Pivotal decompositions of aggregation functions

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1 Preliminaries

A remarkable (though immediate) property of Boolean functions is the so-called Shannon decomposition [9], also called pivotal decomposition [1]. This property states that, for every $n$-ary Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ and every $k \in [n] = \{1, \ldots, n\}$, the following decomposition formula holds

$$f(x) = x_k f(x_0^k) + x_k f(x_1^k), \quad x \in \{0, 1\}^n,$$  \hspace{1cm} (1)

where $x_k = 1 - x_k$ and $x_i^0$ (resp. $x_i^1$) is the $n$-tuple whose $i$-th coordinate is 0 (resp. 1), if $i = k$, and $x_i$, otherwise. Here the ‘+’ sign represents the classical addition for real numbers.

As it is well known, repeated applications of (1) show that any $n$-ary Boolean function can always be expressed as the multilinear polynomial function

$$f(x) = \sum_{S \subseteq [n]} f(1^S) \prod_{i \in S} x_i \prod_{i \in [n] \setminus S} x_i, \quad x \in \{0, 1\}^n,$$  \hspace{1cm} (2)

where $1^S$ is the characteristic vector of $S$ in $\{0, 1\}^n$, that is, the $n$-tuple whose $i$-th coordinate is 1, if $i \in S$, and 0, otherwise.

One can easily show that, if $f$ is nondecreasing (in each variable), decomposition formula (1) reduces to

$$f(x) = \text{med}(f(x_0^k), x_k, f(x_1^k)), \quad x \in \{0, 1\}^n,$$  \hspace{1cm} (3)

or, equivalently,

$$f(x) = x_k (f(x_0^k) \land f(x_1^k)) + x_k (f(x_0^k) \lor f(x_1^k)), \quad x \in \{0, 1\}^n,$$  \hspace{1cm} (4)

where $\land$ (resp. $\lor$) is the minimum (resp. maximum) operation and med is the ternary median operation.

Actually, any of the decomposition formulas (3)–(4) exactly expresses the fact that $f$ should be nondecreasing and hence characterizes the subclass of nondecreasing $n$-ary Boolean functions.

Decomposition property (1) also holds for functions $f : \{0, 1\}^n \rightarrow \mathbb{R}$, called $n$-ary pseudo-Boolean functions. As a consequence, these functions also have the representation given in (2). Moreover, formula (4) clearly characterizes the subclass of nondecreasing $n$-ary pseudo-Boolean functions.
The multilinear extension of an n-ary pseudo-Boolean function \( f : \{0,1\}^n \to \mathbb{R} \) is the function \( \hat{f} : [0,1]^n \to \mathbb{R} \) defined by (see Owen [7, 8])

\[
\hat{f}(x) = \sum_{S \subseteq [n]} f(1_S) \prod_{i \in S} x_i \prod_{i \in [n] \setminus S} (1 - x_i), \quad x \in [0,1]^n.
\]

Thus defined, one can easily see that the class of multilinear extensions and that of nondecreasing multilinear extensions can be characterized as follows.

**Proposition 1.** A function \( f : [0,1]^n \to \mathbb{R} \) is a multilinear extension if and only if it satisfies

\[
f(x) = (1 - x_k) f(x^0_k) + x_k f(x^1_k), \quad x \in [0,1]^n, k \in [n].
\]

**Proposition 2.** A function \( f : [0,1]^n \to \mathbb{R} \) is a nondecreasing multilinear extension if and only if it satisfies

\[
f(x) = x_k (f(x^0_k) \land f(x^1_k)) + x_k (f(x^0_k) \lor f(x^1_k)), \quad x \in [0,1]^n, k \in [n].
\]

The decomposition formulas considered in this introduction share an interesting common feature, namely the fact that any variable, here denoted \( x_k \) and called *pivot*, can be isolated from the others in the evaluation of functions. This feature may be useful when for instance the values \( f(x^0_k) \) and \( f(x^1_k) \) are much easier to compute than that of \( f(x) \). In addition to this, such (pivotal) decompositions may facilitate inductive proofs and may lead to canonical forms such as (2).

In this note we define a more general concept of pivotal decomposition for various functions \( f : [0,1]^n \to \mathbb{R} \), including certain aggregation functions. We also introduce pivotal characterizations of classes of such functions.

## 2 Pivotal decompositions of functions

The examples presented in the previous section motivate the following definition.

**Definition 1.** We say that a function \( f : [0,1]^n \to \mathbb{R} \) is pivotally decomposable if there exists a subset \( D \subseteq \mathbb{R}^3 \) and a function \( \Phi : D \to \mathbb{R} \), called pivotal function, such that

\[
D \supseteq \{(f(x^0_k), z, f(x^1_k)) : z \in [0,1], x \in [0,1]^n\}, \quad k \in [n]
\]

and

\[
f(x) = \Phi(f(x^0_k), x_k, f(x^1_k)), \quad x \in [0,1]^n, k \in [n].
\]

In this case, we say that \( f \) is \( \Phi \)-decomposable.

**Example 1** (Lattice polynomial functions). Recall that a lattice polynomial function is simply a composition of projections, constant functions, and the fundamental lattice operations \( \land \) and \( \lor \); see, e.g., [3, 4]. An \( n \)-ary function \( f : [0,1]^n \to [0,1] \) is a lattice polynomial function if and only if it can be written in the (disjunctive normal) form

\[
f(x) = \bigvee_{S \subseteq [n]} f(1_S) \land \bigwedge_{i \in S} x_i, \quad x \in [0,1]^n.
\]
The so-called discrete Sugeno integrals are exactly those lattice polynomial functions which are idempotent (i.e., \( f(x, \ldots, x) = x \) for all \( x \in [0, 1] \)).

Every lattice polynomial function is \( \Phi \)-decomposable with \( \Phi: [0,1]^3 \to \mathbb{R} \) defined by \( \Phi(r,z,s) = \text{med}(r,z,s) \); see, e.g., [6].

**Example 2 (Lovász extensions).** Recall that the Lovász extension of a pseudo-Boolean function \( f: \{0,1\}^n \to \mathbb{R} \) is the function

\[
L_f(x) = \sum_{S \subseteq [n]} a(S) \bigwedge_{i \in S} x_i,
\]

where the set function \( a: \mathcal{P}[n] \to \mathbb{R} \) is defined by \( a(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} f(T) \); see, e.g., [5]. The so-called discrete Choquet integrals are exactly those Lovász extensions which are nondecreasing and idempotent.

There are ternary Lovász extensions \( L_f: [0,1]^3 \to \mathbb{R} \) that are not pivotally decomposable, e.g., \( L_f(x_1, x_2, x_3) = x_1 \land x_2 + x_2 \land x_3 \).

**Example 3 (T-norms).** A t-norm is a binary function \( T: [0,1]^2 \to [0,1] \) that is symmetric, nondecreasing, associative, and such that \( T(1,x) = x \). Every t-norm \( T: [0,1]^2 \to [0,1] \) is \( \Phi \)-decomposable with \( \Phi: [0,1]^3 \to \mathbb{R} \) defined by \( \Phi(r,z,s) = T(z,s) \).

**Example 4 (Conjugate functions).** Given a function \( f: [0,1]^n \to [0,1] \) and a strictly increasing bijection \( \phi: [0,1] \to [0,1] \), the \( \phi \)-conjugate of \( f \) is the function \( f_\phi = \phi^{-1} \circ f \circ (\phi, \ldots, \phi) \). One can easily show that \( f \) is \( \Phi \)-decomposable for some pivotal function \( \Phi \) if and only if \( f_\phi \) is \( \Phi_\phi \)-decomposable, where \( \Phi_\phi = \phi^{-1} \circ \Phi \circ (\phi, \phi, \phi) \). Combining this for instance with Proposition 2 shows that every quasi-linear mean function (i.e., \( \phi \)-conjugate of a weighted arithmetic mean) is pivotally decomposable.

For every \( k \in [n] \), and every \( a \in [0,1]^n \), we define the unary section \( f_k^a: [0,1] \to \mathbb{R} \) of \( f \) by setting \( f_k^a(x) = f(a^k) \). The \( k \)th argument of \( f \) is said to be **essential** if \( f_k^a \) is constant for every \( a \in [0,1]^n \). Otherwise, it is said to be **essential**. We say that a unary section \( f_k^a \) of \( f \) is **essential** if the \( k \)th argument of \( f \) is essential.

For every function \( f: X^n \to Y \) and every map \( \sigma: [n] \to [m] \), we define the function \( f_\sigma: X^m \to Y \) by \( f_\sigma(x) = f(\sigma(x)) \), where \( \sigma(x) \) denotes the \( n \)-tuple \((a_{\sigma(1)}, \ldots, a_{\sigma(n)})\).

Define on the set \( U = \bigcup_{n \geq 1} \mathbb{R}^{[0,1]^n} \) the equivalence relation \( \equiv \) as follows: For functions \( f: [0,1]^n \to \mathbb{R} \) and \( g: [0,1]^m \to \mathbb{R} \), we write \( f \equiv g \) if there exist maps \( \sigma: [n] \to [m] \) and \( \mu: [m] \to [n] \) such that \( f = g_\sigma \) and \( g = f_\mu \). Equivalently, \( f \equiv g \) means that \( f \) can be obtained from \( g \) by permuting arguments or by adding or deleting inessential arguments.

**Definition 2.** Let \( \Phi: D \to \mathbb{R} \) be a pivotal function. We denote by \( C_\Phi \) the class of all the functions \( f: [0,1]^n \to \mathbb{R} \) (where \( n \geq 0 \)) that are \( \equiv \)-equivalent to a \( \Phi \)-decomposable function with no essential argument or no inessential argument. We say that a class \( C \subseteq U \) is pivotally characterizable if there exists a pivotal function \( \Phi \) such that \( C = C_\Phi \).

In that case, we say that \( C \) is \( \Phi \)-characterized.

**Proposition 3.** Let \( \Phi \) be a pivotal function.

(i) A nonconstant function \( f: [0,1]^n \to \mathbb{R} \) is in \( C_\Phi \) if and only if so are its essential unary sections.
(ii) A constant function \( f : [0, 1]^n \to \{ c \} \) is in \( C_\Phi \) if and only if \( \Phi(c, z, c) = c \) for every \( z \in [0, 1] \).

**Example 5 (Lattice polynomial functions).** The class of lattice polynomial functions is \( \Phi \)-characterized for the pivotal function \( \Phi : [0, 1]^3 \to \mathbb{R} \) defined by \( \Phi(r, z, s) = \text{med}(r, z, s) \).

### 3 Classes characterized by their unary members

Proposition 3 shows that a class \( C_\Phi \) is characterized by the essential unary sections of its members. This observation motivates the following definition, which is inspired from [2].

**Definition 3.** A class \( C \subseteq U \) is characterized by its unary members if it satisfies the following conditions:

(i) A nonconstant function \( f \) is in \( C \) if and only if so are its essential unary sections.

(ii) If \( f \) is a constant function in \( C \) and \( g \equiv f \), then \( g \) is in \( C \).

We denote by \( \text{CU} \) the family of classes characterized by their unary members.

**Theorem 1.** Let \( \Phi \) be a pivotal function. A nonempty subclass of \( C_\Phi \) is characterized by its unary members if and only if it is pivotally characterizable.

**Theorem 2.** The family \( \text{CU} \) can be endowed with a complete and atomic Boolean algebra structure.

### References