Weighted Lattice Polynomials

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Lattice polynomials

Let $L$ be a lattice with lattice operations $\land$ and $\lor$

We assume that $L$ is

- bounded (with bottom $0$ and top $1$)
- distributive

**Definition (Birkhoff 1967)**

An $n$-ary *lattice polynomial* is a well-formed expression involving $n$ variables $x_1, \ldots, x_n \in L$ linked by the lattice operations $\land$ and $\lor$ in an arbitrary combination of parentheses

**Example.**

$$p(x_1, x_2, x_3) = (x_1 \land x_2) \lor x_3$$
Any lattice polynomial naturally defines a *lattice polynomial function* (l.p.f.) \( p : L^n \rightarrow L \).

**Example.**

\[
p(x_1, x_2, x_3) = (x_1 \land x_2) \lor x_3
\]

If \( p \) and \( q \) represent the same function, we say that \( p \) and \( q \) are equivalent and we write \( p = q \).

**Example.**

\[
x_1 \lor (x_1 \land x_2) = x_1
\]
Disjunctive and conjunctive forms of l.p.f.’s

Notation. \([n] := \{1, \ldots, n\}\).

**Proposition (Birkhoff 1967)**

Let \(p : L^n \rightarrow L\) be any l.p.f.

Then there are nonconstant set functions \(v, w : 2^n \rightarrow \{0, 1\}\), with \(v(\emptyset) = 0\) and \(w(\emptyset) = 1\), such that

\[
p(x) = \bigvee_{S \subseteq [n], v(S)=1} \bigwedge_{i \in S} x_i = \bigwedge_{S \subseteq [n], w(S)=0} \bigvee_{i \in S} x_i.
\]

**Example.**

\[
(x_1 \land x_2) \lor x_3 = (x_1 \lor x_3) \land (x_2 \lor x_3)
\]

\[
v(\{3\}) = v(\{1, 2\}) = 1
\]

\[
w(\{1, 3\}) = w(\{2, 3\}) = 0
\]
The set functions $v$ and $w$, which generate $p$, are not unique:

$$x_1 \lor (x_1 \land x_2) = x_1 = x_1 \land (x_1 \lor x_2)$$

**Notation.** $1_S :=$ characteristic vector of $S \subseteq [n]$ in $\{0, 1\}^n$.

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**Proposition (Marichal 2002)**

From among all the set functions $v$ that disjunctively generate the l.p.f. $p$, only one is isotone:

$$v(S) = p(1_S)$$

From among all the set functions $w$ that conjunctively generate the l.p.f. $p$, only one is antitone:

$$w(S) = p(1_{[n]\setminus S})$$
Consequently, any \( n \)-ary l.p.f. can always be written as

\[
p(x) = \bigvee_{S \subseteq [n]} \bigwedge_{i \in S} x_i = \bigwedge_{S \subseteq [n]} \bigvee_{i \in S} x_i
\]

\( p(1_S) = 1 \) \( p(1_{[n]\setminus S}) = 0 \)

**Example.** \( p(x) = (x_1 \land x_2) \lor x_3 \)

<table>
<thead>
<tr>
<th>( S )</th>
<th>( p(1_S) )</th>
<th>( p(1_{[n]\setminus S}) )</th>
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<tbody>
<tr>
<td>( \emptyset )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( {1} )</td>
<td>0</td>
<td>1</td>
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<td>( {2} )</td>
<td>0</td>
<td>1</td>
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<td>( {1, 2} )</td>
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<td>1</td>
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<td>( {1, 3} )</td>
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<td>( {2, 3} )</td>
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<td>0</td>
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<tr>
<td>( {1, 2, 3} )</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
p(x) = x_3 \lor (x_1 \land x_2) \lor (x_1 \land x_3) \lor (x_2 \land x_3) \lor (x_1 \land x_2 \land x_3)
\]

\[
p(x) = (x_1 \lor x_3) \land (x_2 \lor x_3) \land (x_1 \lor x_2 \lor x_3)
\]
Denote by $x_1, \ldots, x_n$ the order statistics resulting from reordering $x_1, \ldots, x_n$ in the nondecreasing order: $x_1 \leq \cdots \leq x_n$.

**Proposition (Ovchinnikov 1996, Marichal 2002)**

$p$ is a symmetric l.p.f. $\iff p$ is an order statistic

**Notation.** Denote by $os_k : L^n \to L$ the $k$th order statistic function.

$$os_k(x) := x_{(k)}$$

Then we have

$$os_k(1_S) = 1 \iff |S| \geq n - k + 1$$

$$os_k(1_{[n] \setminus S}) = 0 \iff |S| \geq k$$
We can generalize the concept of l.p.f. by regarding some variables as parameters.

**Example.** For $c \in L$, we consider

$$p(x_1, x_2) = (c \lor x_1) \land x_2$$

**Definition**

$p : L^n \to L$ is an $n$-ary *weighted lattice polynomial* function (w.l.p.f.) if there exist parameters $c_1, \ldots, c_m \in L$ and a l.p.f. $q : L^{n+m} \to L$ such that

$$p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n, c_1, \ldots, c_m)$$
Disjunctive and conjunctive forms of w.l.p.f.'s

**Proposition (Lausch & Nöbauer 1973)**

Let $p : L^n \to L$ be any w.l.p.f.

Then there are set functions $v, w : 2^{[n]} \to L$ such that

$$p(x) = \bigvee_{S \subseteq [n]} \left[ v(S) \land \bigwedge_{i \in S} x_i \right] = \bigwedge_{S \subseteq [n]} \left[ w(S) \lor \bigvee_{i \in S} x_i \right].$$

**Remarks.**

- $p$ is a l.p.f. if $v$ and $w$ range in $\{0, 1\}$, with $v(\emptyset) = 0$ and $w(\emptyset) = 1$.

- Any w.l.p.f. is entirely determined by $2^n$ parameters, even if more parameters have been considered to construct it.
Proposition (Marichal 2006)

From among all the set functions $v$ that disjunctively generate the w.l.p.f. $p$, only one is isotone:

$$v(S) = p(1_S)$$

From among all the set functions $w$ that conjunctively generate the w.l.p.f. $p$, only one is antitone:

$$w(S) = p(1_{[n]\setminus S})$$
Consequently, any $n$-ary w.l.p.f. can always be written as

$$p(x) = \bigvee_{S \subseteq [n]} \left[ p(1_S) \land \bigwedge_{i \in S} x_i \right] = \bigwedge_{S \subseteq [n]} \left[ p(1_{[n]\setminus S}) \lor \bigvee_{i \in S} x_i \right]$$

**Example.** $p(x) = (c \lor x_1) \land x_2$

<table>
<thead>
<tr>
<th>$S$</th>
<th>$p(1_S)$</th>
<th>$p(1_{[n]\setminus S})$</th>
</tr>
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<tr>
<td>$\emptyset$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>${1}$</td>
<td>0</td>
<td>$c$</td>
</tr>
<tr>
<td>${2}$</td>
<td>$c$</td>
<td>0</td>
</tr>
<tr>
<td>${1, 2}$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$$p(x) = (0 \land 1) \lor (0 \land x_1) \lor (c \land x_2) \lor (1 \land x_1 \land x_2)$$
$$= (c \land x_2) \lor (x_1 \land x_2)$$

$$p(x) = (1 \lor 0) \land (c \lor x_1) \land (0 \lor x_2) \land (0 \lor x_1 \lor x_2)$$
$$= (c \lor x_1) \land x_2$$
Particular case: the Sugeno integral

Let us generalize the concept of discrete Sugeno integral in the framework of bounded distributive lattices.

**Definition (Sugeno 1974)**

An $L$-valued *fuzzy measure* on $[n]$ is an isotone set function $\mu : 2^{[n]} \to L$ such that $\mu(\emptyset) = 0$ and $\mu([n]) = 1$.

The *Sugeno integral* of a function $x : [n] \to L$ with respect to $\mu$ is defined by

$$S_\mu(x) := \bigvee_{S \subseteq [n]} \left[ \mu(S) \land \bigwedge_{i \in S} x_i \right]$$

**Remark.** A function $f : L^n \to L$ is an $n$-ary Sugeno integral if and only if $f$ is a w.l.p.f. fulfilling $f(1_{\emptyset}) = 0$ and $f(1_{[n]}) = 1$. 

Notation. The median function is the function \( os_2 : L^3 \to L \).

Proposition (Marichal 2006)
For any w.l.p.f. \( p : L^n \to L \), there is a fuzzy measure \( \mu : 2^{[n]} \to L \) such that
\[
p(x) = \text{median}[p(\emptyset), S_\mu(x), p([n])]
\]

Corollary (Marichal 2006)
Consider a function \( f : L^n \to L \).
The following assertions are equivalent:
- \( f \) is a Sugeno integral
- \( f \) is an idempotent w.l.p.f., that is such that \( f(x, \ldots, x) = x \)
- \( f \) is a w.l.p.f. fulfilling \( f(\emptyset) = 0 \) and \( f([n]) = 1 \).
Inclusion properties

Weighted lattice polynomials

Sugeno integrals

Lattice polynomials

Order statistics
The median based decomposition formula

Let $f : L^n \rightarrow L$ and $k \in [n]$ and define $f^0_k, f^1_k : L^n \rightarrow L$ as

\[
\begin{align*}
    f^0_k(x) & := f(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) \\
    f^1_k(x) & := f(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_n)
\end{align*}
\]

**Remark.** If $f$ is a w.l.p.f., so are $f^0_k$ and $f^1_k$

Consider the following system of $n$ functional equations, called the *median based decomposition formula*

\[
f(x) = \text{median}[f^0_k(x), x_k, f^1_k(x)] \quad (k = 1, \ldots, n)
\]
Any solution of the median based decomposition formula

\[ f(x) = \text{median} [f_k^0(x), x_k, f_k^1(x)] \quad (k = 1, \ldots, n) \]

is an \( n \)-ary w.l.p.f.

**Example.** For \( n = 2 \) we have

\[ f(x_1, x_2) = \text{median} [f(x_1, 0), x_2, f(x_1, 1)] \]

with

\[ f(x_1, 0) = \text{median} [f(0, 0), x_1, f(1, 0)] \quad (\text{w.l.p.f.}) \]
\[ f(x_1, 1) = \text{median} [f(0, 1), x_1, f(1, 1)] \quad (\text{w.l.p.f.}) \]
The median based decomposition formula characterizes the w.l.p.f.'s

**Theorem (Marichal 2006)**

The solutions of the median based decomposition formula are exactly the \( n \)-ary w.l.p.f.'s.
Thanks for your attention!