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As an application taken from the domain of finance, we then introduce conditioned portfolio optimisation. The optimal control transcription of the two-dimensional mean-variance problem is given and solved using the scheme under consideration. We carry out a backtest using real-world data and confirm that its results validate our proposed numerical scheme.

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A Numerical Scheme for Multisignal Weight Constrained Conditioned Portfolio Optimisation Problems

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Abstract

In this paper, we consider optimal control problems involving a multidimensional objective function integral. We propose a direct collocation discretisation scheme suitable for the numerical solution of problems of this type. A convergence result is established to show that the scheme is consistent with multidimensional Pontryagin Principle relations in several important respects. Whilst the discussion focuses on the two-dimensional case, the simplicity of the scheme allows for easy generalisation.

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1 Introduction

The literature on numerical solution approaches to single dimensional optimal control problems is extremely rich: see e.g [11] for a survey and taxonomy. The case where objective function integrals are multidimensional is less thoroughly explored. Perhaps surprisingly, it appears that the main extant instance, at this point, of a documented numerical solution algorithm for a multidimensional optimal control problem is that given in [7], which discretises a type of formulation examined in e.g. [9] and involving controls defined as the derivatives of the state variables to solve various edge detection problems in the image processing application domain. We propose a direct collocation scheme that can be obtained, for any value of the problem dimension p, as a generalisation of the approach discussed in e.g. [13]. The scheme chosen is as simple as is meaningfully possible, with the approximated control values constant on each two-dimensional surface element and the approximated state values obtained using bilinear interpolation. This choice was made for reasons of stability, performance, simplicity and flexibility: in particular, the linear interpolation approach used allows for an easy generalisation to higher-dimensional problem cases. In the case where $p = 2$, the following sections will initially establish such a generalised scheme in generic terms, and then provide an illustration by applying the scheme to a multiple-signal conditioned portfolio optimisation problem.

The structure of the paper is thus as follows. Section 2 introduces the multidimensional problem format considered and gives a formulation for a Pontryagin Principle applying to problems of this type. Focusing on the two-dimensional case, section 3 proposes a very simple direct collocation discretisation scheme for optimal control problems and establishes consistency (in two central and well-defined respects) of the scheme with the Pontryagin Principle necessary optimality relationships given. Section 4 introduces a conditioned mean-variance portofolio optimisation application of the preceding theory. We give an optimal control formulation for this problem, describe the backtesting logic applied and discuss the results obtained. Section 5 concludes.

2 Higher dimensional optimal control problems

The present paper considers optimal control problems in the following generic format.

Definition 2.1 (Dieudonné-Rashevsky optimal control problem) *Consider the following optimal control problem:*

$$\text{minimise} \qquad J_{I_S}(x(s), u(s)) = \int_{I_S} L(x(s), u(s), s) ds \tag{1}$$

$$\text{subject to} \qquad \frac{\partial x_k}{\partial s^\alpha} = f_k^{(\alpha)}(x_k(s), u(s), s),\ 1 \le k \le m, \tag{2}$$

$$x(s^-) = x^-, x(s^+) = x^+, \tag{3}$$

$$\text{and} \qquad u(s) \in U\ \forall s \in I_S, \tag{4}$$

where $\alpha \in \{1, 2, \dots p\}$ is the signal index, $s^- = (s^{(1)^-}, s^{(2)^-}, \dots s^{(p)^-})$ and $s^+ = (s^{(1)^+}, s^{(2)^+}, \dots s^{(p)^+})$ are the p-dimensional boundaries of the signal support I_S, $p_s(s)$ is the joint p-dimensional probability density of the signal vector $s = (s^{(1)}, s^{(2)}, \dots s^{(p)})$, the state vector x is m-dimensional, the control constraint set $U \subseteq \mathbb{R}^n$ is convex and both $L(x(s), u(s), s)$ and the $f_k^{(\alpha)}(x_k(s), u(s), s)$ are jointly continuous with respect to both the $x_k(s)$ and $u(s)$. The terminal conditions $x(s^+) = x^+$ may or may not apply depending on the problem variant considered.

Given the dimensionality of the integration domain, these problems are in general called multidimensional *optimal control problems, see e.g. [9]. They may also be called* Dieudonné-Rashevsky *problems as the form of the state PDEs in equation* (2) *is sometimes labelled the Dieudonné-Rashevsky form, see Cesari [4].*

The paper [4] also establishes multidimensional necessary relationships which amount to a generalisation of the Pontryagin Principle, and can be restated as in the theorem which follows. Here, we use a multidimensional Hamiltonian $\mathcal{H}_{md}(x(s), u(s), \lambda(s), s)$ in the form

$$\mathcal{H}_{md}(x(s), u(s), \lambda(s), s) = \sum_{i=1}^{m} \sum_{\alpha=1}^{p} \lambda_i^{(\alpha)}(s) f_i^{(\alpha)}(x(s), u(s), s). \tag{5}$$

Theorem 2.2 (Pontryagin Minimum Principle, multidimensional case) *Consider the finite horizon p-dimensional optimal control problem given in definition 2.1.*

Let $(x^*(s), u^*(s))$ be an admissible pair for the above problem: thus $u^*(s)$ is a control, defined on I_S and piecewise continuous with respect to all signal components, with associated state $x^*(s)$. Assume that $(x^*(s), u^*(s))$ solves the given problem i.e. is optimal. If there exists a continuous and piecewise continuously differentiable costate vector function $\lambda^*(s) = (\lambda_1^*(s), \dots, \lambda_m^*(s))$, with

$$\lambda_i^*(s) = \left((\lambda_i^{(1)})^*(s), (\lambda_i^{(2)})^*(s), \dots (\lambda_i^{(p)})^*(s)\right) \forall 1 \le i \le m,$$

such that, $\forall s \in I_S$ and $1 \le i \le m$,

$$\nabla_s \cdot \lambda_i^*(s) = \sum_{\alpha=1}^{p} \frac{\partial (\lambda_i^{(\alpha)})^*}{\partial s^{(\alpha)}} = -\frac{\partial \mathcal{H}_{md}}{\partial x_i} = -\sum_{\alpha=1}^{p} (\lambda_i^{(\alpha)})^* \frac{\partial f_i^{(\alpha)}}{\partial x_i}, \tag{6}$$

$u^*(s)$ minimises the Hamiltonian $\mathcal{H}_{md}(x^*(s), u(s), \lambda^*(s), s)$ for $u(s) \in U$, i.e.

$$\mathcal{H}_{md}(x^*(s), u^*(s), \lambda^*(s), s) \le \mathcal{H}_{md}(x^*(s), u(s), \lambda^*(s), s)\ \forall u(s) \in U, s \in I_S. \tag{7}$$

Proof See [4].

Remark 1) Note that the costate equations are a precondition in this formulation. That is not always the case in related contributions, such as [10], [12] or [9]. However, the generalisations of the Principle obtained in those papers are nonclassical in other ways: for instance, [9] focuses on the variational problem with constrained controls[1], while [12] assumes that the optimal control $u^*(s)$ is contained in the interior of the admissible set U.

[1] The denomination of 'variational' here refers to the problem case where the controls are defined as the derivatives of the state variables. However, variational theory does not support control constraints, such that this problem type still belongs to the domain of optimal control rather than variational calculus.

2) A vector of r pure control path constraints $g(u) = (g_1(u(s)), \dots, g_r(u(s)))$ may be added to the problem and can then most simply be adjoined to the given Hamiltonian of equation (5) without requiring any significant changes to the problem analysis. In that case, a second sum of terms is added to the previous expression to yield

$$\begin{aligned}\mathcal{H}_{md}(x(s), u(s), \lambda(s), s) &= \sum_{i=1}^{m} \sum_{\alpha=1}^{p} \lambda_i^{(\alpha)}(s) f_i^{(\alpha)}(x(s), u(s), s) + \sum_{i=1}^{r} \psi_i(s) g_i(u(s)) \\ &= \sum_{\alpha=1}^{p} \lambda^{(\alpha)}(s) \cdot f^{(\alpha)}(x(s), u(s), s) + \psi(s) \cdot g(u(s)).\end{aligned} \tag{8}$$

3 A discretisation scheme for generic two-dimensional problems

The two-dimensional case will now be discussed in greater detail: this corresponds to the problem format given by equations (1), (2), (3) and (4) where $p = 2$. The two signals will be denoted $s^{(1)}$ and $s^{(2)}$. Either of the individual signal supports $I_{s^{(1)}} = [s^{(1)-}, s^{(1)+}]$ and $I_{s^{(2)}} = [s^{(2)-}, s^{(2)+}]$ are associated with their own respective strict partitions, $\Delta^{(1)} = (s_i^{(1)})_{i=1}^N$ and $\Delta^{(2)} = (s_j^{(2)})_{j=1}^N$. Thus the same number of partition points $N \geq 3$ is used individually for either signal domain. The problem domain then corresponds to the product of the individual supports, that is $I_s = I_{s^{(1)}} \times I_{s^{(2)}}$. Differences between successive points are defined separately for the two signals, giving

$$\begin{cases} h_i^{(1)} = h_i^{(1)}(\Delta^{(1)}) = s_{i+1}^{(1)} - s_i^{(1)}, i \in \{1, 2, \dots N-1\} \\ h_j^{(2)} = h_j^{(2)}(\Delta^{(2)}) = s_{j+1}^{(2)} - s_j^{(2)}, j \in \{1, 2, \dots N-1\}. \end{cases} \tag{9}$$

Define the set of partitions $\Delta = \{\Delta^{(1)}, \Delta^{(2)}\}$, and the maximum difference for the $(i, j)-$surface element as

$$h_{i,j}(\Delta) = \max\left\{h_i^{(1)}, h_j^{(2)}\right\} \forall 1 \leq i, j \leq N-1.$$

Then the effective partition step size, or mesh, for the two-dimensional grid is

$$h = h_{\max}(\Delta) = \max\{h_{i,j}(\Delta) : 1 \leq i, j \leq N-1\}. \tag{10}$$

For the present analysis, the partitions will be assumed regular, and the notations $h_i^{(1)} = h^{(1)}$ as well as $h_j^{(2)} = h^{(2)}$ $\forall (i, j) \in \{1, \dots N-1\}^2$ will be introduced: then $h = \max\{h^{(1)}, h^{(2)}\}$. Leaving out the two restrictions now made with respect to the signal domain partitions used would pose no additional theoretical difficulties in the discussion which follows. However, doing so would further complicate the notation, while adding a degree of flexibility that would be of interest only if, for instance, an adaptive grid algorithm were proposed.

The numerical grid generated is two-dimensional and an index 2-tuple (i, j) has to be used to specify each point: note $s_{i,j}$ the coordinate vector $(s_i^{(1)}, s_j^{(2)})$. The grid surface element delimited by the vertices at $s_{i,j}, s_{i+1,j}, s_{i,j+1}$ and $s_{i+1,j+1}$ will also be called the (i, j) *stage*, where $1 \leq i, j \leq N-1$. Collocation points at each stage are chosen as the surface element midpoints with coordinates

$$\begin{aligned} s_{i+1/2,j+1/2} &= \left(s_{i+1/2}^{(1)}, s_{j+1/2}^{(2)}\right) \\ &= \frac{1}{2}\left(s_{i,j} + s_{i+1,j+1}\right). \end{aligned} \tag{11}$$

A general point within the grid area is located by the coordinates $s = (s_1, s_2)$.
Discretised control values remain constant over each surface element, and equal

$$u_{\text{app}}(s) = \begin{cases} u_{\text{app}}(s_{i+1/2,j+1/2}) \text{ if } s_i^{(1)} \leq s_1 < s_{i+1}^{(1)} \text{ and } s_j^{(2)} \leq s_2 < s_{j+1}^{(2)}, i, j \in \{1, 2, \dots N-2\} \\ u_{\text{app}}(s_{i+1/2,N-1/2}) \text{ if } s_i^{(1)} \leq s_1 < s_{i+1}^{(1)} \text{ and } s_{N-1}^{(2)} \leq s_2 \leq s_N^{(2)}, i \in \{1, 2, \dots N-2\} \\ u_{\text{app}}(s_{N-1/2,j+1/2}) \text{ if } s_{N-1}^{(1)} \leq s_1 \leq s_N^{(1)} \text{ and } s_j^{(2)} \leq s_2 < s_{j+1}^{(2)}, j \in \{1, 2, \dots N-2\} \\ u_{\text{app}}(s_{N-1/2,N-1/2}) \text{ if } s_{N-1}^{(1)} \leq s_1 \leq s_N^{(1)} \text{ and } s_{N-1}^{(2)} \leq s_2 \leq s_N^{(2)}. \end{cases} \tag{12}$$

The state variables are approximated using bilinear interpolation. For simplicity of notation and with no loss in generality, the case of a single state variable $x_1 = x$ will be considered. Thus, given the point $s = (s_1, s_2)$ such that

$s_i^{(1)} \leq s_1 < s_{i+1}^{(1)}$ and $s_j^{(2)} \leq s_2 < s_{j+1}^{(2)}$ with $i, j \in \{1, 2, \ldots N-1\}$, the approximated state value $x_{\text{app}}(s)$ is interpolated as

$$\begin{aligned} x_{\text{app}}(s) = x_{\text{app}}(s_{i,j})\frac{\left(s_{j+1}^{(2)} - s_2\right)\left(s_{i+1}^{(1)} - s_1\right)}{h^{(1)}h^{(2)}} + x_{\text{app}}(s_{i,j+1})\frac{\left(s_2 - s_j^{(2)}\right)\left(s_{i+1}^{(1)} - s_1\right)}{h^{(1)}h^{(2)}} + \\ x_{\text{app}}(s_{i+1,j})\frac{\left(s_{j+1}^{(2)} - s_2\right)\left(s_1 - s_i^{(1)}\right)}{h^{(1)}h^{(2)}} + x_{\text{app}}(s_{i+1,j+1})\frac{\left(s_2 - s_j^{(2)}\right)\left(s_1 - s_i^{(1)}\right)}{h^{(1)}h^{(2)}}. \end{aligned} \tag{13}$$

For clarity, introduce the notations for $i, j \in \{1, 2, \ldots N-1\}$

$$u_{i,j} = u_{i+1/2,j+1/2} = u_{\text{app}}(s_{i+1/2,j+1/2}), \tag{14}$$

$$x_{i,j} = x_{\text{app}}(s_{i,j}) \tag{15}$$

and

$$f_{i,j}^{(\alpha)} = f^{(\alpha)}(x_{i,j}, u_{i,j}, s_{i,j}). \tag{16}$$

For the interpolated state variable value at the midpoint of the (i, j)-th surface element $(1 \leq i, j \leq N-1)$, the interpolation expression (13) then simplifies to

$$x_{\text{app}}(s_{i+1/2,j+1/2}) = \frac{1}{4}\left(x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1}\right). \tag{17}$$

Function values at midpoints are given by

$$\begin{aligned} f_{i+1/2,j+1/2}^{(\alpha)} &= f^{(\alpha)}\left(x_{i+1/2,j+1/2}, u_{i+1/2,j+1/2}, s_{i+1/2,j+1/2}\right) \\ &= f^{(\alpha)}\left(\frac{1}{4}\left(x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1}\right), u_{i,j}, \frac{1}{2}\left(s_{i,j} + s_{i+1,j+1}\right)\right) \end{aligned} \tag{18}$$

for $\alpha \in \{1, 2\}$.

The k-th component of the approximating control vector on the (i, j) stage is now denoted $(u_{i,j})_k$, with $1 \leq i, j \leq N-1$ and $1 \leq k \leq n$.

The resulting discretisation of the generic Dieudonné-Rashevsky problem presented in definition 2.1 and for the case $p = 2$ now corresponds to the following form.

Definition 3.1 (Generic discretisation of 2-D optimal control problem) *The finite dimensional nonlinear optimisation problem that discretises the two-dimensional variant of the previously defined generic Dieudonné-Rashevsky optimal control problem involves minimising the objective function*

$$J(q) = J(x_{N,N}, s_{N,N}) \tag{19}$$

such that, for $\alpha \in \{1, 2\}$ *and* $i, j \in \{1, \ldots N-1\}$,

$$f_{i+1/2,j+1/2}^{(\alpha)} - \frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(\alpha)}} = 0, \tag{20}$$

any pure control path constraints are exactly verified at the collocation points,

$$g_k(u_{i,j}) = 0 \;\forall i, j \in \{1, \ldots N-1\},\; \forall k \in \{1, \ldots r\} \tag{21}$$

and the initial and terminal conditions (if applicable) are verified, i.e.

$$\begin{cases} x_{1,1} = x^- \\ x_{N,N} = x^+ \end{cases} \tag{22}$$

The vector of optimisation variables q contains as its components the approximated state values at the partition points $x_{i,j}$, $1 \leq i, j \leq N$, as well as the approximated control values on each interval $(u_{i,j})_k$, $1 \leq i, j \leq N-1$ and $1 \leq k \leq n$.

Remark We note that possible boundaries on the controls need not be converted into explicit constraints in case the optimiser used directly supports their enforcement. The proof of the following proposition shows that their presence makes no difference to the consistency properties of the discretisation algorithm under discussion: in this proof, each control boundary constraint is given by a function $v_k(u_{i,j})$, $1 \leq k \leq n$, on each grid stage.

A convergence result showing consistency in two important respects of the discretisation algorithm with the multidimensional version of the Pontryagin Principle given in theorem 2.2 can then be established.

Proposition 3.2 (Discretisation algorithm consistency with 2-D Pontryagin conditions) *Consider a generic Dieudonné-Rashevsky optimal control problem as introduced in definition 2.1, as well as its discretisation of the form given in definition 3.1. Let $q = (u, x)'$ be a local optimum for the discretised problem; let a standard (linear independence) constraint qualification be verified for q. Then the costate equations of the multidimensional Pontryagin formulation given in theorem 2.2 are verified at the interior collocation points $s_{i+1/2,j+1/2}$ to order the chosen grid mesh h: that is, $\forall 2 \leq i, j \leq N-1$,*

$$\nabla_s \cdot \lambda = \frac{\partial \lambda^{(1)}_{i+1/2,j+1/2}}{\partial s^{(1)}} + \frac{\partial \lambda^{(2)}_{i+1/2,j+1/2}}{\partial s^{(2)}} = -\sum_{\alpha=1}^{2} \lambda^{(\alpha)}_{i+1/2,j+1/2} \frac{\partial f^{(\alpha)}_{i+1/2,j+1/2}}{\partial x} + O(h). \tag{23}$$

Also, for any optimal control in the interior of the admissible set U, the proposed scheme verifies the first-order condition on the Hamiltonian $\mathcal{H}_{md}$ at each collocation point.

Proof Given the assumptions that q is a local optimum of the problem, and that the constraint qualification is verified at q, the Karush-Kuhn-Tucker (KKT) conditions hold at q. The Lagrange function for the nonlinear optimisation problem is

$$\begin{aligned} L(q,\mu) = J(x_{N,N}, s_{N,N}) - &\\ &\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{\alpha=1}^{2} \mu^{(\alpha)}_{i,j}\left(f^{(\alpha)}_{i+1/2,j+1/2} - \frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(\alpha)}}\right) - \\ &\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{r} \xi^{(k)}_{i,j} g_k(u_{i,j}) - \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{n} \eta^{(k)}_{i,j} v_k(u_{i,j}). \end{aligned} \tag{24}$$

Here the v_k, as mentioned in remark 3, correspond to the individual control boundary inequality constraints.

The KKT gradient condition at q then applies to all of the gradient's components individually, such that, in particular,

$$\frac{\partial L}{\partial x_{i,j}} = 0 \ \forall i, j \in \{1, \ldots N\}. \tag{25}$$

But, for interior points with $2 \leq i, j \leq N-1$,

$$\begin{aligned} \frac{\partial L}{\partial x_{i,j}} = \frac{\partial J(x_{N,N}, s_{N,N})}{\partial x_{i,j}} - &\\ &\sum_{\alpha=1}^{2}\left[\mu^{(\alpha)}_{i-1,j-1}\left(\frac{\partial f^{(\alpha)}_{i-1/2,j-1/2}}{\partial x_{i,j}} - \frac{\partial}{\partial x_{i,j}}\left(\frac{\partial x_{i-1/2,j-1/2}}{\partial s^{(\alpha)}}\right)\right) - \right. \\ &\mu^{(\alpha)}_{i,j-1}\left(\frac{\partial f^{(\alpha)}_{i+1/2,j-1/2}}{\partial x_{i,j}} - \frac{\partial}{\partial x_{i,j}}\left(\frac{\partial x_{i+1/2,j-1/2}}{\partial s^{(\alpha)}}\right)\right) - \\ &\mu^{(\alpha)}_{i-1,j}\left(\frac{\partial f^{(\alpha)}_{i-1/2,j+1/2}}{\partial x_{i,j}} - \frac{\partial}{\partial x_{i,j}}\left(\frac{\partial x_{i-1/2,j+1/2}}{\partial s^{(\alpha)}}\right)\right) - \\ &\left.\mu^{(\alpha)}_{i,j}\left(\frac{\partial f^{(\alpha)}_{i+1/2,j+1/2}}{\partial x_{i,j}} - \frac{\partial}{\partial x_{i,j}}\left(\frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(\alpha)}}\right)\right)\right]. \end{aligned} \tag{26}$$

Note that the Mayer cost term $J(x_{N,N}, s_{N,N})$, the adjoined pure control path constraint terms and the control boundary terms all have zero derivatives with respect to all interior approximated state values. Now consider the individual remaining nonzero terms in this expression. From the bilinear interpolation formula (13), $\forall s = (s_1, s_2)$ such that $s^{(1)-} \leq s_1 \leq s^{(1)+}$ and $s^{(2)-} \leq s_2 \leq s^{(2)+}$,

$$\begin{aligned} \frac{\partial x_{app}}{\partial s^{(1)}}(s) &= -x_{i,j}\frac{s^{(2)}_{j+1} - s_2}{h^{(1)}h^{(2)}} - x_{i,j+1}\frac{s_2 - s^{(2)}_j}{h^{(1)}h^{(2)}} + x_{i+1,j}\frac{s^{(2)}_{j+1} - s_2}{h^{(1)}h^{(2)}} + x_{i+1,j+1}\frac{s_2 - s^{(2)}_j}{h^{(1)}h^{(2)}} \\ &= \left(s^{(2)}_{j+1} - s_2\right)\frac{x_{i+1,j} - x_{i,j}}{h^{(1)}h^{(2)}} + \left(s_2 - s^{(2)}_j\right)\frac{x_{i+1,j+1} - x_{i,j+1}}{h^{(1)}h^{(2)}}. \end{aligned} \tag{27}$$

Hence, at each collocation point,

$$\begin{aligned}\frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(1)}} &= \frac{1}{2}\left[\frac{x_{i+1,j}-x_{i,j}}{h^{(1)}} + \frac{x_{i+1,j+1}-x_{i,j+1}}{h^{(1)}}\right] \\ &= \frac{1}{2h^{(1)}}\left[x_{i+1,j}+x_{i+1,j+1}-x_{i,j}-x_{i,j+1}\right],\end{aligned} \tag{28}$$

where $1 \le i, j \le N-1$. Similarly,

$$\frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(2)}} = \frac{1}{2h^{(2)}}\left[x_{i,j+1}+x_{i+1,j+1}-x_{i,j}-x_{i+1,j}\right]. \tag{29}$$

The cross derivative terms found in equation (26) then equal

$$\begin{aligned}\frac{\partial}{\partial x_{i,j}}\left(\frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(1)}}\right) = -\frac{1}{2h^{(1)}} &= \frac{\partial}{\partial x_{i,j+1}}\left(\frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(1)}}\right) \\ &= -\frac{\partial}{\partial x_{i+1,j}}\left(\frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(1)}}\right) = -\frac{\partial}{\partial x_{i+1,j+1}}\left(\frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(1)}}\right)\end{aligned} \tag{30}$$

and

$$\begin{aligned}\frac{\partial}{\partial x_{i,j}}\left(\frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(2)}}\right) = -\frac{1}{2h^{(2)}} &= \frac{\partial}{\partial x_{i+1,j}}\left(\frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(2)}}\right) \\ &= -\frac{\partial}{\partial x_{i,j+1}}\left(\frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(2)}}\right) = -\frac{\partial}{\partial x_{i+1,j+1}}\left(\frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(2)}}\right)\end{aligned} \tag{31}$$

at the collocation points.

For the partial derivatives of the state PDE functions, use equation (18) to find that

$$\begin{aligned}\frac{\partial}{\partial x_{i,j}} f^{(\alpha)}_{i+1/2,j+1/2} &= \frac{1}{4}\frac{\partial}{\partial x} f^{(\alpha)}_{i+1/2,j+1/2} \\ &= \frac{\partial}{\partial x_{i+1,j}} f^{(\alpha)}_{i+1/2,j+1/2} = \frac{\partial}{\partial x_{i,j+1}} f^{(\alpha)}_{i+1/2,j+1/2} = \frac{\partial}{\partial x_{i+1,j+1}} f^{(\alpha)}_{i+1/2,j+1/2}\end{aligned} \tag{32}$$

$\forall 1 \le i, j \le N-1$, and $\alpha \in \{1,2\}$: this expression is identical for both signals.

The terms in the format of equations (28) and (29) respectively (32) found in expression (26) are then obtained using a change of variable. For instance,

$$\begin{aligned}\frac{\partial}{\partial x_{i,j}}\left(\frac{\partial x_{i+1/2,j-1/2}}{\partial s^{(1)}}\right) &= \frac{\partial}{\partial x_{i,(j-1)+1}}\left(\frac{\partial x_{i+1/2,(j-1)+1/2}}{\partial s^{(1)}}\right) \\ &= -\frac{1}{2h^{(1)}}\end{aligned}$$

and, similarly,

$$\begin{aligned}\frac{\partial f^{(\alpha)}_{i+1/2,j-1/2}}{\partial x_{i,j}} &= \frac{\partial f^{(\alpha)}_{i+1/2,(j-1)+1/2}}{\partial x_{i,(j-1)+1}} \\ &= \frac{1}{4}\frac{\partial f^{(\alpha)}_{i+1/2,(j-1)+1/2}}{\partial x} \\ &= \frac{1}{4}\frac{\partial f^{(\alpha)}_{i+1/2,j-1/2}}{\partial x},\end{aligned}$$

for $\alpha \in \{1,2\}$ and where the variable redefinition $j' = j-1$ is used with the set of expressions given in (28) and (32). This yields the remaining term values

$$\begin{aligned}\frac{\partial}{\partial x_{i,j}}\left(\frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(1)}}\right) &= -\frac{1}{2h^{(1)}} \\ &= -\frac{\partial}{\partial x_{i,j}}\left(\frac{\partial x_{i-1/2,j-1/2}}{\partial s^{(1)}}\right) = -\frac{\partial}{\partial x_{i,j}}\left(\frac{\partial x_{i-1/2,j+1/2}}{\partial s^{(1)}}\right),\end{aligned}$$

$$\frac{\partial}{\partial x_{i,j}}\left(\frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(2)}}\right) = -\frac{1}{2h^{(2)}} = \frac{\partial}{\partial x_{i,j}}\left(\frac{\partial x_{i-1/2,j+1/2}}{\partial s^{(2)}}\right)$$
$$= -\frac{\partial}{\partial x_{i,j}}\left(\frac{\partial x_{i-1/2,j-1/2}}{\partial s^{(2)}}\right) = -\frac{\partial}{\partial x_{i,j}}\left(\frac{\partial x_{i+1/2,j-1/2}}{\partial s^{(2)}}\right)$$

and, identically for both signal indices,

$$\begin{aligned}
\frac{\partial f^{(\alpha)}_{i+1/2,j+1/2}}{\partial x_{i,j}} &= \frac{1}{4}\frac{\partial f^{(\alpha)}_{i+1/2,j+1/2}}{\partial x}\\
\frac{\partial f^{(\alpha)}_{i-1/2,j+1/2}}{\partial x_{i,j}} &= \frac{1}{4}\frac{\partial f^{(\alpha)}_{i-1/2,j+1/2}}{\partial x}\\
\frac{\partial f^{(\alpha)}_{i-1/2,j-1/2}}{\partial x_{i,j}} &= \frac{1}{4}\frac{\partial f^{(\alpha)}_{i-1/2,j-1/2}}{\partial x}.
\end{aligned}$$

Replacing, expression (26) becomes

$$\begin{aligned}
\frac{\partial L}{\partial x_{i,j}} = -\Bigg[&\mu^{(1)}_{i-1,j-1}\left(\frac{1}{4}\frac{\partial f^{(1)}_{i-1/2,j-1/2}}{\partial x} - \frac{1}{2h^{(1)}}\right) + \mu^{(1)}_{i,j-1}\left(\frac{1}{4}\frac{\partial f^{(1)}_{i+1/2,j-1/2}}{\partial x} + \frac{1}{2h^{(1)}}\right) +\\
&\mu^{(1)}_{i-1,j}\left(\frac{1}{4}\frac{\partial f^{(1)}_{i-1/2,j+1/2}}{\partial x} - \frac{1}{2h^{(1)}}\right) + \mu^{(1)}_{i,j}\left(\frac{1}{4}\frac{\partial f^{(1)}_{i+1/2,j+1/2}}{\partial x} + \frac{1}{2h^{(1)}}\right) +\\
&\mu^{(2)}_{i-1,j-1}\left(\frac{1}{4}\frac{\partial f^{(2)}_{i+1/2,j-1/2}}{\partial x} - \frac{1}{2h^{(2)}}\right) + \mu^{(2)}_{i,j-1}\left(\frac{1}{4}\frac{\partial f^{(2)}_{i+1/2,j-1/2}}{\partial x} - \frac{1}{2h^{(2)}}\right) +\\
&\mu^{(2)}_{i-1,j}\left(\frac{1}{4}\frac{\partial f^{(2)}_{i-1/2,j+1/2}}{\partial x} + \frac{1}{2h^{(2)}}\right) + \mu^{(2)}_{i,j}\left(\frac{1}{4}\frac{\partial f^{(2)}_{i+1/2,j+1/2}}{\partial x} + \frac{1}{2h^{(2)}}\right)\Bigg]
\end{aligned}$$

or, grouping terms and using the condition $\frac{\partial L}{\partial x_{i,j}} = 0$,

$$\begin{aligned}
&\frac{\mu^{(1)}_{i,j}}{2h^{(1)}} + \frac{\mu^{(1)}_{i,j-1}}{2h^{(1)}} - \frac{\mu^{(1)}_{i-1,j}}{2h^{(1)}} - \frac{\mu^{(1)}_{i-1,j-1}}{2h^{(1)}} + \frac{\mu^{(2)}_{i,j}}{2h^{(2)}} - \frac{\mu^{(2)}_{i,j-1}}{2h^{(2)}} + \frac{\mu^{(2)}_{i-1,j}}{2h^{(2)}} - \frac{\mu^{(2)}_{i-1,j-1}}{2h^{(2)}} =\\
&\quad -\frac{1}{4}\sum_{\alpha=1}^{2}\left[\mu^{(\alpha)}_{i,j}\frac{\partial f^{(\alpha)}_{i+1/2,j+1/2}}{\partial x} + \mu^{(\alpha)}_{i-1,j}\frac{\partial f^{(\alpha)}_{i-1/2,j+1/2}}{\partial x} + \mu^{(\alpha)}_{i,j-1}\frac{\partial f^{(\alpha)}_{i+1/2,j-1/2}}{\partial x} + \mu^{(\alpha)}_{i-1,j-1}\frac{\partial f^{(\alpha)}_{i-1/2,j-1/2}}{\partial x}\right].
\end{aligned} \tag{33}$$

Now define the quantities

$$\lambda^{(\alpha)}_{i+1/2,j+1/2} = -\frac{\mu^{(\alpha)}_{i,j}}{h^{(1)}h^{(2)}},\ \alpha \in \{1,2\}, 1 \le i, j \le N-1, \tag{34}$$

such that

$$\begin{cases}
\lambda^{(\alpha)}_{i+1/2,j-1/2} = -\frac{\mu^{(\alpha)}_{i,j-1}}{h^{(1)}h^{(2)}}\\
\lambda^{(\alpha)}_{i-1/2,j+1/2} = -\frac{\mu^{(\alpha)}_{i-1,j}}{h^{(1)}h^{(2)}}\\
\lambda^{(\alpha)}_{i-1/2,j-1/2} = -\frac{\mu^{(\alpha)}_{i-1,j-1}}{h^{(1)}h^{(2)}}.
\end{cases} \tag{35}$$

It is then possible to use a bivariate Taylor expansion around each point $s_{i+1/2,j+1/2}$ in order to approximate the values for the λ at the remaining three vertices given in equations (35) for all interior values of i and j. First obtain relations giving the signal values at each of the three points as a function of $s_{i+1/2,j+1/2}$ and the grid differences $h^{(\alpha)}$. Using one-dimensional linear interpolation yields the expressions

$$\begin{aligned} s_{i-1/2,j+1/2} &= s_{i-1,j+1/2} + \frac{1}{2}\left(s_{i,j+1/2} - s_{i-1,j+1/2}\right) \\ &= s_{i+1/2,j+1/2} - \left(s_{i,j+1/2} + \frac{1}{2}s_{i+1,j+1/2} - \frac{1}{2}s_{i,j+1/2}\right) + s_{i-1,j+1/2} + \frac{1}{2}s_{i,j+1/2} - \frac{1}{2}s_{i-1,j+1/2} \\ &= s_{i+1/2,j+1/2} - \frac{1}{2}s_{i+1,j+1/2} + \frac{1}{2}s_{i-1,j+1/2} \\ &= s_{i+1/2,j+1/2} - h^{(1)}, \end{aligned} \tag{36}$$

$$s_{i+1/2,j-1/2} = s_{i+1/2,j+1/2} - h^{(2)} \tag{37}$$

and

$$\begin{aligned} s_{i-1/2,j-1/2} &= s_{i-1/2,j+1/2} - h^{(2)} \\ &= s_{i+1/2,j+1/2} - h^{(1)} - h^{(2)}. \end{aligned} \tag{38}$$

Then Taylor expansions around each $s_{i+1/2,j+1/2}$ to first order of the values for the $\lambda^{(\alpha)}$ at the three collocation points seen in equations (35) result in the costate expressions

$$\begin{cases} \lambda^{(\alpha)}_{i+1/2,j-1/2} = \lambda^{(\alpha)}_{i+1/2,j+1/2} - \frac{1}{2}h^{(2)}\frac{\partial\lambda^{(\alpha)}_{i+1/2,j+1/2}}{\partial s^{(2)}} + O\left[(h^{(2)})^2\right] \\ \lambda^{(\alpha)}_{i-1/2,j+1/2} = \lambda^{(\alpha)}_{i+1/2,j+1/2} - \frac{1}{2}h^{(1)}\frac{\partial\lambda^{(\alpha)}_{i+1/2,j+1/2}}{\partial s^{(1)}} + O\left[(h^{(1)})^2\right] \\ \lambda^{(\alpha)}_{i-1/2,j-1/2} = \lambda^{(\alpha)}_{i+1/2,j+1/2} - \frac{1}{2}h^{(1)}\frac{\partial\lambda^{(\alpha)}_{i+1/2,j+1/2}}{\partial s^{(1)}} - \frac{1}{2}h^{(2)}\frac{\partial\lambda^{(\alpha)}_{i+1/2,j+1/2}}{\partial s^{(2)}} + O\left[(h^{(1)})^2\right] + O\left[(h^{(2)})^2\right] + O\left[h^{(1)}h^{(2)}\right], \end{cases} \tag{39}$$

such that the negative of the left-hand side of equation (33) becomes

$$\begin{aligned} &\frac{1}{2}h^{(2)}\left[\lambda^{(1)}_{i+1/2,j+1/2} + \lambda^{(1)}_{i+1/2,j-1/2} - \lambda^{(1)}_{i-1/2,j+1/2} - \lambda^{(1)}_{i-1/2,j-1/2}\right] + \\ &\qquad \frac{1}{2}h^{(1)}\left[\lambda^{(2)}_{i+1/2,j+1/2} - \lambda^{(2)}_{i+1/2,j-1/2} + \lambda^{(2)}_{i-1/2,j+1/2} - \lambda^{(2)}_{i-1/2,j-1/2}\right] \\ &= \frac{1}{2}h^{(2)}\Bigg\{2\lambda^{(1)}_{i+1/2,j+1/2} - h^{(2)}\frac{\partial\lambda^{(1)}_{i+1/2,j+1/2}}{\partial s^{(2)}} + O\left[\left(h^{(2)}\right)^2\right] - \lambda^{(1)}_{i+1/2,j+1/2} + h^{(1)}\frac{\partial\lambda^{(1)}_{i+1/2,j+1/2}}{\partial s^{(1)}} + \\ &\qquad O\left[\left(h^{(1)}\right)^2\right] - \lambda^{(1)}_{i+1/2,j+1/2} + h^{(1)}\frac{\partial\lambda^{(1)}_{i+1/2,j+1/2}}{\partial s^{(1)}} + h^{(2)}\frac{\partial\lambda^{(1)}_{i+1/2,j+1/2}}{\partial s^{(2)}} + O\left[\left(h^{(1)}h^{(2)}\right)\right]\Bigg\} + \\ &\qquad \frac{1}{2}h^{(1)}\Bigg\{h^{(2)}\frac{\partial\lambda^{(2)}_{i+1/2,j+1/2}}{\partial s^{(2)}} + O\left[\left(h^{(2)}\right)^2\right] + \lambda^{(2)}_{i+1/2,j+1/2} - h^{(1)}\frac{\partial\lambda^{(2)}_{i+1/2,j+1/2}}{\partial s^{(1)}} - O\left[\left(h^{(1)}\right)^2\right] - \\ &\qquad \lambda^{(2)}_{i+1/2,j+1/2} + h^{(1)}\frac{\partial\lambda^{(2)}_{i+1/2,j+1/2}}{\partial s^{(1)}} + h^{(2)}\frac{\partial\lambda^{(2)}_{i+1/2,j+1/2}}{\partial s^{(2)}} + O\left[\left(h^{(1)}h^{(2)}\right)\right]\Bigg\} \\ &= h^{(1)}h^{(2)}\left[\frac{\partial\lambda^{(1)}_{i+1/2,j+1/2}}{\partial s^{(1)}} + \frac{\partial\lambda^{(2)}_{i+1/2,j+1/2}}{\partial s^{(2)}}\right] + O\left[\left(h^{(1)}\right)^3\right] + O\left[\left(h^{(2)}\right)^3\right] + O\left[h^{(1)}\left(h^{(2)}\right)^2\right] + \\ &\qquad O\left[\left(h^{(1)}\right)^2 h^{(2)}\right]. \end{aligned}$$

Using the grid mesh value h, as defined through equation (10), the order terms correspond to $O(h^3)$. At the same time, the negative of the right-hand side of equation (33) equals

$$
\begin{aligned}
&-\frac{1}{4}\sum_{\alpha=1}^{2}\left[h^{(1)}h^{(2)}\lambda^{(\alpha)}_{i+1/2,j+1/2}\frac{\partial f^{(\alpha)}_{i+1/2,j+1/2}}{\partial x}+h^{(1)}h^{(2)}\lambda^{(\alpha)}_{i-1/2,j+1/2}\frac{\partial f^{(\alpha)}_{i-1/2,j+1/2}}{\partial x}+\right.\\
&\qquad\left. h^{(1)}h^{(2)}\lambda^{(\alpha)}_{i+1/2,j-1/2}\frac{\partial f^{(\alpha)}_{i+1/2,j-1/2}}{\partial x}+h^{(1)}h^{(2)}\lambda^{(\alpha)}_{i-1/2,j-1/2}\frac{\partial f^{(\alpha)}_{i-1/2,j-1/2}}{\partial x}\right]\\
&=-\frac{1}{4}\sum_{\alpha=1}^{2}\left[h^{(1)}h^{(2)}\lambda^{(\alpha)}_{i+1/2,j+1/2}\frac{\partial f^{(\alpha)}_{i+1/2,j+1/2}}{\partial x}+\right.\\
&\qquad h^{(1)}h^{(2)}\left(\lambda^{(\alpha)}_{i+1/2,j+1/2}+O\left(h^{(1)}\right)\right)\left(\frac{\partial f^{(\alpha)}_{i+1/2,j+1/2}}{\partial x}+O\left(h^{(1)}\right)\right)+\\
&\qquad h^{(1)}h^{(2)}\left(\lambda^{(\alpha)}_{i+1/2,j+1/2}+O\left(h^{(2)}\right)\right)\left(\frac{\partial f^{(\alpha)}_{i+1/2,j+1/2}}{\partial x}+O\left(h^{(2)}\right)\right)+\\
&\qquad\left. h^{(1)}h^{(2)}\left(\lambda^{(\alpha)}_{i+1/2,j+1/2}+O\left(h^{(1)}\right)+O\left(h^{(2)}\right)\right)\left(\frac{\partial f^{(\alpha)}_{i+1/2,j+1/2}}{\partial x}+O\left(h^{(1)}\right)+O\left(h^{(2)}\right)\right)\right]\\
&=-h^{(1)}h^{(2)}\left[\lambda^{(1)}_{i+1/2,j+1/2}\frac{\partial f^{(1)}_{i+1/2,j+1/2}}{\partial x}+\lambda^{(2)}_{i+1/2,j+1/2}\frac{\partial f^{(2)}_{i+1/2,j+1/2}}{\partial x}\right]+O\left(h^{3}\right).
\end{aligned}
$$

Since $h^{(1)} \neq 0$ and $h^{(2)} \neq 0$, both the above expressions may be divided by $h^{(1)}h^{(2)}$ to show that, at all interior collocation points, the costate equations (6) are verified to order h, as

$$\frac{\partial\lambda^{(1)}_{i+1/2,j+1/2}}{\partial s^{(1)}}+\frac{\partial\lambda^{(2)}_{i+1/2,j+1/2}}{\partial s^{(2)}}=-\left(\lambda^{(1)}_{i+1/2,j+1/2}\frac{\partial f^{(1)}_{i+1/2,j+1/2}}{\partial x}+\lambda^{(2)}_{i+1/2,j+1/2}\frac{\partial f^{(2)}_{i+1/2,j+1/2}}{\partial x}\right)+O(h). \tag{40}$$

We have thus shown that equation (23) holds for the scheme under discussion.

Next, if the optimal control $u(s)$ is contained within the interior of the set of admissible controls U (which is, for instance, the case if U is open), the Hamiltonian minimum condition (7) requires that the first-order condition found in the Euler-Lagrange equations,

$$\frac{\partial}{\partial u(s)}\mathcal{H}_{md}(x(s),u(s),\lambda(s),s)=0\,, \tag{41}$$

be verified $\forall s\in I_S$. A form of consistency of the proposed scheme with this first-order condition can now be shown. From the KKT gradient condition at the optimum point q, the derivative of the Lagrange function with respect to any k-th component of any control variable state vector equals zero, viz.

$$\frac{\partial L}{\partial (u_{i,j})_k}=0\ \forall i,j\in\{1,\dots N-1\}. \tag{42}$$

Since, from equations (28) and (29),

$$\frac{\partial}{\partial (u_{i,j})_k}\left(\frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(\alpha)}}\right)=0\ \forall 1\le i,j\le N-1$$

and the relationships

$$\frac{\partial f}{\partial u_{i,j}}=\frac{\partial f}{\partial u}\text{ and }\frac{\partial g}{\partial u_{i,j}}=\frac{\partial g}{\partial u}$$

hold for any interior surface element, this gives

$$
\begin{aligned}
\frac{\partial L}{\partial (u_{i,j})_k}=&-\sum_{\alpha=1}^{2}\mu^{(\alpha)}_{i,j}\frac{\partial}{\partial u_k(s)}\left(f^{(\alpha)}_{i+1/2,j+1/2}\right)-\sum_{l=1}^{r}\xi^{(l)}_{i,j}\frac{\partial}{\partial u_k}\left(g^{(l)}\left(u_{i,j}\right)\right)-\\
&\sum_{l=1}^{n}\eta^{(l)}_{i,j}\frac{\partial}{\partial u_k}\left(v^{(l)}\left(u_{i,j}\right)\right).
\end{aligned} \tag{43}
$$

Now consider the third sum of terms linked to the control boundary constraints. Additional KKT conditions require that, for all $1 \le l \le n$, $1 \le i, j \le N-1$,

$$\eta^{(l)}_{i,j} v^{(l)}(u_{i,j}) = 0. \tag{44}$$

Thus either $\eta^{(l)}_{i,j} = 0$ or $v^{(l)}(u_{i,j}) = 0$ in each case. The first possibility means that the corresponding term may be dropped from the above first-order condition, while the second possibility implies that the relevant constraint is active, which would contradict the requirement that the candidate solution u under evaluation be interior to the set of admissible controls as enforced by the full set of constraints. Hence all terms involving the boundary constraints $v^{(l)}(u_{i,j})$ must be zero.

Next define variables $\psi_{i,j}$ linked to the pure control path constraints by specifying their value at each collocation point in a way directly analogous to the costates defined in equation (34), viz.

$$\psi^{(l)}_{i+1/2,j+1/2} = -\frac{\xi^{(l)}_{i,j}}{h^{(1)}h^{(2)}},\ l \in \{1, \dots r\}, 1 \le i, j \le N-1.$$

Since both $h^{(1)}$ and $h^{(2)}$ are nonzero, the original KKT condition (42) at every collocation point is then equivalent to

$$\sum_{\alpha=1}^{2} \lambda^{(\alpha)}_{i+1/2,j+1/2} \frac{\partial}{\partial u_k}\left(f^{(\alpha)}_{i+1/2,j+1/2}\right) + \sum_{l=1}^{r} \psi^{(l)}_{i+1/2,j+1/2} \frac{\partial}{\partial u_k}\left(g^{(l)}\left(u_{i,j}\right)\right) = 0. \tag{45}$$

But this is exactly the first-order optimality condition (41) applied to individual control variable components at each interior surface element midpoint $s_{i+1/2,j+1/2}$ and for the Hamiltonian $\mathcal{H}_{md}(x(s), u(s), \lambda(s), s)$ with equation (8) to which the pure control path constraints have been adjoined.

As a further expression of consistency with the continuous-time Pontryagin relations, consider that the above first-order condition is necessarily met by the costates and coefficients and so does not carry any implication on the optimal form of the control if the Hamiltonian is linear in $u(s)$. In that case, consider that the Lagrange function in equation (24) equals

$$\begin{aligned} L(q,\mu) = {} & J(x_{N,N}, s_{N,N}) + \\ & \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{\alpha=1}^{2} h^{(1)}h^{(2)}\lambda^{(\alpha)}_{i+1/2,j+1/2}\left(f^{(\alpha)}_{i+1/2,j+1/2} - \frac{\partial x_{i+1/2,j+1/2}}{\partial s^{(\alpha)}}\right) + \\ & \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{r} h^{(1)}h^{(2)}\psi^{(k)}_{i+1/2,j+1/2} g_k(u_{i,j}). \end{aligned} \tag{46}$$

Here the control boundary constraint terms all equal zero at the candidate solution because of the KKT conditions (44). The above expression contains midpoint integral approximations and so converges towards the expression

$$\begin{aligned} \lim_{h\to 0} L(q,\mu) = {} & J(x(s^+), s^+) + \\ & \sum_{\alpha=1}^{2} \int_{I_S} \left(\lambda^{(\alpha)}(s)\left(f^{(\alpha)}\left(x(s), u(s), s\right) - \frac{\partial x}{\partial s^{(\alpha)}}(s)\right)\right) ds + \\ & \sum_{k=1}^{r} \int_{I_S} \left(\lambda^{(p,k)}(s) g_k(u(s))\right) ds \end{aligned}$$

as the grid mesh h tends toward zero. This expression directly corresponds to the terminal value of the objective function (which $L(q,\mu)$ thus approximates to order h) augmented by integral expressions that implement the various problem constraints. Accordingly, the minimisation of the Lagrange function (46) carried out to solve the discretised nonlinear optimisation problem corresponds to a discretised equivalent of the Pontryagin minimum condition (7). ∎

It can hence be affirmed that the proposed scheme, in its two-dimensional version, provides an optimum which converges to theoretical necessary conditions at the collocation points. In particular, the quantities $\lambda^{(\alpha)}_{i+1/2,j+1/2}$ for $\alpha \in \{1, 2\}$ are theoretically consistent approximations to the problem costates at these same points. Any solutions to the given discretisation scheme will then be candidates (up to the order-h discretisation error, as well as any interpolation error away from the collocation points) to solve the original control problem.

4 Application: Direct discretisation of the multisignal conditioned mean-variance portfolio optimisation problem

The present section will apply the generic theory introduced in the previous section to the specific two-signal mean-variance conditioned portfolio optimisation problem. Section 4.1 introduces the type of problem under discussion. Section 4.2 specialises the discretisation algorithm of section 3. As an empirical illustration, backtest results obtained using a real-world data set are then presented in section 4.3.

4.1 Conditioned portfolio optimisation

Following a theoretical discussion in [8], conditioned portfolio optimisation was introduced in concrete terms in [6]. Just like Markowitz optimisation, conditioned optimisation exists in a discrete-time and myopic framework involving a return series (r_t) for each asset considered, where the time index t is discrete. However, it further considers a second (vector) series (s_t) called the *signal*, and assumes that there is a measurable lagged relationship μ between asset return and signal value observed, i.e. $\forall t \geq 1$,

$$r_{t+1} = \mu(s_t) + \epsilon_{t+1}, \tag{47}$$

where ϵ_t is a noise term such that $E[\epsilon_{t+1}|s_t] = 0$.

The basic intention of the conditioned portfolio optimisation problem is then to use the additional information provided by the signal s in a mathematically correct way to try and improve on the resulting performance of an otherwise classical portfolio strategy. We thus attempt to minimise expected portfolio variance for a given required level of portfolio expected return or, equivalently, to maximise quadratic expected investor utility. Similarly to the classical Markowitz context, a closed-form solution is not available when, as is important in practice, admissible portfolio weights are constrained. In this context, an optimal control formulation of conditioned optimisation problems was proposed in [2]: this covers the basic case in which a single signal is used at any particular time. The numerical scheme introduced in what precedes then allows us to formulate and solve problem variants in which multiple signals are applied at the same time. The next section does so explicitly for the two-signal case.

4.2 Two-signal numerical scheme

Based on [6], the expressions for unconditional expected return and squared return in a conditioned setting where no risk-free asset is available and the relationship between signal and returns (47) holds are

$$E(P) = E[u'(s)\mu(s)] \tag{48}$$

and

$$E(P^2) = E\Big[u'(s)\Big[\mu(s)(\mu(s))' + \Sigma_\epsilon^2\Big]u(s)\Big] \tag{49}$$

for portfolio weight function n-vectors $u(s)$ and a conditional covariance matrix Σ_ϵ^2.

We then aim to maximise the expectation of a quadratic utility function of the form

$$U(x) = a_1 x + a_2 x^2, \tag{50}$$

where a_1 needs to be positive, and a_2 negative, for the problem to make sense. Note that there is no distinction between uncentred second moment and variance in the absence of a risk-free return.

Using a minimisation problem format for consistency with the preceding discussion, the envisaged setting can be presented as follows.

Definition 4.1 (Conditioned mean-variance optimal control problem for two signals) *Consider a market in which no risk-free asset is available. Assume the simultaneous use of two signal series $s^{(1)}$ and $s^{(2)}$, with values denoted s_1 and s_2 and with two-dimensional support I_S, to provide conditioning information. Denote $s = (s_1, s_2)$. The conditioned mean-variance portfolio optimisation problem for an investor utility function in the form of equation* (50) *can be* *formulated as the following optimal control problem:*

$$\text{minimise} \qquad J_{I_S}(x(s),u(s)) = -\int_{I_S}\Big(a_1\frac{\partial^2 x_1}{\partial s^{(1)}\partial s^{(2)}} + a_2\frac{\partial^2 x_2}{\partial s^{(1)}\partial s^{(2)}}\Big)ds \tag{51}$$

$$\text{subject to} \qquad \frac{\partial^2 x_1}{\partial s^{(1)}\partial s^{(2)}} = u'(s)\mu(s)p_s(s) = f_1(x(s),u(s),s), \tag{52}$$

$$\frac{\partial^2 x_2}{\partial s^{(1)}\partial s^{(2)}} = \left(\Big(u'(s)\mu(s)\Big)^2 + u'(s)\Sigma_\epsilon^2 u(s)\right)p_s(s) = f_2(x(s),u(s),s), \tag{53}$$

$$x_1(s^-) = x_2(s^-) = 0 \tag{54}$$

$$\text{and} \qquad u(s)\in U \;\forall s\in I_S. \tag{55}$$

where s^- and s^+ are 2-vectors and the control constraint set $U\subseteq\mathbb{R}^n$ is convex.

The objective functional is thus expressed in Lagrange form, with a surface integral requiring the states to be defined using mixed derivatives. This format does not directly correspond to the Dieudonné-Rashevsky form of definition 2.1 and used in the multidimensional Pontryagin result given in theorem 2.2 as well as the generic discretisation scheme discussed in section 3. However, as suggested initially by Rashevsky (and mentioned in [4], which also gives further examples in the two-dimensional case), any such problem may be written in the Dieudonné-Rashevsky form through the introduction of auxiliary state and control variables: the problem under discussion is thus equivalent to a Dieudonné-Rashevsky formulation. The format of definition 4.1 will be preferred in this instance given it holds the significant advantage over the equivalent Dieudonné-Rashevsky format of not requiring additional auxiliary state variables and hence keeping the size of the discretised nonlinear optimisation problem (in terms of the number of generated variables) as small as possible.

The discretisation scheme previously discussed then generates the following optimisation problem:

$$\text{minimise} \qquad J(q) = J\Big((x_1)_{N,N},(x_2)_{N,N}\Big) \tag{56}$$

$$\text{subject to} \qquad (f_k)_{i+1/2,j+1/2} - \frac{\partial^2 (x_k)_{i+1/2,j+1/2}}{\partial s^{(1)}\partial s^{(2)}} = 0 \;\forall i,j\in\{1,\dots N-1\}, k\in\{1,2\} \tag{57}$$

$$\text{and} \qquad \sum_{k=1}^{n}(u_{i,j})_k - 1 = 0, i,j\in 1,\dots N-1 \tag{58}$$

$$\text{with} \qquad (x_1)_{1,1} = (x_2)_{1,1} = 0 \tag{59}$$

and $U=\mathbb{R}^n_+$ to eliminate the possibility of short positions. All discretised state and control variables are then sampled on a two-dimensional grid, and approximations of the state variable mixed partial derivatives need to be provided at the collocation points. Using a two-dimensional midpoint integration scheme over all stage collocation points, the discretised objective function is calculated as

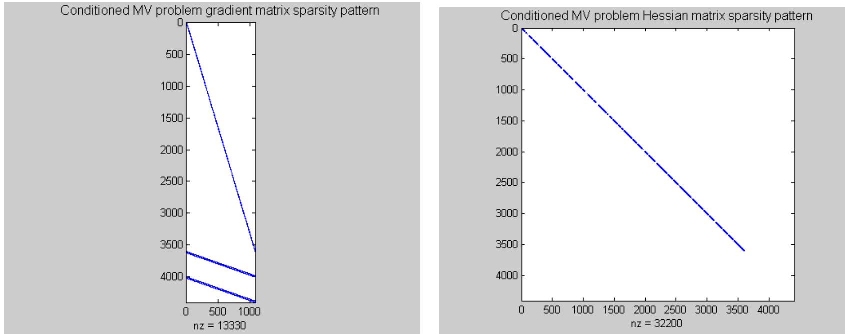
$$\begin{aligned} J(q) = -\Bigg(&a_1\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}u'_{i,j}\mu(s_{i+1/2,j+1/2})p_s(s_{i+1/2,j+1/2}) + \\ &a_2\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\Big((u'_{i,j}\mu(s_{i+1/2,j+1/2}))^2 + u'_{i,j}\Sigma_\epsilon^2 u_{i,j}\Big)p_s(s_{i+1/2,j+1/2})\Bigg)h^{(1)}h^{(2)}. \end{aligned} \tag{60}$$

Given the standard finite difference approximation of second-order mixed partial derivatives (see e.g. [1]) as

$$\frac{\partial^2 x_{i+1/2,j+1/2}}{\partial s^{(1)}\partial s^{(2)}} = \frac{1}{h^{(1)}h^{(2)}}\Big(x_{i,j} + x_{i+1,j+1} - x_{i+1,j} - x_{i,j+1}\Big), \tag{61}$$

at any collocation point $s_{i+1/2,j+1/2}$, the collocation constraints of equations (57) for both expected moments of returns are then for all $i,j\in\{1,\dots N-1\}$

$$\begin{aligned} &\frac{1}{h^{(1)}h^{(2)}}\Big((x_1)_{i,j} + (x_1)_{i+1,j+1} - (x_1)_{i+1,j} - (x_1)_{i,j+1}\Big) - \\ &\qquad u'_{i,j}\mu(s_{i+1/2,j+1/2})p_s(s_{i+1/2,j+1/2}) = 0 \end{aligned} \tag{62}$$



(a) Gradient matrix sparsity pattern.

(b) Hessian matrix sparsity pattern.

Figure 1: Sparsity patterns of gradient (left) and Hessian matrices associated to the two-signal conditioned mean-variance portfolio optimisation problem.

Problem size	Total # (T)	Nonzero # (NZ)	NZ / T	Normal storage (N)	Sparse storage (S)	S / N
$N = 20, n = 10$	4784850	13720	0.29%	38278800	164676	0.43%
$N = 50, n = 10$	$2.09 \cdot 10^8$	91240	0.04%	$1.672 \cdot 10^{9(*)}$	1091676	0.07%
$N = 20, n = 30$	12618550	35380	0.28%	$1.01 \cdot 10^8$	415956	0.41%

Table 1: Matrix sparseness metrics for the gradient matrix associated with conditioned two-dimensional mean-variance optimisation problems of several sizes.

and

$$\frac{1}{h^{(1)}h^{(2)}}\Big((x_2)_{i,j} + (x_2)_{i+1,j+1} - (x_2)_{i+1,j} - (x_2)_{i,j+1}\Big) - u'_{i,j}\Big[\mu(s_{i+1/2,j+1/2})\mu'(s_{i+1/2,j+1/2}) + \Sigma_\epsilon^2\Big]u_{i,j}p_s(s_{i+1/2,j+1/2}) = 0. \tag{63}$$

This two-dimensional problem generates $(N-1)^2 n + 2N^2$ variables when discretised. For a grid of $N = 20$ points per dimension and a market of $n = 10$ assets, this corresponds to 4410 variables. The corresponding dimensions of gradient and Hessian matrices are $(N-1)^2 n + 2N^2 \times 3(N-1)^2 + 2$ and $(N-1)^2 n + 2N^2 \times (N-1)^2 n + 2N^2$, respectively: while the number of variables scales linearly with the size of the asset universe (such that even large portfolio sizes remain manageable using the proposed algorithm), problem dimension shows a power function impact. Figure 1 shows the gradient and Hessian matrix sparsity patterns for the case $N = 20$ and $n = 10$. The nonzero quantities in the gradient and Hessian matrices are obtained through direct differentiation of the collocation constraints (62) and (63).

Sparseness metrics for the gradient and Hessian matrices associated to the two-dimensional problem are given in tables 1 and 2[2]. The respective values NZ giving the number of nonzero elements for both matrices are $2 + (N-1)^2(3n+8)$ for the gradient matrix and $(N-1)^2 n^2$ for the Hessian matrix. We note that optimised use of sparse matrix storage leaves the two-dimensional discretised optimisation problem more manageable, in terms of memory utilisation and computational cost, than the gradient and Hessian matrix sizes on their own might suggest.

[2] In both tables, configurations marked with an asterisk cause out of memory errors on the PC used and are thus computed values based on eight-byte floating point storage.

Problem size	Total # (T)	Nonzero # (NZ)	NZ / T	Normal storage (N)	Sparse storage (S)	S / N
$N = 20, n = 10$	19448100	36100	0.19%	$1.56 \cdot 10^8$	407656	0.26%
$N = 50, n = 10$	$8.42 \cdot 10^8$	240100	0.03%	$6.73 \cdot 10^{9(*)}$	2678656	0.04%
$N = 20, n = 30$	$1.35 \cdot 10^8$	324900	0.24%	$1.082 \cdot 10^{9(*)}$	3556536	0.33%

Table 2: Matrix sparseness metrics for the Hessian matrix associated with conditioned mean-variance optimisation problems of several sizes. Configurations marked with an asterisk cause out of memory errors on the PC used and are thus theoretical values based on eight-byte floating point storage.

4.3 Two-signal empirical analysis

We obtain backtest results by applying the numerical algorithm described to a real-world data set, which collects eleven years of daily returns data chosen to represent a market relevant to investors with domestic currency EUR. This market is made up of ten different funds[3] chosen across both equity and fixed income markets as well as Morningstar style classifications. All funds involved provide EUR return quotes and manage at most a proportion of 30% in non-EUR assets so as to manage the impact of currency risk on the choice of investments. The data covers business days from January 1999 to February 2010: in total, each series contains 2891 returns. Funds rather than individual assets were chosen given they provide a level of built-in diversification and a ten-asset market composed of funds is thus seen as more realistic as an equivalent equity market; additionally, interest-rate exposure is easily achieved through funds. Investment strategies involving funds and requiring frequent portfolio rebalancings, such as the ones being examined, have become realistically achievable even for small investors with the advent of exchange traded funds (ETF). Although the funds listed above are not ETFs, this choice was made purely because of the need to obtain sufficiently exhaustive historical data series: funds comparable to those used are nowadays accessible in an ETF format. The time interval covered by the data is especially interesting as it encompasses two major crises as far as market returns are concerned: the bursting of the dot-com bubble (roughly from spring 2000 to spring 2003) and the initial bear market triggered by the outbreak of the currently ongoing financial crisis (roughly spanning summer 2007 to spring 2009).

Whilst there are many possible candidates that may be used as conditioning signals[4], we will exercise a single pair of signals in the present illustration. This is chosen on the basis that the universe of funds introduced covers equity and interest rates as its main two risky markets: thus use of one signal each for either of these markets is a sensible and realistic choice. Equity risk, or investor attitude to equity risk, is represented by the VDAX index, which is a DAX equivalent of the CBOE volatility index VIX (see e.g. [14]) and may be hoped to contain some information on future equity returns. For the interest rate market, the annualised volatility figures for the Barclays Euro Aggregate bond index are used as a fixed income analogue of the VDAX. Both these indices thus represent the 'pure' market risk (perception) associated with the two main markets considered.

The relationship between returns and signal is represented by a linear regression with intercept, such that we fit $\mu_i(s) = a_i + b_i s$ for every asset i. In each case, we backtest the strategy by working through the full data set, using rolling historical estimation windows of a fixed size of 60 points, determining the optimal portfolio weights and checking at the next date what returns they would have yielded. Portfolio rebalancing is carried out each business day. We estimate the signal densities nonparametrically using kernel density estimates with a Gaussian kernel. The conditional covariance matrix of asset returns is obtained as the sample covariance matrix of the regression residuals. The optimal control problem for each rebalancing day is solved numerically using the algorithm presented and for three different quadratic[5] utility functions in the form of equation (50) and specified by the defining pairs (a_1, a_2), with $a_1 = 1$ in each case and a_2 equal to respectively -0.2, -0.5 and -0.7. Unconditioned optimisation and single-signal (VDAX) backtests used as comparative benchmarks are computed in parallel. We then compare the different strategies using metrics observed both ex ante and ex post. Ex ante results (as is the case with the efficient frontier in Markowitz theory) are obtained in model and thus ignore model risk, while ex post results are obtained after replacing the observed returns. We compute Sharpe ratio and utility as both an ex ante and an ex post metric, along with the additional ex post metrics of mean returns achieved, achieved standard deviations of returns, and maximum drawdowns respectively maximum drawdown durations observed.

Table 3 presents the backtesting results obtained. Firstly, the ex ante results are very impressive. Looking at both the Sharpe ratio and utility figures, the improvements expected from the introduction of the second signal are seen to be comparable to those achieved through use of a single signal relative to classical strategies[6]. The ex ante results given for the two individual moments pertinent to the optimisation show that an increase in return volatility is expected in the two-signal case, but that it is outweighed by the expected improvement in returns for both trade-off metrics used. The relative levels to which this is the case can also be seen to depend on the particular utility function, and its attendant level of investor risk aversion. Thus, the increase in standard deviation (and hence the expected second moment of returns as used in the utility function) is less significant for the two-signal case when the third utility function is used (0.244 from 0.220, i.e. a relative increase of about 10.91%) than when the investor exhibits the greater level of risk

[3] AXA L Fund Equity Europe (AXA), Credit Suisse Bond Fund Management Company Luxembourg Small (CSU), Dekalux Midcap TF (DEK), Dexia Luxpart C (DEX), DWS Euro Bonds Long (DWS), Fidelity Funds Euro Bond Fund A Global Certificate (FIB), Fortis L Fund Equity Socially Responsible Europe (FOB), Invesco Pan European Small Cap Equity Fund Lux (INV), KBC Money Euro Medium Cap (KBC) and Morgan Stanley European Currencies High Yield Bond (MSE). In every case the reinvesting variant of the fund was picked.

[4] [5] is a helpful survey discussing different types of risk aversion indicators that may be suitable. Also, the paper [3] discusses several different signals applied to the same data set used in this section.

[5] The use of quadratic utilities makes sense if investors are concerned only with the first two moments of returns, which any mean-variance optimisation process implicitly assumes.

[6] See [3] for a more detailed discussion of these results.

Utility function a_2	-0.2	-0.5	-0.7
Ex ante			
MV UNC SR	0.343	0.395	0.415
MV UNC utility	0.107	0.070	0.057
MV UNC expected return	0.129	0.090	0.076
MV UNC standard deviation	0.331	0.200	0.163
VDAX SR	0.416	0.489	0.519
VDAX utility	0.167	0.104	0.087
VDAX expected return	0.212	0.147	0.121
VDAX standard deviation	0.475	0.281	0.220
VDAX/BONDIDX SR	0.472	0.553	0.588
VDAX/BONDIDX utility	0.210	0.134	0.108
VDAX/BONDIDX expected return	0.267	0.183	0.150
VDAX/BONDIDX standard deviation	0.536	0.315	0.244
Ex post			
MV UNC SR	0.127	0.158	0.172
MV UNC utility	0.015	0.003	0.001
MV UNC MD	22.664%	15.235%	11.675%
MV UNC MDD	811	687	677
MV UNC mean return	0.060	0.046	0.041
MV UNC standard deviation	0.472	0.293	0.237
VDAX SR	0.162	0.207	0.216
VDAX utility	0.029	0.012	0.007
VDAX MD	12.824%	7.224%	6.032%
VDAX MDD	474	191	191
VDAX mean return	0.088	0.071	0.059
VDAX standard deviation	0.543	0.342	0.272
VDAX/BONDIDX SR	0.155	0.205	0.215
VDAX/BONDIDX utility	0.023	0.010	0.005
VDAX/BONDIDX MD	12.540%	6.135%	5.010%
VDAX/BONDIDX MDD	260	157	157
VDAX/BONDIDX mean return	0.089	0.073	0.061
VDAX/BONDIDX standard deviation	0.571	0.360	0.284

Table 3: Mean (ex ante and ex post) metrics of portfolio returns obtained for unconditioned (MV UNC) and 1-signal conditioned (VDAX) as well as 2-signal conditioned (VDAX/BONDIDX) quadratic polynomial utility function based mean-variance optimisation, VDAX and BONDIDX signals.

appetite implied by the first utility function (0.536 from 0.475, that is an increase of 12.84%). It can then be said that the mechanics of the two-dimensional problem work, and that the numerical solution procedure introduced in the previous sections generates solutions that are consistent with expected results: indeed, within the ex ante model, the additional degrees of freedom afforded by the move to vector relationships between signals and returns should by construction allow for performance increases as additional signals are introduced.

However, the situation with respect to ex post results is somewhat different. Looking at the Sharpe ratios and utilities observed, it is apparent that the two-signal strategy now performs marginally worse than the single-signal optimum for all three utility functions applied. Whilst the ex post standard deviation is indeed found to be slightly larger in the two-signal case, the expected improvement in return has failed to appear to any significant extent: even though the average observed return is higher when two signals are used, the increase is almost negligible, such that the risk-return trade-off is preferable in the single signal case. This observation is marginal, and likely not to be statistically significant across the population of possible asset universes. Figure 2 plots the time paths for the additive simple returns function obtained in the case of the second utility function (with risk aversion coefficient $a_2 = -0.5$) to illustrate that the two cases are largely indifferent as far as a preference ordering is concerned.

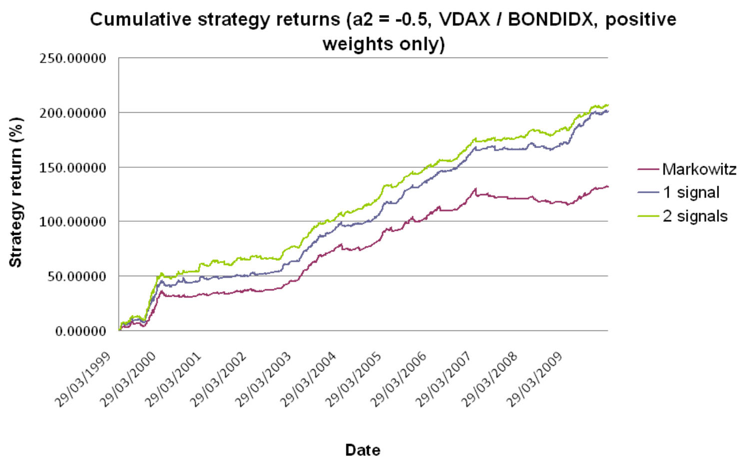


Figure 2: Time paths of additive returns observed for both the Markowitz and conditioned strategies, one and two simultaneous (VDAX and BONDIDX) signals, weights constrained to be positive, 60 day estimation windows.

The figure shows that the two-signal strategy returns curve very slightly dominates its single-signal equivalent over the entire backtest period. Apparent differences between the two curves concentrate on the two crisis periods covered by the data set. For the dot-com bubble deflation period, the divergence between the two strategies' performances is unclear in terms of preferential order: while the two-signal strategy initially extends its advance, it then loses that ground toward the end of 2002. This is also the only interval over which volatility of the two-signal returns is visibly larger than is the case for the single-signal optima. With respect to the bear market spanning 2007 to 2009, however, the two-signal curve is unquestionably preferable. The slight subjective preference for the two-signal strategy suggested from this figure (as opposed to the measured Sharpe ratio and utility values) is consistent with the maximum drawdown and drawdown duration figures given in table 3.

By way of a conclusion, then, it seems legitimate to state that no significant preference for either the single-signal or the two-signal strategy can be inferred from the backtest results just discussed. On the other hand, the ex ante improvements seen when the second signal is exercised are sizeable. This discrepancy may be due to an averaging effect caused by the simultaneous use of different signals, show up the estimation risk attached to the use of a simple linear regression model for the calibration of the signal to return relationship (47) or constitute the result of overfitting, given that the number of degrees of freedom available through the discretised optimisation problem increases as a power function of problem dimension. Seen from the viewpoint of a finance practitioner, the usefulness of the given multiple-signal optimal control problem formulation has thus not been shown conclusively, and further investigations are needed in this respect. However, the behaviour of ex ante results shows that the approach is able to solve the two-dimensional problem case robustly and in line with theoretical expectations. The relevance of multisignal conditioned optimisation as an illustration for the discretisation scheme under discussion is therefore beyond doubt.

5 Conclusion

The present paper has introduced a new direct discretisation algorithm for multidimensional optimal control problems, with the two-dimensional case discussed in detail. The general two-dimensional problem format is initially presented and a standard associated set of necessary conditions for optimality are given. The algorithm is then introduced and a convergence result with respect to the necessary conditions is presented and shown. The geometrical simplicity and symmetry of the given scheme implies that generalisation to higher dimensions is technically straightforward.

We then introduce conditioned portfolio optimisation as an example application. After specialising the discretisation algorithm and describing a backtest approach, we discuss the results obtained and find that introduction of a second signal significantly improves ex ante performance metrics. The two-signal case is further seen to marginally improve ex post performance during crisis periods. In this way, the ex ante observations validate the algorithm presented and the application serves to illustrate the type of context in which the scheme can profitably be put to use.

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