Associative and Quasitrivial Operations on Finite Sets
Characterizations and Enumeration

Jean-Luc Marichal

University of Luxembourg
Luxembourg

in collaboraton with Miguel Couceiro and Jimmy Devillet
Part I: Single-peaked orderings
Motivating example (Romero, 1978)

Suppose you are asked to order the following six objects in decreasing preference:

\[ a_1 : 0 \text{ sandwich} \]
\[ a_2 : 1 \text{ sandwich} \]
\[ a_3 : 2 \text{ sandwiches} \]
\[ a_4 : 3 \text{ sandwiches} \]
\[ a_5 : 4 \text{ sandwiches} \]
\[ a_6 : \text{ more than 4 sandwiches} \]

We write \( a_i \prec a_j \) if \( a_i \) is preferred to \( a_j \)
Single-peaked orderings

\[ a_1 : \quad 0 \text{ sandwich} \]
\[ a_2 : \quad 1 \text{ sandwich} \]
\[ a_3 : \quad 2 \text{ sandwiches} \]
\[ a_4 : \quad 3 \text{ sandwiches} \]
\[ a_5 : \quad 4 \text{ sandwiches} \]
\[ a_6 : \quad \text{more than 4 sandwiches} \]

- after a good lunch: \( a_1 \prec a_2 \prec a_3 \prec a_4 \prec a_5 \prec a_6 \)
- if you are starving: \( a_6 \prec a_5 \prec a_4 \prec a_3 \prec a_2 \prec a_1 \)
- a possible intermediate situation: \( a_4 \prec a_3 \prec a_5 \prec a_2 \prec a_1 \prec a_6 \)
- a quite unlikely preference: \( a_6 \prec a_5 \prec a_2 \prec a_1 \prec a_3 \prec a_4 \)
Single-peaked orderings

Let us represent graphically the latter two preferences with respect to the reference ordering $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$

$\begin{align*}
\text{a}_4 &\prec \text{a}_3 \prec \text{a}_5 \prec \text{a}_2 \prec \text{a}_1 \prec \text{a}_6 \\
\text{a}_6 &\prec \text{a}_5 \prec \text{a}_2 \prec \text{a}_1 \prec \text{a}_3 \prec \text{a}_4
\end{align*}$
Single-peaked orderings

Single-peakedness

\[ a_i < a_j < a_k \implies a_j < a_i \text{ or } a_j < a_k \]

Forbidden patterns
Single-peaked orderings

Definition (Black, 1948)

Let $\leq$ and $\preceq$ be total orderings on $X_n = \{a_1, \ldots, a_n\}$. Then $\preceq$ is said to be **single-peaked for** $\leq$ if for any $a_i, a_j, a_k \in X_n$ such that $a_i < a_j < a_k$ we have $a_j \prec a_i$ or $a_j \prec a_k$.

Let us assume that $X_n = \{a_1, \ldots, a_n\}$ is endowed with the ordering $a_1 < \cdots < a_n$

For $n = 4$

$$
\begin{align*}
& a_1 \prec a_2 \prec a_3 \prec a_4 & a_4 \prec a_3 \prec a_2 \prec a_1 \\
& a_2 \prec a_1 \prec a_3 \prec a_4 & a_3 \prec a_2 \prec a_1 \prec a_4 \\
& a_2 \prec a_3 \prec a_1 \prec a_4 & a_3 \prec a_2 \prec a_4 \prec a_1 \\
& a_2 \prec a_3 \prec a_4 \prec a_1 & a_3 \prec a_4 \prec a_2 \prec a_1
\end{align*}
$$

There are $2^{n-1}$ total orderings $\preceq$ on $X_n$ that are single-peaked for $\leq$
Single-peaked orderings

Recall that a *weak ordering* (or *total preordering*) on $X_n$ is a binary relation $\preceq$ on $X_n$ that is total and transitive.

Defining a weak ordering on $X_n$ amounts to defining an ordered partition of $X_n$

$$C_1 \prec \cdots \prec C_k$$

where $C_1, \ldots, C_k$ are the equivalence classes defined by $\sim$

For $n = 3$, we have 13 weak orderings

\[
\begin{align*}
    a_1 &\prec a_2 \prec a_3 & a_1 &\sim a_2 \prec a_3 & a_1 &\sim a_2 \sim a_3 \\
    a_1 &\prec a_3 \prec a_2 & a_1 &\prec a_2 \sim a_3 \\
    a_2 &\prec a_1 \prec a_3 & a_2 &\prec a_1 \sim a_3 \\
    a_2 &\prec a_3 \prec a_1 & a_3 &\prec a_1 \sim a_2 \\
    a_3 &\prec a_1 \prec a_2 & a_1 &\sim a_3 \prec a_2 \\
    a_3 &\prec a_2 \prec a_1 & a_2 &\sim a_3 \prec a_1
\end{align*}
\]
Single-peaked orderings

**Definition**

Let $\leq$ be a total ordering on $X_n$ and let $\prec$ be a weak ordering on $X_n$. We say that $\prec$ is **weakly single-peaked for** $\leq$ if for any $a_i, a_j, a_k \in X_n$ such that $a_i \prec a_j \prec a_k$ we have $a_j \prec a_i$ or $a_j \prec a_k$ or $a_i \sim a_j \sim a_k$.

Let us assume that $X_n$ is endowed with the ordering $a_1 < \cdots < a_n$.

For $n = 3$

$$
\begin{align*}
    a_1 &\prec a_2 \prec a_3 & a_1 \sim a_2 \prec a_3 & a_1 \sim a_2 \sim a_3 \\
    a_1 &\prec a_3 \prec a_2 & a_1 \prec a_2 \sim a_3 \\
    a_2 &\prec a_1 \prec a_3 & a_2 \prec a_1 \sim a_3 \\
    a_2 &\prec a_3 \prec a_1 & a_3 \prec a_1 \sim a_2 \\
    a_3 &\prec a_1 \prec a_2 & a_1 \sim a_3 \prec a_2 \\
    a_3 &\prec a_2 \prec a_1 & a_2 \sim a_3 \prec a_1
\end{align*}
$$
Single-peaked orderings

Examples

\[ a_3 \sim a_4 \prec a_2 \prec a_1 \sim a_5 \prec a_6 \]
\[ a_3 \sim a_4 \prec a_2 \sim a_1 \prec a_5 \prec a_6 \]

Forbidden patterns
Part II: Associative and quasitrivial operations
Let $F : X_n^2 \to X_n$ be an operation on $X_n = \{1, \ldots, n\}$

**Definition**

- The points $(u, v)$ and $(x, y)$ of $X_n^2$ are said to be $F$-connected if
  \[ F(u, v) = F(x, y) \]

- The point $(x, y)$ of $X_n^2$ is said to be $F$-isolated if it is not $F$-connected to another point.
Connectedness and Contour plots

Examples
Definition

For any \( x \in X_n \), the \textit{F-degree of} \( x \), denoted \( \text{deg}_F(x) \), is the number of points \((u, v) \neq (x, x)\) such that \( F(u, v) = F(x, x) \).

Remark. The point \((x, x)\) is \( F \)-isolated iff \( \text{deg}_F(x) = 0 \).
Connectedness and Contour plots

Examples
Quasitriviality

Definition

$F : X_n^2 \rightarrow X_n$ is said to be

- *quasitrivial* (or *conservative*) if

  $$F(x, y) \in \{x, y\} \quad (x, y \in X_n)$$

- *idempotent* if

  $$F(x, x) = x \quad (x \in X_n)$$

**Fact.** If $F$ is quasitrivial, then it is idempotent

**Fact.** If $F$ is idempotent and if $(x, y)$ is $F$-isolated, then $x = y$

$$F(x, y) = F(F(x, y), F(x, y))$$
Quasitriviality

Let $\Delta_{X_n} = \{(x, x) \mid x \in X_n\}$

**Fact**

$F : X_n^2 \to X_n$ is quasitrivial iff

- it is idempotent
- every point $(x, y) \notin \Delta_{X_n}$ is $F$-connected to either $(x, x)$ or $(y, y)$

**Corollary.** If $F$ is quasitrivial, then it has at most one $F$-isolated point.
Neutral and annihilator elements

Definition

- $e \in X_n$ is said to be a **neutral element** of $F : X_n^2 \to X_n$ if

  $$F(x, e) = F(e, x) = x, \quad x \in X_n$$

- $a \in X_n$ is said to be an **annihilator element** of $F : X_n^2 \to X_n$ if

  $$F(x, a) = F(a, x) = a, \quad x \in X_n$$
Neutral and annihilator elements

Proposition

Assume that $F : X_n^2 \rightarrow X_n$ is quasitrivial.

- $e \in X_n$ is a neutral element of $F$ iff $\deg_F(e) = 0$
- $a \in X_n$ is an annihilator element of $F$ iff $\deg_F(a) = 2n - 2$. 
Associative, quasitrivial, and commutative operations

Theorem

Let $F : X_n^2 \rightarrow X_n$. The following assertions are equivalent.

(i) $F$ is associative, quasitrivial, and commutative
(ii) $F = \max_{\leq}$ for some total ordering $\leq$ on $X_n$

The total ordering $\leq$ is uniquely determined as follows:

$$x \leq y \iff \deg_F(x) \leq \deg_F(y)$$

Fact. There are exactly $n!$ such operations.
Theorem

Let $F : X_n^2 \to X_n$. The following assertions are equivalent.

(i) $F$ is associative, quasitrivial, and commutative

(ii) $F = \max_{\preceq}$ for some total ordering $\preceq$ on $X_n$

(iii) $F$ is quasitrivial and $\{\deg_F(x) \mid x \in X_n\} = \{0, 2, 4, \ldots, 2n - 2\}$
Associative, quasitrivial, and commutative operations

Definition.

\( F : X_n^2 \rightarrow X_n \) is said to be \( \leq \)-preserving for some total ordering \( \leq \) on \( X_n \) if for any \( x, y, x', y' \in X_n \) such that \( x \leq x' \) and \( y \leq y' \), we have \( F(x, y) \leq F(x', y') \).

Theorem

Let \( F : X_n^2 \rightarrow X_n \). The following assertions are equivalent.

(i) \( F \) is associative, quasitrivial, and commutative

(ii) \( F = \max_\leq \) for some total ordering \( \leq \) on \( X_n \)

(iii) \( F \) is quasitrivial and \( \{\deg_F(x) \mid x \in X_n\} = \{0, 2, 4, \ldots, 2n-2\} \)

(iv) \( F \) is quasitrivial, commutative, and \( \leq \)-preserving for some total ordering \( \leq \) on \( X_n \)
Definition.

A **uninorm on** $X_n$ is an operation $F : X_n^2 \rightarrow X_n$ that

- has a neutral element $e \in X_n$

and is

- associative
- commutative
- $\leq$-preserving for some total ordering $\leq$ on $X_n$
Theorem

Let $F : X_n^2 \rightarrow X_n$. The following assertions are equivalent.

(i) $F$ is associative, quasitrivial, and commutative

(ii) $F = \max_{\preceq}$ for some total ordering $\preceq$ on $X_n$

(iii) $F$ is quasitrivial and $\{\text{deg}_F(x) | x \in X_n\} = \{0, 2, 4, \ldots, 2n - 2\}$

(iv) $F$ is quasitrivial, commutative, and $\leq$-preserving for some total ordering $\leq$ on $X_n$

(v) $F$ is an idempotent uninorm on $X_n$ for some total ordering $\leq$ on $X_n$
Associative, quasitrivial, and commutative operations

Assume that $X_n = \{1, \ldots, n\}$ is endowed with the usual total ordering $\leq_n$ defined by $1 <_n \cdots <_n n$.

**Theorem**

Let $F : X_n^2 \to X_n$. The following assertions are equivalent.

(i) $F$ is quasitrivial, commutative, and $\leq_n$-preserving ($\Rightarrow$ associative)

(ii) $F = \max_{\preceq}$ for some total ordering $\preceq$ on $X_n$ that is single-peaked for $\leq_n$.
Associative, quasitrivial, and commutative operations

Remark.

- There are $n!$ operations $F : X_n^2 \to X_n$ that are associative, quasitrivial, and commutative.
- There are $2^{n-1}$ of them that are $\leq_n$-preserving
Associative and quasitrivial operations

Examples of noncommutative operations
Definition.

The *projection operations* \( \pi_1 : X_n^2 \to X_n \) and \( \pi_2 : X_n^2 \to X_n \) are respectively defined by

\[
\pi_1(x, y) = x, \quad x, y \in X_n
\]

\[
\pi_2(x, y) = y, \quad x, y \in X_n
\]
Assume that $X_n = \{1, \ldots, n\}$ is endowed with a weak ordering $\preceq$

**Ordinal sum of projections**

$$\text{osp}_{\preceq} : X_n^2 \to X_n$$

Permuting the elements related to a box does not change the graph of $F$
Theorem (Länger 1980)

Let $F : X_n^2 \rightarrow X_n$. The following assertions are equivalent.

(i) $F$ is associative and quasitrivial

(ii) $F = \text{osp}_{\lesssim}$ for some weak ordering $\lesssim$ on $X_n$

The weak ordering $\lesssim$ is uniquely determined as follows:

$$x \lesssim y \iff \deg_F(x) \leq \deg_F(y)$$
Examples

\[ 1 < 2 < 3 < 4 \]

\[ 2 \sim 1 \sim 3 \sim 4 \]

\[ 1 < 2 < 3 < 4 \]

\[ 1 \sim 4 \sim 2 \sim 3 \]
Assessing and quasitrivial operations

How to check whether a quasitrivial operation $F : X_n^2 \rightarrow X_n$ is associative?

1. Order the elements of $X_n$ according to the weak ordering $\preceq$ defined by
   
   $x \preceq y \iff \deg_F(x) \leq \deg_F(y)$

2. Check whether the resulting operation is one of the corresponding ordinal sums
Associative and quasitrivial operations

Which ones are $\leq$-preserving?

1 $<$ 2 $<$ 3 $<$ 4

2 $\asymp$ 1 $\sim$ 3 $\asymp$ 4

1 $\asymp$ 4 $\asymp$ 2 $\sim$ 3
Assume that $X_n = \{1, \ldots, n\}$ is endowed with the usual total ordering $\leq_n$ defined by $1 <_n \cdots <_n n$.

**Theorem**

Let $F : X_n^2 \to X_n$. The following assertions are equivalent.

(i) $F$ is associative, quasitrivial, and $\leq_n$-preserving

(ii) $F = \text{osp}_{\preceq}$ for some weak ordering $\preceq$ on $X_n$ that is weakly single-peaked for $\leq_n$. 

Associative and quasitrivial operations
Associative and quasitrivial operations
1. We have introduced and identified a number of integer sequences in http://oeis.org
   - Number of associative and quasitrivial operations: A292932
   - Number of associative, quasitrivial, and $\leq_n$-preserving operations: A293005
   - Number of weak orderings on $X_n$ that are weakly single-peaked for $\leq_n$: A048739
   - ...

2. Most of our characterization results still hold on arbitrary sets $X$ (not necessarily finite)
Some references

N. L. Ackerman.
A characterization of quasitrivial \( n \)-semigroups.
To appear in Algebra Universalis.

S. Berg and T. Perlinger.
Single-peaked compatible preference profiles: some combinatorial results.

D. Black.
On the rationale of group decision-making.

M. Couceiro, J. Devillet, and J.-L. Marichal.
Quasitrivial semigroups: characterizations and enumerations.

H. Länger.
The free algebra in the variety generated by quasi-trivial semigroups.

N. J. A. Sloane (editor).
The On-Line Encyclopedia of Integer Sequences.
http://oeis.org/