The Chaining Interaction Index among Players in Cooperative Games

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Revised version

Abstract

This paper introduces a new interaction index among players of a cooperative game or criteria in a decision making problem: the chaining interaction whose definition is based on the maximal chains of the lattice related to the set of coalitions. This interaction index as well as the Shapley and Banzhaf interaction indices belong to the class of cardinal-probabilistic interaction indices. Conversion formulas between some representations are pointed out. Moreover, links with multilinear extensions and potentials are also investigated.

Keywords: game theory, multicriteria decision making, Shapley and Banzhaf values, interaction indices.

1 Introduction

Let v^N be a cooperative game on the finite set of players N that is a set function v called capacity from 2^N to $\mathbb{R}$ such that $v(\emptyset) = 0$. For each coalition $S \subseteq N$ of players, the real number $v(S)$ represents the *worth* or the *power* of S.

Since the work of Shapley [13] the concept of value of a game has been widely used in cooperative game theory. Taking the Shapley value related to i, $\phi^v(i)$, as a typical example, the fact that in general this value is different from $v(i)$ shows that the players interact. Players i and j have interest to cooperate when the worth of coalition $\{i, j\}$ is more than the sum of individual worths. Players i and j have no interest to cooperate when the worth of $\{i, j\}$ is less than the sum of individual worths. Players i and j can act independently in case of equality.

The interaction between players i and j can be considered as the average of the marginal contributions of j in the presence of i minus the average of the marginal contributions of j

in the absence of i which corresponds to the weighted sum over all coalitions $T \subseteq N \setminus \{i,j\}$ of

$$[v(T \cup \{i,j\}) - v(T \cup \{i\})] - [v(T \cup \{j\}) - v(T)] = \delta_{\{i,j\}} v(T \cup \{i,j\})$$

that is, with $t = |T|, n = |N|$,

$$I^v(\{i,j\}) = \sum_{T \subseteq N \setminus \{i,j\}} p_t^2(n)\, \delta_{\{i,j\}} v(T \cup \{i,j\}) \quad \text{with} \quad p_t^2(n) \geq 0, \quad \sum_{T \subseteq N \setminus \{i,j\}} p_t^2(n) = 1.$$

The interaction between s members of a coalition S is obtained as a natural extension of the case $s = |S| = 2$:

$$I^v(S) = \sum_{T \subseteq N \setminus S} p_t^s(n)\, \delta_S v(T \cup S) \quad \text{with} \quad p_t^s(n) \geq 0, \quad \sum_{T \subseteq N \setminus S} p_t^s(n) = 1,$$

where $\delta_S v(T)$ is the S-derivative of v at T, defined recursively by $\delta_i v(T) = v(T) - v(T \setminus i)$, $\delta_{\{i,j\}} v(T) = \delta_i[\delta_j v(T)] = \delta_j[\delta_i v(T)]$,...

This family of *cardinal-probabilistic* interaction indices has been introduced recently by Grabisch and Roubens [7] and contains:

- the interaction *à la* Shapley I^v_{Sh} if

  $$p_t^s(n) = p_t^1(n-s+1) = \frac{(n-t-s)!\, t!}{(n-s+1)!}, \quad t = 0, \ldots, n-s.$$

  For one-membered coalitions, we get the Shapley value [13]:

  $$I^v_{\text{Sh}}(i) = \phi^v_{\text{Sh}}(i) = \sum_{T \subseteq N \setminus \{i\}} \gamma_t\, [v(T \cup \{i\}) - v(T)]$$

  with $\gamma_t = p_t^1(n) = \frac{(n-t-1)!\, t!}{n!}$.

  The Shapley interaction between a pair of players $\{i,j\}$ was already defined in 1993 by Murofushi and Soneda [10]. Borrowing concepts from multiattribute utility theory, they proposed to define the interaction index of elements $i, j \in N$ by

  $$I^v(\{i,j\}) := \sum_{T \subseteq N \setminus \{i,j\}} p_t^2(n)\, [v(T \cup \{i,j\}) - v(T \cup \{i\}) - v(T \cup \{j\}) + v(T)],$$

  with $p_t^2(n) = \frac{(n-t-2)!\, t!}{(n-1)!}$. They interpreted such an expression as a kind of average value of the *added value* given by putting i and j together, all coalitions being considered.

- the interaction *à la* Banzhaf I^v_{B} if

  $$p_t^s(n) = p_t^1(n-s+1) = \frac{1}{2^{n-s}}, \quad t = 0, \ldots, n-s.$$

  For one-membered coalitions, we get the Banzhaf value [2]:

  $$I^v_{\text{B}}(i) = \phi^v_{\text{B}}(i) = \frac{1}{2^{n-1}} \sum_{T \subseteq N \setminus \{i\}} [v(T \cup \{i\}) - v(T)].$$

In multicriteria decision making, when N represents a set of criteria, v is a *weight function* and the real number $v(S)$ represents the *weight* related to the combination S of criteria. Typically, such a weight function is a *fuzzy measure* on N, that is a monotonic set function $v : 2^N \to [0,1]$ fulfilling the boundary conditions $v(\emptyset) = 0$ and $v(N) = 1$. Monotonicity means that $v(R) \leq v(S)$ whenever $R \subseteq S \subseteq N$.

In this context, the parallelism between game theory and multicriteria decision making is clear. The overall importance of a criterion $i \in N$ is not solely determined by the number $v(i)$, but also by all $v(T)$ such that $i \in T$. Thus, a coefficient of importance can be represented by a power index, which could be given for example by the Shapley value.

Considering a pair of criteria $\{i,j\} \subseteq N$, we easily see that the difference

$$v(\{i,j\}) - v(\{i\}) - v(\{j\})$$

seems to reflect the degree of interaction between i and j. This difference is positive if there is a synergy effect between i and j. These two criteria then interfere in a positive way. The difference is negative in case of overlap effect between i and j. The criteria then interfere in a negative way. Finally, the difference is zero when the individual importances $v(\{i\})$ and $v(\{j\})$ are adding without interfering. In this case, there is no interaction between i and j.

As for importance, a proper definition of interaction should consider not only $v(\{i\})$, $v(\{j\})$, $v(\{i,j\})$ but also the measures of all subsets containing i and j. Thus the interaction between criteria i an j, and more generally, among a combination S of criteria can be represented by a cardinal-probabilistic interaction index.

The purpose of this paper is to introduce a third interaction index belonging to the family of cardinal-probabilistic interaction indices: the *chaining interaction index* I^v_{R}, for which one has

$$p^s_t(n) = s\,\frac{(n-s-t)!\,(s+t-1)!}{n!}. \tag{1}$$

We can already notice that $I^v_{\mathrm{R}}(i) = I^v_{\mathrm{Sh}}(i)$ for all $i \in N$. Thus $I^v_{\mathrm{R}}(S)$ can be considered as an extension of the Shapley value to determine the interaction between the players of the coalition $S \subseteq N$.

2 The chaining interaction index

Let us consider the lattice $L(N)$ related to the power set of N. We can represent $L(N)$ as a graph called Hasse Diagram $H(N)$ whose nodes correspond to the coalitions $S \subseteq N$ and the edges represent adding a player to the bottom coalition to get the top coalition. A *maximal chain* of $H(N)$ is an ordered collection of $n+1$ nested distinct coalitions

$$M = (\emptyset = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = N).$$

The set of maximal chains of $H(N)$ is denoted $C(N)$. Let M be an element of $C(N)$ and M^S the minimal coalition belonging to M that contains S. The cardinality of $C(N)$ is equal to $n!$ and we define

$$I^v_{\mathrm{R}}(S) = \frac{1}{n!} \sum_{M \subseteq C(N)} \delta_S v(M^S), \quad \emptyset \neq S \subseteq N.$$

The value $I_{\mathrm{R}}^v(i)$ corresponds to the Shapley value related to i as it was mentioned by Edelman [4] dealing with cooperative games in which only certain coalitions are allowable.

We now prove that I_{R}^v is a cardinal-probabilistic interaction for which $p_t^s(n)$ is defined by (1).

If $C^{S,S\cup T}$ represents the subclass included in $C(N)$ of maximal chains that have $\{S\cup T\}$ as minimal coalition including S, we have

$$\begin{aligned} I_{\mathrm{R}}^v(S) &= \frac{1}{n!}\sum_{T\subseteq N\setminus S}\left|C^{S,S\cup T}\right|\,\delta_S v(S\cup T)\\ &= \sum_{T\subseteq N\setminus S} p_t^s(n)\,\delta_S v(T\cup S)\end{aligned}$$

with $p_t^s(n)=\frac{1}{n!}\left|C^{S,S\cup T}\right|$, $s=1,\dots,n$, $t=0,\dots,n-s$.

For example, if $N=\{1,2,3\}$, we have

$$\begin{aligned} C^{\{1\},\{1\}} &= \{(\emptyset\subsetneq\{1\}\subsetneq\{1,2\}\subsetneq\{1,2,3\}),(\emptyset\subsetneq\{1\}\subsetneq\{1,3\}\subsetneq\{1,2,3\})\}\\ C^{\{1\},\{1,2\}} &= \{(\emptyset\subsetneq\{2\}\subsetneq\{1,2\}\subsetneq\{1,2,3\})\}\\ C^{\{1\},\{1,3\}} &= \{(\emptyset\subsetneq\{3\}\subsetneq\{1,3\}\subsetneq\{1,2,3\})\}\\ C^{\{1\},\{1,2,3\}} &= \{(\emptyset\subsetneq\{2\}\subsetneq\{2,3\}\subsetneq\{1,2,3\}),(\emptyset\subsetneq\{3\}\subsetneq\{2,3\}\subsetneq\{1,2,3\})\}\end{aligned}$$

$$p_0^1(3)=\frac{2}{6},\quad p_1^1(3)=\frac{1}{6},\quad p_2^1(3)=\frac{2}{6}$$

$$\begin{aligned} C^{\{1,2\},\{1,2\}} &= \{(\emptyset\subsetneq\{1\}\subsetneq\{1,2\}\subsetneq\{1,2,3\}),(\emptyset\subsetneq\{2\}\subsetneq\{1,2\}\subsetneq\{1,2,3\})\}\\ C^{\{1,2\},\{1,2,3\}} &= \{(\emptyset\subsetneq\{1\}\subsetneq\{1,3\}\subsetneq\{1,2,3\}),(\emptyset\subsetneq\{2\}\subsetneq\{2,3\}\subsetneq\{1,2,3\}),\\ &\quad\ (\emptyset\subsetneq\{3\}\subsetneq\{1,3\}\subsetneq\{1,2,3\}),(\emptyset\subsetneq\{3\}\subsetneq\{2,3\}\subsetneq\{1,2,3\})\}.\end{aligned}$$

It is easy to observe that $C^{S,S\cup T}$ corresponds to a disjoint union of sets of maximal chains defined in sublattices of $L(N)$. In particular, we can see that

$$\left|C^{S,S\cup T}\right| = \left|C(N\setminus(S\cup T))\right|\times\Big|\bigcup_{i\in S} C((S\setminus\{i\})\cup T)\Big|.$$

We then have, for all $s=1,\dots,n$ and all $t=0,\dots,n-s$,

$$\begin{aligned} p_t^s(n) &= \frac{1}{n!}\left|C^{S,S\cup T}\right| = \frac{s}{n!}\,|C(N\setminus(S\cup T))|\times|C((S\setminus\{1\})\cup T)|\\ &= s\,\frac{(n-s-t)!\,(s+t-1)!}{n!}.\end{aligned}$$

The chaining interaction index I_{R}^v is of cardinal-probabilistic type since, if $S\neq\emptyset$,

$$\begin{aligned} \sum_{T\subseteq N\setminus S} p_t^s(n) &= \sum_{t=0}^{n-s}\binom{n-s}{t}\,s\,\frac{(n-s-t)!\,(s+t-1)!}{n!}\\ &= \frac{s!\,(n-s)!}{n!}\sum_{t=0}^{n-s}\binom{s+t-1}{s-1}=1.\end{aligned}$$

3 Equivalent representations

Any game v^N has a canonical representation in terms of unanimity games that determine a linear basis for v.

If $\{v_T\}_{T\subseteq N}$ is such that $v_T(S) = 1$ for $S \supseteq T$ and 0 otherwise then

$$v(S) = \sum_{T\subseteq N} a(T)\, v_T(S) = \sum_{T\subseteq S} a(T) \tag{2}$$

and the real coefficients $\{a(T)\}_{T\subseteq N}$ are called the *dividends* of the coalitions in game v. In combinatorics, a viewed as a set function on N is called the Möbius transform of v. The interaction indices $I_{\rm Sh}$ and $I_{\rm B}$ can be expressed from this set function as follows (see [6]):

$$\begin{aligned} I_{\rm Sh}(S) &= \sum_{T\supseteq S} \frac{1}{t+s-1}\, a(T), \quad \emptyset \neq S \subseteq N \\ I_{\rm B}(S) &= \sum_{T\supseteq S} \frac{1}{2^{t-s}}\, a(T), \quad \emptyset \neq S \subseteq N. \end{aligned}$$

The set function a is a representation of v since there is a bijection between the set of games and the set of dividends, i.e. defining one of the two allows to compute the other without ambiguity. More formally, a set function $w : 2^N \to \mathbb{R}$ is a representation of v if there exists an invertible transform $\mathcal{T}$ such that

$$w = \mathcal{T}(v) \quad \text{and} \quad v = \mathcal{T}^{-1}(w).$$

It has been proved in [6] that $I^v_{\rm Sh}$ and $I^v_{\rm B}$ are also representations of v.

We now prove that $I^v_{\rm R}$ is a representation of v. More precisely, we prove the following two identities.

$$I_{\rm R}(S) = \sum_{T\supseteq S} \frac{s}{t}\, a(T), \quad \emptyset \neq S \subseteq N \tag{3}$$

$$a(S) = \sum_{T\supseteq S} (-1)^{t-s}\, \frac{s}{t}\, I_{\rm R}(T), \quad \emptyset \neq S \subseteq N. \tag{4}$$

On the one hand, we have (see [6])

$$\sum_{T\subseteq N\setminus S} p^s_t(n)\, \delta_S v(T\cup S) = \sum_{T\supseteq S} \beta^s_t(n)\, a(T)$$

with

$$\beta^s_t(n) = \sum_{k=0}^{n-t} \binom{n-t}{k} p^s_{k+t-s}(n), \quad t = s, \ldots, n.$$

When $p^s_t(n)$ is given by (1), the previous identity becomes

$$\beta^s_t(n) = \frac{s\,(n-t)!\,(t-1)!}{n!} \sum_{k=0}^{n-t} \binom{k+t-1}{t-1} = \frac{s}{t}, \quad t = s, \ldots, n,$$

which proves (3). On the other hand, we have, by (3),

$$\sum_{T\supseteq S} (-1)^{t-s}\, \frac{s}{t}\, I_{\rm R}(T) = \sum_{T\supseteq S} (-1)^{t-s}\, \frac{s}{t} \sum_{K\supseteq T} \frac{t}{k}\, a(K)$$

$$
\begin{aligned}
&= \sum_{K\supseteq S} \frac{s}{k}\Big[\sum_{T:S\subseteq T\subseteq K}(-1)^{t-s}\Big]\,a(K)\\
&= \sum_{K\supseteq S} \frac{s}{k}\Big[\sum_{t=s}^{k}\binom{k-s}{t-s}(-1)^{t-s}\Big]\,a(K)\\
&= \sum_{K\supseteq S} \frac{s}{k}\,(1-1)^{k-s}\,a(K) = a(S),
\end{aligned}
$$

which proves (4).

It is interesting to note how formula (3) is simple. Moreover, comparing I^v_{R} and I^v_{Sh} one can see that the terms of the summation are weighted by elements which are linearly decreasing with the argument $t=|T|$, whereas these elements are exponentially decreasing for I^v_{B}.

The conversion formulas between I_{R} and v can be given as follows:

$$
I_{\mathrm{R}}(S) = \sum_{T\subseteq N} \frac{s\,(-1)^{|S\setminus T|}}{n\binom{n-1}{|S\cup T|-1}}\,v(T), \quad \emptyset\neq S\subseteq N, \tag{5}
$$

$$
v(S) = \sum_{T\subseteq N\setminus S} \frac{(-1)^t}{t+1}\sum_{i\in S} I_{\mathrm{R}}(T\cup i), \quad \emptyset\neq S\subseteq N. \tag{6}
$$

Indeed, since $\delta_S v(S\cup T) = \sum_{L\subseteq S}(-1)^{s-l}v(L\cup T)$, we obtain, by setting $T' := T\cup L$ (which implies $L = T'\cap S$ and $T = T'\setminus S$):

$$
\begin{aligned}
I_{\mathrm{R}}(S) &= \sum_{T\subseteq N\setminus S} s\frac{(n-s-t)!\,(s+t-1)!}{n!}\sum_{L\subseteq S}(-1)^{s-l}v(L\cup T)\\
&= \sum_{T'\subseteq N} \frac{s}{n}\,\frac{(n-|S\cup T'|)!\,(|S\cup T'|-1)!}{(n-1)!}\,(-1)^{|S\setminus T'|}v(T')
\end{aligned}
$$

which proves (5). On the other hand, we have, by (2) and (4),

$$
\begin{aligned}
v(S) &= \sum_{T\subseteq S}\sum_{K\supseteq T}(-1)^{k-t}\,\frac{t}{k}\,I_{\mathrm{R}}(K)\\
&= \sum_{K\subseteq N}\frac{1}{k}\,I_{\mathrm{R}}(K)\sum_{T\subseteq K\cap S}(-1)^{k-t}t\\
&= \sum_{K\subseteq N}\frac{1}{k}\,I_{\mathrm{R}}(K)\underbrace{\sum_{t=0}^{|K\cap S|}\binom{|K\cap S|}{t}(-1)^{k-t}t}_{(*)}
\end{aligned}
$$

where the second sum $(*)$ equals $(-1)^{k+1}$ if $|K\cap S|=1$ and 0 otherwise. Therefore

$$
v(S) = \sum_{\substack{K\subseteq N\\ |K\cap S|=1}} \frac{(-1)^{k+1}}{k}\,I_{\mathrm{R}}(K)
$$

which proves (6).

The conversion formulas between I_{R} and I_{Sh} can also be given. We have

$$I_{\mathrm{R}}(S) = \sum_{T\supseteq S} \gamma_t^s \, I_{\mathrm{Sh}}(T), \quad \emptyset \neq S \subseteq N, \tag{7}$$

$$I_{\mathrm{Sh}}(S) = I_{\mathrm{R}}(S) + \sum_{\substack{T\supseteq S\\ T\neq S}} (-1)^{t-s} \frac{s-1}{t\,(t-s+1)} \, I_{\mathrm{R}}(T), \quad \emptyset \neq S \subseteq N, \tag{8}$$

with

$$\gamma_t^s = \int_0^1 s\, x^{s-1} \, B_{t-s}(x)\, dx = \sum_{k=0}^{t-s} \binom{t-s}{k} \frac{s}{t-k} \, B_k, \quad t = s, \ldots, n,$$

where $\{B_n\}_{n\in\mathbb{N}}$ is the sequence of Bernoulli numbers

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, \ldots$$

and $B_n(x)$ is the nth Bernoulli polynomial defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k \, x^{n-k}.$$

First values of γ_t^s are:

$s\backslash t$	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2		1	1/6	0	−1/60	0	1/126
3			1	1/4	1/60	−1/40	−1/210
4				1	3/10	1/30	−1/35
5					1	1/3	1/21
6						1	5/14
7							1

Let us prove (7). It has been established in [6] that

$$a(S) = \sum_{T\supseteq S} B_{t-s} \, I_{\mathrm{Sh}}(T), \quad S \subseteq N.$$

Therefore, by (3),

$$I_{\mathrm{R}}(S) = \sum_{K\supseteq S} \frac{s}{k} \sum_{T\supseteq K} B_{t-k} \, I_{\mathrm{Sh}}(T) = \sum_{T\supseteq S} \gamma_t^s \, I_{\mathrm{Sh}}(T)$$

with

$$\begin{aligned}
\gamma_t^s &= \sum_{K: S\subseteq K\subseteq T} \frac{s}{k} \, B_{t-k} = \sum_{k=s}^{t} \binom{t-s}{k-s} \frac{s}{k} \, B_{t-k} \\
&= \sum_{k=0}^{t-s} \binom{t-s}{k} \frac{s}{k+s} \, B_{t-s-k} = \sum_{k=0}^{t-s} \binom{t-s}{k} \frac{s}{t-k} B_k \\
&= \int_0^1 s \sum_{k=0}^{t-s} \binom{t-s}{k} x^{t-k-1} B_k \, dx \\
&= \int_0^1 s\, x^{s-1} \, B_{t-s}(x)\, dx.
\end{aligned}$$

Let us prove (8). It has been established in [6] that

$$I_{\rm Sh}(S) = \sum_{T\supseteq S} \frac{1}{t-s+1}\, a(T), \quad S\subseteq N.$$

Therefore, by (4),

$$\begin{aligned} I_{\rm Sh}(S) &= \sum_{K\supseteq S} \frac{1}{k-s+1} \sum_{T\supseteq K} (-1)^{t-k}\,\frac{k}{t}\, I_{\rm R}(T) \\ &= \sum_{T\supseteq S} \frac{(-1)^t}{t} \Big[\sum_{K:S\subseteq K\subseteq T} (-1)^k\, \frac{k}{k-s+1}\Big]\, I_{\rm R}(T) \end{aligned}$$

and

$$\begin{aligned} \sum_{K:S\subseteq K\subseteq T} (-1)^k\,\frac{k}{k-s+1} &= \sum_{k=s}^{t} \binom{t-s}{k-s} (-1)^k\,\frac{k}{k-s+1} \\ &= (-1)^s \sum_{k=0}^{t-s} \binom{t-s}{k} (-1)^k\,[1+\frac{s-1}{k+1}] \\ &= (-1)^s\,\Big[(1-1)^{t-s} + (s-1)\int_0^1 \sum_{k=0}^{t-s}\binom{t-s}{k}(-x)^k\,dx\Big] \\ &= (-1)^s\,\Big[(1-1)^{t-s} + (s-1)\int_0^1 (1-x)^{t-s}\,dx\Big] \\ &= (-1)^s\,\Big[(1-1)^{t-s} + \frac{s-1}{t-s+1}\Big]. \end{aligned}$$

Hence the result.

Using similar arguments, we can also prove that

$$I_{\rm B}(S) = \sum_{T\supseteq S} \Big(-\frac{1}{2}\Big)^{t-s}\,\frac{2s-t}{t}\, I_{\rm R}(T), \quad \emptyset\neq S\subseteq N;$$

indeed, we simply have

$$\begin{aligned} I_{\rm B}(S) &= \sum_{K\supseteq S} \Big(\frac{1}{2}\Big)^{k-s} a(K) \\ &= \sum_{K\supseteq S} \Big(\frac{1}{2}\Big)^{k-s} \sum_{T\supseteq K} (-1)^{t-k}\,\frac{k}{t}\, I_{\rm R}(T) \\ &= \sum_{T\supseteq S} \frac{(-1)^t}{t} \Big[\sum_{K:S\subseteq K\subseteq T} (-1)^k \frac{k}{2^{k-s}}\Big]\, I_{\rm R}(T) \end{aligned}$$

and

$$\begin{aligned} \sum_{K:S\subseteq K\subseteq T} (-1)^k \frac{k}{2^{k-s}} &= \sum_{k=s}^{t} \binom{t-s}{k-s} (-1)^k \frac{k}{2^{k-s}} \\ &= (-1)^s \sum_{k=0}^{t-s} \binom{t-s}{k} (-1)^k\,\frac{k+s}{2^k} \\ &= (-1)^s\, 2^{s-t}(2s-t). \end{aligned}$$

4 Links with multilinear extensions and potentials

The *multilinear extension* of v (MLE) is a multilinear polynomial defined by (see e.g. [11])

$$g(x_1,\dots,x_n) := \sum_{S\subseteq N} v(S)\prod_{i\in S} x_i \prod_{i\in N\setminus S}(1-x_i) = \sum_{S\subseteq N} a(S)\prod_{i\in S} x_i,$$

where $x_i \in [0,1]$.

Introducing the notation $\underline{c} = (c,\dots,c) \in [0,1]^n$ for any constant c belonging to $[0,1]$ we define the S-derivative of g as

$$\Delta_S g(x_1,\dots,x_n) = \frac{\partial^s g(x_1,\dots,x_n)}{\partial x_{i_1}\cdots\partial x_{i_s}}, \quad \text{where} \quad S=\{i_1,\dots,i_s\},$$

and we obtain that [6]

$$\begin{aligned} I^v_{\mathrm{Sh}}(S) &= \int_0^1 (\Delta_S g)(\underline{x})\,dx, \quad \emptyset\neq S\subseteq N,\\ I^v_{\mathrm{B}}(S) &= (\Delta_S g)(\underline{1/2}), \quad \emptyset\neq S\subseteq N. \end{aligned}$$

We thus see that the Banzhaf interaction index related to S is the value of the S-derivative of the MLE of game v on the center of the hypercube $[0,1]^n$, while the Shapley interaction index related to S is obtained by integrating the S-derivative of the MLE of game v along the main diagonal of the hypercube.

There is also a close link between $\Delta_S g$ and I_{R}. We can easily prove that

$$I^v_{\mathrm{R}}(S) = \int_0^1 s\,x^{s-1}\,(\Delta_S g)(\underline{x})\,dx, \quad \emptyset\neq S\subseteq N; \tag{9}$$

indeed, it has been proved in [6] that

$$(\Delta_S g)(\underline{x}) = \sum_{T\supseteq S} x^{t-s}\,a(T),$$

and hence we have

$$\begin{aligned} I^v_{\mathrm{R}}(S) = \sum_{T\supseteq S}\frac{s}{t}\,a(T) &= \int_0^1 \sum_{T\supseteq S} s\,x^{t-1}\,a(T)\,dx\\ &= \int_0^1 s\,x^{s-1}\sum_{T\supseteq S} x^{t-s}\,a(T)\,dx\\ &= \int_0^1 s\,x^{s-1}\,(\Delta_S g)(\underline{x})\,dx. \end{aligned}$$

We also have

$$I_{\mathrm{R}}(S) = \sum_{T\supseteq S}\Big[\int_0^1 s\,x^{s-1}\,(x-\frac{1}{2})^{t-s}\,dx\Big]\,I_{\mathrm{B}}(T), \quad \emptyset\neq S\subseteq N;$$

indeed, it has been shown in [6] that

$$(\Delta_S g)(\underline{x}) = \sum_{T\supseteq S}(x-\frac{1}{2})^{t-s}\,I_{\mathrm{B}}(T),$$

and hence we can conclude by (9).

It is worth noting that formula (7) can be retrieved by using the same argument; indeed, it has been shown in [6] that

$$(\Delta_S g)(\underline{x}) = \sum_{T \supseteq S} B_{t-s}(x)\, I_{\mathrm{Sh}}(T),$$

which is sufficient.

The chaining interaction index I_{R}^v can be expressed easily in term of *potential.* If P^v is a functional that represents the potential for the Shapley value (see Hart and Mas-Colell [8, 9]) defined as $P^v(\emptyset) = 0$ and $\sum_{i \in N} \delta_i P^v(N) = v(N)$, then we have

$$P^v(N) = \sum_{T \subseteq N} \frac{1}{t}\, a(T),$$

and

$$\delta_i P^v(N) = P^v(N) - P^v(N \setminus \{i\}) = \phi_{\mathrm{Sh}}^v(i), \quad i \in N.$$

More generally, one can easily see that

$$s\, \delta_S P^v(N) = I_{\mathrm{R}}(S), \quad \emptyset \neq S \subseteq N.$$

Likewise, if Q^v represents the potential for the Banzhaf value (see Dragan [3]) defined as $Q^v(\emptyset) = 0$ and $\sum_{i \in N} \delta_i Q^v(N) = \sum_{i \in N} \phi_{\mathrm{B}}^v(i)$, then we have

$$Q^v(N) = \sum_{T \subseteq N} \frac{1}{2^{t-1}}\, a(T),$$

and

$$\delta_i Q^v(N) = \phi_{\mathrm{B}}^v(i), \quad i \in N.$$

More generally, we can readily verify that

$$2^{s-1}\, \delta_S Q^v(N) = I_{\mathrm{B}}(S), \quad \emptyset \neq S \subseteq N.$$

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