On the cardinality index of fuzzy measures and the signatures of coherent systems

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Boolean and pseudo-Boolean functions

Boolean functions:

\[ f: \{0, 1\}^n \rightarrow \{0, 1\} \]

Pseudo-Boolean functions:

\[ f: \{0, 1\}^n \rightarrow \mathbb{R} \]

Set functions:

\[ [n] = \{1, \ldots, n\} \]

\[ f: 2^n \rightarrow \{0, 1\} \]
\[ f: 2^n \rightarrow \mathbb{R} \]
Set functions

A *discrete fuzzy measure* on the finite set $X = \{1, \ldots, n\}$ is a nondecreasing set function $\mu : 2^X \to [0, 1]$ satisfying the conditions $\mu(\emptyset) = 0$ and $\mu(X) = 1$

**Interpretation:**
For any subset $S \subseteq X$, the number $\mu(S)$ can be interpreted as the certitude that we have that a variable will take on its value in the set $S \subseteq X$
A *cooperative game* on a finite set of players $N = \{1, \ldots, n\}$ is a set function $v: 2^N \rightarrow \mathbb{R}$ which assigns to each coalition $S \subseteq N$ of players a real number $v(S)$ which represents the *worth* of $S$.

The game is said to be *simple* if $v$ takes on its values in $\{0, 1\}$.

The *structure of a semicoherent system* made up of $n$ components is a set function $\phi: 2^\mathbb{n} \rightarrow \{0, 1\}$ ...
Power indexes

Let $v: 2^N \to \mathbb{R}$ be a game on a set $N = \{1, \ldots, n\}$ of players.
Let $j \in N$ be a player.

**Banzhaf power index** (Banzhaf, 1965)

$$
\psi_B(v, j) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{j\}} (v(S \cup \{j\}) - v(S))
$$

**Shapley power index** (Shapley, 1953)

$$
\psi_{Sh}(v, j) = \sum_{S \subseteq N \setminus \{j\}} \frac{1}{n \binom{n-1}{|S|}} (v(S \cup \{j\}) - v(S))
$$
Let $\mu: 2^X \to \mathbb{R}$ be a fuzzy measure on a set $X = \{1, \ldots, n\}$ of values.

Let $k \in \{0, \ldots, n - 1\}$

**Cardinality index** (Yager, 2002)

$$C_k = \frac{1}{(n-k)\binom{n}{k}} \sum_{S \subseteq X, |S| = k} \sum_{x \in X \setminus S} (\mu(S \cup \{x\}) - \mu(S))$$

**Interpretation:**

$C_k$ is the average gain in certitude that we obtain by adding an arbitrary element to an arbitrary $k$-element subset.
Cardinality index

**Alternative formulation** (game theory notation)

\[ C_k = \frac{1}{(n - k)\binom{n}{k}} \sum_{|S|=k} \sum_{j \in N \setminus S} \left( v(S \cup \{j\}) - v(S) \right) \]

**Interpretation:**

\[ C_k \] is the average gain that we obtain by adding an arbitrary player to an arbitrary \( k \)-player coalition

...when compared with the Banzhaf power index...

\[ \psi_B(v,j) = \frac{1}{2^{n-1}} \sum_{S \ni j} v(S) - \frac{1}{2^{n-1}} \sum_{S \not\ni j} v(S) \]
Introduction to network reliability
**System**

**Definition.** A *system* consists of several interconnected units

**Assumptions:**

1. The system and the units are of the crisply *on/off* kind
2. A serially connected segment of units is functioning if and only if every single unit is functioning

![Diagram of serial connection]

3. A system of parallel units is functioning if and only at least one unit is functioning

![Diagram of parallel connection]
**System**

**Example.** Home video system

1. Blu-ray player
2. DVD player
3. LCD monitor
4. Amplifier
5. Speaker A
6. Speaker B
Structure function

Definition.
The state of a component \( j \in [n] = \{1, \ldots, n\} \) can be represented by a Boolean variable

\[
x_j = \begin{cases} 
1 & \text{if component } j \text{ is functioning} \\
0 & \text{if component } j \text{ is in a failed state}
\end{cases}
\]

The state of the system is described from the component states through a Boolean function \( \phi : \{0, 1\}^n \rightarrow \{0, 1\} \)

\[
\phi(x_1, \ldots, x_n) = \begin{cases} 
1 & \text{if the system is functioning} \\
0 & \text{if the system is in a failed state}
\end{cases}
\]

This function is called the structure function of the system
**Structure function**

**Series structure**

\[ \phi(x_1, x_2, x_3) = x_1 \cdot x_2 \cdot x_3 = \prod_{j=1}^{3} x_j \]

**Parallel structure**

\[ \phi(x_1, x_2, x_3) = 1 - (1 - x_1)(1 - x_2)(1 - x_3) = \bigcup_{j=1}^{3} x_j \]
Structure function

Home video system

\[ \phi(x_1, \ldots, x_6) = (x_1 \cup x_2) \cdot x_3 \cdot x_4 \cdot (x_5 \cup x_6) \]
Lifetimes

Notation

1. $T_j =$ *lifetime* of component $j \in [n]$
2. $T =$ *lifetime* of the system

We assume that the component lifetimes $T_1, \ldots, T_n$ are continuous and i.i.d.
Barlow-Proschan importance index

**Importance index** (Barlow-Proschan, 1975)

\[
I_{\text{BP}}^{(j)} = \Pr(T = T_j) \quad j \in [n]
\]

\(I_{\text{BP}}^{(j)}\) is a measure of importance of component \(j\)

**In the i.i.d. case:**

\[
I_{\text{BP}}^{(j)} = \sum_{S \subseteq [n]\setminus\{j\}} \frac{1}{n \binom{n-1}{|S|}} \left( \phi(S \cup \{j\}) - \phi(S) \right) = \psi_{\text{Sh}}(\phi, j)
\]
Let \( T_1 \leq \cdots \leq T_n \) be the order statistics obtained from the variables \( T_1, \ldots, T_n \)

**System signature** (Samaniego, 1985)

\[
s_k = \Pr(T = T_{(k)}) \quad k = 1, \ldots, n
\]

**In the i.i.d. case** (Boland, 2001)

\[
s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{S \subseteq [n], |S| = n-k+1} \phi(S) - \frac{1}{\binom{n}{n-k}} \sum_{S \subseteq [n], |S| = n-k} \phi(S) = C_{n-k}
\]
B-P importance index and system signature

Series structure

\[ I_{BP} = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \quad s = (1, 0, 0) \]
B-P importance index and system signature

Home video system

\[ I_{BP} = \left( \frac{1}{15}, \frac{1}{15}, \frac{11}{30}, \frac{11}{30}, \frac{1}{15}, \frac{1}{15} \right) \]

\[ s = \left( \frac{1}{3}, \frac{2}{5}, \frac{4}{15}, 0, 0, 0 \right) \]
Manual computation of the cardinality index
Any set function \( f: 2^{[n]} \rightarrow \mathbb{R} \) can be represented as a multilinear polynomial

\[
f(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} f(S) \prod_{j \in S} x_j \prod_{j \in [n] \setminus S} (1 - x_j)
\]

\[
f(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} c(S) \prod_{j \in S} x_j
\]
The *multilinear extension* of $f$ (Owen, 1972) is the function $\overline{f} : [0, 1]^n \to \mathbb{R}$ defined by

$$
\overline{f}(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} f(S) \prod_{j \in S} x_j \prod_{j \in [n] \setminus S} (1 - x_j)
$$

$$
\overline{f}(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} c(S) \prod_{j \in S} x_j
$$

**Example:**

$$
\max(x_1, x_2) = x_1 \cup x_2 = 1 - (1 - x_1)(1 - x_2) = x_1 + x_2 - x_1 x_2
$$
Manual computation: Banzhaf and Shapley power indexes

\[
\psi_B(f, j) = \left( \frac{\partial}{\partial x_j} \bar{f} \right)(\frac{1}{2}, \ldots, \frac{1}{2})
\]

\[
\psi_{Sh}(f, j) = \int_0^1 \left( \frac{\partial}{\partial x_j} \bar{f} \right)(x, \ldots, x) \, dx
\]

(Owen, 1972)
With any $n$-degree polynomial $p : \mathbb{R} \rightarrow \mathbb{R}$ we associate the reflected polynomial $R^n p : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$ (R^n p)(x) = x^n p\left(\frac{1}{x}\right) $$

$p(x) = a_0 + a_1 x + \cdots + a_n x^n \Rightarrow (R^n p)(x) = a_n + a_{n-1} x + \cdots + a_0 x^n$

(M. and Mathonet, 2013)

Setting $p(x) = \frac{d}{dx}\left(\bar{f}(x, \ldots, x)\right)$, we have

$$ (R^{n-1} p)(x + 1) = \sum_{k=1}^{n} s_k \left(\begin{array}{c} n \\ k \end{array}\right) k x^{k-1} $$
Example. Home video system

\[ \phi(x_1, \ldots, x_6) = (x_1 \cup x_2) \ x_3 \ x_4 (x_5 \cup x_6) \]

\[ \overline{\phi}(x_1, \ldots, x_6) = x_1 x_3 x_4 x_5 + x_2 x_3 x_4 x_5 + x_1 x_3 x_4 x_6 + x_2 x_3 x_4 x_6 - x_1 x_2 x_3 x_4 x_5 - x_1 x_2 x_3 x_4 x_6 - x_1 x_3 x_4 x_5 x_6 - x_2 x_3 x_4 x_5 x_6 + x_1 x_2 x_3 x_4 x_5 x_6 \]

\[ \overline{\phi}(x, \ldots, x) = 4x^4 - 4x^5 + x^6 \]
Manual computation: Cardinality index

\[ \overline{\phi}(x, \ldots, x) = 4x^4 - 4x^5 + x^6 \]

\[ p(x) = \frac{d}{dx}(\overline{\phi}(x, \ldots, x)) = 16x^3 - 20x^4 + 6x^5 \]

\[ (R^5 p)(x) = 6 - 20x + 16x^2 \]

\[ (R^5 p)(x+1) = 2 + 12x + 16x^2 = s_1 \binom{6}{1} + s_2 \binom{6}{2} 2x + s_2 \binom{6}{3} 3x^2 + \ldots \]

\[ \Rightarrow \quad s = \left( \frac{1}{3}, \frac{2}{5}, \frac{4}{15}, 0, 0, 0 \right) \]

\[ \Rightarrow \quad C = \left( 0, 0, 0, \frac{4}{15}, \frac{2}{5}, \frac{1}{3} \right) \]
Least squares approximation problems
Least squares approximation problems

Denote by $V$ the set of games $g$ on $N$ of the form

\[ g(x) = c_0 + \sum_{j \in N} c_j x_j, \quad c_0, c_1, \ldots, c_n \in \mathbb{R} \]

**Approximation problem** (Hammer and Holzman, 1992)

Given a game $f$ on $N$, the *best first-degree approximation* of $f$ is the game $f^*$ on $N$ that minimizes the square distance

\[ \| f - g \|^2 = \sum_{x \in \{0,1\}^n} (f(x) - g(x))^2 \]

from among all games $g \in V$.

We have

\[ c_j^* = \psi_B(f, j) \quad j \in N \]
Least squares approximation problems

Denote by $V_c$ the set of games $g$ on $N$ of the form

$$g(x) = \sum_{j \in N} c_j x_j$$

such that $g(1, \ldots, 1) = f(1, \ldots, 1)$

**Approximation problem** (Charnes et al., 1988)

For a given game $f$ on $N$, the *best $c$-approximation* of $f$ is the unique game $f^*$ on $N$ that minimizes the square distance

$$\|f - g\|^2_c = \sum_{T \subseteq N} \frac{1}{\binom{n-2}{|T|-1}} \left( f(T) - g(T) \right)^2$$

from among all games $g \in V_c$.

We have

$$c_j^* = \psi_{\text{Sh}}(f, j) \quad j \in N$$
Least squares approximation problems

Denote by $V_s$ the set of *symmetric* games $g$ on $N$, that is, of the form

$$g(x) = c_0 + \sum_{k=1}^{n} c_k x(k), \quad c_0 = 0$$

**Approximation problem** (M. and Mathonet, 2012)

For a given game $f$ on $N$, the *best symmetric approximation* of $f$ is the unique game $f^*$ on $N$ that minimizes the square distance

$$\| f - g \|_s^2 = \sum_{T \subseteq N} \frac{1}{|T|} (f(T) - g(T))^2$$

from among all games $g \in V_s$.

We have

$$c_k^* = s_k = C_{n-k} \quad k = 1, \ldots, n$$
Thank you for your attention!
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