Comparison meaningful aggregation functions

A state of the art

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Let $F$ be an aggregation function from $\mathbb{R}^n$ to $\mathbb{R}$:

$$x_{n+1} = F(x_1, \ldots, x_n)$$

where $x_1, \ldots, x_n$ are the independent variables and $x_{n+1}$ is the dependent variable.

The general form of $F$ is restricted if we know the scale type of the variables $x_1, \ldots, x_n$ and $x_{n+1}$ (Luce 1959).

A scale type is defined by the class of admissible transformations, transformations which change the scale into an alternative acceptable scale.

$x_i$ defines an ordinal scale if the class of admissible transformations consists of the increasing bijections (automorphisms) of $\mathbb{R}$ onto $\mathbb{R}$. 
Example:
Suppose $x$ defines an ordinal scale and consider some of its values:

\[ 4 \quad 5 \quad 7 \quad 8 \quad 10 \]

Let $\phi : \mathbb{R} \to \mathbb{R}$ be any increasing bijection.
Then $\phi(x)$ defines an alternative acceptable scale.

\[ -3 \quad 1.5 \quad 14.2 \quad 58 \quad 263 \]
Suppose $x_1, \ldots, x_n$ define the same ordinal scale.
What are the possible aggregation functions $F(x_1, \ldots, x_n)$?

**Examples:**

- The *arithmetic mean* is meaningless:
  \[
  \frac{3 + 5}{2} < \frac{1 + 8}{2}
  \]
  Choose $\phi$ such that $\phi(1) = 1$, $\phi(3) = 4$, $\phi(5) = 7$, $\phi(8) = 8$.
  \[
  \frac{\phi(3) + \phi(5)}{2} > \frac{\phi(1) + \phi(8)}{2}
  \]

- The *min* and *max* functions are meaningful:
  \[
  \min(3, 5) > \min(1, 8) \quad \text{and} \quad \min(\phi(3), \phi(5)) > \min(\phi(1), \phi(8))
  \]
Principle of theory construction (Luce 1959)

Admissible transformations of the independent variables should lead to an admissible transformation of the dependent variable.

Suppose that

$$x_{n+1} = F(x_1, \ldots, x_n)$$

where $x_{n+1}$ is an ordinal scale and $x_1, \ldots, x_n$ are independent ordinal scales.

Let $A(\mathbb{R})$ be the automorphism group of $\mathbb{R}$.

For any $\phi_1, \ldots, \phi_n \in A(\mathbb{R})$, there is $\Phi_{\phi_1, \ldots, \phi_n} \in A(\mathbb{R})$ such that

$$F[\phi_1(x_1), \ldots, \phi_n(x_n)] = \Phi_{\phi_1, \ldots, \phi_n}[F(x_1, \ldots, x_n)]$$
Assume $x_1, \ldots, x_n$ define the same ordinal scale. Then the functional equation simplifies into

$$F[\phi(x_1), \ldots, \phi(x_n)] = \Phi_\phi[F(x_1, \ldots, x_n)]$$

Equivalently, $F$ fulfills the condition (Orlov 1981)

$$F(x_1, \ldots, x_n) \leq F(x'_1, \ldots, x'_n)$$

$$\Leftrightarrow$$

$$F[\phi(x_1), \ldots, \phi(x_n)] \leq F[\phi(x'_1), \ldots, \phi(x'_n)]$$

$F$ is said to be *comparison meaningful* (Ovchinnikov 1996)
Assume $x_1, \ldots, x_n$ are independent ordinal scales. Recall that the functional equation is

$$F[\phi_1(x_1), \ldots, \phi_n(x_n)] = \Phi_{\phi_1, \ldots, \phi_n}[F(x_1, \ldots, x_n)]$$

Equivalently, $F$ fulfills the condition

$$F(x_1, \ldots, x_n) \leq F(x'_1, \ldots, x'_n)$$

$$\iff$$

$$F[\phi_1(x_1), \ldots, \phi_n(x_n)] \leq F[\phi_1(x'_1), \ldots, \phi_n(x'_n)]$$

We say that $F$ is strongly comparison meaningful.
Purpose of the presentation

To provide a complete description of comparison meaningful functions

To provide a complete description of strongly comparison meaningful functions
First result (Osborne 1970, Kim 1990)

$F : \mathbb{R}^n \to \mathbb{R}$ is continuous and strongly comparison meaningful

\[
\exists k \in \{1, \ldots, n\} \\
\exists g : \mathbb{R} \to \mathbb{R} \quad - \text{continuous} \\
- \text{strictly monotonic or constant} \\
\text{such that} \\
F(x_1, \ldots, x_n) = g(x_k)
\]

+ idempotent, i.e., $F(x, \ldots, x) = x$

\[
\Leftrightarrow \quad \exists k \in \{1, \ldots, n\} \text{ such that} \\
F(x_1, \ldots, x_n) = x_k
\]
The nondecreasing case

Second result (Marichal & Mesiar & Rückschlossová 2004)

$F : \mathbb{R}^n \rightarrow \mathbb{R}$ is nondecreasing and strongly comparison meaningful

\[ \iff \begin{cases} 
\exists k \in \{1, \ldots, n\} \\
\exists \ g : \mathbb{R} \rightarrow \mathbb{R} \text{ strictly increasing or constant} \\
such that \\
F(x_1, \ldots, x_n) = g(x_k) 
\end{cases} \]

+ idempotent

\[ \iff \begin{cases} 
\exists k \in \{1, \ldots, n\} \text{ such that} \\
F(x_1, \ldots, x_n) = x_k 
\end{cases} \]
Third result (Marichal & Mesiar & Rückschlossová 2004)

$F : \mathbb{R}^n \rightarrow \mathbb{R}$ is strongly comparison meaningful

\[
\exists k \in \{1, \ldots, n\} \\
\exists g : \mathbb{R} \rightarrow \mathbb{R} \text{ strictly monotonic or constant}
\] such that

\[
F(x_1, \ldots, x_n) = g(x_k)
\]

+ idempotent

\[
\exists k \in \{1, \ldots, n\} \text{ such that}
\]

\[
F(x_1, \ldots, x_n) = x_k
\]
Comparison meaningful functions

First result (Orlov 1981)

\( F : \mathbb{R}^n \rightarrow \mathbb{R} \) is
- symmetric
- continuous
- internal, i.e., \( \min_i x_i \leq F(x_1, \ldots, x_n) \leq \max_i x_i \)
- comparison meaningful

\[ \iff \exists k \in \{1, \ldots, n\} \text{ such that } \]
\[ F(x_1, \ldots, x_n) = x(k) \]

where \( x(1), \ldots, x(n) \) denote the order statistics resulting from reordering \( x_1, \ldots, x_n \) in the nondecreasing order.

Next step: suppress symmetry and relax internality into idempotency
Lattice polynomials

Definition (Birkhoff 1967)
An $n$-variable lattice polynomial is any expression involving $n$ variables $x_1, \ldots, x_n$ linked by the lattice operations

$$\land = \min \quad \text{and} \quad \lor = \max$$

in an arbitrary combination of parentheses.

For example,

$$L(x_1, x_2, x_3) = (x_1 \lor x_3) \land x_2$$

is a 3-variable lattice polynomial.
Proposition (Ovchinnikov 1998, Marichal 2002)
A lattice polynomial on $\mathbb{R}^n$ is **symmetric** iff it is an order statistic.

We have

$$x^{(k)} = \bigvee_{T \subseteq \{1, \ldots, n\} \atop |T| = n-k+1} \bigwedge_{i \in T} x_i = \bigwedge_{T \subseteq \{1, \ldots, n\} \atop |T| = k} \bigvee_{i \in T} x_i$$

Define the **kth order statistic function**

$$\text{OS}_k : x \mapsto x^{(k)}$$
The nonsymmetric case

Second result (Yanovskaya 1989)

\( F : \mathbb{R}^n \to \mathbb{R} \) is
- continuous
- idempotent
- comparison meaningful

\[ \iff \exists \text{ a lattice polynomial } L : \mathbb{R}^n \to \mathbb{R} \text{ such that } F = L. \]

\( + \) symmetric

\[ \iff \exists k \in \{1, \ldots, n\} \text{ such that } F = OS_k \text{ (kth order statistic)}. \]

Next step: suppress idempotency
The nonidempotent case

Third result (Marichal 2002)

\( F : \mathbb{R}^n \rightarrow \mathbb{R} \) is
- continuous
- comparison meaningful

\[ \exists L : \mathbb{R}^n \rightarrow \mathbb{R} \text{ lattice polynomial} \]
\[ \exists g : \mathbb{R} \rightarrow \mathbb{R} \text{ continuous} \]
\[ \exists g : \mathbb{R} \rightarrow \mathbb{R} \text{ strictly monotonic or constant} \]
\[ \Leftrightarrow \left\{ \begin{array}{l}
\exists L : \mathbb{R}^n \rightarrow \mathbb{R} \\
\exists g : \mathbb{R} \rightarrow \mathbb{R} \\
\text{ such that } \quad F = g \circ L
\end{array} \right. \]

+ symmetric

\[ F = g \circ \text{OS}_k \]
Towards the noncontinuous case

**Fourth result (Marichal 2002)**

$F : \mathbb{R}^n \rightarrow \mathbb{R}$ is
- nondecreasing
- idempotent
- comparison meaningful

$\Leftrightarrow \exists$ a lattice polynomial $L : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $F = L$.

**Note:** These functions are continuous!

+ symmetric

$F = OS_k$

**Next step:** suppress idempotency
The nondecreasing case

Fifth result (Marichal & Mesiar & Rückschlossová 2004)

\( F : \mathbb{R}^n \to \mathbb{R} \) is
- nondecreasing
- comparison meaningful

\[ \begin{align*}
\exists \ L : \mathbb{R}^n \to \mathbb{R} \text{ lattice polynomial} \\
\exists \ g : \mathbb{R} \to \mathbb{R} \text{ strictly increasing or constant} \\
such \text{that} \\
F = g \circ L
\end{align*} \]

These functions are continuous up to possible discontinuities of function \( g \)

Final step: suppress nondecreasing monotonicity (a hard task!)
The general case

... is much more complicated to describe

- We lose the concept of lattice polynomial
- The description of $F$ is done through a partition of the domain $\mathbb{R}^n$ into particular subsets, called invariant subsets
Invariant subsets

Let us consider the subsets of $\mathbb{R}^n$ of the form

$$I = \{ x \in \mathbb{R}^n \mid x_{\pi(1)} \triangleleft_1 \cdots \triangleleft_{n-1} x_{\pi(n)} \}$$

where $\pi$ is any permutation on $\{1, \ldots, n\}$ and $\triangleleft_i \in \{<,=\}$.

Denote this class of subsets by $\mathcal{I}(\mathbb{R}^n)$.

Example: $\mathbb{R}^2$

Description of $\mathcal{I}(\mathbb{R}^2)$:

- $I_1 = \{ (x_1, x_2) \mid x_1 = x_2 \}$
- $I_2 = \{ (x_1, x_2) \mid x_1 < x_2 \}$
- $I_3 = \{ (x_1, x_2) \mid x_1 > x_2 \}$
Invariant subsets

**Proposition (Bartłomiejczyk & Drewniak 2004)**
The class $\mathcal{I}(\mathbb{R}^n)$ consists of the *minimal invariant* subsets of $\mathbb{R}^n$.

That is,

- Each subset $I \in \mathcal{I}(\mathbb{R}^n)$ is *invariant* in the sense that
  
  $$(x_1, \ldots, x_n) \in I \implies (\phi(x_1), \ldots, \phi(x_n)) \in I \quad \forall \phi \in A(\mathbb{R})$$

- Each subset $I \in \mathcal{I}(\mathbb{R}^n)$ is *minimal* in the sense that it has no proper invariant subset
The general case

**Sixth result (Marichal & Mesiar & Rückschlossová 2004):**

\( F : \mathbb{R}^n \rightarrow \mathbb{R} \) is **comparison meaningful**

\[ \Leftrightarrow \forall I \in \mathcal{I}(\mathbb{R}^n), \begin{cases} \exists k_I \in \{1, \ldots, n\} \\
\exists g_I : \mathbb{R} \rightarrow \mathbb{R} \text{ strictly monotonic or constant} \\
\text{such that} \\
F|_I(x_1, \ldots, x_n) = g_I(x_{k_I}) \end{cases} \]

where \( \forall I, I' \in \mathcal{I}(\mathbb{R}^n), \)

- either \( g_I = g_{I'} \)
- or \( \text{ran}(g_I) = \text{ran}(g_{I'}) \) is a singleton
- or \( \text{ran}(g_I) < \text{ran}(g_{I'}) \)
- or \( \text{ran}(g_I) > \text{ran}(g_{I'}) \)
Invariant functions

Now, assume that

\[ x_{n+1} = F(x_1, \ldots, x_n) \]

where \( x_1, \ldots, x_n \) and \( x_{n+1} \) define the same ordinal scale.

Then the functional equation simplifies into

\[ F[\phi(x_1), \ldots, \phi(x_n)] = \phi[F(x_1, \ldots, x_n)] \]

(introduced in Marichal & Roubens 1993)

\( F \) is said to be \textit{invariant} (Bartłomiejczyk & Drewniak 2004)
The symmetric case

First result (Marichal & Roubens 1993)

\( F : \mathbb{R}^n \rightarrow \mathbb{R} \) is

- symmetric
- continuous
- nondecreasing
- invariant

\[ \Leftrightarrow \exists k \in \{1, \ldots, n\} \text{ such that } F = \text{OS}_k \]

Next step: suppress symmetry and nondecreasing monotonicity
The nonsymmetric case

Second result (Ovchinnikov 1998)

\[ F : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is } \begin{cases} \text{continuous} \\ \text{invariant} \end{cases} \]

\[ \Leftrightarrow \exists \text{ a lattice polynomial } L : \mathbb{R}^n \rightarrow \mathbb{R} \text{ such that } F = L \]

**Note:** These functions are nondecreasing!

+ symmetric

\[ F = \text{OS}_k \]

**Next step:** suppress continuity
The nondecreasing case

Third result (Marichal 2002)
$F : \mathbb{R}^n \to \mathbb{R}$ is
- nondecreasing
- invariant

$\Leftrightarrow \exists$ a lattice polynomial $L : \mathbb{R}^n \to \mathbb{R}$ such that $F = L$

Note: These functions are continuous!

+ symmetric

$F = OS_k$

Final step: suppress nondecreasing monotonicity
The general case

The general case was first described by Ovchinnikov (1998)

A simpler description in terms of invariant sets is due to Bartłomiejczyk & Drewniak (2004)

**Fourth result (Ovchinnikov 1998)**

\[ F : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is invariant} \]

\[ \iff \forall I \in \mathcal{I}(\mathbb{R}^n), \quad \left\{ \begin{array}{l} \exists k_I \in \{1, \ldots, n\} \\
\text{such that} \\
F|_I(x_1, \ldots, x_n) = x_{k_I} \end{array} \right. \]
Conclusion

We have described all the possible merging functions

\[ F : \mathbb{R}^n \to \mathbb{R}, \]

which map \( n \) ordinal scales into an ordinal scale.

These results hold true when \( F \) is defined on \( E^n \), where \( E \) is any open real interval.

The cases where \( E \) is a non-open real interval all have been described and can be found in

J.-L. Marichal, R. Mesiar, and T. Rückschlossová,
A Complete Description of Comparison Meaningful Functions,
Thank you for your attention