Derivative relationships between volume and surface area of compact regions in $\mathbb{R}^p$

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Introductory Examples

• Sphere in $\mathbb{R}^3$ of radius $r > 0$:

\[ V = \frac{4}{3} \pi r^3 \]
\[ A = 4\pi r^2 \]

\[ \frac{dV}{dr} = A \]

The rate of change in volume is the surface area

• Circle in $\mathbb{R}^2$ of radius $r > 0$:

\[ A = \pi r^2 \]
\[ P = 2\pi r \]

\[ \frac{dA}{dr} = P \]

The rate of change in area is the perimeter
• Cube in $\mathbb{R}^3$ of edge length $s > 0$:

\[
V = s^3 \\
A = 6s^2
\]

\[
\frac{dV}{ds} = 3s^2 \neq A !!!
\]

• Square in $\mathbb{R}^2$ of side length $s > 0$:

\[
A = s^2 \\
P = 4s
\]

\[
\frac{dA}{ds} = 2s \neq P !!!
\]
Cube of edge length $s > 0$

Express volume and area in terms of the inradius

$$r = \frac{s}{2} \quad \Leftrightarrow \quad s = 2r$$

$$V = 8r^3$$

$$A = 24r^2$$

Increasing the inradius $r$ makes $V$ increase at a rate $A$

Appropriate notation:

$$V \rightarrow V(s) \rightarrow V[s(r)]$$

$$A \rightarrow A(s) \rightarrow A[s(r)]$$

$$\frac{d}{dr}V[s(r)] = A[s(r)]$$
Let us formalize the problem...

One-parameter family of compact regions in $\mathbb{R}^p$

$$\mathcal{R} := \{ R(s) \subset \mathbb{R}^p \mid s \in E \} \quad (E = \text{real interval})$$

With $\mathcal{R}$ is associated:

- $V : E \to \mathbb{R}_+$ differentiable
- $A : E \to \mathbb{R}_+$ continuous

$V(s)$ is the volume of $R(s)$
$A(s)$ is the area of $R(s)$
Example: Family of cubes in $\mathbb{R}^3$

Edge length of $R(s)$: $s$

$$V(s) = s^3$$
$$A(s) = 6s^2$$

Alternative representation:

Edge length of $R(s)$: $\phi(s)$

e.g. $s =$ diameter of $R(s)$

$$\Rightarrow \quad \phi(s) = \frac{s}{\sqrt{3}}$$

$$V_\phi(s) = \phi(s)^3$$
$$A_\phi(s) = 6\phi(s)^2$$
We search for a change of variable $s \mapsto r(s)$ so that

$$\frac{d}{dr} V[s(r)] = A[s(r)] \quad (r \in r(E))$$

Note: $r$ represents a linear dimension (a length)

**Questions:**

Given a family $\mathcal{R}$,

1. When does such a change of variable exists?
2. When it exists, how can we calculate it?
3. When it exists, can we provide a geometric interpretation of it?
Proposition

Suppose \( V(s) \) is a strictly monotone and differentiable function in \( E \) and \( A(s) \) is a continuous function in \( E \). Then there is a differentiable change of variable

\[
    r(s) : E \to r(E),
\]

defined as

\[
    r(s) = \int \frac{V'(s)}{A(s)} \, ds \quad (s \in E)
\]

and unique within an additive constant \( C \in \mathbb{R} \), such that

\[
    \frac{d}{dr} V[s(r)] = A[s(r)] \quad (r \in r(E)).
\]
Stability under any change of representation

If $V(s)$ and $A(s)$ are replaced with

$$V_\phi(s) = V[\phi(s)] \quad \text{and} \quad A_\phi(s) = A[\phi(s)]$$

respectively, where $\phi$ is a differentiable function from $E$ into itself, then $r(s)$ is simply replaced with

$$r_\phi(s) = \int \frac{V'_\phi(s)}{A_\phi(s)} \, ds = \int \frac{V'[\phi(s)] \phi'(s)}{A[\phi(s)]} \, ds$$

$$= \int \frac{V'(t)}{A(t)} \, dt \bigg|_{t=\phi(s)}$$

$$= r[\phi(s)]$$
Example: Family of cubes in $\mathbb{R}^3$

\[ V(s) = s^3 \]
\[ A(s) = 6s^2 \]

\[ \Rightarrow r(s) = \int \frac{3s^2}{6s^2} ds = \frac{s}{2} + C \]

If $C = 0$ then $r(s) = \frac{s}{2}$ (inradius)

We can consider $C \neq 0$:

e.g. $r(s) = \frac{s}{2} - r_0$

\[ V[s(r)] = 8(r + r_0)^3 \]
\[ A[s(r)] = 24(r + r_0)^2 \]
Family of rhombi in $\mathbb{R}^2$

Sides of fixed length $a > 0$

A diagonal of variable length $s \in [0, 2a[$

$$A(s) = s \sqrt{a^2 - s^2/4}$$

$$P(s) = 4a$$

$$r(s) = \int \frac{A'(s)}{P(s)} \, ds = \frac{1}{4a} \int A'(s) \, ds = \frac{A(s)}{4a} + C$$

If $C = 0$ then $r(s) = \frac{A(s)}{4a}$.

$$A[s(r)] = 4ar$$

$$P[s(r)] = 4a$$
Interpretation:
Let $r^*(s)$ be the inradius of rhombus $R(s)$

$$\frac{A(s)}{4} = \frac{ar^*(s)}{2}$$

$$\Rightarrow r(s) = \frac{A(s)}{4a} = \frac{r^*(s)}{2}$$

(half of the inradius)
Family of rectangles in $\mathbb{R}^2$

Fixed length $a > 0$
Variable width $s > 0$

\[
A(s) = as \\
P(s) = 2s + 2a
\]

\[
r(s) = \int \frac{A'(s)}{P(s)} \, ds = \int \frac{a}{2s + 2a} \, ds = \frac{a}{2} \ln(2s + 2a) + C
\]

Interpretation ?
Family of similar rectangles in $\mathbb{R}^2$

Width $s > 0$

Length $2s > 0$

\[ A(s) = 2s^2 \]
\[ P(s) = 6s \]

\[ r(s) = \int \frac{A'(s)}{P(s)} \, ds = \int \frac{4s}{6s} \, ds = \frac{2}{3} s + C \]

Interpretation?

Setting $r_1(s) = s$ and $r_2(s) = s/2$, we have

\[ r(s) = \frac{2}{3} s = \frac{2}{\frac{1}{s} + \frac{2}{s}} = H[r_1(s), r_2(s)]. \]
Case of Similar Regions

Suppose that $R$ is made up of similar regions and $s \in \mathbb{R}_+$ is a characteristic linear dimension

Then, there are $k_1, k_2 > 0$ such that

\[
V(s) = k_1 s^p \\
A(s) = k_2 s^{p-1}
\]

\[
\Rightarrow \quad r(s) = p \frac{V(s)}{A(s)} + C
\]


Conversely,...
**Proposition**

Suppose $V(s)$ is a strictly monotone and differentiable function in $E$ and $A(s)$ is a continuous function in $E$. Let

$$r(s) = \int \frac{V'(s)}{A(s)} ds \quad (s \in E).$$

Then there exists a constant $C \in \mathbb{R}$ such that

$$r(s) = p \frac{V(s)}{A(s)} + C \quad (s \in E)$$

if and only if there exists a constant $k > 0$ such that

$$A(s)^p = kV(s)^{p-1} \quad (s \in E).$$

In this case, $\mathcal{R}$ is said to be a homogeneous family
Isoperimetric Ratio

The isoperimetric ratio (Pólya, 1954) of a compact region $R$ in $\mathbb{R}^p$ is given by $Q = \frac{A^p}{V^{p-1}}$.

The previous proposition says that $\mathcal{R}$ is homogeneous iff the isoperimetric ratio

$$Q(s) = \frac{A(s)^p}{V(s)^{p-1}} \quad (s \in E)$$

is constant in $E$.

Example: Family of cubes in $\mathbb{R}^3$ $\Rightarrow$ $Q(s) = 216$
Immediate Corollary

*If the regions of $\mathcal{R}$ are all similar then $\mathcal{R}$ is a homogeneous family.*

**Converse false:** Consider the hexagons $R(s)$ whose inner angles all have a fixed amplitude $2\pi/3$ and the consecutive sides have lengths $a(s), b(s), c(s), a(s), b(s),$ and $c(s),$ respectively. Then

\[
A(s) = \frac{\sqrt{3}}{2}[a(s)b(s) + b(s)c(s) + c(s)a(s)],
\]
\[
P(s) = 2[a(s) + b(s) + c(s)].
\]

By choosing $a(s) = 1, b(s) = s^2,$ and $c(s) = (s + 1)^2,$ where $s \in \mathbb{R}_+,$ we obtain a homogeneous family.
Proposition

R is a homogeneous family if and only if there exists a differentiable change of variable $\phi : E \to \phi(E)$ and constants $k_1, k_2 > 0$ such that

\[ V(s) = k_1 \phi(s)^p \quad \text{and} \quad A(s) = k_2 \phi(s)^{p-1} \quad (s \in E). \]

$V(s)$ and $A(s)$ are homogeneous functions of degrees $p$ and $p - 1$, respectively, up to the same change of variable $\phi(s)$.
Elasticity

Define the area elasticity of volume as the proportional change in volume relative to the proportional change in area, that is,

\[ e_{V,A}(s) = \frac{\frac{dV(s)}{V(s)}}{\frac{dA(s)}{A(s)}} = \frac{V'(s)}{V(s)} \frac{A(s)}{A'(s)} \frac{V(s)}{A(s)} = \frac{V'(s)}{V'(s)} \frac{A(s)}{A(s)} \frac{V(s)}{V(s)}. \]

Proposition

*R is a homogeneous family if and only if*

\[ e_{V,A}(s) = \frac{p}{p-1} \quad (s \in E). \]
Open Questions

• Characterize geometrically homogeneous families

• Given a class of compact regions in $\mathbb{R}^p$, find homogeneous subfamilies, if any.
Geometric Interpretation of $r$?

**Theorem**  For any family of similar circumscribing polytopes, the variable $r$ represents the radius of the inscribed sphere


**Corollary**  If a $p$-dimensional sphere of radius $r$ is inscribed in a polytope, then

$$V = \frac{r}{p} A.$$

Proposition

Let $R(s)$ be a homogeneous family of $n$-faced polyhedra $R(s)$ that are star-like with respect to a point $T(s)$ in the interior of $R(s)$. Let $P_i(s)$ be the pyramid whose base is the $i$th facet of $R(s)$ and whose vertex is $T(s)$. Then

$$r(s) = \sum_{i=1}^{n} \frac{A_i(s)}{A(s)} r_i(s)$$

and

$$\frac{1}{r(s)} = \sum_{i=1}^{n} \frac{V_i(s)}{V(s)} \frac{1}{r_i(s)}$$

where $V_i(s)$, $A_i(s)$, and $r_i(s)$ are respectively the volume of $P_i(s)$, the surface area of the base of $P_i(s)$, and the altitude from $T(s)$ of $P_i(s)$. 
Case of triangle

The centroid $T$ of any triangle provides an equal-area triangulation.

So we have

$$\frac{1}{r} = \sum_{i=1}^{3} \frac{V_i}{V} \cdot \frac{1}{r_i} = \frac{1}{3} \sum_{i=1}^{3} \frac{1}{r_i}$$

that is

$$r = H(r_1, r_2, r_3).$$

Setting $h_i := 3r_i$ (triangle altitudes), we get

$$3r = H(h_1, h_2, h_3)$$

For any triangle, the harmonic mean of its altitudes is three times the inradius of the triangle.
Open Questions

• Generalize the previous proposition to any star-like region (cones, cylinders...)

• Generalize the previous proposition to any region (torus...)

Some results on similar regions

1. Any convex region $R$ in $\mathbb{R}^2$ having an inscribed circle $S$ of radius $r$ has the property

$$\frac{d}{dr} A = P$$

2. Let $R \subset \mathbb{R}^2$ be a region as in (1) above and which is symmetric w.r.t. an axis through the center of $S$. For the solid formed by revolving $R$ about that axis of symmetry, we have

$$\frac{d}{dr} V = A$$

The same for the solid formed by lifting $R$ to a height of $2r$.

M. Dorff and L. Hall, Solids in $\mathbb{R}^n$ whose area is the derivative of the volume, submitted.
Singular Case
(non similar regions)

Let $R \subset \mathbb{R}^2$ be a disc or a regular polygonal region with in-radius $r$. For any solid formed by revolving $R$ about an axis that does not intersect $R$, we have

$$\frac{d}{dr} V = A$$

**Example**: Torus obtained by rotating a circle centered at the fixed point $(a, 0)$ and of radius $r < a$:

$$V = (2\pi a)(\pi r^2)$$
$$A = (2\pi a)(2\pi r)$$

$$\frac{d}{dr} V = A$$
Another open problem: the case of $n$-parameter families

Example: Consider a family of rectangles $R(s_1, s_2)$ with length $s_1 > 0$ and width $s_2 > 0$. Consider also the linear change of variables

$$r_1(s) = \frac{s_1}{2} \quad \text{and} \quad r_2(s) = \frac{s_2}{2}$$

which inverts into

$$s_1(r) = 2r_1 \quad \text{and} \quad s_2(r) = 2r_2.$$ 

Then we clearly have

$$A(s) = 4r_1(s)r_2(s).$$
$$P(s) = 4r_1(s) + 4r_2(s).$$

Finally,

$$\frac{\partial}{\partial r_1} A[s(r)] + \frac{\partial}{\partial r_2} A[s(r)] = P[s(r)].$$

In the general case, we consider the following derivative relationship:

$$\sum_{j=1}^{n} \frac{\partial}{\partial r_j} V[s(r)] = A[s(r)],$$

where $r(s)$ is an appropriate change of variables.

to be continued...