On bisymmetric and quasitrivial operations
56th ISFE

Jimmy Devillet

University of Luxembourg
Let $X$ be a nonempty set and let $F : X^2 \to X$

**Definition**

- The points $(x, y), (u, v) \in X^2$ are *F-connected* if
  \[ F(x, y) = F(u, v) \]

- The point $(x, y) \in X^2$ is *F-isolated* if it is not F-connected to another point in $X^2$
For any integer $n \geq 1$, let $X_n = \{1, \ldots, n\}$ endowed with $\leq$

**Example.** $F(x, y) = \max\{x, y\}$ on $(X_4, \leq)$
Bisymmetry and quasitriviality

Definition

$F : X^2 \rightarrow X$ is said to be

- **bisymmetric** if

  \[ F(F(x, y), F(u, v)) = F(F(x, u), F(y, v)) \quad x, y, u, v \in X \]

- **quasitrivial** if

  \[ F(x, y) \in \{x, y\} \quad x, y \in X \]

Lemma (Kepka, 1981)

If $F$ is bisymmetric and quasitrivial then it is associative
Weak orderings

Recall that a binary relation $R$ on $X$ is said to be

- **total** if $\forall x, y: xRy$ or $yRx$
- **transitive** if $\forall x, y, z: xRy$ and $yRz$ implies $xRz$

A *weak ordering on* $X$ *is a binary relation* $\preceq$ *on* $X$ *that is total and transitive.*

- symmetric part: $\sim$
- asymmetric part: $\prec$

Recall that $\sim$ is an equivalence relation on $X$ and that $\prec$ induces a linear ordering on the quotient set $X/\sim$. 
Motivation

**Theorem (Länger, 1980)**

$F$ is associative and quasitrivial iff there exists a weak ordering $\preceq$ on $X$ such that

$$F|_{A \times B} = \begin{cases} 
\max_{\preceq} |A \times B|, & \text{if } A \neq B, \\
\pi_1|_{A \times B} \text{ or } \pi_2|_{A \times B}, & \text{if } A = B,
\end{cases}$$

for all $A, B \in X/\sim$.

If $X = X_n = \{1, \ldots, n\}$, then

$$x \preceq y \iff |F^{-1}(\{x\})| \leq |F^{-1}(\{y\})|$$
Motivation

\[ F_{|A \times B} = \begin{cases} \max_{\sim} |A \times B|, & \text{if } A \neq B, \\ \pi_1|A \times B \text{ or } \pi_2|A \times B|, & \text{if } A = B, \end{cases} \quad \forall A, B \in X/\sim \]

\[ x \preceq y \iff |F^{-1}({\{x\}})| \leq |F^{-1}({\{y\}})| \]

\[ 1 < 2 < 3 \quad 2 \prec 1 \sim 3 \]
Motivation

\[ F(F(2, 3), F(1, 2)) = 3 \neq 1 = F(F(2, 1), F(3, 2)) \]

\[ \implies F \text{ is not bisymmetric} \]
Quasilinearity

Let \( \preccurlyeq \) be a weak ordering on \( X \)

**Definition**

\( \preccurlyeq \) is *quasilinear* if there exist no pairwise distinct \( a, b, c \in X \) such that \( a \prec b \sim c \)

**Example.** On \( X = \{1, 2, 3, 4\} \), consider the \( \preccurlyeq \)

\[
2 \sim 3 \prec 1 \prec 4
\]
Quasilinearity

\[ F|_{A \times B} = \begin{cases} 
\max_{\sim} |_{A \times B}, & \text{if } A \neq B, \\
\pi_1|_{A \times B} \text{ or } \pi_2|_{A \times B}, & \text{if } A = B,
\end{cases} \quad \forall A, B \in X/\sim
\]

\[ \sim \] is not quasilinear and \( F \) is not bisymmetric
Quasilinearity

\[ x \preceq y \iff |F^{-1}(\{x\})| \leq |F^{-1}(\{y\})| \]

\preceq is quasilinear and \( F \) is bisymmetric
A characterization

\[ F|_{A \times B} = \begin{cases} \max \preceq |_{A \times B}, & \text{if } A \neq B, \\ \pi_1|_{A \times B} \text{ or } \pi_2|_{A \times B}, & \text{if } A = B, \end{cases} \quad \forall A, B \in X/\sim \quad (\ast) \]

**Theorem**

For any \( F: X^2 \to X \), the following are equivalent.

(i) \( F \) is bisymmetric and quasitrivial

(ii) \( F \) is of the form \((\ast)\) for some quasilinear \( \preceq \)
A characterization

We denote the *set of minimal elements of $X$ for $\preceq$* by $\text{min}_{\preceq} X$
Bisymmetric and quasitrivial operations
The nondecreasing case

Let $\leq$ be a linear ordering on $X$

**Definition.** $F : X^2 \to X$ is *nondecreasing for $\leq$* if

$$F(x, y) \leq F(x', y') \quad \text{whenever} \quad x \leq x' \text{ and } y \leq y'$$
The nondecreasing case

\[ F|_{A \times B} = \begin{cases} \max_{\prec} |_{A \times B}, & \text{if } A \neq B, \\ \pi_1|_{A \times B} \text{ or } \pi_2|_{A \times B}, & \text{if } A = B, \end{cases} \quad \forall A, B \in X/\sim \]
Weakly single-peaked weak orderings

**Definition** (Couceiro et al., 2018)
\( \preceq \) is said to be *weakly single-peaked for \( \leq \)* if for any \( a, b, c \in X \),
\[
a < b < c \implies b \prec a \text{ or } b \prec c \text{ or } a \sim b \sim c
\]

**Example.** The weak ordering \( \preceq \) on
\[
X_4 = \{1 < 2 < 3 < 4\}
\]
defined by
\[
2 \prec 1 \sim 3 \prec 4
\]
is weakly single-peaked for \( \leq \)
Weakly single-peaked weak orderings

$x \preceq y \iff |F^{-1} (\{x\})| \leq |F^{-1} (\{y\})|$
Associative quasitrivial and nondecreasing operations

\[ F |_{A \times B} = \begin{cases} 
\max_{\lesssim} |A \times B|, & \text{if } A \neq B, \\
\pi_1 |_{A \times B} \text{ or } \pi_2 |_{A \times B}, & \text{if } A = B, 
\end{cases} \quad \forall A, B \in X / \sim \quad (*) \]

**Theorem (Couceiro et al., 2018)**

For any \( F : X^2 \to X \), the following are equivalent.

(i) \( F \) is associative, quasitrivial, and nondecreasing

(ii) \( F \) is of the form (*) for some \( \lesssim \) that is weakly single-peaked for \( \leq \)
Bisymmetric quasitrivial and nondecreasing operations

For any $F : X^2 \to X$, the following are equivalent.

(i) $F$ is bisymmetric, quasitrivial, and nondecreasing

(ii) $F$ is of the form $(\ast)$ for some $\preceq$ that is quasilinear and weakly single-peaked for $\leq$
Final remarks

In arXiv: 1712.07856

1. Characterizations of the class of bisymmetric and quasitrivial operations as well as the subclass of those operations that are nondecreasing

2. New integer sequences (http://www.oeis.org)
   - Number of bisymmetric and quasitrivial operations: A296943
   - Number of bisymmetric, quasitrivial, and nondecreasing operations: A296953
   - ...
Selected references

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