_i(y) \prod_{i \in [n] \setminus S} [1 - F_i(y)]
$$

(see e.g. David & Nagaraja 2003)

Examples.

$$
F_{os_1}(y) = 1 - \prod_{i=1}^{n} [1 - F_i(y)]
$$

$$
F_{os_n}(y) = \prod_{i=1}^{n} F_i(y)
$$
Example: order statistics on $\mathbb{R}$

\[ \text{os}_k(x_1, \ldots, x_n) = x_{(k)} \]

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Graph of $F_{os1}(y)$

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It can be extended to an aggregation function from $\mathbb{R}^n$ to $\mathbb{R}$.

$$p(x_1, \ldots, x_n) = \bigvee_{S \subseteq [n]} \bigwedge_{i \in S} x_i = \bigwedge_{S \subseteq [n]} \bigvee_{i \in S} x_i$$

Note. $\wedge = \min$, $\vee = \max$

$$F_p(y) = 1 - \sum_{S \subseteq [n]} \prod_{i \in [n] \setminus S} F_i(y) \prod_{i \in S} [1 - F_i(y)]$$

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New results: lattice polynomial functions on \( \mathbb{R} \)

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Let \( p : L^n \rightarrow L \) be a l.p.f. on \( L = [0, 1] \)
It can be extended to an aggregation function from \( \mathbb{R}^n \) to \( \mathbb{R} \).

\[
p(x_1, \ldots, x_n) = \bigvee_{S \subseteq [n]} \bigwedge_{i \in S} x_i = \bigwedge_{S \subseteq [n]} \bigvee_{i \in S} x_i
\]

Note. \( \wedge = \min, \ \vee = \max \)

\[
F_p(y) = 1 - \sum_{S \subseteq [n]} \prod_{i \in [n] \setminus S} F_i(y) \prod_{i \in S} [1 - F_i(y)]
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New results: lattice polynomial functions on $\mathbb{R}$

**Example.** $p(x) = (x_1 \land x_2) \lor x_3$

Uniform random variables $X_1, X_2, X_3$ on $[0, 1]$

Graph of $F_p(y)$  
Graph of $f_p(y)$
New results: lattice polynomial functions on $\mathbb{R}$

Consider

- $v_p : 2^{[n]} \to \mathbb{R}$, defined by $v_p(S) := p(1_S)$
- $v_p^* : 2^{[n]} \to \mathbb{R}$, defined by $v_p^*(S) = 1 - v_p([n] \setminus S)$
- $m_v : 2^{[n]} \to \mathbb{R}$, the Möbius transform of $v$, defined by
  
  $$m_v(S) := \sum_{T \subseteq S} (-1)^{|S| - |T|} v(T)$$

Alternate expressions of $F_p(y)$

$$F_p(y) = 1 - \sum_{S \subseteq [n]} m_{v_p}(S) \prod_{i \in S} [1 - F_i(y)]$$

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New results: lattice polynomial functions on $\mathbb{R}$

Consider

- $\nu_p : 2^{[n]} \to \mathbb{R}$, defined by $\nu_p(S) := p(1_S)$
- $\nu_p^* : 2^{[n]} \to \mathbb{R}$, defined by $\nu_p^*(S) = 1 - \nu_p([n] \setminus S)$
- $m_\nu : 2^{[n]} \to \mathbb{R}$, the Möbius transform of $\nu$, defined by

$$m_\nu(S) := \sum_{T \subseteq S} (-1)^{|S| - |T|} \nu(T)$$

### Alternate expressions of $F_p(y)$

$$F_p(y) = 1 - \sum_{S \subseteq [n]} m_{\nu_p}(S) \prod_{i \in S} [1 - F_i(y)]$$

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Consider

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New results: weighted lattice polynomial functions on $\mathbb{R}$

Let $p : \mathbb{R}^n \rightarrow \mathbb{R}$ be a w.l.p.f. on $\mathbb{R} = [-\infty, +\infty]$

**Notation.** $e_S :=$ characteristic vector of $S$ in $\{-\infty, +\infty\}^n$

\[
p(x) = \bigvee_{S \subseteq [n]} [p(e_S) \wedge \bigwedge_{i \in S} x_i] = \bigwedge_{S \subseteq [n]} [p(e_{[n]\setminus S}) \vee \bigvee_{i \in S} x_i]
\]

\[
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New results: weighted lattice polynomial functions on \( \overline{\mathbb{R}} \)

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New results: weighted lattice polynomial functions on $\mathbb{R}$

**Example.** $p(x) = (c \land x_1) \lor x_2$

Uniform random variables $X_1, X_2$ on $[0, 1]$
$F(y) = \text{median}[0, y, 1]$

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$F_p(y) = F(y) \left( F(y) + H_c(y)[1 - F(y)] \right)$

Graph of $F_p(y)$ for $c = 1/2$
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$$\int_{[0,1]^n} g[p(x)] \, dx$$

Solution. The integral is given by $E[g(Y_p)]$, where the variables $X_1, \ldots, X_n$ are uniform on $[0, 1]$

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**Sugeno integral**

$$\int_{[0,1]^n} S_\mu(x) \, dx = \sum_{S \subseteq [n]} \int_0^{\mu(S)} y^{n-|S|} (1 - y)^{|S|} \, dy$$

**Example.**

$$\int_{[0,1]^2} [(c \land x_1) \lor x_2] \, dx = \frac{1}{2} + \frac{1}{2} c^2 - \frac{1}{3} c^3$$

**Note.** Recall the expected value of the Choquet integral

$$\int_{[0,1]^n} C_\mu(x) \, dx = \sum_{S \subseteq [n]} \mu(S) \int_0^1 y^{n-|S|} (1 - y)^{|S|} \, dy$$

(Marichal 2004)
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\[
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(Marichal 2004)
Application: reliability of systems

Consider a system made up of $n$ indep. components $C_1, \ldots, C_n$

Each component $C_i$ has
- a lifetime $X_i$
- a reliability $r_i(t)$ at time $t > 0$

$$r_i(t) = \Pr[X_i > t] = 1 - F_i(t)$$

Assumptions:

1. The lifetime of a series subsystem is the minimum of the component lifetimes.
2. The lifetime of a parallel subsystem is the maximum of the component lifetimes.
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- The lifetime of a parallel subsystem is the maximum of the component lifetimes
**Question.** What is the lifetime of the following system?

![Diagram of a system with nodes X1, X2, and X3 connected in a series and parallel configuration.]

**Solution.** \( Y = (X_1 \land X_2) \lor X_3 \)
Question. What is the lifetime of the following system?

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Application: reliability of systems

For a system mixing series and parallel connections:

*System lifetime:*

\[ Y_p = p(X_1, \ldots, X_n) \]

where \( p \) is

- an \( n \)-ary l.p.f.
- an \( n \)-ary w.l.p.f. if some \( X_i \)'s are constant

We then have explicit formulas for

- the c.d.f. of \( Y_p \)
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System reliability at time $t > 0$

\[ R_p(t) := \Pr[Y_p > t] = 1 - F_p(t) \]

For any measurable function $g : [0, \infty] \rightarrow \mathbb{R}$ such that

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\[ \mathbb{E}[g(Y_p)] = g(0) + \int_0^\infty R_p(t) \, dg(t) \]

Mean time to failure:

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Application: reliability of systems

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Thanks for your attention!