Strategic Exploitation of a Common-Property Resource under Uncertainty★

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Running Title: Uncertainty and Strategic Resource Exploitation

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Abstract

We construct a game of noncooperative common-resource exploitation which delivers analytical solutions for its symmetric Markov-perfect Nash equilibrium. We examine how introducing uncertainty to the natural law of resource reproduction affects strategic exploitation. We show that the commons problem is always present in our example and we identify cases in which increases in risk amplify or mitigate the commons problem. For a specific class of games which imply Markov-perfect strategies that are linear in the resource stock (our example belongs to this class), we provide general results on how payoff-function features affect the responsiveness of exploitation strategies to changes in riskiness. These broader characterizations of games which imply linear strategies (appearing in an Online Appendix) can be useful in future work, given the technical difficulties that may arise from the possible nonlinearity of Markov-perfect strategies in more general settings.

Keywords: Renewable resource exploitation, stochastic non-cooperative dynamic games, the commons

JEL Classification Codes: C73, C72, Q20, O13, D43

1. Introduction

In games of common-property renewable resource exploitation each player partly controls the future evolution of the resource, given the strategies of other players. Models in which there is a dynamic element and in which resources are shared, play an important role in economics, e.g., industrial organization models or models with natural resources. The fundamental, infinite-horizon setup, in which all players have full information about the economic environment, has been studied in the economics literature almost exclusively within the deterministic framework. The main finding of this literature is that the equilibrium is characterized by a "commons problem". Namely, the higher the number of non-cooperating players, the higher the aggregate exploitation rate, so the lower the level of the resource in the long run.[1] Our goal in this paper is to examine how noncooperative strategic interaction is affected by uncertainty in the natural law of resource reproduction.

Our focus on randomness in resource reproduction is a natural starting point for the study of uncertainty in resource games. In the real world, resources evolve according to stochastic laws of motion. Especially in the context of natural resources, as is the case with biological populations such as forests and fish species, these evolve subject to the existence of predators or climate, that are affected by random disturbances.[2]

Stochastic dynamic games can be particularly complex and difficult to characterize when the law of resource reproduction, the payoffs and the distributions of random disturbances are all given by general functions.[3] At the same time, the task of characterizing decisions in

[1] See, for example, Mirman (1979) and Levhari and Mirman (1980), Levhari, Michener and Mirman (1981), Benhabib and Radner (1992), Dockner and Sorger (1996), Sorger (1998 and 2005) and Koulovatianos and Mirman (2007).

[2] In cases of governmental provisions of infrastructure for companies, such as railroads, electricity grids, telecommunication networks, etc., financing and maintenance is also subject to random shocks, such as business cycles or political cycles.

[3] An example of a study examining the link between extraction decisions and uncertain reproduction outcomes under perfect competition, and also optimal resource preservation policies, is the fishery application

the presence of uncertainty in a general framework can be demanding even in the case of a single decision maker.[4] We discuss why technical problems arise in multiple-player dynamic games which use general functional forms. Specifically, in resource games problems arise because each player's objective function directly contains the strategies of other players. When the Markov-perfect Nash strategies of other players are strictly concave, a player's objective function may lose key properties, such as concavity, differentiability, and continuity. These technical difficulties are discussed in Mirman (1979).

A special class of dynamic games avoids such technical difficulties related to the concavity of Markov-perfect Nash strategies. It is the class of games which possess primitives such that symmetric Markov-perfect Nash strategies are linear decision rules with respect to the common resource.[5] A game that falls in this class is the parametric example of Levhari and Mirman (1980). Yet, strategies in the Levhari and Mirman (1980) example are unaffected by introducing uncertainty. Here we provide a new example that nests and extends the Levhari and Mirman (1980) example, in which introducing uncertainty and risk changes (in terms of first- or second-order stochastic dominance) affect exploitation strategies.

Our analysis is extended to the case of N players, where N can be more than two players. A study that extended the Levhari-Mirman (1980) model to $N > 2$ players is Okuguchi (1981), who has also emphasized the effects of entry (or exit) in fish war in comparison with cooperative solutions (joint resource management by all players). Understanding how

of Mirman and Spulber (1985). For a paper studying uncertainty and games see Amir (1996). For studies pointing out technical issues in *deterministic* differential resource games, such as multiplicity of equilibrium strategies, arising even in setups with some simplifying assumptions on primitives, see Dockner and Sorger (1996), and Sorger (1998), while for fundamental proofs of equilibrium existence see Sundaram (1989) and Dutta and Sundaram (1992, 1993).

[4] For example, Mirman (1971) analyzes uncertainty in a model with a single controller, providing a general result about the role of uncertainty on decisions in two-period models, and discussing issues arising in the infinite-horizon setup.

[5] Our analysis does not restrict the search for optimal strategies within the linear class. Rather, it restricts attention to those dynamic games which have a symmetric equilibrium in linear Markov-perfect strategies, among all possible Markov-perfect strategies.

noncooperative strategic behavior changes as we add players to a game is the key to understanding whether, under particular forms of regulation, cooperation is sustainable as a subgame-perfect equilibrium.[6] Here, apart from presenting our example and performing some comparative analysis of strategies as we increase risk, we do not provide extensions to resource regulations. We do, however, provide an Online Appendix which proposes theoretical tools in order to analyze the comprehensive class of games with primitives that allow for Markov-perfect Nash exploitation strategies which are linear in the resource stock.[7] These tools and theoretical results which are based on a stochastic-dominance analysis of how risk changes affect strategies, are applicable to other examples of linear-Markov-perfect-Nash-strategy games that one may discover along the way.

2. The general framework

Time is discrete and the horizon is infinite, i.e. $t = 0, 1, \ldots$. Let the state variable, x, evolve naturally (in the case of no exploitation) according to the law of motion,

$$x_{t+1} = \theta_t f(x_t) . \tag{1}$$

We assume $f' > 0$, $f'' \leq 0$. The random variable θ_t is i.i.d., independent of x_t and,

$$\theta_t \sim \Theta(\theta_t), \qquad t = 0, 1, \ldots .$$

We assume that Θ, the distribution function of θ_t, has support $S_\theta \subseteq \mathbb{R}_+$ and mean $E(\theta_t) < \infty$, for all t.

[6] A comprehensive analysis of how regulatory rules may affect the linkup between noncooperative strategies and the outside option for cooperation is provided by the book by Ostrom et al. (1994). In recent work, Polasky et al. (2006) have built a resource game that clarifies conditions under which cooperation can become a subgame-perfect-equilibrium outcome. More interestingly, Tarui et al. (2008) have extended the Polasky et al. (2006) framework in order to include imperfect monitoring of each player's harvest.

[7] The Online Appendix is available at the journal's repository of supplementary material which can be accessed via http://www.jeem-supplemental.org/.

We consider $N \geq 1$ identical players. In period t, each player $j \in \{1, \ldots, N\}$ consumes $c_{j,t} \geq 0$ units of the available stock, and then a realization of the random shock takes place. Next period's level of x is given by

$$x_{t+1} = \theta_t f\left(x_t - \sum_{i=1}^{N} c_{i,t}\right).$$

Each player $j \in \{1, \ldots, N\}$ maximizes his expected discounted utility over an infinite-period horizon,

$$E_0\left[\sum_{t=0}^{\infty} \delta^t u\left(c_{j,t}\right)\right].$$

We assume $u : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable with $u' > 0$, $u'' < 0$. All players have the same momentary utility function, u, and discount factor $\delta \in (0, 1)$.

We compare decisions made in the stochastic model with decisions made in a version of the deterministic model in which the shock θ is always equal to its mean, $E(\theta) = \bar{\theta}$.[8] We distinguish between the stochastic model carrying the subscript "s", and the deterministic model with subscript "d". We focus on Markov-perfect Nash equilibrium strategies for this stochastic environment, i.e. strategies of the form $\{c_i = C_{s,i}(x)\}_{i=1}^{N}$. The problem of player $j \in \{1, \ldots, N\}$ can be expressed using a Bellman equation, namely,

$$V_{s,j}(x) = \max_{0 \leq c_j \leq x - \sum_{\substack{i=1 \\ i \neq j}}^{N} C_{s,i}(x)} \left\{ u(c_j) + \delta E\left[V_{s,j}\left(\theta f\left(x - c_j - \sum_{\substack{i=1 \\ i \neq j}}^{N} C_{s,i}(x) \right)\right)\right]\right\}, \tag{2}$$

in which $V_{s,j}(x)$ is the value function of player j in the stochastic setup. In the deterministic model, player j's problem, conditional upon the strategies of all other players is,

$$V_{d,j}(x) = \max_{0 \leq c_j \leq x - \sum_{\substack{i=1 \\ i \neq j}}^{N} C_{d,i}(x)} \left[u(c_j) + \delta V_{d,j}\left(\bar{\theta} f\left(x - c_j - \sum_{\substack{i=1 \\ i \neq j}}^{N} C_{d,i}(x) \right)\right)\right]. \tag{3}$$

[8] See Hahn (1969), Stiglitz (1970) and Mirman (1971).

Let $\mathbb{S}_s$ (resp. $\mathbb{S}_d$) denote the Markov-perfect Nash equilibrium strategies of the stochastic (resp. deterministic) game, i.e.

$$\mathbb{S}_s = \left\{ \left\{ C^*_{s,i}(x) \right\}_{i=1}^N \;\middle|\; \begin{array}{c} \text{for all } j \in \{1,\dots,N\} \quad C^*_{s,j}(x) \text{ solves problem (2)} \\ \text{given } C_i(x) = C^*_{s,i}(x),\ i \in \{1,\dots,N\} \text{ with } i \neq j \end{array} \right\},$$

and

$$\mathbb{S}_d = \left\{ \left\{ C^*_{d,i}(x) \right\}_{i=1}^N \;\middle|\; \begin{array}{c} \text{for all } j \in \{1,\dots,N\} \quad C^*_{d,j}(x) \text{ solves problem (3)} \\ \text{given } C_i(x) = C^*_{d,i}(x),\, i \in \{1,\dots,N\} \text{ with } i \neq j \end{array} \right\}.$$

Since players have identical utility functions in our analysis, and since we focus on state-dependent (Markov-perfect-Nash) strategies, it suffices to denote a game only by the tuple $\langle u, f, \Theta \rangle$ for simplicity.

With the assumptions imposed so far, it is not possible to analyze the complete equilibrium sets, nor to give conditions for the existence of globally unique equilibrium in either the stochastic or the deterministic game. The presence of other players' strategies in problems (2) and (3) gives rise to various potential problems.

Suppose, for the moment, that the value function is twice continuously differentiable.[9] The interior first-order condition, from the dynamic program (2), is

$$u'(c_j) = \delta f'(y)\, E\left[\theta V'_{s,j}(\theta f(y))\right], \quad \text{with} \quad y = x - c_j - \sum_{\substack{i=1 \\ i \neq j}}^N C_{s,i}(x). \tag{4}$$

Using the envelope theorem on (2) yields,

[9] This is not a valid assumption for the general setup, as we discuss below.

$$V'_{s,j}(x) = \delta \left[1 - \sum_{\substack{i=1 \\ i \neq j}}^{N} C'_{s,i}(x) \right] f'(y) E\left[\theta V'_{s,j}(\theta f(y)) \right]. \tag{5}$$

Combining (5) with (4),

$$V'_{s,j}(x) = u'(C_{s,j}(x)) \left[1 - \sum_{\substack{i=1 \\ i \neq j}}^{N} C'_{s,i}(x) \right]. \tag{6}$$

A crucial technical difficulty is revealed by (6). While $u(\cdot)$ is strictly concave, the strategies $C_{s,i}(\cdot)$ of the other players need not be convex, and the value function $V_{s,j}(\cdot)$ need not be concave.[10] In fact, non-concavity is not the only problem that may arise in a player's value function due to the presence of the strategies of other players in this setup. Mirman (1979) provides examples of games using functions u and f, which meet our general assumptions, but lead to value functions that are not concave, not differentiable, not even continuous, and may admit multiple solutions.[11] Mirman (1979) demonstrates by example that using the same functions u and f in a single-controller optimization problem would lead to twice continuously differentiable and strictly concave value functions. However, these desirable value-function properties can be lost in games of the form given by (2) or (3).

The example by Levhari and Mirman (1980) does not suffer from the potential technical difficulties that may arise from the nonlinearity of Markov-perfect Nash strategies of other players: the fundamentals u and f in the Levhari-Mirman (1980) example are such that the resulting Markov-perfect Nash strategies are linear in the resource stock. In light of

[10]For example, even if we assume, for a moment, that the second derivatives of strategies $C_{s,i}(\cdot)$ exist, then differentiating both sides of (6) with respect to x leads to,

$$V''_{s,j}(x) = u''(C_{s,j}(x)) \left[1 - \sum_{\substack{i=1 \\ i \neq j}}^{N} C'_{s,i}(x) \right] C'_{s,j}(x) - u'(C_{s,j}(x)) \sum_{\substack{i=1 \\ i \neq j}}^{N} C''_{s,i}(x) \;,$$

in which $V''_{s,j}(x)$ may be strictly positive if $\Sigma_{i \neq j} C''_{s,i}(x) < 0$ for some x.

[11]Such problems do not preclude the existence of equilibrium (see, the specific example in Mirman (1979, pp. 65-72)).

this technical property, here we contribute an example which extends the Levhari-Mirman (1980) analysis (see Section 5): our example in this paper accommodates the study of how uncertainty affects strategic exploitation by players while the Markov-perfect Nash strategies are linear in the resource stock, too.

3. Games with a Markov-Nash equilibrium in linear-symmetric strategies

Since games $\langle u, f, \Theta \rangle$ in which fundamentals u and f lead to linear Markov-perfect Nash strategies avoid technical problems, Theorem 1 below shows a way to identify whether a pair of functions u and f implies that a certain game $\langle u, f, \Theta \rangle$ allows symmetric linear Markov-perfect Nash strategies of the form,[12]

$$C_i^*(x) = \omega x, \text{ with } \omega \in (0, 1/N) \text{ for all } x > 0, \text{ all } i \in \{1, \ldots, N\} \ . \tag{7}$$

The linearity of strategies given by (7) guarantees the differentiability of the value function: these strategies satisfy conditions (4) and (5), and therefore condition (6), which becomes,

$$V'(x) = [1 - (N-1)\,\omega]\, u'(\omega x) \ . \tag{8}$$

Therefore, in this equilibrium $V' > 0$, while V'' exists and is strictly negative. In other words, a Markov-perfect Nash equilibrium with linear symmetric strategies ensures that V is twice continuously differentiable, strictly increasing and strictly concave. Notice that, since we focus on symmetric strategies, the index j is dropped from the value function.

A game $\langle u, f, \Theta \rangle$ can have at most one linear-symmetric strategy in the equilibrium set $\mathbb{S}_s$, and similarly a game $\left\langle u, f, \bar{\theta} \right\rangle$ can have at most one linear-symmetric strategy in the equilibrium set $\mathbb{S}_d$. To see this, consider the stochastic case first. Fix $x > 0$, and consider

[12] Levhari and Mirman (1980) also examine cases of non-symmetric strategies by allowing the discount factors of players to differ. Because here we are interested in the N-player case in which N can be bigger than 2, we restrict attention to symmetric strategies.

(4) *assuming* (interior) linear symmetric strategies $C_{s,i}(x) = \omega x$. This necessary condition implies,[13]

$$\psi_s(\omega) = -u'(\omega x) + \delta f'((1 - N\omega)x) E[\theta V_s'(\theta f((1 - N\omega)x))] = 0 \quad \text{for all } x. \tag{9}$$

Similarly in the case of $\langle u, f, \bar{\theta} \rangle$, the necessary condition from (3) implies,

$$\psi_d(\omega) = -u'(\omega x) + \delta \bar{\theta} f'((1 - N\omega)x) V_d'\left(\bar{\theta} f((1 - N\omega)x)\right) = 0 \quad \text{for all } x. \tag{10}$$

Notice that,

$$\psi_s'(\omega) > 0 \quad \text{and} \quad \psi_d'(\omega) > 0 \quad \text{for all } \omega \in \left(0, \frac{1}{N}\right), \tag{11}$$

so there can be at most one solution in each case.

In the games we study, the equilibrium strategies, ω_s and ω_d of the games $\langle u, f, \Theta \rangle$ and $\langle u, f, \bar{\theta} \rangle$ respectively, are the unique solutions to,[14]

$$\psi_s(\omega_s) = 0 \text{ and } \psi_d(\omega_d) = 0 . \tag{12}$$

While functions ψ_s and ψ_d are convenient to work with in order to characterize properties of the Markov-Nash equilibrium in linear strategies, they do not provide the best way to verify the existence of a solution, since the value functions themselves enter the expressions in (9) and (10). Theorem 1 gives necessary and sufficient conditions so that there is precisely one

[13]Notice that since (9) must be met for all $x > 0$ in the case of linear-symmetric strategies, the function $\psi_s(\cdot)$ does not depend on x in equilibrium (i.e., when $\omega = \omega_s$). Even if the left-hand side of (9) depends on x whenever (9) is not met with equality (i.e., when $\omega \neq \omega_s$), this potential dependence on x does not affect our analysis, so we discard x for the sake of simplicity.

[14]Amir (1996) shows that for some games an equilibrium does not exist in the deterministic case, while there is at least one equilibrium for the stochastic version of the same model. In our extended example, presented in Section 5, linear-symmetric equilibrium strategies exist in both the deterministic and the stochastic game.

such symmetric equilibrium in linear strategies in the stochastic and the deterministic games respectively, based on the data of the corresponding game.[15]

Theorem 1 *Game $\langle u, f, \Theta\rangle$ has a unique Markov-perfect Nash equilibrium in linear symmetric strategies, if and only if there exists only one $\omega_s \in (0, 1/N)$ such that*

$$\Phi^s(\omega_s, x) = \kappa_s \quad \text{some } \kappa_s \in \mathbb{R}, \text{ for all } x > 0$$

in which

$$\Phi^s(\omega, x) = u(\omega x) - \delta \frac{1-(N-1)\omega}{1-N\omega} E\left[u\left(\omega\theta f\left((1-N\omega)x\right)\right)\right]$$

and Θ is such that all players' value functions are well-defined.[16] *Game $\langle u, f, \bar{\theta}\rangle$ has a unique Nash equilibrium in linear symmetric strategies if and only if there exists only one $\omega_d \in (0, 1/N)$ such that*

$$\Phi^d(\omega_d, x) = \kappa_d \quad \text{some } \kappa_d \in \mathbb{R}, \text{ for all } x > 0$$

in which

$$\Phi^d(\omega, x) = u(\omega x) - \delta \frac{1-(N-1)\omega}{1-N\omega} u\left(\omega\bar{\theta} f\left((1-N\omega)x\right)\right)$$

and $\bar{\theta}$ is such that all players' value functions are well-defined.

Proof

[15]For the approach followed in the proof of Theorem 1, see Chang (1988), Xie (2003), and Shimomura and Xie (2008). Note that this result is not a global-uniqueness equilibrium result.

[16]Whether a player's optimization problem is well-defined in a stochastic Markovian game can depend on the nature of the shock. For example, if the support of the shock is unbounded, conditions must be placed on the distribution of the shock in order to guarantee that value functions of players exist. In the context of the general single-controller stochastic growth model (which is the same as the model of Brock and Mirman (1972)), Stachurski (2002) identifies a simple condition on the mean of some monotonic transformation of the random shock that is sufficient to guarantee a well-behaved optimization problem and a well-defined long-run stationary distribution. Unlike Stachurski (2002), we do not provide such a condition for the general game, but we do so in the context of our more specific analysis in Section 5.

See Appendix A. □

In our parametric example of Section 5 we use Theorem 1 in order to identify appropriate functions u, f, Θ, and the level of $\bar{\theta}$ that guarantee linear Markov-perfect Nash strategies.

4. Why introducing multiplicatively-separable shocks to the resource-renewal law of the Levhari-Mirman (1980) model does not affect symmetric Markov-perfect Nash strategies

Using the Levhari-Mirman (1980) functions, namely $u(c) = \ln(c)$ and $f(x) = x^{\alpha}$, the value function of each player in the stochastic model is of the form,

$$V_s(x) = \frac{\alpha}{1-\alpha\delta}\ln(x) + b_s\ , \tag{13}$$

whereas the value function in the deterministic case is,

$$V_d(x) = \frac{\alpha}{1-\alpha\delta}\ln(x) + b_d\ , \tag{14}$$

in which b_s and b_d are constants. The resulting symmetric Markov-perfect Nash strategies are linear, namely, $C_{s,i}(x) = \omega_s x$ and $C_{d,i}(x) = \omega_d x$ for all $i \in \{1, ..., N\}$. Yet, it is straightforward to show from (2) and (3) that,

$$\omega_s = \omega_d = \frac{1-\alpha\delta}{N(1-\alpha\delta)+\alpha\delta}\ . \tag{15}$$

Therefore, the presence of uncertainty does not alter the rate of consumption in the Levhari-Mirman (1980) model. The difference between the stochastic and the deterministic model is that in the stochastic case the state variable evolves randomly and approaches a long-run stationary distribution. In the stochastic model, the mathematical expectation of algebraic expressions involving random disturbances enter the value function in an additively-separable manner and become part of the constant term b_s in (13), which does not affect optimization. This additive separability of terms b_s and b_d in (13) and (14) justifies why $\omega_s = \omega_d$, while, generally, $b_s \neq b_d$. Below we extend the Levhari-Mirman (1980) example, with uncertainty affecting Markov-perfect Nash strategies.

5. An extended example

Let,

$$u(c) = \frac{c^{1-\frac{1}{\eta}} - 1}{1 - \frac{1}{\eta}} \ , \tag{16}$$

and

$$f(x) = \left[\alpha x^{1-\frac{1}{\eta}} + (1-\alpha)\,\phi^{1-\frac{1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \ , \tag{17}$$

with $\eta > 0$, $\phi \geq 0$ and $\alpha \in (0,1]$. Notice that for $\eta = 1$, $u(c) = \ln(c)$ and $f(x) = \phi^{1-\alpha}x^{\alpha}$, i.e. the Levhari-Mirman (1980) model.[17] A similar example, applied to problems of Cournot oligopoly, has been presented in Koulovatianos and Mirman (2007).[18]

5.1 Existence of linear-symmetric equilibrium strategies

As in Stachurski (2002), we identify a single sufficient condition on the mean of the distribution of the transformed random variable, $\theta_t^{1-\frac{1}{\eta}}$, in order that the stochastic equilibrium be

[17] To show that $\lim_{\eta\to1}\left(c^{1-1/\eta} - 1\right)/(1-1/\eta) = \ln(c)$, re-write the limit as $\lim_{\eta\to1}\left(e^{(1-1/\eta)\ln(c)} - 1\right)/(1-1/\eta) = \lim_{\eta\to1} e^{(1-1/\eta)\ln(c)} \cdot \ln(c) = \ln(c)$, after having applied L'Hôpital's rule. A similar proof for the fact that $\lim_{\eta\to1}\left[\alpha x^{1-1/\eta} + (1-\alpha)\,\phi^{1-1/\eta}\right]^{\eta/(\eta-1)} = \phi^{1-\alpha}x^{\alpha}$ can be found in Chiang (1984, pp. 428-430).

[18] For a game with linear symmetric strategies, take (16) and consider a general CES production function

$$f(x) = \left[\alpha x^{1-\frac{1}{\gamma}} + (1-\alpha)\,\phi^{1-\frac{1}{\gamma}}\right]^{\frac{\gamma}{\gamma-1}} \ .$$

From the general first-order conditions of a game with linear strategies,

$$u'(\omega x) = \delta\left[1-(N-1)\,\omega\right] f'\left((1-N\omega)\,x\right) E\left[\theta u'\left(\omega f\left((1-N\omega)\,x\right)\theta\right)\right]$$

(see (4) and (8)). Using (16) and the CES production function,

$$x^{-\frac{1}{\eta}} = \delta\left[1-(N-1)\,\omega\right]\alpha\left(\frac{y}{x}\right)^{\frac{1}{\gamma}}(1-N\omega)^{-\frac{1}{\gamma}}\,y^{-\frac{1}{\eta}}E\left(\theta^{1-\frac{1}{\eta}}\right) \ ,$$

with $y \equiv f((1-N\omega)\,x)$. Setting $\eta = \gamma$ is the only way to obtain linear strategies with these functions. The single-controller version of this example with uncertainty (i.e. $N = 1$), is similar to this presented by Benhabib and Rustichini (1994).

well-defined.

Proposition 1 *If u and f are given by (16) and (17) respectively, and Θ is such that,*

$$E\left(\theta^{1-\frac{1}{\eta}}\right) \equiv \zeta < \frac{1}{\alpha\delta} \quad \text{and} \quad \left[E\left(\theta\right)\right]^{1-\frac{1}{\eta}} \equiv \bar{\zeta} < \frac{1}{\alpha\delta}, \tag{18}$$

then ω_s satisfying,

$$\omega_s = \frac{1}{\alpha\delta\zeta}\left[\left(1-N\omega_s\right)^{\frac{1}{\eta}} - \alpha\delta\zeta\left(1-N\omega_s\right)\right], \tag{19}$$

and ω_d satisfying,

$$\omega_d = \frac{1}{\alpha\delta\bar{\zeta}}\left[\left(1-N\omega_d\right)^{\frac{1}{\eta}} - \alpha\delta\zeta\left(1-N\omega_d\right)\right], \tag{20}$$

define unique linear-symmetric strategies that are Markov-perfect Nash equilibrium strategies of the infinite-horizon stochastic game and the deterministic game, respectively.

Proof of Proposition 1 From Theorem 1, by simple substitution we can see that $\Phi^s\left(\omega,x\right)$ takes the form,

$$\Phi^s\left(\omega,x\right) = \frac{\omega^{1-\frac{1}{\eta}}\left(1-N\omega\right)^{-\frac{1}{\eta}}}{1-\frac{1}{\eta}}\left\{\left(1-N\omega\right)^{\frac{1}{\eta}} - \alpha\delta\zeta\left[\left(1-\left(N-1\right)\omega\right)\right]\right\}x^{1-\frac{1}{\eta}} + g\left(\omega\right)\ ,$$

in which $g\left(\omega\right)$ does not depend on x. Therefore, if $\left(1-N\omega\right)^{\frac{1}{\eta}} - \alpha\delta\zeta\left[\left(1-\left(N-1\right)\omega\right)\right] = 0$ for a unique $\omega \in \left(0,1/N\right)$, then the sufficiency part of Theorem 1 can be applied. This condition is clearly equivalent to (19). By analogous argument, in the deterministic case the sufficiency part of Theorem 1 is satisfied if there is a unique $\omega \in \left(0,1/N\right)$ satisfying condition (20). Now it remains to show that ω_s and ω_d are unique, and also that the optimization problem of each player is well-defined. Let the two value functions,

$$V_s(x) = \frac{\alpha\omega_s^{1-\frac{1}{\eta}}}{1-\alpha\delta\zeta\left(1-N\omega_s\right)^{1-\frac{1}{\eta}}}\frac{x^{1-\frac{1}{\eta}}-1}{1-\frac{1}{\eta}}+b_s, \tag{21}$$

$$V_d(x) = \frac{\alpha\omega_d^{1-\frac{1}{\eta}}}{1-\alpha\delta\bar{\zeta}\left(1-N\omega_d\right)^{1-\frac{1}{\eta}}}\frac{x^{1-\frac{1}{\eta}}-1}{1-\frac{1}{\eta}}+b_d, \tag{22}$$

in which b_s and b_d are constants. If $N\omega_s = \Omega_s \in (0,1)$ and $N\omega_d = \Omega_d \in (0,1)$, then (21) and (22) imply that the problem of each player is well-defined. To show that $\Omega_s \in (0,1)$, we examine two cases.

Case 1: $\eta > 1$

Focusing on aggregate consumption rates, $\Omega \equiv N\omega$, we express (19) as,

$$N = \frac{\alpha\delta\zeta\Omega}{\left(1-\Omega\right)^{\frac{1}{\eta}} - \alpha\delta\zeta\left(1-\Omega\right)} \equiv H(\Omega)\ . \tag{23}$$

Here,

$$H(0) = 0 \quad \text{and} \quad H(1) = \infty\ ,$$

while

$$H'(\Omega) = \alpha\delta\zeta\frac{\frac{1-\left(1-\frac{1}{\eta}\right)\Omega}{\left(1-\Omega\right)^{1-\frac{1}{\eta}}} - \alpha\delta\zeta}{\left[\left(1-\Omega\right)^{\frac{1}{\eta}} - \alpha\delta\zeta\left(1-\Omega\right)\right]^2} > 0\ , \quad \text{for all } \Omega \in (0,1)\ .$$

To see the last statement, notice that $1-\left(1-\frac{1}{\eta}\right)\Omega \geq \left(1-\Omega\right)^{1-\frac{1}{\eta}}$ for all $\Omega \in [0,1)$, with equality if and only if $\Omega = 0$. So, applying the intermediate-value theorem to (23) shows that $\Omega_s \in (0,1)$, a unique symmetric equilibrium in linear strategies. This case is depicted by Figure 1. For $\Omega_d \in (0,1)$ and uniqueness, replace ζ with $\bar{\zeta}$ in the above argument.

Case 2: $\eta < 1$

For this case it is useful to express (19) as,

$$\Gamma(\Omega) \equiv \frac{N}{\Omega} = \frac{\alpha\delta\zeta}{\left(1-\Omega\right)^{\frac{1}{\eta}} - \alpha\delta\zeta\left(1-\Omega\right)} \equiv \Xi(\Omega)\ , \tag{24}$$

in which $\Gamma(0) = \infty$ and $\Xi(0) = \alpha\delta\zeta / (1 - \alpha\delta\zeta)$, and

$$\Xi(\Omega) > 0 \ \text{ if } \ \Omega \in \left[0\,,\, 1 - (\alpha\delta\zeta)^{\frac{\eta}{1-\eta}}\right] \ .$$

Due to (18) this interval is non-empty. Moreover,

$$\Xi'(\Omega) = \alpha\delta\zeta \frac{\frac{1}{\eta}(1-\Omega)^{\frac{1}{\eta}-1} - \alpha\delta\zeta}{\left[(1-\Omega)^{\frac{1}{\eta}} - \alpha\delta\zeta(1-\Omega)\right]^2} > 0 \quad \text{for all } \ \Omega \in \left[0\,,\, 1 - (\alpha\delta\zeta)^{\frac{\eta}{1-\eta}}\right]$$

as $0 < (1-\Omega)^{\frac{1}{\eta}} - \alpha\delta\zeta(1-\Omega) < \left[\frac{1}{\eta}(1-\Omega)^{\frac{1}{\eta}-1} - \alpha\delta\zeta\right](1-\Omega)$ for all $\Omega \in \left[0\,,\, 1 - (\alpha\delta\zeta)^{\frac{\eta}{1-\eta}}\right]$. Finally,

$$\Xi\left(1 - (\alpha\delta\zeta)^{\frac{\eta}{1-\eta}}\right) = \infty \ .$$

Given that $\Gamma\left(1 - (\alpha\delta\zeta)^{\frac{\eta}{1-\eta}}\right) < \infty$ and $\Gamma'(\Omega) < 0$ for all $\Omega \in \left[0\,,\, 1 - (\alpha\delta\zeta)^{\frac{\eta}{1-\eta}}\right]$, it follows that $\Omega_s \in \left(0\,,\, 1 - (\alpha\delta\zeta)^{\frac{\eta}{1-\eta}}\right) \subset (0,1)$, and it is unique. These properties of functions $\Gamma(\Omega)$ and $\Xi(\Omega)$ are depicted by Figure 2 which graphically demonstrates the uniqueness of Ω. For $\Omega_d \in (0,1)$ and the uniqueness of it replace ζ with $\bar{\zeta}$ in the above argument.

For the last case of $\eta = 1$, see Section 4 above, to verify that (15) satisfies both (19) and (20), and also that $\Omega_s = \Omega_d \in (0,1)$. $\square$

Note that Proposition 1 shows that the Levhari-Mirman (1980) model is indeed a knife-edge case for $\eta = 1$ in our extended example. According to our analysis in Section 4, in the case of $\eta = 1$ of the extended example, uncertainty plays no role.[19] However, for other values of η, uncertainty has a profound impact on each player's exploitation strategy.

5.2 Impact of uncertainty on strategies

Based on the analytical solution given by Proposition 1, we can check how strategic exploitation rates are affected by changes in risk. By the term "changes in risk" we mean changing

[19] Notice also that, for $\eta = 1$, (21) and (22) yield (13) and (14).

the distribution function Θ according to the notions of first-order stochastic dominance (FSD) and second-order stochastic dominance (SSD).

The concept of FSD involves changing Θ so that any monotonic transformation of the random variable θ, $h(\theta)$ leads to a change in its expected value $E[h(\theta)]$. Definition 1 formally states the definition of FSD (see, for example, Levy (1992, p. 557).

Definition 1 *Let two random variables, $\tilde{X}$ and X, in a common probability space, with both supports being subsets of $Z \subseteq \mathbb{R}_+$. Then X first-order stochastically dominates $\tilde{X}$, or $\tilde{X} \preceq_{FSD} X$, if and only if*[20]

$$E[h(X)] \geq E\left[h\left(\tilde{X}\right)\right] \quad \text{for all non-decreasing functions } h \; .$$

The concept of SSD can accommodate the notion of a "mean-preserving spread", according to which two distribution functions, Θ and $\tilde{\Theta}$ with corresponding realizations θ and $\tilde{\theta}$ can have $E(\theta) = E\left(\tilde{\theta}\right)$, but $E[h(\theta)] \neq E\left[h\left(\tilde{\theta}\right)\right]$ for all concave functions h. The formal definition of SSD is given by Definition 2 (again cf. Levy (1992, p. 557).

Definition 2 *Let two random variables, $\tilde{X}$ and X, defined on a common probability space, with both supports being subsets of $Z \subseteq \mathbb{R}_+$. Then X second-order stochastically dominates $\tilde{X}$, or $\tilde{X} \preceq_{SSD} X$, if and only if*

$$E[h(X)] \geq E\left[h\left(\tilde{X}\right)\right] \quad \text{for all concave functions } h \; .$$

Proposition 2 provides a comparison among exploitation strategies after changing the stochastic structure according to the FSD and SSD concepts.

[20]Letting $\tilde{F}$ and F be the distribution functions of $\tilde{X}$ and X,respectively, then $\tilde{X} \preceq_{FSD} X$ if and only if $\tilde{F}(z) \geq F(z)$ for all $z \in Z$.

Proposition 2 *Let u and f be given by (16) and (17), and Θ, $\tilde{\Theta}$, and $\hat{\Theta}$ with corresponding realizations θ, $\tilde{\theta}$, and $\hat{\theta}$ be such that (18) is met, and with ζ, $\tilde{\zeta}$, $\hat{\zeta}$, and $\bar{\zeta}$ corresponding to the definitions given by (18). Let also $\tilde{\theta} \preceq_{FSD} \theta$ and $\hat{\theta} \preceq_{SSD} \theta$. If ω_s, $\tilde{\omega}_s$, and $\hat{\omega}_s$ characterize the solutions to games $\langle u, f, \Theta \rangle$, $\left\langle u, f, \tilde{\Theta} \right\rangle$, and $\left\langle u, f, \hat{\Theta} \right\rangle$ respectively, then*

$$\eta \lesseqqgtr 1 \Leftrightarrow \tilde{\omega}_s \lesseqqgtr \omega_s \ , \tag{25}$$

$$\eta \lesseqqgtr 1 \Leftrightarrow \hat{\omega}_s \lesseqqgtr \omega_s \ , \tag{26}$$

$$\eta \lesseqqgtr 1 \Leftrightarrow \omega_s \lesseqqgtr \omega_d \ . \tag{27}$$

Proof

See Appendix B. □

Equivalence (25) of Proposition 2 which refers to FSD states that if risk changes so that the expectation of any nondecreasing transformation of the shock increases (this would be a shift from $\tilde{\Theta}$ to Θ in Proposition 2), then the exploitation rate of each player will decrease ($\omega_s < \tilde{\omega}_s$) if the elasticity of intertemporal substitution, η, is higher than 1. The FSD change in risk described by a shift from $\tilde{\Theta}$ to Θ is perceived as an improvement in the efficiency of resource reproduction and as a source of future utility gains for any value of $\eta > 0$.[21] A high elasticity of intertemporal substitution ($\eta > 1$) means more tolerance to any intertemporal consumption profile (c_t, c_{t+1}) in which c_t and c_{t+1} may be quite different. The intertemporal flexibility implied by $\eta > 1$ combined with expectations of higher future gains imply dominance of the (intertemporal) substitution effect over the wealth effect which generates an incentive for each player to conserve the resource (by decreasing the exploitation rate, i.e. $\omega_s < \tilde{\omega}_s$).

[21]To see this perceived improvement consider Definition 1, function $V_s(x)$ given by equation (21), and j-th player's expectation one period ahead, i.e., $E\left[V_s^j(\theta f(z))\right]$, with $z = x - \Sigma_{i \neq j} C_{s,i}(x)$, which show that a shift from $\tilde{\Theta}$ to Θ with $\tilde{\theta} \preceq_{FSD} \theta$ implies $E\left[V_s^j(\theta f(z))\right] \geq E\left[V_s^j\left(\tilde{\theta} f(z)\right)\right]$ for all η.

5.3 General results concerning games with primitives that imply Markov-perfect Nash Strategies which are linear in the resource stock and their relevance with our extended example

The result given by (25) seems specific to our example, i.e. particularly specific to restricting η to being equal to another parameter of the resource-reproduction function. Yet, in the Online Appendix (see Theorem 7 and Remark 1 therein) we show that, in the comprehensive class of games $\langle u, f, \Theta \rangle$ which imply linear Markov-perfect Nash exploitation strategies, the concept of FSD and the equivalence given by (25) hinge upon the comparison of the elasticity of intertemporal substitution with unity (i.e., $-u'(c) / [c \cdot u''(c)]$, in which u is not necessarily given by (16)). Regarding the role of "increasing risk" through a mean-preserving spread, equivalence (26) implies that if the coefficient of relative risk aversion is higher than unity ($\eta < 1$), then an increase in risk (shifting from Θ to $\hat{\Theta}$ with $\hat{\theta} \preceq_{SSD} \theta$) leads to a drop in the exploitation rate for each player ($\hat{\omega}_s < \omega_s$ – see Theorem 6 in the Online Appendix). In essence, the impact of increasing risk when $\eta < 1$ is an effort to conserve the resource, in a similar fashion to precautionary-savings investment analyses. In the Online Appendix we show that for the comprehensive class of games $\langle u, f, \Theta \rangle$ which imply linear Markov-perfect Nash exploitation strategies, SSD and the equivalence given by (26) depend on the convexity of the term $u'(c) \cdot c$.[22]

[22] Our results in the Online Appendix can be useful to researchers who are active in theory and/or applications of dynamic resource games. Theorists can see why (and how) the mechanics uncovered by our example survive in more general setups in which Markov-perfect Nash strategies are linear in the resource stock. Researchers who conduct applied research through numerical simulations of dynamic resource games may be helped by a conjecture: for setups in which Markov-perfect Nash strategies are *close to linear in the resource stock*, the role of primitives in strategic extraction under uncertainty that we have uncovered in the Online Appendix is likely to survive to some extent as well.

5.4 Changes in risk and the commons problem in our example

In this subsection we investigate whether increasing risk has an impact on the rate at which aggregate exploitation rates increase with the number of players. This investigation requires a comparison of aggregate exploitation rates along two dimensions, the number of players and the degree of riskiness. In performing such a comparison, if we employ two random variables, θ and $\tilde{\theta}$, with a discrete difference in the degree of riskiness, the comparison becomes unclear, as the initial aggregate exploitation rates can also be discretely different before increasing the number of players. For this reason, we employ a concept of changing risk that allows for marginal increases in riskiness. Consider a lognormally distributed shock,

$$\ln(\theta) \sim N\left(\mu - \frac{\sigma^2}{2}, \sigma^2\right). \tag{28}$$

The expectation of θ is, $E(\theta) = e^{\mu}$, whereas its variance is, $Var(\theta) = e^{2\mu}\left(e^{\sigma^2} - 1\right)$, i.e., the parameter σ has an impact only on the variance of θ and not on its mean (but the parameter μ has an impact on both the mean and the variance of θ). So, any two distributions given by (28) with different values of parameter σ are linked through second-order stochastic dominance (increase in risk). In particular, if $\theta \sim \Theta(\theta)$ with parameters (μ, σ) and $\tilde{\theta} \sim \tilde{\Theta}\left(\tilde{\theta}\right)$ with parameters $(\mu, \tilde{\sigma})$, then $\tilde{\sigma} > \sigma$ implies that $\tilde{\theta} \preceq_{SSD} \theta$ ($\tilde{\theta}$ represents an increase in risk from θ).

Proposition 3 examines the impact of increases in risk on the intensity of the tragedy of the commons.

Proposition 3 *If u and f are given by (16) and (17), Θ obeys (28) and condition (18) is met, then, (i) the commons problem is amplified by an increase in riskiness if and only if $\eta < 1$, (ii) the commons problem is mitigated by an increase in riskiness if and only if $\eta > 1$, (iii) the commons problem is unaffected by an* *increase in riskiness if and only if* $\eta = 1$.

Proof

See Appendix B. □

Proposition 3 shows that, if, $\eta < 1$, the overexploitation tendency is exacerbated as both riskiness and the number of players increase. Note, however, Proposition 2 states that whenever $\eta < 1$, for a fixed number of players, all players would tend to conserve the resource by decreasing their consumption rates as uncertainty increases. So, in our example, if players are highly risk-averse, then the commons problem dominates any conservation incentives that Proposition 2 has suggested to arise after increasing risk. These results indicate the complex, yet interesting, strategic behavior in Markov-perfect Nash equilibrium outcomes and call for further investigation in future research.

6. Concluding Remarks

The impact of uncertainty on strategic behavior in games of common-resource exploitation is not adequately understood. One reason for this lack of progress is that technical anomalies may arise if Markov-perfect-Nash strategies are nonlinear. One specific class of games which overcomes such anomalies is games that produce Markov-perfect Nash exploitation strategies which are linear in the resource stock. While the famous Levhari-Mirman (1980) example falls within this particular class, it nevertheless implies no impact of uncertainty on the strategic behavior of players. In this paper we have contributed another analytical example with linear Markov-perfect Nash exploitation strategies which nests the Levhari-Mirman (1980) example as a special case and which offers insights on how changes in uncertainty affects players' strategic behavior.

Our example involves additively-separable utility with constant relative risk aversion.

There are two key findings within our example. First, if the coefficient of relative risk aversion is higher than unity, then, for a given number of symmetric players, each player will conserve the resource once uncertainty (or an increase in risk) is introduced. This conservation response of players to increases in risk resembles investment literatures in which a high coefficient of relative risk aversion can be one of the ingredients leading to precautionary savings. Second, our example suggests that strategic interaction is more complex: we find that the addition of players leads to exacerbated increases in aggregate exploitation rates as we simultaneously increase risk, if the coefficient of relative risk aversion is higher than unity. It seems that for highly risk-averse players the commons problem dominates any conservation incentives that our first finding has suggested to arise after increasing risk. So, in our example, increasing risk exacerbates the commons problem if players are highly risk-averse.

In an Online Appendix we also develop technical tools for studying the comprehensive class of games that produce Markov-perfect Nash exploitation strategies which are linear in the resource stock. This more general analysis shows that the conservation response of players to increases in risk indeed hinges upon properties of the momentary utility function of players, always within the linear-strategy class of games. Our results regarding the impact of increasing risk on the intensity of the commons problem may be more specific to our example which links the elasticity of intertemporal substitution to parameters of the natural law of resource reproduction. Nevertheless, both our example and our more general analysis in the Online Appendix give shape to specific questions regarding the impact of uncertainty on strategic interaction in common-resource games for further investigation in more general setups.

Our contribution is related to the understanding of how rules for regulating the commons (see, for example, Ostrom et al. (1994)) may depend on the magnitude of risk borne by players in a common-resource-exploitation environment. Although we do not examine any forms of regulation, our example and the tools we develop in the Online Appendix may help in formally understanding how uncertainty affects strategic exploitation in more general settings in future work. For example, Polasky et al. (2006) study whether cooperation can become a subgame-perfect-equilibrium outcome, and Tarui et al. (2008) extend this analysis to including imperfect monitoring of each player's harvest. These two papers are closer to tackling questions of regulation than ours. Yet, introducing informational uncertainty, e.g., through Bayesian learning, to such studies, may be natural extensions in order to understand regulation in more realistic environments. Our work here may be a starting point in order to pursue such extensions.

7. Appendix A – Proof of Theorem 1

Proof of Theorem 1

For the necessity part in the case of the stochastic game, suppose game $\langle u, f, \Theta \rangle$ has an equilibrium in linear-symmetric strategies, $C^*_{s,i}(x) = \omega_s x$ all i, $\left\{C^*_{s,i}(x)\right\}_{i=1}^{N} \in \mathbb{S}_s$. Consider the first-order condition given by (9) and combine it with (6), which gives

$$u'(\omega_s x) - \delta[1 - (N-1)\omega_s] f'((1 - N\omega_s) x) E[\theta u'(\omega_s f((1 - N\omega_s) x) \theta)] = 0 \ . \tag{29}$$

Condition (29) holds for all $x > 0$. Therefore, integrating with respect to x, yields the expression corresponding to $\Phi^s(\omega_s, x)$ on the left-hand side. Hence $\Phi^s(\omega_s, x) = \kappa_s$ for some $\kappa_s \in \mathbb{R}$, for all $x > 0$.

For the sufficiency part, assume that there is only one $\omega_s \in (0, 1/N)$ such that $\Phi^s(\omega_s, x) = \kappa_s$ for some $\kappa_s \in \mathbb{R}$, for all $x > 0$, and let Θ be such that the value function of each player is well-defined. Then, differentiating both sides of $\Phi^s(\omega_s, x) = \kappa_s$ with respect to x leads to (29), which is the necessary condition for a linear-symmetric optimum. Differentiating both sides of (8) with respect to x, yields $V''(x) = [1 - (N-1)\omega_s] \omega_s u''(\omega_s x) < 0$, due to that $\omega_s \in (0, 1/N)$. So, $C^*_{s,j}(x) = \omega_s x$ for all $j \in \{1, ..., N\}$ is the only Nash equilibrium in symmetric linear strategies.

For the deterministic case suppose that game $\left\langle u, f, \bar{\theta} \right\rangle$ has an equilibrium in linear-symmetric strategies, $C^*_{d,i}(x) = \omega_d x$ all i, $\left\{C^*_{d,i}(x)\right\}_{i=1}^{N} \in \mathbb{S}_d$. The necessary first order condition becomes:

$$u'(\omega_d x) = \delta[1 - (N-1)\omega_d] f'((1 - N\omega_d) x) \bar{\theta} u'\left(\omega_d f((1 - N\omega_d) x) \bar{\theta}\right) \ . \tag{30}$$

The remainder of the proof is analogous to that in the stochastic game so we omit it. □

8. Appendix B – Proofs of Propositions 2 and 3

Proof of Proposition 2 By applying the implicit function theorem to (23) and (24), with the aid of Figures 1 and 2, it follows that

$$\omega_s = Z(\zeta) \ , \quad \tilde{\omega}_s = Z\left(\tilde{\zeta}\right) \ , \quad \hat{\omega}_s = Z\left(\hat{\zeta}\right) \quad \text{and} \quad \omega_d = Z\left(\bar{\zeta}\right), \tag{31}$$

in which $Z'(z) < 0$ for all $z \in (0, 1/(\alpha\delta))$. The monotonicity of $Z(\cdot)$ together with Definitions 1 and 2 imply,

$$\tilde{\omega}_s \lesseqqgtr \omega_s \Leftrightarrow \tilde{\zeta} = E\left(\tilde{\theta}^{1-\frac{1}{\eta}}\right) \gtreqqless E\left(\theta^{1-\frac{1}{\eta}}\right) \Leftrightarrow \eta \lesseqqgtr 1 \ ,$$

and

$$\hat{\omega}_s \lesseqqgtr \omega_s \Leftrightarrow \hat{\zeta} = E\left(\hat{\theta}^{1-\frac{1}{\eta}}\right) \gtreqqless E\left(\theta^{1-\frac{1}{\eta}}\right) \Leftrightarrow \eta \lesseqqgtr 1 \ ,$$

which prove (25), and (26) respectively. The equivalence given by (27) is a direct implication of equivalence (26), since the comparison between the deterministic and the stochastic model is a special case of SSD. Alternatively, equivalence (27) can be proved through Jensen's inequality since

$$\omega_s \lesseqqgtr \omega_d \Leftrightarrow E\left(\theta^{1-\frac{1}{\eta}}\right) \gtreqqless [E(\theta)]^{1-\frac{1}{\eta}} \Leftrightarrow \eta \lesseqqgtr 1 \ ,$$

which completes the proof of the Proposition. □

Proof of Proposition 3 We denote the aggregate exploitation rate by $\Omega(\sigma) = N\omega(\sigma)$. Equation (38) implies (after some algebra) that

$$\frac{\partial \Omega(\sigma)}{\partial N} = \frac{\eta\zeta(\sigma)\,\omega(\sigma)\,\xi(\sigma)^{\eta-1}}{N\left[1 - \frac{N-1}{N}\eta\zeta(\sigma)\,\xi(\sigma)^{\eta-1}\right]}. \tag{32}$$

in which

$$\zeta(\sigma) \equiv \left[\alpha\delta E\left(\theta(\sigma)^{1-\frac{1}{\eta}}\right)\right]^{\eta},$$

and

$$\xi(\sigma) \equiv [1-(N-1)\,\omega(\sigma)] \ .$$

Equation (32) can be re-written as,

$$\frac{\frac{\partial\Omega(\sigma)}{\partial N}}{\Omega(\sigma)} = \frac{\partial \ln[\Omega(\sigma)]}{\partial N} = \frac{\eta\zeta(\sigma)\,\xi(\sigma)^{\eta-1}}{N^2\left[1-\frac{N-1}{N}\eta\zeta(\sigma)\,\xi(\sigma)^{\eta-1}\right]}. \tag{33}$$

Taking the partial derivative of this last expression with respect to σ, yields,

$$\frac{\partial^2 \ln[\Omega(\sigma)]}{\partial N\partial\sigma} = \frac{\eta\zeta'(\sigma)\,\xi(\sigma)^{\eta-2}\left[\xi(\sigma)+(\eta-1)\,\zeta(\sigma)\,\frac{\xi'(\sigma)}{\zeta'(\sigma)}\right]}{N^2\left[1-\frac{N-1}{N}\eta\zeta(\sigma)\,\xi(\sigma)^{\eta-1}\right]^2}. \tag{34}$$

Since,

$$\frac{\xi'(\sigma)}{\zeta'(\sigma)} = \frac{d\xi(\sigma)}{d\zeta(\sigma)},$$

we can calculate $d\xi(\sigma)/d\zeta(\sigma)$ through (19), by applying the implicit function theorem. In particular, (19) can be written as,

$$\frac{N-1}{N}\zeta(\sigma)\,\xi(\sigma)^{\eta} + \frac{1}{N} - \xi(\sigma) = 0 \ , \tag{35}$$

so,

$$\frac{\xi'(\sigma)}{\zeta'(\sigma)} = \frac{\frac{N-1}{N}\xi(\sigma)^{\eta}}{1-\frac{N-1}{N}\eta\zeta(\sigma)\,\xi(\sigma)^{\eta-1}} \ . \tag{36}$$

After substituting (36), the expression $\xi(\sigma)+(\eta-1)\,\zeta(\sigma)\,\frac{\xi'(\sigma)}{\zeta'(\sigma)}$ on the RHS of (34), becomes,

$$\xi(\sigma)+(\eta-1)\,\zeta(\sigma)\,\frac{\xi'(\sigma)}{\zeta'(\sigma)} = \frac{\xi(\sigma)-\frac{N-1}{N}\zeta(\sigma)\,\xi(\sigma)^{\eta}}{1-\frac{N-1}{N}\eta\zeta(\sigma)\,\xi(\sigma)^{\eta-1}}.$$

Yet, (35) implies that, $\xi(\sigma)-\frac{N-1}{N}\zeta(\sigma)\,\xi(\sigma)^{\eta}=\frac{1}{N}$, so,

$$\xi(\sigma)+(\eta-1)\,\zeta(\sigma)\,\frac{\xi'(\sigma)}{\zeta'(\sigma)} = \frac{1}{N\left[1-\frac{N-1}{N}\eta\zeta(\sigma)\,\xi(\sigma)^{\eta-1}\right]},$$

and (34) gives,

$$\frac{\partial^2 \ln\left[\Omega\left(\sigma\right)\right]}{\partial N \partial \sigma} = \frac{\eta \zeta'\left(\sigma\right) \xi\left(\sigma\right)^{\eta-2}}{N^3 \left[1 - \frac{N-1}{N} \eta \zeta\left(\sigma\right) \xi\left(\sigma\right)^{\eta-1}\right]^3}. \tag{37}$$

Moreover,

$$\frac{\partial \Omega\left(\sigma\right)}{\partial N} = \frac{\eta \zeta\left(\sigma\right) \omega\left(\sigma\right) \xi\left(\sigma\right)^{\eta-1}}{N \left[1 - \frac{N-1}{N} \eta \zeta\left(\sigma\right) \xi\left(\sigma\right)^{\eta-1}\right]} > 0 \Rightarrow 1 - \frac{N-1}{N} \eta \zeta\left(\sigma\right) \xi\left(\sigma\right)^{\eta-1} > 0 \ ,$$

so, (37) implies that,

$$\frac{\partial^2 \ln\left[\Omega\left(\sigma\right)\right]}{\partial N \partial \sigma} \gtreqqless 0 \Leftrightarrow \zeta'\left(\sigma\right) \gtreqqless 0 \Leftrightarrow \eta \lesseqqgtr 1 \ ,$$

which proves the Proposition. $\square$

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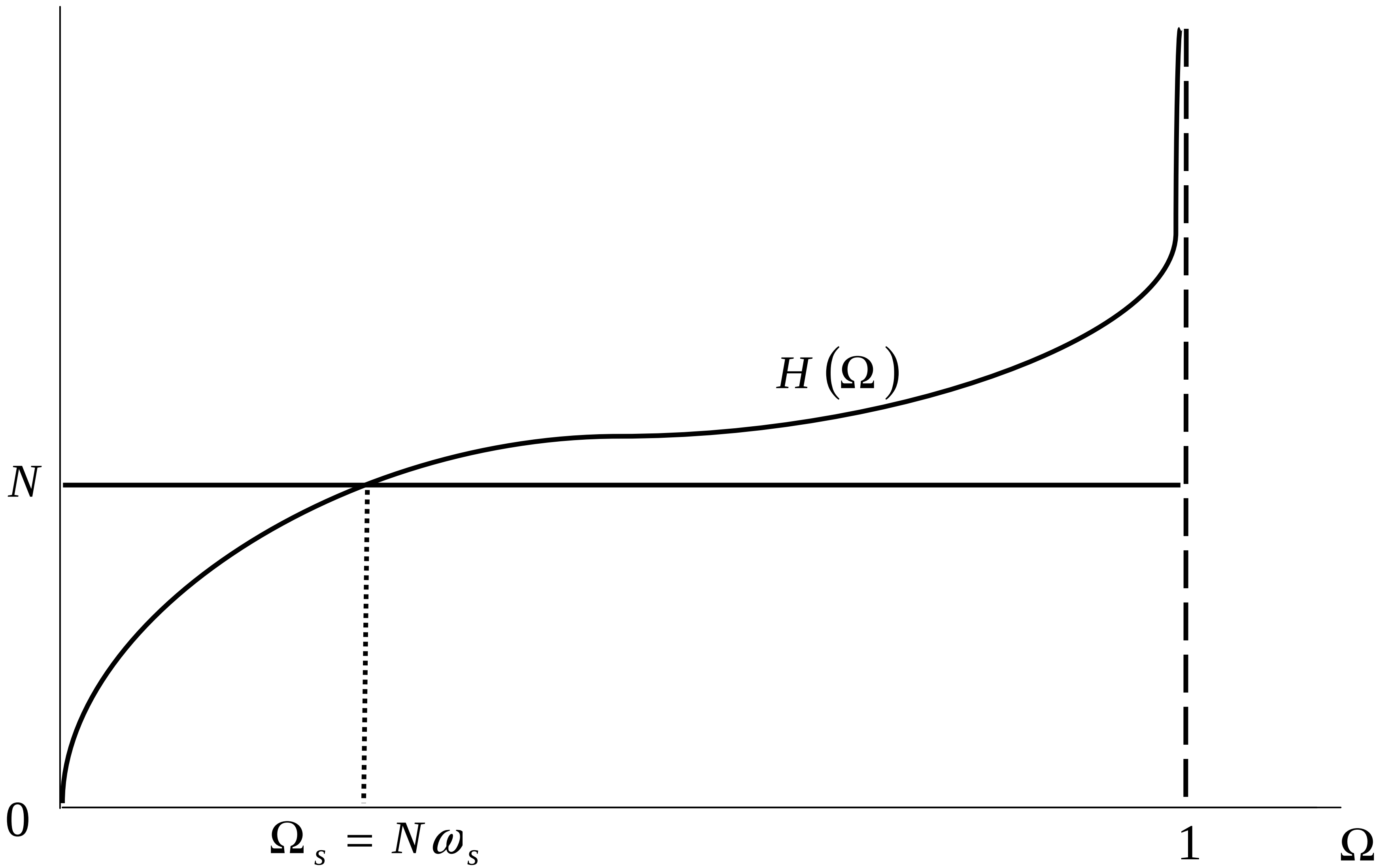

Figure 1 Equilibrium strategies when $\eta > 1$

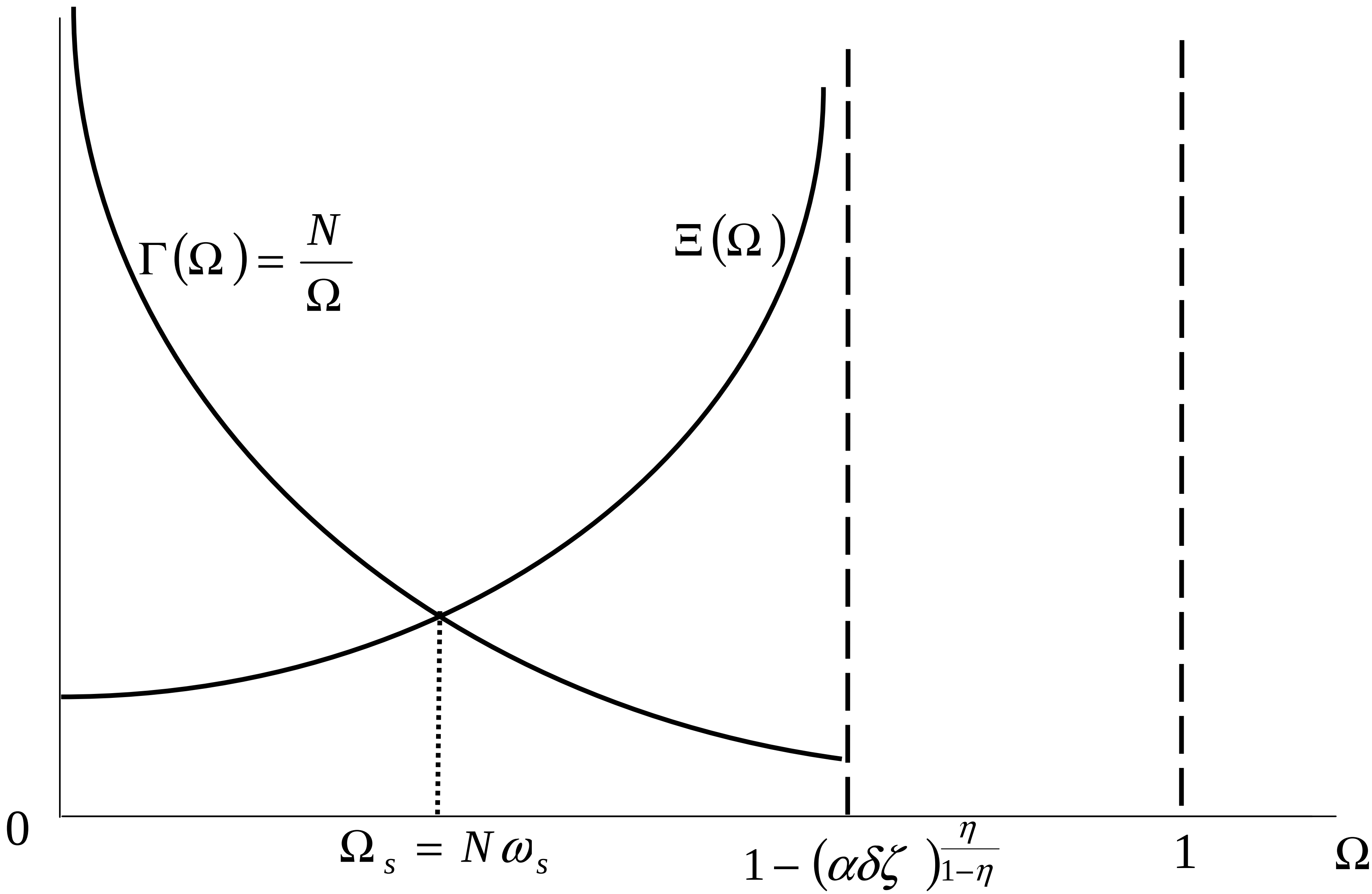

Figure 2 Equilibrium strategies when $\eta < 1$

Supplementary Material: Online Appendix for

Strategic Exploitation of a Common Property Resource under Uncertainty

by

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General Results Concerning Games with Primitives that Imply Markov-Perfect Nash Strategies which are Linear in the Resource Stock

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The commons problem and uncertainty: a first look

For the general results we concentrate our analysis on the unique equilibrium in linear-symmetric Markov strategies for games which satisfy the conditions of Theorem 1. We use the short-hand notation $\langle u, f, \Theta \rangle_{SULS}^{\infty}(\omega_s)$ and $\langle u, f, \bar{\theta} \rangle_{DULS}^{\infty}(\omega_d)$ to denote the game and the equilibrium we are studying in the stochastic and deterministic game respectively. In addition, since the main body of the paper contains Theorem 1, we name all theorems starting from "Theorem 2" in this Online Appendix.

Theorem 2 demonstrates a global result about the strategic behavior of players. We show that for all games $\langle u, f, \Theta \rangle_{SULS}^{\infty}(\omega_s)$ and $\langle u, f, \bar{\theta} \rangle_{DULS}^{\infty}(\omega_d)$ the commons problem holds.

Theorem 2 *Suppose games $\langle u, f, \Theta \rangle$ and $\langle u, f, \bar{\theta} \rangle$ satisfy the conditions of Theorem 1 and consider $\langle u, f, \Theta \rangle_{SULS}^{\infty}(\omega_s)$ and $\langle u, f, \bar{\theta} \rangle_{DULS}^{\infty}(\omega_d)$. As the number of players, N, increases, the aggregate exploitation rates, $\Omega_s \equiv N\omega_s$ and $\Omega_d \equiv N\omega_d$ increase.*

Proof Consider $\langle u, f, \Theta \rangle_{SULS}^{\infty}(\omega_s)$. For any $x > 0$, a player's necessary condition (9) in the paper can be expressed as a function of the aggregate exploitation rate $\Omega \equiv N\omega$, as,

$$\begin{aligned} \hat{\Psi}^s(\Omega, N) &= -u'\left(\frac{\Omega}{N}x\right) + h_s(\Omega) = 0, \text{ in which} \\ h_s(\Omega) &= \delta f'((1-\Omega)x)\, E\left[\theta V_s'(\theta f((1-\Omega)x))\right] . \end{aligned}$$

Given that $V_s'' < 0$, and $f'' \leq 0$, $h_s'(\Omega) > 0$. Applying the implicit function theorem on the equilibrium condition

$$-u'\left(\frac{\Omega_s}{N}x\right) + h_s(\Omega_s) = 0,$$

we obtain

$$\frac{d\Omega_s}{dN} = \frac{-u''\left(\frac{\Omega_s}{N}x\right)}{-u''\left(\frac{\Omega_s}{N}x\right)\frac{x}{N} + h_s'(\Omega_s)} \frac{\Omega_s x}{N^2} > 0, \tag{38}$$

which proves the result. The argument for the deterministic case is the analogous and we omit it. $\square$

Theorem 2 is sharp for the class of games we examine. It indicates that the commons problem is robust for games of joint exploitation.[23] However, Theorem 2 does not address how the introduction of uncertainty affects strategic exploitation and the commons problem. Theorem 3 shows that uncertainty may increase or decrease the equilibrium aggregate level of exploitation, by comparing the equilibria of pairs of games $\langle u, f, \Theta\rangle_{SULS}^{\infty}(\omega_s)$ and $\langle u, f, \bar{\theta}\rangle_{DULS}^{\infty}(\omega_d)$. Theorem 3 is similar to a theorem appearing in Mirman (1971, cf. Theorem 2 p. 181), appropriately adjusted in order to accommodate a symmetric equilibrium in linear strategies.

Theorem 3 *Suppose games $\langle u, f, \Theta\rangle$ and $\langle u, f, \bar{\theta}\rangle$ satisfy the conditions of Theorem 1 and consider $\langle u, f, \Theta\rangle_{SULS}^{\infty}(\omega_s)$ and $\langle u, f, \bar{\theta}\rangle_{DULS}^{\infty}(\omega_d)$. Then, for all $x > 0$*

$$\omega_s \gtreqless \omega_d \Longleftrightarrow E\left[\Lambda_s\left(\theta\rho_d\right)\right] \lesseqgtr \Lambda_d\left(\bar{\theta}\rho_d\right) \tag{39}$$

where,$\Lambda_s(z) \equiv zV_s'(z)$, $\Lambda_d(z) \equiv zV_d'(z)$ and $\rho_d \equiv f\left((1 - N\omega_d)\,x\right)$.

Proof

Fix $x > 0$. Given that both the deterministic and the stochastic strategies are interior, $\omega_s, \omega_d \in (0\,,\,1/N)$, then $\psi_d(\omega_d) = \psi_s(\omega_s) = 0$. Since ψ_s is strictly increasing on $(0\,,\,1/N)$ (see (11) in the paper),

$$\omega_s \gtreqless \omega_d \Longleftrightarrow \psi_s(\omega_d) \lesseqgtr 0 \Longleftrightarrow \psi_s(\omega_d) \lesseqgtr \psi_d(\omega_d) \tag{40}$$

[23]Other studies that examine the issue of the tragedy of the commons include Dutta and Sundaram (1993), Sorger (1998, 2005) , and Dockner and Sorger (1996).

Moreover, from (9) in the paper

$$\psi_s(\omega_d) = -u'(\omega_d x) + \delta \frac{f'((1-N\omega_d)x)}{f((1-N\omega_d)x)} E\left[\theta f((1-N\omega_d)x) V_s'(\theta f((1-N\omega_d)x))\right]$$

or

$$\psi_s(\omega_d) = -u'(\omega_d x) + \delta \frac{f'((1-N\omega_d)x)}{\rho_d} E\left[\Lambda_s(\theta\rho_d)\right]. \tag{41}$$

Similarly, from (10),

$$\psi_d(\omega_d) = -u'(\omega_d x) + \delta \frac{f'((1-N\omega_d)x)}{\rho_d} \Lambda_d\left(\bar{\theta}\rho_d\right). \tag{42}$$

Combining (41) and (42) with (40), (39) holds, for all $x > 0$. □

Theorem 3 gives *necessary and sufficient conditions* for the direction of the impact of uncertainty on consumption. However, these are based on characteristics (relative curvature) of the two value functions and not on the primitives of the games.[24] In order to find the link to the primitives of the games, notice that from (8)

$$V_s'(x) = [1-(N-1)\omega_s]\, u'(\omega_s x) \tag{43}$$

and

$$V_d'(x) = [1-(N-1)\omega_d]\, u'(\omega_d x). \tag{44}$$

Hence,

$$E\left[\Lambda_s(\theta\rho_d)\right] = [1-(N-1)\omega_s]\, f'((1-N\omega_d)x)\, E\left[\theta u'(\omega_s f((1-N\omega_d)x)\theta)\right] \tag{45}$$

[24]Within the class of games we discuss, explicit value functions can be derived, thus making the application of Theorem 3 straightforward. In our parametric examples in Section 5, we have provided such explicit formulas for the value functions. The necessity part of Theorem 3 proves useful in characterizing parameter choices that determine the effect of uncertainty on strategies.

and

$$\Lambda_d\left(\bar{\theta}\rho_d\right) = \left[1-(N-1)\,\omega_d\right] f'\left(\left(1-N\omega_d\right)x\right)\bar{\theta}u'\left(\omega_d f\left(\left(1-N\omega_d\right)x\right)\bar{\theta}\right). \tag{46}$$

This connection between the value functions and the primitives of the model offered by equations (45) and (46) proves useful. In particular, this connection helps in distinguishing how changes in the model's primitives affect the impact of uncertainty on players' exploitation strategies.

Some of our proofs in the remainder of the paper rely on the existence of a well-behaved recursive mapping for calculating the equilibrium. This calculation procedure is the solution technique suggested by Levhari and Mirman (1980). We next present some key results about this procedure.

The Levhari-Mirman recursive procedure

Levhari and Mirman (1980) start from the static, one period, symmetric equilibrium, in which consumption rates of players are equal to $1/N$ in symmetric equilibrium. They use this strategy in order to form the value function and then they continue with the two-period problem, calculate the symmetric strategies again, generalizing the process to the n-period problem. In general, if a linear symmetric Markov-perfect strategy exists for the n-period problem, then a tractable recursive mapping on the consumption rates $\omega^{(n)}$ can be constructed, in which $\omega^{(n)}$ denotes the symmetric-equilibrium consumption strategy of the n-period problem. Characterizing the evolution of $\omega^{(n)}$, as n increases, is sufficient for characterizing the evolution of the linear-symmetric consumption functions in our class of models.

For $n = 2, 3, \ldots$, the n-period problem of player $j \in \{1, \ldots, N\}$ in Bellman form, is

$$V_{s,j}^{(n)}(x) = \max_{0 \le c_j \le x - \sum_{\substack{i=1 \\ i \ne j}}^{N} C_{s,i}^{(n)}(x)} \left\{ u(c_j) + \delta E\left[V_{s,j}^{(n-1)}\left(\theta f\left(x - c_j - \sum_{\substack{i=1 \\ i \ne j}}^{N} C_{s,i}^{(n)}(x) \right)\right)\right]\right\}, \quad (47)$$

for the stochastic case, in which $V_{s,j}^{(n)}$ and $V_{s,j}^{(n-1)}$ are the n- and $(n-1)$-period value functions of player j, and $C_{s,i}^{(n)}$ is the n-period strategy of player i. Similarly, in the deterministic case, for $n = 2, 3, \ldots$, the n-period problem of player $j \in \{1, \ldots, N\}$ is

$$V_{d,j}^{(n)}(x) = \max_{0 \le c_j \le x - \sum_{\substack{i=1 \\ i \ne j}}^{N} C_{d,i}^{(n)}(x)} \left[u(c_j) + \delta V_{d,j}^{(n-1)}\left(\bar{\theta} f\left(x - c_j - \sum_{\substack{i=1 \\ i \ne j}}^{N} C_{d,i}^{(n)}(x) \right)\right)\right]. \quad (48)$$

Again, we focus on (interior) linear-symmetric Markov-perfect Nash strategies. For the stochastic game, $C_{s,i}^{(n)^*}(x) = \omega_s^{(n)} x$ for all $x > 0$, with $\omega_s^{(n)} \in (0, 1/N)$, all i, and for the deterministic game, $C_{d,i}^{(n)^*}(x) = \omega_d^{(n)} x$ for all $x > 0$, with $\omega_d^{(n)} \in (0, 1/N)$, all i.

Assuming $C_{s,i}^{(n)^*}(x) = \omega_s^{(n)} x$ in (47), the necessary first order condition of the $(n+1)$-period problem of player $j \in \{1, \ldots, N\}$ is,

$$-u'(c_j) + \delta\left[1 - (N-1)\omega_s^{(n)}\right] f'(y) E\left[\theta u'\left(\omega_s^{(n)} \theta f(y)\right)\right] = 0, \quad (49)$$

in which $y = x - c_j - \sum_{\substack{i=1 \\ i \ne j}}^{N} C_i(x)$ and $C_i(x)$ is the strategy of a player $i \ne j$.[25] Therefore, linear symmetric equilibrium strategies, $\omega_s^{(n+1)}$, for the $(n+1)$-period problem, solve,[26]

$$\Psi^s\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right) = 0, \quad (50)$$

[25] This necessary condition is derived from (47) following the same steps as in the infinite-horizon case, i.e., after applying the envelope theorem on (47). Notice that 49 does not restrict the player to a search for an optimum among linear strategies only in the $(n+1)$-period problem.

[26] As above, with function $\psi_s(\cdot)$, (50) must be met for all $x > 0$ in the case of linear-symmetric strategies, so the function $\Psi^s(\cdot)$ does not depend on x in equilibrium (i.e., when the expression given by (51) is evaluated at $\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right)$, $n = 1, 2, \ldots$). Even if the expression given by (51) depends on x whenever (50) is not met with equality (i.e., when this expression is evaluated at some $\left(\omega, \omega_s^{(n)}\right)$, with $\omega \ne \omega_s^{(n+1)}$, $n = 1, 2, \ldots$), this potential dependence on x does not affect our analysis, so we discard x for the sake of simplicity.

in which

$$\Psi^s\left(\omega,\omega_s^{(n)}\right)=g\left(\omega\right)+h^s\left(\omega,\omega_s^{(n)}\right), \tag{51}$$

with $g\left(\omega\right)=-u'\left(\omega x\right)$, and

$$h^s\left(\omega,\omega_s^{(n)}\right)=\delta\left[1-\left(N-1\right)\omega_s^{(n)}\right]f'\left(\left(1-N\omega\right)x\right)E\left[\theta u'\left(\omega_s^{(n)}\theta f\left(\left(1-N\omega\right)x\right)\right)\right] \ .$$

It follows that, due to the assumptions $u''<0$ and $f''\leq 0$,

$$\Psi_1^s\left(\omega,\omega_s^{(n)}\right)>0 \ , \ \Psi_2^s\left(\omega,\omega_s^{(n)}\right)<0 \ , \ \text{for all } x>0, \ \omega\in\left(0,\frac{1}{N}\right), \ \ \omega_s^{(n)}\in\left(0,\frac{1}{N}\right] \ . \tag{52}$$

Therefore, if $\omega_s^{(n+1)}$, the solution to (50) exists, and lies in the open interval $(0,1/N)$, after applying the implicit function theorem to (50),

$$\frac{d\omega_s^{(n+1)}}{d\omega_s^{(n)}}=-\frac{\Psi_2^s\left(\omega_s^{(n+1)},\omega_s^{(n)}\right)}{\Psi_1^s\left(\omega_s^{(n+1)},\omega_s^{(n)}\right)}>0 \ , \ \text{for all } x>0, \ \omega_s^{(n)}\in\left(0,\frac{1}{N}\right] \ , \tag{53}$$

as implied by (52). The same remarks hold for the deterministic case, for the deterministic game. Given $\omega_d^{(n)}$, $\omega_d^{(n+1)}$ is the solution to

$$\Psi^d\left(\omega_d^{(n+1)},\omega_d^{(n)}\right)=0 \ , \tag{54}$$

in which

$$\Psi^d\left(\omega,\omega_d^{(n)}\right)=g\left(\omega\right)+h^d\left(\omega,\omega_d^{(n)}\right) \ , \tag{55}$$

with

$$h^d\left(\omega,\omega_d^{(n)}\right)=\delta\left[1-\left(N-1\right)\omega_d^{(n)}\right]f'\left(\left(1-N\omega\right)x\right)\bar{\theta}u'\left(\omega_d^{(n)}\bar{\theta}f\left(\left(1-N\omega\right)x\right)\right) \ .$$

The two conditions, (50) and (54), define two recursive mappings.

Definition 3 *Consider* $\langle u, f, \Theta \rangle$ *and* $\left\langle u, f, \bar{\theta} \right\rangle$. *The mapping* $M_s : [0, 1/N] \rightarrow [0, 1/N]$ *is given by* $\Psi^s (M_s (\omega), \omega) = 0$. *The mapping* $M_d : [0, 1/N] \rightarrow [0, 1/N]$ *is given by* $\Psi^d (M_d (\omega), \omega) = 0$.

The following results provide a characterization for these two mappings. In particular, Lemma 1 shows the importance of imposing an Inada condition on the utility function for obtaining interior solutions (this Inada condition is a sufficient condition).

Lemma 1 *If* $\lim_{c \to 0} u'(c) = \infty$, *for all* $\omega_s^{(1)} \in (0,\, 1/N]$ *and all* $\omega_d^{(1)} \in (0,\, 1/N]$, *the sequences* $\left\{ \omega_s^{(n)} \right\}_{n=2}^{\infty}$ *generated by the mapping* M_s, *and* $\left\{ \omega_d^{(n)} \right\}_{n=2}^{\infty}$ *generated by* M_d, *are such that* $\omega_s^{(n)}$ *and* $\omega_d^{(n)}$ *are unique with* $\omega_s^{(n)}, \omega_d^{(n)} \in (0,\, 1/N)$, $n = 2, 3, \ldots$.

Proof

Fix any $x > 0$ and any $\omega_s^{(n)} \in (0,\, 1/N]$. Then, from (51), $\Psi^s \left(\omega, \omega_s^{(n)} \right) = g(\omega) + h^s(\omega)$, in which

$$\begin{aligned} h^s(\omega) &> 0 \quad \text{and} \quad g(\omega) < 0 \text{ for all } \omega \in (0, 1/N), \text{ and} \\ \lim_{\omega \to 0} g(\omega) &= -\infty \quad \text{and} \quad \lim_{\omega \to 1/N} h^s(\omega) = \infty \end{aligned}$$

so $\Psi^s \left(\omega, \omega_s^{(n)} \right)$ intersects the zero axis within the interval $(0, 1/N)$ at least once. Yet, the fact that, for all $\omega \in (0, 1/N)$, $h^{s\prime}(\omega) > 0$, and $g'(\omega) > 0$, implies that $\Psi^s \left(\omega, \omega_s^{(n)} \right) = 0$ has a unique solution, $\omega^{(n+1)} \in (0, 1/N)$. The same argument can be used for the deterministic analogue of the stochastic model. $\square$

Lemma 1 is crucial in the proof of Theorem 4, which establishes convergence of the Levhari-Mirman (1980) recursive procedure.

Theorem 4 *Consider* $\langle u, f, \Theta \rangle_{SULS}^{\infty}(\omega_s)$ *and* $\langle u, f, \bar{\theta} \rangle_{DULS}^{\infty}(\omega_d)$ *and suppose the one-period games have a symmetric equilibrium strategy,* $\omega_s^{(1)} = \omega_d^{(1)} = 1/N$*. If* $\lim_{c \to 0} u'(c) = \infty$*, then starting from* $\omega_s^{(1)} = \omega_d^{(1)} = 1/N$*, the recursive mappings* M_s *and* M_d *are convergent with* $\lim_{n \to \infty} \omega_s^{(n)} = \omega_s$ *and* $\lim_{n \to \infty} \omega_d^{(n)} = \omega_d$*.*

Proof

Fix $x > 0$ and consider the stochastic game. Using the same argument as in the proof of Lemma 1, starting from $\omega_s^{(1)}$ there exists a unique $\omega_s^{(2)} \in (0, 1/N)$, i.e. $\omega_s^{(2)} < \omega_s^{(1)}$. Moreover,

$$\omega_s < \omega_s^{(2)} < \omega_s^{(1)} = \frac{1}{N}. \tag{56}$$

In order to show (56), suppose that $\omega_s^{(2)} \leq \omega_s$. Since $\Psi_1^s\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right) > 0$, $\Psi_2^s\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right) < 0$, and $\Psi^s(\omega_s, \omega_s) = 0$, $0 = \Psi^s(\omega_s, \omega_s) > \Psi^s\left(\omega_s, \omega_s^{(1)}\right) \geq \Psi^s\left(\omega_s^{(2)}, \omega_s^{(1)}\right)$, which contradicts that $\Psi^s\left(\omega_s^{(2)}, \omega_s^{(1)}\right) = 0$. From Lemma 1, $\omega_s^{(1)} = 1/N$ gives rise to a unique sequence $\left\{\omega_s^{(n)}\right\}_{n=1}^{\infty}$ that is generated from (50). Given that $\omega_s \in (0, 1/N)$ is unique, and the mapping M_s is a continuous function with $M_s'\left(\omega_s^{(n)}\right) = -\frac{\Psi_2^s\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right)}{\Psi_1^s\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right)} > 0$ for all $x > 0$, (56) and the intermediate value theorem imply that $\omega_s < \omega_s^{(n+1)} < \omega_s^{(n)}, n = 1, 2, \ldots$. Therefore the sequence $\left\{\omega_s^{(n)}\right\}_{n=1}^{\infty}$ converges and $\lim_{n \to \infty} \omega_s^{(n)} = \omega_s$. The same argument holds for the deterministic analogue of the stochastic model. $\square$

Theorem 4 shows that the procedure that Levhari and Mirman (1980) suggested and implemented in their example, leads to recursive computability of the infinite-horizon strategies for a more general class of games, i.e., the class of games $\langle u, f, \Theta \rangle_{SULS}^{\infty}(\omega_s)$ and $\langle u, f, \bar{\theta} \rangle_{DULS}^{\infty}(\omega_d)$, which in addition have a unique symmetric equilibrium in the stage game, and which satisfy $\lim_{c \to 0} u'(c) = \infty$. While this result is interesting in its own right (as we have identified a reliable calculation procedure), some properties of the mappings M_s and M_d are useful in the analysis of uncertainty. Lemma 2 states these properties of M_s and M_d.

Lemma 2 *Consider* $\langle u, f, \Theta\rangle_{SULS}^{\infty}(\omega_s)$ *and* $\langle u, f, \bar{\theta}\rangle_{DULS}^{\infty}(\omega_d)$ *and suppose the one-period games have a symmetric equilibrium strategy,* $\omega_s^{(1)} = \omega_d^{(1)} = 1/N$. *If* $\lim_{c\to 0} u'(c) = \infty$, *then for any interval* $\mho_s = [\breve{\omega}_s, \hat{\omega}_s] \subseteq (0,\, 1/N]$,

$$\begin{aligned} M_s(\breve{\omega}_s) &\geq \breve{\omega}_s \text{ and } M_s(\hat{\omega}_s) \leq \hat{\omega}_s \Rightarrow \omega_s \in \mho_s\ , \\ M_s(\breve{\omega}_s) &> \breve{\omega}_s \text{ and } M_s(\hat{\omega}_s) < \hat{\omega}_s \Rightarrow \omega_s \in (\breve{\omega}_s, \hat{\omega}_s) \end{aligned}$$

and

$$\lim_{n\to\infty} \omega_s^{(n)} = \omega_s \text{ for all } \omega_s^{(1)} \in \mho_s\ .$$

Moreover, for any interval $\mho_d = [\breve{\omega}_d, \hat{\omega}_d] \subseteq (0,\, 1/N]$,

$$\begin{aligned} M_d(\breve{\omega}_d) &\geq \breve{\omega}_d \text{ and } M_d(\hat{\omega}_d) \leq \hat{\omega}_d \Rightarrow \omega_d \in \mho_d\ , \\ M_d(\breve{\omega}_d) &> \breve{\omega}_d \text{ and } M_d(\hat{\omega}_d) < \hat{\omega}_d \Rightarrow \omega_d \in (\breve{\omega}_d, \hat{\omega}_d)\ , \end{aligned}$$

and

$$\lim_{n\to\infty} \omega_d^{(n)} = \omega_d \ \ for\ all \ \ \omega_s^{(1)} \in \mho_d.$$

Proof

Fix any $x > 0$ and let $\mho_s = [\breve{\omega}_s, \hat{\omega}_s] \subseteq (0,\, 1/N]$ with $M_s(\breve{\omega}_s) \geq \breve{\omega}_s$ and $M_s(\hat{\omega}_s) \leq \hat{\omega}_s$. Since M_s is a continuous function with $M_s'\left(\omega_s^{(n)}\right) = -\frac{\Psi_2^s\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right)}{\Psi_1^s\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right)} > 0$ for all $x > 0$, the intermediate value theorem implies that $\omega_s \in \mho_s$, since ω_s is by assumption unique. As $M_s(\breve{\omega}_s) \geq \breve{\omega}_s$, strict equality holds only if $\breve{\omega}_s = \omega_s$. If $M_s(\breve{\omega}_s) > \breve{\omega}_s$, then $\breve{\omega}_s < \omega_s$. From Lemma 1, any $\omega_s^{(1)} \in (0, 1/N]$ gives rise to a unique sequence $\left\{\omega_s^{(n)}\right\}_{n=1}^{\infty}$, so $\omega_s^{(n)} < \omega_s^{(n+1)} < \omega_s$, $n = 1, 2, ...,$ for all $\omega_s^{(1)} \in [\breve{\omega}_s, \omega_s)$. Thus, M_s is stable for all $\omega_s^{(1)} \in [\breve{\omega}_s, \omega_s)$. By a similar argument, if $M_s(\hat{\omega}_s) < \hat{\omega}_s$, $\omega_s^{(n)} > \omega_s^{(n+1)} > \omega_s$, $n = 1, 2, ...$, for all $\omega_s^{(1)} \in (\omega_s, \hat{\omega}_s]$, so M_s is stable for all $\omega_s^{(1)} \in (\omega_s, \hat{\omega}_s]$. Finally, for $M_s(\hat{\omega}_s) \leq \hat{\omega}_s$, equality holds only if $\hat{\omega}_s = \omega_s$, completing the proof. The same argument can be used for the deterministic analogue of the stochastic model. $\square$

Lemma 2 is crucial for the comparisons that follow. Specifically, in comparing two distinct models (e.g. the stochastic and the deterministic, or two models with different stochastic structures), we can view the solution to one model as a starting point for calculating the solution to the other model. If this starting point drives the necessary condition of the second model to be positive or negative, then we can identify the direction in which the strategy must be updated. The results in Lemma 2, yield a method for identifying where the fixed point (infinite-horizon equilibrium) of the second model lies.

With Lemma 2 we are able to identify the primitive features of the model that are responsible for the impact of uncertainty on strategies. For games that satisfy the conditions of the lemma, we show that the result of Theorem 2 hinges on features of the utility function, u, alone, and not on f. Of course, f plays an implicit role, since features of both f and u interact for the game to belong to this class of games. Nevertheless, our results identify simple conditions on u that lead to specific effects of uncertainty on strategies.

In addition to analyzing the comparison between the stochastic and the deterministic game, we study the effect of changes in risk under (i) second-order stochastic dominance (SSD) change, and, (ii) first-order stochastic dominance (FSD) change in the distribution of the shock. We find that, within the class of games we study, only the structure of the utility function is needed to explain the impact of changing risk on strategies. In particular, in the case of first-order stochastic dominance, it is the coefficient of relative risk aversion that is responsible for the effect of changes in risk on strategies.

The commons problem and uncertainty: a second look

Theorem 5 *Suppose games* $\langle u, f, \Theta\rangle$ *and* $\langle u, f, \bar{\theta}\rangle$ *satisfy the conditions of Theorem 1 and consider* $\langle u, f, \Theta\rangle^{\infty}_{SULS}(\omega_s)$ *and* $\langle u, f, \bar{\theta}\rangle^{\infty}_{DULS}(\omega_d)$. *Further suppose the one-period games have a symmetric equilibrium strategy,* $\omega_s^{(1)} = \omega_d^{(1)} = 1/N$. *If* $\lim_{c\to 0} u'(c) = \infty$, *then: (i)* $\omega_s < \omega_d$ *if and only if* $\lambda(z)$ *is strictly convex, (ii)* $\omega_s > \omega_d$ *if and only if* $\lambda(z)$ *is strictly concave, and (iii)* $\omega_s = \omega_d$ *if and only if* $\lambda(z)$ *is affine, in which*

$$\lambda(z) = zu'(z) \ . \tag{57}$$

Proof

Fix any $x > 0$. $\Psi^s\left(\omega, \omega^{(n)}\right)$ and $\Psi^d\left(\omega, \omega^{(n)}\right)$ can be expressed as,

$$\begin{aligned}\Psi^s\left(\omega, \omega^{(n)}\right) &= -u'(\omega x) \\ &\quad +\delta\frac{\left[1-(N-1)\,\omega^{(n)}\right]}{\omega^{(n)}}\frac{f'((1-N\omega)\,x)}{f((1-N\omega)\,x)}E\left[\lambda\left(\omega^{(n)}\theta f((1-N\omega)\,x)\right)\right]\end{aligned} \tag{58}$$

and

$$\begin{aligned}\Psi^d\left(\omega, \omega^{(n)}\right) &= -u'(\omega x) \\ &\quad +\delta\frac{\left[1-(N-1)\,\omega^{(n)}\right]}{\omega^{(n)}}\frac{f'((1-N\omega)\,x)}{f((1-N\omega)\,x)}\lambda\left(\omega^{(n)}\bar{\theta}f((1-N\omega)\,x)\right)\end{aligned} \tag{59}$$

From (58), (59), and Jensen's inequality,

(a) $\Psi^s\left(\omega,\omega^{(n)}\right) > \Psi^d\left(\omega,\omega^{(n)}\right)$ for all $\omega,\omega^{(n)} \in (0,1/N)$, if and only if, $\lambda(\cdot)$ is strictly convex,

(b) $\Psi^s\left(\omega,\omega^{(n)}\right) < \Psi^d\left(\omega,\omega^{(n)}\right)$ for all $\omega,\omega^{(n)} \in (0,1/N)$, if and only if, $\lambda(\cdot)$ is strictly concave,

(c) $\Psi^s\left(\omega,\omega^{(n)}\right) = \Psi^d\left(\omega,\omega^{(n)}\right)$ for all $\omega,\omega^{(n)} \in (0,1/N)$, if and only if, $\lambda(\cdot)$ is affine.

So, in case (a), $\Psi^d(\omega_s,\omega_s) < \Psi^s(\omega_s,\omega_s) = 0$, so $M_d(\omega_s) > \omega_s$. In the proof of Theorem 4 it is shown that $M_d(1/N) < 1/N$. Then, by Lemma 2, $\omega_d \in (\omega_s, 1/N)$, which proves statement (i). In case (b), $\Psi^s(\omega_d,\omega_d) < \Psi^d(\omega_d,\omega_d) = 0$, so $M_s(\omega_d) > \omega_d$. Since $M_s(1/N) < 1/N$ (from the proof of Theorem 4), Lemma 2 implies that $\omega_s \in (\omega_d, 1/N)$, which proves statement (ii). Finally, statement (iii) is straightforward. □

Theorem 5 implies that, for the class of games with linear symmetric equilibrium strategies, the model's characteristics behind the result of Theorem 2 are given solely by a condition that pertains to the utility function, u. This does not mean that f does not play any role in the linkup between uncertainty and strategic behavior. Together with u, the function f is crucial for placing a game $\langle u, f, \Theta\rangle$ in the class of games with linear strategies. The implication of Theorem 5 is that, in this class of games, the effect of uncertainty on strategic behavior is determined by a condition on the utility function alone. Moreover, notice that Theorem 5 does not require that $u(\cdot)$ be thrice continuously differentiable.[27]

We provide two additional characterizations based on comparisons of games using the notions of second- and first-order stochastic dominance.

[27]For a discussion of this point see Mirman (1971, p. 182) in his analysis of a two-period problem.

Resource Exploitation and Second-Order Stochastic Dominance (SSD)

We examine two games, $\langle u, f, \Theta \rangle$ and $\left\langle u, f, \tilde{\Theta} \right\rangle$, that have linear-symmetric equilibrium strategies, ω_s and $\tilde{\omega}_s$, with shocks denoted by $\theta \sim \Theta(\theta)$ and $\tilde{\theta} \sim \tilde{\Theta}\left(\tilde{\theta}\right)$, and such that one shock is riskier than the other, in the sense that one distribution dominates the other with respect to Second-Order Stochastic Dominance (SSD). Theorem 6 provides conditions that dictate the effect of increasing risk on strategic decisions.

Theorem 6 *Suppose games $\langle u, f, \Theta \rangle$ and $\left\langle u, f, \tilde{\Theta} \right\rangle$ satisfy the conditions of Theorem 1 and consider $\langle u, f, \Theta \rangle_{SULS}^{\infty}(\omega_s)$ and $\left\langle u, f, \tilde{\Theta} \right\rangle_{SULS}^{\infty}(\tilde{\omega}_s)$. Further suppose the one-period games have a symmetric equilibrium strategy, $\omega_s^{(1)} = \omega_d^{(1)} = 1/N$ and $\lim_{c \to 0} u'(c) = \infty$. If $\tilde{\theta} \preceq_{SSD} \theta$ then: (i) if $\lambda(z)$ is strictly convex, then $\tilde{\omega}_s < \omega_s$, (ii) if $\lambda(z)$ is strictly concave, then $\tilde{\omega}_s > \omega_s$, and, (iii) if $\lambda(z)$ is affine, then $\tilde{\omega}_s = \omega_s$, where $\lambda(z)$ is as defined in (57).*

Proof

Fix any $x > 0$. The necessary conditions of the two problems, $\Psi^s\left(\omega, \omega^{(n)}\right)$ and $\tilde{\Psi}^s\left(\omega, \omega^{(n)}\right)$, can be expressed as,

$$\Psi^s\left(\omega, \omega^{(n)}\right) = -u'(\omega x) + \delta \frac{\left[1 - (N-1)\,\omega^{(n)}\right]}{\omega^{(n)}} \frac{f'((1-N\omega)\,x)}{f((1-N\omega)\,x)} E\left[\lambda\left(\omega^{(n)} \theta f((1-N\omega)\,x)\right)\right], \tag{60}$$

and

$$\tilde{\Psi}^s\left(\omega,\omega^{(n)}\right) = -u'(\omega x)$$
$$+\delta\frac{\left[1-(N-1)\,\omega^{(n)}\right]}{\omega^{(n)}}\frac{f'((1-N\omega)\,x)}{f((1-N\omega)\,x)}E\left[\lambda\left(\omega^{(n)}\tilde{\theta}f((1-N\omega)\,x)\right)\right] \ . \tag{61}$$

Using the expressions (60) and (61), and Definition 2,

(a) if $\lambda(\cdot)$ is strictly convex, then $\tilde{\Psi}^s\left(\omega,\omega^{(n)}\right) > \Psi^s\left(\omega,\omega^{(n)}\right)$ for all $\omega,\omega^{(n)} \in (0,1/N)$,

(b) if $\lambda(\cdot)$ is strictly concave, then $\tilde{\Psi}^s\left(\omega,\omega^{(n)}\right) < \Psi^s\left(\omega,\omega^{(n)}\right)$ for all $\omega,\omega^{(n)} \in (0,1/N)$,

(c) if $\lambda(\cdot)$ is affine, then $\tilde{\Psi}^s\left(\omega,\omega^{(n)}\right) = \Psi^s\left(\omega,\omega^{(n)}\right)$ for all $\omega,\omega^{(n)} \in (0,1/N)$.

So, in case (a), $\Psi^s(\tilde{\omega}_s,\tilde{\omega}_s) < \tilde{\Psi}^s(\tilde{\omega}_s,\tilde{\omega}_s) = 0$, so $M_s(\tilde{\omega}_s) > \tilde{\omega}_s$. In the proof of Theorem 4 it was shown that $M_s(1/N) < 1/N$. By Lemma 2, $\omega_s \in (\tilde{\omega}_s, 1/N)$, which proves statement (i). In case (b), $\tilde{\Psi}^s(\omega_s,\omega_s) < \Psi^s(\omega_s,\omega_s) = 0$, so $\tilde{M}_s(\omega_s) > \omega_s$. Since $\tilde{M}_s(1/N) < 1/N$ (see the proof of Theorem 4), Lemma 2 implies that $\tilde{\omega}_s \in (\omega_s, 1/N)$, which proves statement (ii). Finally, statement (iii) is straightforward. □

Resource Exploitation and First-Order Stochastic Dominance (FSD)

We examine two games, $\langle u,f,\Theta\rangle$ and $\left\langle u,f,\tilde{\Theta}\right\rangle$, that have linear-symmetric equilibrium strategies, ω_s and $\tilde{\omega}_s$, with shocks denoted by $\theta \sim \Theta(\theta)$ and $\tilde{\theta} \sim \tilde{\Theta}\left(\tilde{\theta}\right)$, and such that one distribution dominates the other with respect to First-Order Stochastic Dominance (FSD).

Theorem 7 *Suppose games $\langle u,f,\Theta\rangle$ and $\left\langle u,f,\tilde{\Theta}\right\rangle$ satisfy the conditions of Theorem 1 and consider $\langle u,f,\Theta\rangle^{\infty}_{SULS}(\omega_s)$ and $\left\langle u,f,\tilde{\Theta}\right\rangle^{\infty}_{SULS}(\tilde{\omega}_s)$. Further suppose the one-period games have a symmetric equilibrium strategy, $\omega_s^{(1)} = \omega_d^{(1)} = 1/N$ and $\lim_{c\to 0} u'(c) = \infty$. If $\tilde{\theta} \preceq_{FSD} \theta$ then,*

$$\lambda'(z) \gtreqless 0 \quad \text{for all } z>0 \Rightarrow \tilde{\omega}_s \gtreqless \omega_s \ . \tag{62}$$

Proof

Integration by parts yields,

$$E\left[h\left(\theta\right)\right]-E\left[h\left(\tilde{\theta}\right)\right]=\int_{S_{\theta}}\left[\tilde{\Theta}\left(z\right)-\Theta\left(z\right)\right]h'\left(z\right)dz$$

for all differentiable functions h. So, setting $h\left(z\right)=\lambda\left(\omega^{(n)}zf\left(\left(1-N\omega\right)x\right)\right)$, for any $\omega^{(n)}\in(0,1/N]$ and any $\omega\in(0,1/N)$, the fact that $\tilde{\theta}\preceq_{FOSD}\theta$ implies,

$$\lambda'\left(z\right)\gtreqqless 0\ \forall\ z>0\Rightarrow E\left[\lambda\left(\omega^{(n)}\tilde{\theta}f\left(\left(1-N\omega\right)x\right)\right)\right]\lesseqqgtr E\left[\lambda\left(\omega^{(n)}\theta f\left(\left(1-N\omega\right)x\right)\right)\right]\ . \quad (63)$$

Based on (63), (60), and (61),

(a) $\lambda'\left(z\right)<0$ for all $z>0\Rightarrow\Psi^{s}\left(\tilde{\omega}_{s},\tilde{\omega}_{s}\right)<\tilde{\Psi}^{s}\left(\tilde{\omega}_{s},\tilde{\omega}_{s}\right)=0$,

(b) $\lambda'\left(z\right)>0$ for all $z>0\Rightarrow\tilde{\Psi}^{s}\left(\omega_{s},\omega_{s}\right)<\Psi^{s}\left(\omega_{s},\omega_{s}\right)=0$,

(c) $\lambda'\left(z\right)=0$ for all $z>0\Rightarrow\tilde{\Psi}^{s}\left(\omega,\omega^{(n)}\right)=\Psi^{s}\left(\omega,\omega^{(n)}\right)=0$ for all $\omega,\omega^{(n)}\in(0,1/N)$.

The rest of the proof follows exactly as in Theorem 6. □

Remark 1

$$\lambda'\left(z\right)\gtreqqless 0\ \text{ for all }\ z>0\Leftrightarrow-\frac{u'\left(z\right)}{zu''\left(z\right)}\gtreqqless 1\ \text{ for all }\ z>0,$$

i.e. the sufficient condition (62) is tightly linked with the value of the elasticity of intertemporal substitution.

Application of General Theoretical Results on the Specific Example of the Paper

Clearly our parametric example satisfies the conditions identified in Lemma 1 and Theorem 4, i.e. $\lim_{c\to 0} u'(c) = \infty$, and also the one-period games have a unique symmetric equilibrium strategy, $\omega_s^{(1)} = \omega_d^{(1)} = 1/N$. Furthermore, substituting (16) and (17) into the necessary conditions of the finite-horizon problem given by (50) and (54), we find,

$$M_s(\omega) = \frac{1}{N + (\alpha\delta\zeta)^{\eta} \frac{[1-(N-1)\omega]^{\eta}}{\omega}} \tag{64}$$

$$M_d(\omega) = \frac{1}{N + \left(\alpha\delta\bar{\zeta}\right)^{\eta} \frac{[1-(N-1)\omega]^{\eta}}{\omega}} \tag{65}$$

It is straightforward to verify from (64) and (65) that $M_s(\omega), M_d(\omega) \in (0, 1/N)$ for all $\omega \in (0, 1/N]$. Therefore, we can apply Theorems 3, 5, 6 and 7 to our model.

We can apply Theorem 3 as follows. Let $\rho_d = f((1 - N\omega_d)x)$, and recall the definitions of Λ_s and Λ_d given in Theorem 3. Notice that

$$E[\Lambda_s(\theta\rho_d)] = \frac{\alpha\zeta\omega_s^{1-\frac{1}{\eta}}}{1 - \alpha\delta\zeta(1 - N\omega_s)^{1-\frac{1}{\eta}}} \rho_d^{1-\frac{1}{\eta}} . \tag{66}$$

while

$$\Lambda_d\left(\bar{\theta}\rho_d\right) = \frac{\alpha\bar{\zeta}\omega_d^{1-\frac{1}{\eta}}}{1 - \alpha\delta\bar{\zeta}(1 - N\omega_d)^{1-\frac{1}{\eta}}} \rho_d^{1-\frac{1}{\eta}} . \tag{67}$$

From (19),

$$\frac{\alpha\zeta\omega_s^{1-\frac{1}{\eta}}}{1 - \alpha\delta\zeta(1 - N\omega_s)^{1-\frac{1}{\eta}}} = \left(\frac{1}{\omega_s} - N\right)^{\frac{1}{\eta}},$$

so substituting this expression into (66),

$$E[\Lambda_s(\theta\rho_d)] = \left(\frac{1}{\omega_s} - N\right)^{\frac{1}{\eta}} \rho_d^{1-\frac{1}{\eta}} . \tag{68}$$

After using (20), as was done with (19), (67) becomes,

$$\Lambda_d\left(\bar{\theta}\rho_d\right) = \left(\frac{1}{\omega_d} - N\right)^{\frac{1}{\eta}} \rho_d^{1-\frac{1}{\eta}} . \tag{69}$$

Proposition 2 in the paper identifies the parameters that are behind the comparison given in Theorem 3.

In our example,

$$\lambda\left(c\right)=u'\left(c\right)c=c^{1-\frac{1}{\eta}}\Rightarrow\begin{cases}\lambda'\left(c\right)\gtreqless 0\Leftrightarrow\eta\gtreqless 1\\ \lambda''\left(c\right)\lesseqgtr 0\Leftrightarrow\eta\gtreqless 1\end{cases}. \tag{70}$$

Since u is thrice differentiable, the concavity of λ can be examined through the sign of its second derivative. Notice that, (70) illustrates the connection between Theorem 3 and the group of Theorems 5, 6 and 7 together with the results stated by Proposition 2. Most importantly, it shows how the properties of u can affect the role of uncertainty on strategies.

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