Derivative relationships between volume and surface area of compact regions in $\mathbb{R}^p$

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Introductory Examples

- **Sphere in $\mathbb{R}^3$ of radius $r > 0$:**

$$V = \tfrac{4}{3}\pi r^3 \qquad A = 4\pi r^2 \qquad \boxed{\frac{dV}{dr} = A}$$

The rate of change in volume is the surface area

- **Circle in $\mathbb{R}^2$ of radius $r > 0$:**

$$A = \pi r^2 \qquad P = 2\pi r \qquad \boxed{\frac{dA}{dr} = P}$$

The rate of change in area is the perimeter

- **Cube in $\mathbb{R}^3$ of edge length $s > 0$:**

$$V = s^3 \qquad A = 6s^2$$

$$\boxed{\frac{dV}{ds} = 3s^2 \neq A \text{ !!!}}$$

- **Square in $\mathbb{R}^2$ of side length $s > 0$:**

$$A = s^2 \qquad P = 4s$$

$$\boxed{\frac{dA}{ds} = 2s \neq P \text{ !!!}}$$

Cube of edge length $s > 0$

Express volume and area in terms of the inradius

$$r = \frac{s}{2} \quad \Leftrightarrow \quad s = 2r$$

$$V = 8r^3 \qquad \boxed{\frac{dV}{dr} = A}$$

$$A = 24r^2$$

Increasing the inradius r makes V increase at a rate A

Appropriate notation:

$$V \to V(s) \to V[s(r)] \qquad \boxed{\frac{d}{dr}V[s(r)] = A[s(r)]}$$

$$A \to A(s) \to A[s(r)]$$

Let us formalize the problem...

One-parameter family of compact regions in $\mathbb{R}^p$

$$\mathcal{R} := \{R(s) \subset \mathbb{R}^p \mid s \in E\} \qquad (E = \text{real interval})$$

With $\mathcal{R}$ is associated:

$V : E \to \mathbb{R}_+$ differentiable

$A : E \to \mathbb{R}_+$ continuous

$V(s)$ is the volume of $R(s)$

$A(s)$ is the area of $R(s)$

Example: Family of cubes in $\mathbb{R}^3$

Edge length of $R(s)$: s

$$\begin{aligned} V(s) &= s^3 \\ A(s) &= 6s^2 \end{aligned}$$

Alternative representation:

Edge length of $R(s)$: $\phi(s)$

e.g. $s =$ diameter of $R(s)$

$$\Rightarrow \quad \phi(s) = \frac{s}{\sqrt{3}}$$

$$\begin{aligned} V_\phi(s) &= \phi(s)^3 \\ A_\phi(s) &= 6\phi(s)^2 \end{aligned}$$

We search for a change of variable $s \mapsto r(s)$ so that

$$\frac{d}{dr} V[s(r)] = A[s(r)] \qquad (r \in r(E))$$

Note : r represents a linear dimension (a length)

Questions:

Given a family $\mathcal{R}$,

1. When does such a change of variable exists ?
2. When it exists, how can we calculate it ?
3. When it exists, can we provide a geometric interpretation of it ?

Proposition

Suppose $V(s)$ is a strictly monotone and differentiable function in E and $A(s)$ is a continuous function in E. Then there is a differentiable change of variable

$$r(s) : E \to r(E),$$

defined as

$$r(s) = \int \frac{V'(s)}{A(s)}\, ds \qquad (s \in E)$$

and unique within an additive constant $C \in \mathbb{R}$, such that

$$\frac{d}{dr} V[s(r)] = A[s(r)] \qquad (r \in r(E)).$$

Stability under any change of representation

If $V(s)$ and $A(s)$ are replaced with

$$V_\phi(s) = V[\phi(s)] \qquad \text{and} \qquad A_\phi(s) = A[\phi(s)]$$

respectively, where ϕ is a differentiable function from E into itself, then $r(s)$ is simply replaced with

$$\begin{aligned} r_\phi(s) &= \int \frac{V'_\phi(s)}{A_\phi(s)}\, ds = \int \frac{V'[\phi(s)]\,\phi'(s)}{A[\phi(s)]}\, ds \\ &= \int \frac{V'(t)}{A(t)}\, dt\Big|_{t=\phi(s)} \\ &= r[\phi(s)] \end{aligned}$$

Example: Family of cubes in $\mathbb{R}^3$

$$\begin{aligned} V(s) &= s^3 \\ A(s) &= 6s^2 \end{aligned}$$

$$\Rightarrow \quad r(s) = \int \frac{3s^2}{6s^2}\, ds = \frac{s}{2} + C$$

If $C = 0$ then $r(s) = \frac{s}{2}$ (inradius)

We can consider $C \neq 0$:

e.g. $r(s) = \dfrac{s}{2} - r_0$

$$\begin{aligned} V[s(r)] &= 8(r + r_0)^3 \\ A[s(r)] &= 24(r + r_0)^2 \end{aligned}$$

Family of rhombi in $\mathbb{R}^2$

Sides of fixed length $a > 0$

A diagonal of variable length $s \in]0, 2a[$

$$\begin{aligned} A(s) &= s\sqrt{a^2 - s^2/4} \\ P(s) &= 4a \end{aligned}$$

$$r(s) = \int \frac{A'(s)}{P(s)}\, ds = \frac{1}{4a} \int A'(s)\, ds = \frac{A(s)}{4a} + C$$

If $C = 0$ then $r(s) = \frac{A(s)}{4a}$.

$$\begin{aligned} A[s(r)] &= 4ar \\ P[s(r)] &= 4a \end{aligned}$$

Interpretation:

Let $r^*(s)$ be the inradius of rhombus $R(s)$

$$\frac{A(s)}{4} = \frac{ar^*(s)}{2}$$

$$\Rightarrow \quad r(s) = \frac{A(s)}{4a} = \frac{r^*(s)}{2}$$

(half of the inradius)

Family of rectangles in $\mathbb{R}^2$

Fixed length $a > 0$

Variable width $s > 0$

$$\begin{aligned} A(s) &= as \\ P(s) &= 2s + 2a \end{aligned}$$

$$r(s) = \int \frac{A'(s)}{P(s)}\, ds = \int \frac{a}{2s + 2a}\, ds = \frac{a}{2}\, \ln(2s + 2a) + C$$

Interpretation ?

Family of similar rectangles in $\mathbb{R}^2$

Width $s > 0$

Length $2s > 0$

$$\begin{aligned} A(s) &= 2s^2 \\ P(s) &= 6s \end{aligned}$$

$$r(s) = \int \frac{A'(s)}{P(s)}\, ds = \int \frac{4s}{6s}\, ds = \frac{2}{3}\, s + C$$

Interpretation ?

Setting $r_1(s) = s$ and $r_2(s) = s/2$, we have

$$r(s) = \frac{2}{3}\, s = \frac{2}{\frac{1}{s} + \frac{2}{s}} = H[r_1(s), r_2(s)].$$

Case of Similar Regions

Suppose that $\mathcal{R}$ is made up of similar regions and $s \in \mathbb{R}_+$ is a characteristic linear dimension
Then, there are $k_1, k_2 > 0$ such that

$$\begin{aligned} V(s) &= k_1 s^p \\ A(s) &= k_2 s^{p-1} \end{aligned}$$

$$\Rightarrow \quad \boxed{r(s) = p\,\frac{V(s)}{A(s)} + C}$$

J. Tong, Area and perimeter, volume and surface area, *College Math. J.* **28** (1) (1997) 57.

Conversely,...

Proposition

Suppose $V(s)$ is a strictly monotone and differentiable function in E and $A(s)$ is a continuous function in E. Let

$$r(s) = \int \frac{V'(s)}{A(s)}\, ds \qquad (s \in E).$$

Then there exists a constant $C \in \mathbb{R}$ such that

$$r(s) = p\,\frac{V(s)}{A(s)} + C \qquad (s \in E)$$

if and only if there exists a constant $k > 0$ such that

$$A(s)^p = kV(s)^{p-1} \qquad (s \in E).$$

In this case, $\mathcal{R}$ is said to be a *homogeneous* family

Isoperimetric Ratio

The isoperimetric ratio (Pólya, 1954) of a compact region R in $\mathbb{R}^p$ is given by $Q = A^p/V^{p-1}$.

The previous proposition says that $\mathcal{R}$ is homogeneous iff the isoperimetric ratio

$$Q(s) = A(s)^p/V(s)^{p-1} \qquad (s \in E)$$

is constant in E.

Example : Family of cubes in $\mathbb{R}^3 \quad \Rightarrow \quad Q(s) = 216$

Immediate Corollary

If the regions of $\mathcal{R}$ are all similar then $\mathcal{R}$ is a homogeneous family.

Converse false: Consider the hexagons $R(s)$ whose inner angles all have a fixed amplitude $2\pi/3$ and the consecutive sides have lengths $a(s)$, $b(s)$, $c(s)$, $a(s)$, $b(s)$, and $c(s)$, respectively. Then

$$\begin{aligned} A(s) &= \frac{\sqrt{3}}{2}[a(s)b(s) + b(s)c(s) + c(s)a(s)], \\ P(s) &= 2[a(s) + b(s) + c(s)]. \end{aligned}$$

By choosing $a(s) = 1$, $b(s) = s^2$, and $c(s) = (s+1)^2$, where $s \in \mathbb{R}_+$, we obtain a homogeneous family.

Proposition

R is a homogeneous family if and only if there exists a differentiable change of variable $\phi : E \to \phi(E)$ and constants $k_1, k_2 > 0$ such that

$$V(s) = k_1 \phi(s)^p \quad \textit{and} \quad A(s) = k_2 \phi(s)^{p-1} \qquad (s \in E).$$

$V(s)$ and $A(s)$ are homogeneous functions of degrees p and $p-1$, respectively, up to the same change of variable $\phi(s)$.

Elasticity

Define the *area elasticity of volume* as the proportional change in volume relative to the proportional change in area, that is,

$$e_{V,A}(s) = \frac{\frac{dV(s)}{V(s)}}{\frac{dA(s)}{A(s)}} = \frac{V'(s)}{A'(s)}\,\frac{A(s)}{V(s)}.$$

Proposition

R is a homogeneous family if and only if

$$e_{V,A}(s) = \frac{p}{p-1} \qquad (s \in E).$$

Open Questions

- Characterize geometrically homogeneous families
- Given a class of compact regions in $\mathbb{R}^p$, find homogeneous subfamilies, if any.

Geometric Interpretation of r ?

Theorem *For any family of similar circumscribing polytopes, the variable r represents the radius of the inscribed sphere*

J. Emert and R. Nelson, Volume and surface area for polyhedra and polytopes, *Math. Mag.* **70** (1997) 365–371.

Corollary *If a p-dimensional sphere of radius r is inscribed in a polytope, then*

$$V = \frac{r}{p} A.$$

M.J. Cohen, Ratio of volume of inscribed sphere to polyhedron, *Amer. Math. Monthly* **72** (1965) 183–184.

Proposition

Let $\mathcal{R}$ be a homogeneous family of n-faced polyhedra $R(s)$ that are star-like with respect to a point $T(s)$ in the interior of $R(s)$. Let $P_i(s)$ be the pyramid whose base is the ith facet of $R(s)$ and whose vertex is $T(s)$. Then

$$r(s) = \sum_{i=1}^{n} \frac{A_i(s)}{A(s)} \, r_i(s)$$

and

$$\frac{1}{r(s)} = \sum_{i=1}^{n} \frac{V_i(s)}{V(s)} \, \frac{1}{r_i(s)}$$

where $V_i(s)$, $A_i(s)$, and $r_i(s)$ are respectively the volume of $P_i(s)$, the surface area of the base of $P_i(s)$, and the altitude from $T(s)$ of $P_i(s)$.

Case of triangle

The centroid T of any triangle provides an equal-area triangulation.

So we have

$$\frac{1}{r} = \sum_{i=1}^{3} \frac{V_i}{V} \frac{1}{r_i} = \frac{1}{3} \sum_{i=1}^{3} \frac{1}{r_i}$$

that is

$$r = H(r_1, r_2, r_3).$$

Setting $h_i := 3r_i$ (triangle altitudes), we get

$$3r = H(h_1, h_2, h_3)$$

For any triangle, the harmonic mean of its altitudes is three times the inradius of the triangle

Open Questions

- Generalize the previous proposition to any star-like region (cones, cylinders...)
- Generalize the previous proposition to any region (torus...)

Some results on similar regions

1. *Any convex region R in $\mathbb{R}^2$ having an inscribed circle S of radius r has the property*

$$\frac{d}{dr}A = P$$

2. *Let $R \subset \mathbb{R}^2$ be a region as in (1) above and which is symmetric w.r.t. an axis through the center of S. For the solid formed by revolving R about that axis of symmetry, we have*

$$\frac{d}{dr}V = A$$

   The same for the solid formed by lifting R to a height of $2r$.

M. Dorff and L. Hall, Solids in $\mathbb{R}^n$ whose area is the derivative of the volume, submitted.

Singular Case

(non similar regions)

Let $R \subset \mathbb{R}^2$ be a disc or a regular polygonal region with in-radius r. For any solid formed by revolving R about an axis that does not intersect R, we have

$$\frac{d}{dr} V = A$$

Example : Torus obtained by rotating a circle centered at the fixed point $(a, 0)$ and of radius $r < a$:

$$V = (2\pi a)(\pi r^2)$$
$$A = (2\pi a)(2\pi r)$$

$$\frac{d}{dr} V = A$$

Another open problem : the case of n-parameter families

Example: Consider a family of rectangles $R(s_1, s_2)$ with length $s_1 > 0$ and width $s_2 > 0$. Consider also the linear change of variables

$$r_1(s) = \frac{s_1}{2} \quad \text{and} \quad r_2(s) = \frac{s_2}{2}$$

which inverts into

$$s_1(r) = 2r_1 \quad \text{and} \quad s_2(r) = 2r_2.$$

Then we clearly have

$$A(s) = 4r_1(s)r_2(s)$$

and

$$P(s) = 4r_1(s) + 4r_2(s).$$

Finally,

$$\frac{\partial}{\partial r_1} A[s(r)] + \frac{\partial}{\partial r_2} A[s(r)] = P[s(r)].$$

In the general case, we consider the following derivative relationship:

$$\sum_{j=1}^{n} \frac{\partial}{\partial r_j} V[s(r)] = A[s(r)],$$

where $r(s)$ is an appropriate change of variables.

to be continued...