On the infinitesimal rigidity of weakly convex polyhedra

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Abstract

The main motivation here is a question: whether any polyhedron which can be subdivided into convex pieces without adding a vertex, and which has the same vertices as a convex polyhedron, is infinitesimally rigid. We prove that it is indeed the case for two classes of polyhedra: those obtained from a convex polyhedron by "denting" at most two edges at a common vertex, and suspensions with a natural subdivision.

1 A question on the rigidity of polyhedra

A question. The rigidity of Euclidean polyhedra has a long and interesting history. Legendre [LegII] and Cauchy [Cau13] proved that convex polyhedra are rigid: if there is a continuous map between the surfaces of two convex polyhedra that is a congruence when restricted to each face, then the map is a congruence between the polyhedra (see [Sab04]). However the rigidity of non-convex polyhedra remained an open question until the first example of flexible (non-convex) polyhedra were discovered [Con77].

We say that a polyhedral surface is *weakly strictly convex* if for every vertex $\mathbf{p}_i$ there is a (support) plane that intersects the surface at exactly $\mathbf{p}_i$. If it is also true that every edge e of the triangulated surface has a (support) plane that intersects the surface at exactly e, we say the surface is *strongly strictly convex*. If there is an edge such that the internal dihedral angle is greater than $180°$, we say that edge is a *non-convex edge* of the surface.

In addition to being rigid, strongly strictly convex polyhedra with all faces triangles are *infinitesimally rigid*: there is no non-trivial first-order deformation that is an infinitesimal congruence on each triangular face. This point, which was first proved by Dehn [Deh16], is important in Alexandrov's subsequent theory concerning the induced metrics on convex polyhedra (and from there on convex bodies, see [Ale58]). Alexandrov also showed that Dehn's Theorem can be extended to the case when the polyhedral surface is weakly strictly convex, as well as being convex. In other words, vertices of the subdivision can only be vertices of the convex set, and they cannot appear in the interior of faces, for example. If vertices of a convex polyhedral surface do lie in the interior of a face, then the surface is rigid, but not infinitesimally rigid. This shows that the underlying framework is what determines infinitesimal rigidity, rather than simply the surface as a space.

Our main motivation here is a question concerning the infinitesimal rigidity of a class of frameworks determined by polyhedra which are weakly strictly convex.

Question 1.1. *Let $P \subset \mathbb{E}^3$ be a polyhedral surface, with vertices $\mathbf{p}_1, \cdots, \mathbf{p}_n$, such that:*

i.) P is weakly convex;

ii.) P is decomposable*, i.e., it can be written as the union of non-overlapping convex polyhedra, such that any two intersect in a common face, without adding any new vertices.*

Is P then necessarily infinitesimally rigid ?

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This question comes from [Sch05], where it is proved that the answer is positive if the condition, that there exists an ellipsoid which contains no vertex of P but intersects all its edges, is added. The goal pursued here is to prove that the answer is also positive for two classes of polyhedra which are by construction decomposable.

Denting polyhedra. There is an easy way to construct many examples of polyhedra for which Condition i.) above holds: start from a convex polyhedron in $\mathbb{E}^3$ and "dent" it at some of its edges, in the following manner.

Definition 1.2. *Let $P \subset \mathbb{E}^3$ be a polyhedron. P is obtained by* denting *a convex polyhedron Q at an edge e if P has the same vertices as Q, and the same faces, except that the two faces of Q adjacent to e (which are required to be triangles) are replaced by the two other triangles, sharing an edge, so that the union of the two new triangles has the same boundary as the union of the two triangles which were removed.*

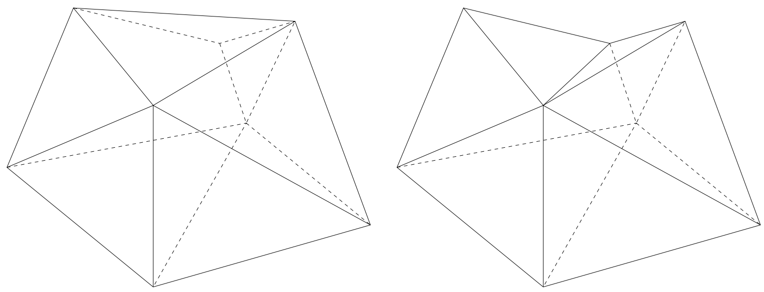

Figure 1: Denting a polyhedron.

Simple dented polyhedra are decomposable. Clearly, polyhedra obtained by denting a strongly strictly convex polyhedron at any set of edges have a convex set of vertices (Condition i.) in the question above). Moreover, those obtained by denting at one edge are decomposable, and this remains true when denting has occured at two edges which are adjacent to a vertex.

Remark 1.3. *Let $P \subset \mathbb{E}^3$ be a convex polyhedron, let $\mathbf{p}_i$ be a vertex of P, and let e, e' be two edges of P containing $\mathbf{p}_i$ as one of their endpoints, but which are not two edges of a face of P. Let Q be the polyhedron obtained by denting P at e and at e'. Then Q is decomposable.*

Proof. Clearly Q is star-like with respect to $\mathbf{p}_i$, so it can decomposed as a union of pyramids, each one corresponding to one of the faces of P which are not adjacent to $\mathbf{p}_i$. □

However there is no reason to believe that denting a convex polyhedron at more than two edges, or at two edges not containing a vertex, yields a decomposable polyhedron.

Infinitesimal rigidity of simple dented polyhedra. The first result of this paper is that the answer to Question 1.1 is positive for polyhedra constructed in this simple manner.

Theorem 1.4. *Let $Q \subset \mathbb{E}^3$ be a polyhedron obtained by denting a strongly convex triangulated polyhedral surface at one edge, or at two edges sharing a vertex (but which are not both contained in a face). Then Q is infinitesimally rigid.*

The proof, given in Section 2, uses all the excess strength of the Legendre-Cauchy argument.

Suspensions. A *suspension* P is a polyhedral surface obtained from a closed polygonal curve $(\mathbf{p}_1, \dots, \mathbf{p}_n)$ connected to two vertices $\mathbf{N}$, the north pole, and $\mathbf{S}$, the south pole. The edges of P are $[\mathbf{p}_i, \mathbf{p}_{i+1}]$, where i is taken mod n, and $[\mathbf{N}, \mathbf{p}_i], [\mathbf{S}, \mathbf{p}_i]$ for $i = 1, \dots, n$.

Our second task here is to verify Question 1.1 in the case of a suspension, where the natural choice of a decomposition is taken, namely by using the decomposition $[\mathbf{N}, \mathbf{S}, \mathbf{p}_i, \mathbf{p}_{i+1}]$, for $i = 1, \dots, n$.

Theorem 1.5. *Let P be a weakly strictly convex suspension such that the segment $[\mathbf{N}, \mathbf{S}]$ is contained in P. Then P is infinitesimally rigid.*

The proof, which is given in Section 3, uses the notion of tensegrity.

Corollary 1.6. *Let P be a polyhedron with a convex set of vertices. Suppose that P can be cut into simplices (with disjoint interior) so that only one new edge, interior to P, appears. Then P is infinitesimally rigid.*

Proof. Let e be the edge interior to P which appears in the simplicial decomposition. Let P' be the union of the simplices, appearing in the simplicial decomposition of P, which contain e. Any isometric first-order deformation of P clearly restricts to an isometric first-order deformation of P'. But P' is by construction a suspension, to which Theorem 1.5 applies. So the isometric first-order deformation of P' is trivial, and so is the isometric first-order deformation fo P. □

Another type of argument. In Section 4 we give another proof of a special case of Theorem 1.5, based on a simple idea: given a suspension we compute the first order variation, under a variation of the $\mathbf{N}-\mathbf{S}$ distance, of the sum of the angles of the simplices at the $\mathbf{N}-\mathbf{S}$ line. This leads us in Section 5 to define a symmetric matrix attached to a polyhedron along with a simplicial decomposition, which is non-singular if and only if the polyhedron is infinitesimally rigid. We then formulate another question, for which a positive answer would imply a positive answer to Question 1.1.

2 Rigidity of dented polyhedra

Outline. The proof of Theorem 1.4 follows quite precisely the arguments of the original Cauchy proof, but with slightly sharper estimates. We consider a first-order deformation of Q, and associated to each edge e of Q a "sign", which is 0 if the dihedral angle at e does not vary (at first order), + if it increases, − if it decreases. The first lemma is of a geometrical nature:

Lemma 2.1. *The following hold:*

1. *For each vertex $\mathbf{q}_i$ of Q where Q is not convex, either all the signs attached to the edges of Q containing $\mathbf{q}_i$ are 0, or some have a sign + and others have a sign −. In that case there are at least 3 edges containing $\mathbf{q}_i$ with a sign which is not 0.*
2. *At each vertex $\mathbf{q}_i$ of Q where Q is convex, either the signs assigned to all edges are 0, or there are at least 4 changes of signs when one considers the edges containing $\mathbf{q}_i$ in cyclic order.*

Here a "sign change" means a sequence of edges such that the sign on the first edge is + (resp. −), the sign on the last edge is − (resp. +), and the signs on the other edges are all 0.

The second lemma is of a topological nature.

Lemma 2.2. *Let Γ be a graph embedded in the sphere, which is the 1-skeleton of a cellular decomposition of $\mathbf{S}^2$. It is not possible to assign a sign + or − to each edge of Γ such that:*

- *the signs assigned to all the edges containing a given vertex are never all the same,*
- *there are at least 4 changes of sign at all vertices except at most three.*

Proof of the geometric lemma. The geometric lemma 2.1 follows from a remark concerning infinitesimal deformations of spherical polygon (see e.g. [Glu75, Sch04]).

Lemma 2.3. *Let $\mathbf{p} \subset \mathbf{S}^2$ be a polygon, with vertices $\mathbf{p}_1, \cdots, \mathbf{p}_n$. Let $\theta_1, \cdots, \theta_n$ be the angle of $\mathbf{p}$ at the vertices. The possible first-order variations of those angles under the infinitesimal deformations of $\mathbf{p}$ are the n-uples $(\theta'_1, \cdots, \theta'_n)$ characterized by the relation:*

$$\sum_{i=1}^{n} \theta'_i \mathbf{p}_i = 0 \ .$$

Lemma 2.1 now follows by considering the link of each vertex of Q. If $\mathbf{q}_i$ is such a vertex, then, since the set of vertices of Q is convex, the link $L(\mathbf{q}_i)$ of Q at $\mathbf{q}_i$ is contained in an open hemisphere, so it is clear from Lemma 2.3 that the signs associated to the edges of Q containing $\mathbf{q}_i$ — which correspond to the vertices of $L(\mathbf{q}_i)$, with a sign + if the angle of $L(\mathbf{q}_i)$ increases and − if it decreases — cannot be all non-positive or all non-negative. Moreover, if Q is convex at $\mathbf{q}_i$, then there are at least 4 changes of signs; otherwise there would be exactly 2 changes of signs, and, if $\mathbf{u} \in \mathbb{R}^3$ were a vector such that the plane orthogonal to $\mathbf{u}$ containing $\mathbf{q}_i$

separated the edges with a + from the edges with a $-$, then the scalar product of $\mathbf{u}$ with the left-hand side of the equation in Lemma 2.3 would be non-zero.

It is interesting to observe that Lemma 2.2 also easily follows from an argument using stresses as in Section 3.

Proof of the combinatorial lemma. The proof of Lemma 2.2 follows quite directly the argument given by Cauchy. Let v, e, f be the number of vertices, edges and faces of Γ, respectively, and let s be the number of changes of signs, i.e., pairs $\{e, e'\}$ of edges which are adjacent edges of a face, with a sign + on one and $-$ on the other.

By the hypothesis of the Lemma, there are at least 4 changes of signs at each vertex except perhaps at 3 vertices where there are at least 2 changes of signs, so that:

$$4v - 6 \le s \ .$$

However there is an upper bound on the number of changes of signs on the faces of Γ: there can be at most 2 changes of signs on a triangular face, at most 4 changes of signs on a face with 4 or 5 edges, at most 6 changes of signs on a face with 6 or 7 edges, etc. Calling f_k the number of faces with k edges, this means that:

$$s \le 2f_3 + 4f_4 + 4f_5 + 6f_6 + 6f_7 + 8f_8 + 8f_9 + \cdots \ .$$

However each edge of Γ bounds two faces, which shows that:

$$2e = 3f_3 + 4f_4 + 5f_5 + 6f_6 + 7f_7 + \cdots \ .$$

Taking twice this equation and substracting:

$$4f = 4f_3 + 4f_3 + 4f_4 + 4f_5 + 4f_6 + 4f_7 + \cdots \ ,$$

we obtain that:

$$4e - 4f = 2f_3 + 4f_4 + 6f_5 + 8f_6 + 10f_7 + \cdots \ ,$$

so that:

$$4e - 4f \ge s \ .$$

Putting together the two inequalities on s yields that $4e - 4f \ge 4v - 6$, which contradicts the Euler relation, $v - e + f = 2$.

Proof of the rigidity theorem. The proof of Theorem 1.4 now follows as in the original Cauchy proof. Suppose that Q has a non-trivial infinitesimal deformation, and assign a sign +, $-$ or 0 to each edge depending on whether the angle at that edge increases, decreases or stays constant in the deformation. Then consider the graph Γ obtained from the 1-skeleton of Q by removing all edges with a 0 and all vertices of Q which are contained only in edges with sign 0. It follows from Lemma 2.3 that Γ is still the 1-skeleton of a cell decomposition of the sphere, because any vertex of Γ is contained in at least 3 edges of Γ. Lemma 2.1 shows that there are at least 4 changes of sign at each vertex of Γ except perhaps at 3 vertices where there are at least 2 changes of sign, and Lemma 2.2 shows that this is impossible.

3 Suspensions

We introduce the definition of infinitesimal rigidity in the context of a tensegrity. Consider a finite collection points $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ in $\mathbb{E}^d$ and a graph G with those points as vertices, and with edges, called bars, cables or struts, between some pairs of those points. Consider vectors $\mathbf{p}' = (\mathbf{p}'_1, \ldots, \mathbf{p}'_n)$ in $\mathbb{E}^d$, $\mathbf{p}'_i$ regarded as an infinitesimal motion, a velocity, associated to $\mathbf{p}_i$ for each $i = 1, \ldots, n$. We say that $\mathbf{p}'$ is an *infinitesimal flex* of the tensegrity if the following equation of vector inner products holds:

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) \begin{cases} \le 0 & \text{if } \{i,j\} \text{ is a cable,} \\ = 0 & \text{if } \{i,j\} \text{ is a bar,} \\ \ge 0 & \text{if } \{i,j\} \text{ is a strut.} \end{cases} \tag{1}$$

A bar framework is *infinitesimally rigid* if the only infinitesimal flexes are the trivial ones that come as the derivative of a family of rigid congruences of all of Euclidean space restricted to the configuration $\mathbf{p}$.

One useful tool in showing infinitesimal rigidity is the concept of a *stress*, which are scalars ω_{ij} associated to each edge $\{i,j\}$. We write this as a single vector $\omega = (\dots, \omega_{ij}, \dots)$. We say ω is an *equilibrium stress* if the following vector equation holds for each i:

$$\sum_j \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0 \quad \text{for every } \{i,j\} \text{ an edge of } G. \tag{2}$$

We say the the stress $\omega = (\dots, \omega_{ij}, \dots)$ is *proper* if ω_{ij} is non-negative for cables and non-positive for struts. (There is no sign condition for bars.)

One important way to use this concept is the following:

Lemma 3.1. *If $\mathbf{p}'$ is infinitesimal flex of a tensegrity framework $G(\mathbf{p})$ and $\omega = (\dots, \omega_{ij}, \dots)$ is a proper equilibrium stress for $\mathbf{p}$, then $\sum_{ij} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0$ and thus if $\omega_{ij} \neq 0$, then $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0$.*

Proof. The equation $\sum_{ij} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0$ follows by taking the inner product of Equation (2) with $\mathbf{p}'_i$ and summing over all i. The last part of the conclusion follows from the condition that ω is proper and the inequality in Condition (1). □

One way to look at Lemma 3.1 is that a proper equilibrium stress in a tensegrity "blocks" the infinitesimal motion that would decrease a cable or increase a strut in the sense that the inequalities of Condition (1) would be strict.

Corollary 3.2. *Suppose that a tensegrity has a proper equilibrium stress and is infinitesimally rigid. Then the bar framework obtained by removing any of the edges with a non-zero stress, and converting all the other edges to bars, is infinitesimally rigid.*

Proof. Any infinitesimal flex that is non-zero on the removed edge is blocked by the stress. So the removed edge can be put back as far as the infinitesimal flex is concerned. □

One consequence of Corollary 3.2 is that one bar framework can be exchanged for another, by adding a bar and removing another when there is an equilibrium stress that is non-zero on both bars.

Suppose that we have a suspension $G(\mathbf{p})$ regarded as polyhedral surface. We say that a suspension is $\mathbf{N} - \mathbf{S}$ *decomposable* if the projection on the plane orthogonal to the line through $\mathbf{N}$ and $\mathbf{S}$ of the equator is one-to-one and the projection of the point $\mathbf{N}$ (and $\mathbf{S}$) lies inside the projection of the equator. So an $\mathbf{N} - \mathbf{S}$ decomposable suspension can be decomposed into non-overlaping tetrahedra, as in the Main Question 1.1.

For any suspension we create a tensegrity by labeling the $\{\mathbf{N}, \mathbf{S}\}$ edge and the equatorial edges as cables. The lateral edges are simply bars. Call this a *tensegrity suspension.*

Lemma 3.3. *A strongly strictly convex tensegrity suspension has a proper equilibrium stress.*

Proof. A convex suspension with n vertices (including $\mathbf{N}$ and $\mathbf{S}$) has $3(n-2) = 3n-6$ edges, and the associated tensegrity has $3n - 6 + 1 = 3n - 5$ edges. The equilibrium conditions involve $3n - 5$ variables and $3n$ linear equations, one for each coordinate of each vertex. There is always a 6-dimensional linear subspace of the $3n$-dimensional space that is orthogonal to the space of all possible stresses. (This corresponds to the trivial infinitesimal flexes.) So there must be a one-dimensional space of equilibrium stresses. The signs of the stresses are proper by the Cauchy-Dehn argument. There are only four edges incident to an equatorial vertex, and so they alternate in sign. Note also, in this case, that the sign of the stresses on the lateral edges is opposite from the sign on the stresses of the equator and $\mathbf{N} - \mathbf{S}$ edge. □

Lemma 3.4. *Any $\mathbf{N} - \mathbf{S}$ decomposable strictly weakly convex tensegrity suspension $G(\mathbf{p})$ has a proper equilibrium stress.*

Proof. The proof is by induction on n, the number of equatorial vertices. If there are only 3 vertices on the equator, the decomposability condition implies that the suspension is strongly strictly convex, so Lemma 3.3 implies that it has a proper equilibrium stresss. Whenever there is a non-convex lateral edge with $n+1$ equatorial edges, we will show how to create the proper equilibrium stress from another $\mathbf{N} - \mathbf{S}$ decomposable strictly weakly convex tensegrity suspension with n equatorial edges.

We assume that the tensegrity suspension on the vertices $\mathbf{p}_1, \dots, \mathbf{p}_n$ has a proper equilibrium stress, and we wish to show that the tensegrity suspension on the vertices $\mathbf{p}_1, \dots, \mathbf{p}_n, \mathbf{p}_{n+1}$ also has a proper equilibrium stress, where, say, the lateral edge $[\mathbf{N}, \mathbf{p}_{n+1}]$ is not strictly convex.

Consider the wedge W determined by the two planes $[\mathbf{N}, \mathbf{S}, \mathbf{p}_1]$ and $[\mathbf{N}, \mathbf{S}, \mathbf{p}_n]$ between $\mathbf{p}_1$ and $\mathbf{p}_n$ cyclicly around the $\mathbf{N} - \mathbf{S}$ axis. Since the suspension is weakly strictly convex at $\mathbf{p}_{n+1}$, this point can not be in the tetrahedron determined by $[\mathbf{N}, \mathbf{S}, \mathbf{p}_1, \mathbf{p}_n]$, yet it has to be in W. So the five points $\mathbf{N}, \mathbf{S}, \mathbf{p}_1, \mathbf{p}_{n+1}, \mathbf{p}_n$ determine a (small) tensegrity suspension over a triangle, where $[\mathbf{N}, \mathbf{p}_{n+1}]$ is the axis since it is non-convex. In this case $[\mathbf{N}, \mathbf{S}]$ is a lateral edge, and $[\mathbf{p}_1, \mathbf{p}_{n+1}]$ is an equatorial edge. Thus we can choose an equilibrium stress for the small suspension such that the stress on $[\mathbf{p}_1, \mathbf{p}_{n+1}]$ is exactly the negative of the equilibrium stress of the suspension on $\mathbf{p}_1, \ldots, \mathbf{p}_n$ on $[\mathbf{p}_1, \mathbf{p}_{n+1}]$. When these two stresses are added, they cancel. So the stresses on $[\mathbf{N}, \mathbf{S}]$ for both the small suspension and the large one are positive, and hence their sum is positive. So this sum is a proper equilibrium stress for the larger suspension as desired. Figure 2 shows this situation.

If the four vertices $\mathbf{N}, \mathbf{p}_1, \mathbf{p}_{n+1}, \mathbf{p}_n$ are coplanar, then there is an equilibrium stress as before, except that it is 0 on $[\mathbf{N}, \mathbf{S}]$ and the same argument applies.

If there are no non-convex lateral edges in the suspension, since the equatorial edges are always convex, the whole suspension is convex and Lemma 3.3 applies. □

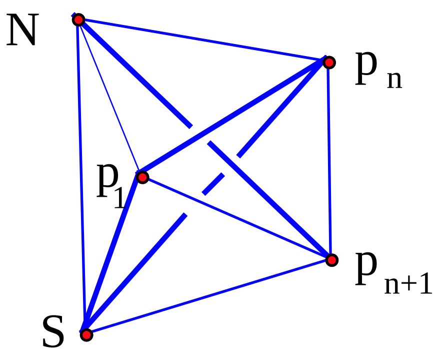

Figure 2: This shows the case when the edge $[\mathbf{N}, \mathbf{p}_{n+1}]$ is non-convex. The thick edges are struts with a negative stress and the thin edges are cables with a positive stress.

Corollary 3.5. *Any $\mathbf{N} - \mathbf{S}$ decomposable strictly weakly convex suspension is infinitesimally rigid.*

Proof. Since the bar framework obtained by adding the $[\mathbf{N}, \mathbf{S}]$ bar is infinitesimally rigid and there is an equilibrium stress non-zero on $[\mathbf{N}, \mathbf{S}]$ when that bar is added, Corollary 3.2 implies that the suspension itself is infinitesimally rigid. □

4 Rigidity of suspensions through variations of angles

Suspensions over convex polygons. Consider a suspension P with vertices $\mathbf{N}, \mathbf{S}$ and $\mathbf{p}_1, \cdots, \mathbf{p}_n$, as defined above. There is a natural subclass of suspensions, which are all weakly convex and decomposable.

Definition 4.1. *P is a* **suspension over a convex polygon** *if the projection of the closed polygonal line with vertices $\mathbf{p}_1, \cdots, \mathbf{p}_n$ (in this cyclic order), along $(\mathbf{N}, \mathbf{S})$, on any plane transverse to the line $(\mathbf{N}, \mathbf{S})$ is convex and contains in its interior the intersection of the plane with $(\mathbf{N}, \mathbf{S})$.*

Theorem 4.2. *Let P be a suspension over a convex polygon. Then P is infinitesimally rigid.*

Note that, under the hypothesis of the theorem, P is always decomposable (because it is a suspension) and it is also weakly convex. However, the hypothesis in Theorem 4.2 is stronger than in Theorem 1.5 since the projection of the equator on any plane transverse to the $\mathbf{N} - \mathbf{S}$ axis is required to be a convex polygon. So the statement of this theorem is not very interesting in itself, we include it here because its proof is different from the proof given for Theorem 4.2 an can be interesting as an indication of possible ways to tackle Question 1.1 (as seen in the next section).

The first step in the proof is to apply a projective transformation to P so that the planes orthogonal to $[\mathbf{N}, \mathbf{S}]$ at $\mathbf{N}$ and at $\mathbf{S}$ are support planes of P. This does not change the infinitesimal rigidity or flexibility of

P since it is well known (at least since works of Darboux [Dar93], Sauer [Sau35], and J. Clerk Maxwell in the 19-th Century, that infinitesimal rigidity is a projectively invariant property). From here on we suppose that this additional property is satisfied.

Deformations of simplices. The argument given below is based on a computation concerning the first-order deformations of Euclidean simplices. We consider a simplex with two vertices called $\mathbf{N}$ and $\mathbf{S}$ of coordinates $(0,0,1)$ and $(0,0,0)$ in $\mathbb{R}^3$, and with two other vertices $\mathbf{p}_1$ and $\mathbf{p}_2$ of coordinates $(r_1\cos(\alpha_1), r_1\sin(\alpha_1), z_1)$ and $(r_2\cos(\alpha_2), r_2\sin(\alpha_2), z_2)$.

Lemma 4.3. *There is a unique first-order deformation of this simplex under which the distance between* $\mathbf{N}$ *and* $\mathbf{S}$ *varies at speed* 1*, and the lengths of all the other edges remain constant. Under this deformation, the first-order variation of the angle* $\theta = \alpha_2 - \alpha_1$ *at the edge* $\mathbf{N}-\mathbf{S}$ *is:*

$$\theta' = \frac{1}{r_1 r_2 \sin\theta}\left((z_1-z_2)^2 + z_1(1-z_1)\left(1-\frac{r_2}{r_1}\cos\theta\right) + z_2(1-z_2)\left(1-\frac{r_1}{r_2}\cos\theta\right)\right) .$$

Proof. The square of the distance between $\mathbf{S}$ and $\mathbf{p}_1$ is equal to $z_1^2 + r_1^2$. Since this remains constant under the deformation we have:

$$z_1 z_1' + r_1 r_1' = 0 .$$

Similarly, calling z the third coordinate of $\mathbf{N}$ (so that $z = 1$ "before" the deformation takes place), the square of the distance between $\mathbf{N}$ and $\mathbf{p}_1$ is equal to $(z-z_1)^2 + r_1^2$, and it is constant under the deformation, so that:

$$r_1 r_1' + (1-z_1)(1-z_1') = 0 .$$

It follows that

$$z_1' = 1 - z_1 , \quad r_1' = -\frac{z_1}{r_1}(1-z_1) ,$$

and similarly:

$$z_2' = 1 - z_2 , \quad r_2' = -\frac{z_2}{r_2}(1-z_2) .$$

Furthermore the square of the distance between $\mathbf{p}_1$ and $\mathbf{p}_2$ is equal to $(z_2-z_1)^2 + r_1^2 + r_2^2 - 2r_1r_2\cos\theta$. Since it remains constant in the deformation we have that:

$$(z_2-z_1)(z_2'-z_1') + r_1r_1' + r_2r_2' - r_1'r_2\cos\theta - r_1r_2'\cos\theta + r_1r_2\theta'\sin\theta = 0 ,$$

so that:

$$-(z_2-z_2)^2 + \left(1-\frac{r_2}{r_1}\cos\theta\right) r_1r_1' + \left(1-\frac{r_1}{r_2}\cos\theta\right) r_2r_2' + r_1r_2\theta'\sin\theta = 0 ,$$

and thus:

$$r_1r_2\theta'\sin\theta = (z_2-z_2)^2 + z_1(1-z_1)\left(1-\frac{r_2}{r_1}\cos\theta\right) + z_2(1-z_2)\left(1-\frac{r_1}{r_2}\cos\theta\right) ,$$

from which the result follows. □

An invariant controling the infinitesimal rigidity of suspensions. Let $\mathbf{p}_1, \cdots, \mathbf{p}_n$ be the vertices of P different from $\mathbf{N}$ and $\mathbf{S}$, in the cyclic order on which they appear on the "equator" of P. Suppose that the coordinates of $\mathbf{p}_i$ are $(r_i\cos\alpha_i, r_i\sin\alpha_i, z_i)$ for $1 \le i \le n$. Below we use cyclic notation, so that $\mathbf{p}_{n+1} = \mathbf{p}_1$.

We define a quantity $\Lambda(P)$ which is the sum of the terms appearing in Lemma 4.3 above for the simplices $[\mathbf{N}, \mathbf{S}, \mathbf{p}_i, \mathbf{p}_{i+1}]$. We will see that $\Lambda(P) = 0$ if and only if P is infinitesimally flexible. Below we will also give simpler geometric expressions of Λ, leading in particular to the proof of Theorem 4.2. It is defined as:

$$\Lambda(P) := \sum_{i=1}^{n} \frac{1}{r_i r_{i+1}\sin\theta_i}\left((z_{i+1}-z_i)^2 + z_i(1-z_i)\left(1-\frac{r_{i+1}}{r_i}\cos\theta_i\right) + z_{i+1}(1-z_{i+1})\left(1-\frac{r_i}{r_{i+1}}\cos\theta_i\right)\right) .$$

Consider any first-order deformation of P. If the distance between $\mathbf{N}$ and $\mathbf{S}$ does not vary, then the deformation is trivial, because each simplex S_i of vertices $\mathbf{N}, \mathbf{S}, \mathbf{p}_i$ and $\mathbf{p}_{i+1}$ would then remain the same. It follows that the angle θ_i of S_i at the edge $\mathbf{N}-\mathbf{S}$ varies accordingly to Lemma 4.3. Since the sum of the angles of the simplices S_i at the edge $\mathbf{N}-\mathbf{S}$ has to be equal to 2π, a first-order deformation of P is trivial unless the sum of the terms corresponding to Lemma 4.3 for the simplices $S_i, 1 \le i \le n$, sum to 0. It is not difficult to check that the converse is true, too: if the sum of the first-order variations of the angle at $\mathbf{N}-\mathbf{S}$ of the simplices S_i vanishes, then there is non-trivial first-order deformation of P. This shows the following statement:

Lemma 4.4. *P is infinitesimally rigid if and only if $\Lambda(P) \neq 0$.*

Another expression of $\Lambda(P)$. Now consider the polygon $\mathbf{p}$ which is the orthogonal projection of the "equator" of P on the plane $z=0$. Let $\mathbf{u}_1,\cdots,\mathbf{u}_n$ be its vertices, with $\mathbf{u}_i$ equal to the orthogonal projection of $\mathbf{p}_i$ on $z=0$. We call a_i twice the area of the triangle $(\mathbf{0},\mathbf{u}_i,\mathbf{u}_{i+1})$, and b_i twice the *oriented* area of the triangle $(\mathbf{u}_{i-1},\mathbf{u}_i,\mathbf{u}_{i+1})$.

Lemma 4.5. *We have:*

$$\Lambda(P)=\sum_{i=1}^n \frac{(z_{i+1}-z_i)^2}{a_i}+\frac{z_i(1-z_i)b_i}{a_{i-1}a_i}~.$$

Proof. A simple computation shows that:

$$\begin{aligned}\Lambda(P) &= \sum_{i=1}^n \frac{(z_{i+1}-z_i)^2}{r_ir_{i+1}\sin\theta_i}+\frac{z_i(1-z_i)}{r_i^2}\left(\frac{r_i-r_{i+1}\cos\theta_i}{r_{i+1}\sin\theta_i}+\frac{r_i-r_{i-1}\cos\theta_{i-1}}{r_{i-1}\sin\theta_{i-1}}\right)\\ &= \sum_{i=1}^n \frac{(z_{i+1}-z_i)^2}{a_i}+\frac{z_i(1-z_i)}{r_i^2}\left(\frac{r_i-r_{i+1}\cos\theta_i}{r_{i+1}\sin\theta_i}+\frac{r_i-r_{i-1}\cos\theta_{i-1}}{r_{i-1}\sin\theta_{i-1}}\right)~.\end{aligned}$$

Let β_i be the angle at $\mathbf{u}_i$ between $(\mathbf{u}_i,\mathbf{u}_{i-1})$ and $(\mathbf{u}_i,\mathbf{0})$ and let γ_i be the angle at $\mathbf{u}_i$ between the oriented lines $(\mathbf{u}_i,0)$ and $(\mathbf{u}_i,\mathbf{u}_{i+1})$. It is easy to check on a diagram that $r_{i+1}\sin\theta_i/(r_i-r_{i+1}\cos\theta_i)$ is the tangent of γ_i, while $r_{i-1}\sin\theta_{i-1}/(r_i-r_{i-1}\cos\theta_{i-1})$ is the tangent of β_i. It follows that:

$$\begin{aligned}\Lambda(P) &= \sum_{i=1}^n \frac{(z_{i+1}-z_i)^2}{a_i}+\frac{z_i(1-z_i)}{r_i^2}(\cot(\gamma_i)+\cot(\beta_i))\\ &= \sum_{i=1}^n \frac{(z_{i+1}-z_i)^2}{a_i}+\frac{z_i(1-z_i)\sin(\alpha_i)}{r_i^2\sin(\beta_i)\sin(\gamma_i)}~,\end{aligned}$$

where α_i is the interior angle of $\mathbf{p}$ at $\mathbf{u}_i$, i.e., $\alpha_i=\beta_i+\gamma_i$. Note that $a_i=r_i\|\mathbf{u}_{i+1}-\mathbf{u}_i\|\sin\theta_i$ and that $a_{i-1}=r_{i-1}\|u_i-u_{i-1}\|\sin\theta_{i-1}$, it follows that

$$\Lambda(P)=\sum_{i=1}^n \frac{(z_{i+1}-z_i)^2}{a_i}+\frac{z_i(1-z_i)\sin(\alpha_i)\|\mathbf{u}_{i+1}-\mathbf{u}_i\|.\|\mathbf{u}_i-\mathbf{u}_{i-1}\|}{a_{i-1}a_i}~,$$

and the expression given in the Lemma follows because $b_i=\sin(\alpha_i)\|\mathbf{u}_{i+1}-\mathbf{u}_i\|.\|\mathbf{u}_i-\mathbf{u}_{i-1}\|$. □

The proof of Theorem 4.2 is a direct consequence of Lemma 4.5 since, under the hypothesis of the theorem, all terms involved in the sum are non-negative (and some of them are strictly positive).

5 A more technical version of Question 1.1

The arguments in the previous section suggest a way to transform Question 1.1 in a way which makes it much less elementary but perhaps more precise in a fairly technical manner. For a general decomposable polyhedron P, along with a decomposition as a union of simplices with disjoint interior (and with no vertex beyond those of P) it is still possible to define an invariant generalizing the number Λ defined above for suspensions. However in this more general case $\Lambda(P)$ is a $r\times r$ matrix, where r is the number of "interior" edges in the simplicial decomposition of P.

After defining $\Lambda(P)$, we will show here that it is a symmetric matrix, and that P is infinitesimally rigid if and only if $\Lambda(P)$ is non-singular. In addition, the content of the previous section shows that, under some mild assumptions, its diagonal has positive entries. This suggests that $\Lambda(P)$ might be positive definite whenever P is weakly convex; this would at least imply a positive answer to Question 1.1.

An extension of the invariant Λ. In this section we consider a polyhedron P, along with a simplicial decomposition of P with no vertex except the vertices of P. We call $S_1,\cdots,S_q$ the simplices in the decomposition of P, and $e_1,\cdots,e_r$ the interior edges of the decomposition, i.e., the segments which are edges of the simplicial decomposition but are contained in the interior of P.

Clearly any isometric first-order deformation of P is uniquely determined by the first-order variation of the lengths of the e_i, because the other edges of the simplicial decomposition have fixed length.

Definition 5.1. *Let $E \subset \mathbb{R}^r$ be the set of lengths $l_1, \cdots, l_r$ so that, for each of the simplices S_i, if the lengths of the edges of S_i which are in the interior of P are fixed at the values determined by the l_j and the length of the other edges are equal to their length in P, the resulting 6 numbers are indeed the lengths of the edges of an Euclidean simplex.*

There is a "special" element $l^0 = (l_1^0, \cdots, l_r^0)$ in E, it is the r-uple of lengths of the edges e_i in the polyhedron P.

To each choice of $(l_1, \cdots, l_r) \in E$ we can associate an Euclidean structure on the simplicial decomposition of P used here, but with cone singularities at the edges e_i. So to each interior edge e_i is attached a number, the sum of the angles at e_i of the simplices containing it (or in other terms the angle around the cone singularity corresponding to e_i), which we call θ_i. This defines a map:

$$\begin{array}{rrcl} \phi: & E & \mapsto & (0,\infty)^r \\ & (l_1, \cdots, l_r) & \to & (\theta_1, \cdots, \theta_r) \end{array}$$

It clearly follows from the construction that $\phi(l^0) = (2\pi, \cdots, 2\pi)$.

Definition 5.2. *Let $\Lambda(P)$ be the Jacobian matrix of ϕ at l^0, i.e.:*

$$\Lambda(P) := \left(\frac{\partial \theta_i}{\partial l_j}\right)_{1\le i,j\le r} .$$

In the special case of suspensions considered in Section 3 and 4, $\Lambda(P)$ is a 1×1 matrix, and its unique entry is the quantity called $\Lambda(P)$ there.

$\Lambda(P)$ and the infinitesimal rigidity of P. As for suspensions, $\Lambda(P)$ can be used to determine when P is infinitesimally rigid.

Lemma 5.3. *P is infinitesimally flexible if and only if $\Lambda(P)$ is singular (i.e., its kernel has dimension at least 1).*

Proof. Suppose first that P is not infinitesimally rigid, and consider a first-order edge-length preserving deformation. Let $(l_1', \cdots, l_r')$ be the corresponding first-order variations of the lengths of the interior edges of the simplicial decomposition. Under the same deformation, the sum of the angles at each interior edge e_i remains equal to 2π (at first order) so that $\Lambda(P)(l_1', \cdots, l_r') = 0$, and this shows that $\Lambda(P)$ is singular.

Suppose conversely that $\Lambda(P)$ is singular, and let $(l_1', \cdots, l_r')$ be a non-zero element in the kernel of $\Lambda(P)$. It defines a first-order variation of the Euclidean metric on each of the simplices appearing in the decomposition of P, and therefore of the Euclidean structure, with singularities at the edges, naturally defined on this simplicial complex. However the first-order variation under $(l_1', \cdots, l_r')$ of the angle at each of the interior edges vanishes precisely because $(l_1', \cdots, l_r')$ is in the kernel of $\Lambda(P)$. This means that the Euclidean structure remains associated to an Euclidean polyhedron, and therefore that P is not infinitesimally rigid. □

$\Lambda(P)$ is symmetric. This is another simple property of $\Lambda(P)$. To prove it we need some additional notation. We call $\overline{e}_1, \cdots, \overline{e}_{\overline{r}}$ the edges of the simplicial decomposition which are contained in the boundary of P – i.e., those which are not among the e_i – and $\overline{l}_1, \cdots, \overline{l}_{\overline{r}}$ their lengths. For each simplex S_i and each edge e_j (resp. $\overline{e}_j$) which is an edge of S_i we call $\alpha_{i,j}$ (resp. $\overline{\alpha}_{i,j}$) the angle of S_i at e_j (resp. at $\overline{e}_j$).

We then introduce the sum of the "mean curvatures" of the simplices:

$$H := \sum_{i,j} l_j \alpha_{i,j} + \sum_{i,j} \overline{l}_j \overline{\alpha}_{i,j} ,$$

where the first sum is over all simplices S_i and edges e_j such that e_j is an edge of S_i, while the second sum is the corresponding quantity with $\overline{e}_j$ instead of e_j. Then, under a first-order deformation:

$$dH = \sum_{i,j} (l_j d\alpha_{i,j} + \alpha_{i,j} dl_j) + \sum_{i,j} (\overline{l}_j d\overline{\alpha}_{i,j} + \overline{\alpha}_{i,j} d\overline{l}_j) .$$

Now we consider H as a function over E, which means that we fix the values of the $\overline{l}_j$, the lengths of the edges of the simplicial decomposition of P which are on the boundary of P. The formula for the first-order variation of H simplifies and becomes:

$$dH = \sum_{i,j} \alpha_{i,j} dl_j + \sum_{i,j} l_j d\alpha_{i,j} + \sum_{i,j} \overline{l}_j d\overline{\alpha}_{i,j} .$$

But the celebrated Schläfli formula states that

$$\sum_{i,j} \bar{l}_j d\bar{\alpha}_{i,j} + \sum_{i,j} l_j d\alpha_{i,j} = 0 ,$$

so that

$$dH = \sum_{i,j} \alpha_{i,j} dl_j = \sum_j \theta_j dl_j .$$

This means that $\Lambda(P)$, as defined above, is the Hessian matrix of H, considered as a function of the l_j (which are coordinates on E). So it is a symmetric matrix.

Another question. It follows from Section 4 that, under some fairly simple geometric hypothesis on P, the diagonal of $\Lambda(P)$ is positive. This leads to the

Question 5.4. *Is $\Lambda(P)$ positive definite whenever P is weakly convex ?*

A positive answer would imply that weakly convex and decomposable polyhedra are infinitesimally rigid, i.e., a positive answer to Question 1.1.

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