Dynamic many-valued logics for searching games with errors

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Introduction

Providing concrete interpretations of many-valued logics has always been an intriguing problem. In [6], Mundici develops a model of the Rényi-Ulam searching games with lies in terms of Łukasiewicz logic and MV-algebras. In this game, a liar picks out a number in a given search space $M$. A detective has to guess this number by asking Yes/No questions to the liar who is allowed to lie a maximum given number of times.

In his model of the game, Mundici interprets the states of knowledge of the detective at a given step of the game as an element of an MV-algebra. Even though this model provides a way to interpret the effect of the liar’s answers on the states of knowledge of the game, its language (the language of MV-algebras) is not rich enough to state specifications about a whole round of the game.

The starting point of this talk is the will to add a ‘dynamic’ layer to this ‘static’ interpretation of the game. We actually develop finitely-valued generalizations of Propositional Dynamic Logic, which is a multi-modal logic designed to reason about programs (see [2, 5]). Informally, these new logics are a mixture of many-valued modal logics (as introduced in [1, 3, 4]) and algebras of regular programs.

$n + 1$-valued Kripke models

We fix $n \geq 1$ for the remainder of the paper and we denote by $\mathbb{L}_n$ the sub-MV-algebra $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ of $[0, 1]$.

We denote by $\Pi$ a set of programs and by $\text{Form}$ a set of formulas defined from a countable set $\text{Prop}$ of propositional variables $p, q \ldots$ and a countable set $\Pi_0$ of atomic programs $a, b, \ldots$ by the following Backus-Naur forms (where $\phi$ are formulas and $\alpha$ are programs):

$$
\phi ::= p \mid 0 \mid \neg \phi \mid \phi \to \phi \mid [\alpha]\phi \\
\alpha ::= a \mid \phi? \mid \alpha ; \alpha \mid \alpha \cup \alpha \mid \alpha^* .
$$
**Definition 1.** An \( n + 1 \)-valued Kripke model \( M = \langle W, R, Val \rangle \) is given by a non empty set \( W \), a map \( R : \Pi_0 \to 2^{W \times W} \) that assigns a binary relation \( R_a \) to any \( a \) of \( \Pi_0 \) and a map \( \text{Val} : W \times \text{Prop} \to \mathbb{L}_n \) that assigns a truth value to any propositional variable \( p \) of \( \text{Prop} \) in any world \( w \) of \( W \).

The maps \( R \) and \( \text{Val} \) are extended by mutual induction to formulas and programs by the following rules (where \( \lnot^{[0,1]} \) and \( \to^{[0,1]} \) denote \( \text{ŁUKASIEWICZ's} \) interpretation of \( \lnot \) and \( \to \) on \( [0,1] \)):

1. \( R_{\alpha;\beta} = R_{\alpha} \circ R_{\beta} \);
2. \( R_{\alpha;\beta} = R_{\alpha} \cap R_{\beta} \);
3. \( R_{\psi?} = \{(u,u) \mid \text{Val}(u,\psi) = 1\} \);
4. \( R_{\alpha}^* = \bigcup_{n \in \omega} (R_{\alpha})^n \);
5. \( \text{Val}(w,\phi \to \psi) = \text{Val}(w,\phi)^{[0,1]} \text{Val}(w,\psi) \);
6. \( \text{Val}(w,\lnot \psi) = \lnot^{[0,1]} \text{Val}(w,\psi) \);
7. \( \text{Val}(w,[\alpha]\psi) = \bigwedge \{\text{Val}(v,\psi) \mid (w,v) \in R_{\alpha}\} \)

If \( w \) is a world of a Kripke model \( M \) and if \( \text{Val}(w,\phi) = 1 \), we write \( M,w \models \phi \) and say that \( \phi \) is true in \( w \). If \( \phi \) is a formula that is true in each world of a model \( M \) then \( \phi \) is true in \( M \). A formula that is true in every Kripke model is called a tautology.

Hence, we intend to interpret the operator ‘;’ as the concatenation program operator, the operator ‘\( \cup \)’ as the alternative program operator and the operator ‘\( * \)’ as the Kleene program operator.

**n + 1-valued propositional dynamic logics**

The purpose of the talk is to characterize the theory of the \( n + 1 \)-valued Kripke models (Theorem 5).

**Definition 2.** An \( n + 1 \)-valued propositional dynamic logic (or simply a logic) is a subset \( L \) of \( \text{Form} \) that is closed under the rules of modus ponens, uniform substitution and necessitation (generalization) and that contains the following axioms:

1. tautologies of the \( n + 1 \)-valued \( \text{ŁUKASIEWICZ} \) logic;
2. for any program \( \alpha \), axioms defining modality \([\alpha]\):
   (a) \( [\alpha](p \to q) \to ([\alpha]p \to [\alpha]q) \),
   (b) \( [\alpha](p \oplus p) \leftrightarrow [\alpha]p \oplus [\alpha]p \),
   (c) \( [\alpha](p \odot p) \leftrightarrow [\alpha]p \odot [\alpha]p \),
3. the axioms that define the program operators: for any programs \( \alpha \) and \( \beta \) of \( \Pi \):

(a) $[\alpha \cup \beta]p \leftrightarrow [\alpha]p \land [\beta]p$,
(b) $[\alpha; \beta]p \leftrightarrow [\alpha]([\beta]p)$,
(c) $[q?]p \leftrightarrow (\neg q^n \lor p)$,
(d) $[\alpha^*]p \leftrightarrow (p \land [\alpha][\alpha^*]p)$,
(e) $[\alpha^*]p \rightarrow [\alpha^*][\alpha^*]p$,

4. the induction axiom $(p \land [\alpha^*](p \rightarrow [\alpha][p]^n)) \rightarrow [\alpha^*]p$ for any program $\alpha$.

We denote by PDL$_n$ the smallest $n + 1$-valued propositional dynamic logic.

As usual, a formula $\phi$ that belongs to a logic $L$ is called a theorem of $L$.

**Completeness result**

The classical construction of the canonical model can be adapted for PDL$_n$. We denote by $F_n$ the Lindenbaum - Tarski algebra of PDL$_n$. The reduct of $F_n$ to the language of MV-algebras is an MV-algebra. We denote by $\mathcal{MV}(F_n, L_n)$ the set of MV-homomorphisms from the MV-reduct of $F_n$ to $L_n$.

**Definition 3.** The canonical model of PDL$_n$ is defined as the model $M^c = (W^c, R^c, Val^c)$ where

1. $W^c = \mathcal{MV}(F_n, L_n)$;
2. if $\alpha \in \Pi$, the relation $R^c_\alpha$ is defined by
   
   $$R^c_\alpha = \{(u, v) \mid \forall \phi \in F_n \ (u([\alpha]\phi) = 1 \Rightarrow v(\phi) = 1)\};$$
3. the map $Val^c$ is defined by
   
   $$Val^c : W^c \times \text{Form} : (u, \phi) \mapsto u(\phi).$$

Even though the valuation in $M^c$ is defined for any formula, it turns out that it is compatible with the inductive definition of a valuation in a Kripke model.

**Proposition 4.**

1. If $\phi \in \text{Form}$, if $\alpha \in \Pi$ and if $u$ is a world of $W^c$ then $Val^c(u, [\alpha]\phi) = \bigwedge \{Val^c(v, \phi) \mid v \in R^c_\alpha u\}$.
2. For any $\alpha \in \Pi$, the relation $R^c_\alpha$ is a reflexive and transitive extension of $R_\alpha$.

According to the second item of the previous proposition, the canonical model may not be Kripke model. Nevertheless, it is possible to use a filtration lemma in order to use the canonical model to obtain a completeness result for PDL$_n$.

**Theorem 5.** The logic PDL$_n$ is complete with respect to the $n + 1$-valued Kripke models, i.e., a formula $\phi$ is a theorem of PDL$_n$ if and only if $\phi$ is a tautology.
References


