# A Complete Description of Comparison Meaningful Functions

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# Aggregation of measurement scales

We consider

- $S = \{a, b, c, \ldots\}$ : a set of *alternatives*
- $N = \{1, \ldots, n\}$ : a set of *attributes*

For any $a \in S$ and any $i \in N$, let $f_i(a) \in \mathbb{R}$ be the *score* of $a \in S$ according the $i$th attribute.

$f_i : S \to \mathbb{R}$ is a *scale of measurement*

We want to obtain an *overall evaluation* of $a \in S$ by means of an aggregation function $F_{f_1,\ldots,f_n} : S \to \mathbb{R}$, which depends on $f_1, \ldots, f_n$.

We assume that

$$F_{f_1,\ldots,f_n}(a) = F[f_1(a), \ldots, f_n(a)] \qquad (a \in S)$$

Thus, $F$ is regarded as an aggregation function from $\mathbb{R}^n$ to $\mathbb{R}$ :

$$x_{n+1} = F(x_1, \ldots, x_n)$$

where $x_1, \ldots, x_n$ are the independent variables and $x_{n+1}$ is the dependent variable.

The general form of $F$ is restricted if we know the *scale type* of the variables $x_1, \ldots, x_n$ and $x_{n+1}$ (Luce 1959).

A scale type is defined by the class of *admissible transformations*, transformations which change the scale into an alternative acceptable scale.

$x_i$ defines an *ordinal scale* if the class of admissible transformations consists of the increasing bijections (automorphisms) of $\mathbb{R}$ onto $\mathbb{R}$.

**Principle of theory construction (Luce 1959)**

Admissible transformations of the independent variables should lead to an admissible transformation of the dependent variable.

Suppose that

$$x_{n+1} = F(x_1, \ldots, x_n)$$

where $x_{n+1}$ is an ordinal scale and $x_1, \ldots, x_n$ are independent ordinal scales.

Let $A(\mathbb{R})$ be the automorphism group of $\mathbb{R}$.

For any $\phi_1, \ldots, \phi_n \in A(\mathbb{R})$, there is $\Phi_{\phi_1,\ldots,\phi_n} \in A(\mathbb{R})$ such that

$$F[\phi_1(x_1), \ldots, \phi_n(x_n)] = \Phi_{\phi_1,\ldots,\phi_n}[F(x_1, \ldots, x_n)]$$

Assume $x_1, \ldots, x_n$ define the *same* ordinal scale.
Then the functional equation simplifies into

$$F[\phi(x_1), \ldots, \phi(x_n)] = \Phi_\phi[F(x_1, \ldots, x_n)]$$

Equivalently, $F$ fulfills the condition (Orlov 1981)

$$F(x_1, \ldots, x_n) \leqslant F(x_1', \ldots, x_n') \\ \Updownarrow \\ F[\phi(x_1), \ldots, \phi(x_n)] \leqslant F[\phi(x_1'), \ldots, \phi(x_n')]$$

$F$ is said to be *comparison meaningful* (Ovchinnikov 1996)

Assume $x_1, \ldots, x_n$ are *independent* ordinal scales.
Recall that the functional equation is

$$F[\phi_1(x_1), \ldots, \phi_n(x_n)] = \Phi_{\phi_1,\ldots,\phi_n}[F(x_1, \ldots, x_n)]$$

Equivalently, $F$ fulfills the condition

$$\begin{gathered} F(x_1, \ldots, x_n) \leqslant F(x_1', \ldots, x_n') \\ \Updownarrow \\ F[\phi_1(x_1), \ldots, \phi_n(x_n)] \leqslant F[\phi_1(x_1'), \ldots, \phi_n(x_n')] \end{gathered}$$

We say that $F$ is *strongly comparison meaningful*

## Purpose of the presentation

To provide a complete description of comparison meaningful functions

To provide a complete description of strongly comparison meaningful functions

# The continuous case

**First result (Osborne 1970, Kim 1990)**

$F:\mathbb{R}^n\to\mathbb{R}$ is continuous and strongly comparison meaningful

$$\Leftrightarrow \begin{cases} \exists\, k\in\{1,\ldots,n\} \\ \exists\, g:\mathbb{R}\to\mathbb{R} \quad \text{- continuous} \\ \qquad\qquad\qquad \text{- strictly monotonic or constant} \\ \text{such that} \\ \qquad F(x_1,\ldots,x_n)=g(x_k) \end{cases}$$

$+$ idempotent (agreeing), i.e., $F(x,\ldots,x)=x$

$$\Leftrightarrow \begin{cases} \exists\, k\in\{1,\ldots,n\} \text{ such that} \\ \qquad F(x_1,\ldots,x_n)=x_k \end{cases}$$

# The nondecreasing case

**Second result (Marichal & Mesiar & Rückschlossová 2004)**
$F : \mathbb{R}^n \to \mathbb{R}$ is nondecreasing and strongly comparison meaningful

$$\Leftrightarrow \begin{cases} \exists\, k \in \{1,\ldots,n\} \\ \exists\, g : \mathbb{R} \to \mathbb{R} \ \ \text{strictly increasing or constant} \\ \text{such that} \\ \qquad F(x_1,\ldots,x_n) = g(x_k) \end{cases}$$

$+$ idempotent

$$\Leftrightarrow \begin{cases} \exists\, k \in \{1,\ldots,n\} \text{ such that} \\ \qquad F(x_1,\ldots,x_n) = x_k \end{cases}$$

# The general case

**Third result (Marichal & Mesiar & Rückschlossová 2004)**

$F : \mathbb{R}^n \to \mathbb{R}$ is strongly comparison meaningful

$$\Leftrightarrow \left\{ \begin{array}{l} \exists\, k \in \{1,\ldots,n\} \\ \exists\, g : \mathbb{R} \to \mathbb{R} \;\text{ strictly monotonic or constant} \\ \text{such that} \\ \qquad F(x_1,\ldots,x_n) = g(x_k) \end{array} \right.$$

$+$ idempotent

$$\Leftrightarrow \left\{ \begin{array}{l} \exists\, k \in \{1,\ldots,n\} \text{ such that} \\ \qquad F(x_1,\ldots,x_n) = x_k \end{array} \right.$$

# Comparison meaningful functions

**First result (Orlov 1981)**

$F : \mathbb{R}^n \to \mathbb{R}$ is
- symmetric
- continuous
- internal, i.e., $\min_i x_i \leqslant F(x_1, \ldots, x_n) \leqslant \max_i x_i$
- comparison meaningful

$$\Leftrightarrow \begin{cases} \exists\, k \in \{1, \ldots, n\} \text{ such that} \\ F(x_1, \ldots, x_n) = x_{(k)} \end{cases}$$

where $x_{(1)}, \ldots, x_{(n)}$ denote the *order statistics* resulting from reordering $x_1, \ldots, x_n$ in the nondecreasing order.

**Next step** : suppress symmetry and relax internality into idempotency

# Lattice polynomials

**Definition (Birkhoff 1967)**

An $n$-variable *lattice polynomial* is any expression involving $n$ variables $x_1, \ldots, x_n$ linked by the lattice operations

$$\wedge = \min \quad \text{and} \quad \vee = \max$$

in an arbitrary combination of parentheses.

For example,

$$L(x_1, x_2, x_3) = (x_1 \vee x_3) \wedge x_2$$

is a 3-variable lattice polynomial.

# Lattice polynomials

**Proposition (Ovchinnikov 1998, Marichal 2002)**

A lattice polynomial on $\mathbb{R}^n$ is symmetric iff it is an order statistic.

We have

$$x_{(k)} = \bigvee_{\substack{T\subseteq\{1,\ldots,n\}\\ |T|=n-k+1}} \bigwedge_{i\in T} x_i = \bigwedge_{\substack{T\subseteq\{1,\ldots,n\}\\ |T|=k}} \bigvee_{i\in T} x_i$$

# The nonsymmetric case

**Second result (Yanovskaya 1989)**

$F : \mathbb{R}^n \to \mathbb{R}$ is
- continuous
- idempotent
- comparison meaningful

$\Leftrightarrow \quad \exists$ a lattice polynomial $L : \mathbb{R}^n \to \mathbb{R}$ such that $F = L$.

$+$ symmetric

$\Leftrightarrow \quad \exists\ k \in \{1, \ldots, n\}$ such that $F = \mathrm{OS}_k$ ($k$th order statistic).

**Next step** : suppress idempotency

# The nonidempotent case

**Third result (Marichal 2002)**

$F : \mathbb{R}^n \to \mathbb{R}$ is - continuous
- comparison meaningful

$$\Leftrightarrow \begin{cases} \exists\, L : \mathbb{R}^n \to \mathbb{R} \text{ lattice polynomial} \\ \exists\, g : \mathbb{R} \to \mathbb{R} \quad \text{- continuous} \\ \qquad\qquad\qquad\quad \text{- strictly monotonic or constant} \\ \text{such that} \\ \qquad F = g \circ L \end{cases}$$

$+$ symmetric

$$F = g \circ \mathrm{OS}_k$$

# Towards the noncontinuous case

**Fourth result (Marichal 2002)**

$F : \mathbb{R}^n \to \mathbb{R}$ is
- nondecreasing
- idempotent
- comparison meaningful

$\Leftrightarrow$ $\exists$ a lattice polynomial $L : \mathbb{R}^n \to \mathbb{R}$ such that $F = L$.

**Note :** These functions are continuous !

$+$ symmetric

$$F = \mathrm{OS}_k$$

**Next step** : suppress idempotency

# The nondecreasing case

**Fifth result (Marichal & Mesiar & Rückschlossová 2004)**

$F : \mathbb{R}^n \to \mathbb{R}$ is - nondecreasing
- comparison meaningful

$$\Leftrightarrow \begin{cases} \exists\, L : \mathbb{R}^n \to \mathbb{R} \text{ lattice polynomial} \\ \exists\, g : \mathbb{R} \to \mathbb{R} \text{ strictly increasing or constant} \\ \text{such that} \\ \qquad F = g \circ L \end{cases}$$

These functions are continuous up to possible discontinuities of function $g$

**Final step** : suppress nondecreasing monotonicity (a hard task !)

# The general case

**... is much more complicated to describe**

- We loose the concept of lattice polynomial
- The description of $F$ is done through a partition of the domain $\mathbb{R}^n$ into particular subsets, called *invariant subsets*

# Invariant sets

**Definition (Bartłomiejczyk & Drewniak 2004)**

- A nonempty set $I \subseteq \mathbb{R}^n$ is *invariant* if

$$(x_1, \ldots, x_n) \in I \quad \Rightarrow \quad (\phi(x_1), \ldots, \phi(x_n)) \in I \qquad \forall\, \phi \in A(\mathbb{R})$$

- An invariant set $I$ is *minimal* if it has no proper invariant subset

Let $\mathcal{I}(\mathbb{R}^n)$ denote the family of minimal invariant subsets of $\mathbb{R}^n$

The family $\mathcal{I}(\mathbb{R}^n)$ partitions $\mathbb{R}^n$ into equivalence classes :

$$x \sim y \quad \Leftrightarrow \quad \exists\, \phi \in A(\mathbb{R}) \ : \ y_i = \phi(x_i) \ \ \forall\, i$$

# Description of the family $\mathcal{I}(\mathbb{R}^n)$

**Proposition (Bartłomiejczyk & Drewniak 2004)**

$$I \in \mathcal{I}(\mathbb{R}^n) \Leftrightarrow \begin{cases} \exists \text{ a permutation } \pi \text{ on } \{1,\ldots,n\} \\ \exists \text{ a sequence } \{\triangleleft_i\}_{i=0}^n \text{ of symbols } \triangleleft_i \in \{<,=\} \\ \text{such that} \\ \qquad I = \{x \in \mathbb{R}^n \mid x_{\pi(1)} \triangleleft_1 \cdots \triangleleft_{n-1} x_{\pi(n)}\} \end{cases}$$

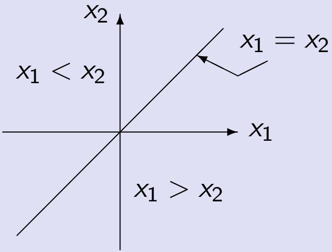
**Example :** $\mathbb{R}^2$

Minimal invariant sets :

$I_1 = \{(x_1,x_2) \mid x_1 = x_2\}$

$I_2 = \{(x_1,x_2) \mid x_1 < x_2\}$

$I_3 = \{(x_1,x_2) \mid x_1 > x_2\}$



# The general case

**Sixth result (Marichal & Mesiar & Rückschlossová 2004)**

$F : \mathbb{R}^n \to \mathbb{R}$ is comparison meaningful

$$\Leftrightarrow \quad \forall\, I \in \mathcal{I}(\mathbb{R}^n), \quad \begin{cases} \exists\, k_I \in \{1,\ldots,n\} \\ \exists\, g_I : \mathbb{R} \to \mathbb{R} \ \text{ strictly monotonic or constant} \\ \text{such that} \\ \qquad F|_I(x_1,\ldots,x_n) = g_I(x_{k_I}) \\ \\ \quad \text{where } \forall\, I, I' \in \mathcal{I}(\mathbb{R}^n), \\ \qquad \bullet \ \text{either } g_I = g_{I'} \\ \qquad \bullet \ \text{or } ran(g_I) = ran(g_{I'}) \text{ is a singleton} \\ \qquad \bullet \ \text{or } ran(g_I) < ran(g_{I'}) \\ \qquad \bullet \ \text{or } ran(g_I) > ran(g_{I'}) \end{cases}$$

# Invariant functions

Now, assume that

$$x_{n+1} = F(x_1, \ldots, x_n)$$

where $x_1, \ldots, x_n$ and $x_{n+1}$ define the *same* ordinal scale.

Then the functional equation simplifies into

$$F[\phi(x_1), \ldots, \phi(x_n)] = \phi[F(x_1, \ldots, x_n)]$$

(introduced in Marichal & Roubens 1993)

$F$ is said to be *invariant* (Bartłomiejczyk & Drewniak 2004)

# The symmetric case

**First result (Marichal & Roubens 1993)**

$F:\mathbb{R}^n\to\mathbb{R}$ is
- symmetric
- continuous
- nondecreasing
- invariant

$\Leftrightarrow \quad \exists\ k\in\{1,\ldots,n\}$ such that $F=\mathrm{OS}_k$

**Next step** : suppress symmetry and nondecreasing monotonicity

# The nonsymmetric case

**Second result (Ovchinnikov 1998)**

$F:\mathbb{R}^n\to\mathbb{R}$ is
- continuous
- invariant

$\Leftrightarrow$ $\exists$ a lattice polynomial $L:\mathbb{R}^n\to\mathbb{R}$ such that $F=L$

**Note :** These functions are nondecreasing !

$+$ symmetric

$$F=\mathrm{OS}_k$$

**Next step** : suppress continuity

# The nondecreasing case

**Third result (Marichal 2002)**

$F : \mathbb{R}^n \to \mathbb{R}$ is - nondecreasing
- invariant

$\Leftrightarrow$ $\exists$ a lattice polynomial $L : \mathbb{R}^n \to \mathbb{R}$ such that $F = L$

**Note :** These functions are continuous !

+ symmetric

$$F = \mathrm{OS}_k$$

**Final step** : suppress nondecreasing monotonicity

# The general case

The general case was first described by Ovchinnikov (1998)

A simpler description in terms of invariant sets is due to Bartłomiejczyk & Drewniak (2004)

**Fourth result (Ovchinnikov 1998)**

$F:\mathbb{R}^n\to\mathbb{R}$ is invariant

$$\Leftrightarrow\ \forall\, I\in\mathcal{I}(\mathbb{R}^n),\ \begin{cases}\exists\, k_I\in\{1,\ldots,n\}\\ \text{such that}\\ \qquad F|_I(x_1,\ldots,x_n)=x_{k_I}\end{cases}$$

# Conclusion

We have described all the possible merging functions

$$F : \mathbb{R}^n \to \mathbb{R},$$

which map $n$ ordinal scales into an ordinal scale.

These results hold true when $F$ is defined on $E^n$, where $E$ is any open real interval.

The cases where $E$ is a non-open real interval all have been described and can be found in

J.-L. Marichal, R. Mesiar, and T. Rückschlossová,
A Complete Description of Comparison Meaningful Functions,
*Aequationes Mathematicae*, in press.

Thank you for your attention