Clones of pivotally decomposable operations

Bruno Teheux
joint work with Miguel Couceiro

Mathematics Research Unit
University of Luxembourg
Motivation

Shannon decomposition of operations $f : \{0, 1\}^n \rightarrow \{0, 1\}$:

$$f(x) = x_k f(x^1_k) + (1 - x_k) f(x^0_k),$$

where

- $x^a_k$ is obtained from $x$ by replacing its $k^{th}$ component by $a$. 

Motivation

Shannon decomposition of operations $f : \{0, 1\}^n \rightarrow \{0, 1\}$:

$$f(x) = x_k f(x_k^1) + (1 - x_k) f(x_k^0),$$

Median decomposition of polynomial operations over bounded DL:

$$f(x) = \text{med}(x_k, f(x_k^1), f(x_k^0)),$$

where

- $x_k^a$ is obtained from $x$ by replacing its $k^{th}$ component by $a$.
- $\text{med}(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$
Motivation

Shannon decomposition of operations \( f : \{0, 1\}^n \rightarrow \{0, 1\} \):

\[
f(x) = x_k f(x^1_k) + (1 - x_k) f(x^0_k),
\]

Median decomposition of polynomial operations over bounded DL:

\[
f(x) = \text{med}(x_k, f(x^1_k), f(x^0_k)),
\]

where

\[
\text{\cdot } x^a_k \text{ is obtained from } x \text{ by replacing its } k^{th} \text{ component by } a.
\]
\[
\text{\cdot } \text{med}(x, y, z) = (x \land y) \lor (x \land z) \lor (y \land z)
\]

Goal: Uniform approach of these decomposition schemes.
Pivotal decomposition

A set and $0, 1 \in A$

Let $\Pi: A^3 \rightarrow A$ an operation

**Definition.** An operation $f: A^n \rightarrow A$ is $\Pi$-decomposable if

$$f(x) = \Pi(x_k, f(x_k^1), f(x_k^0))$$

for all $x \in A^n$ and all $k \leq n$. 
Pivotal decomposition

A set and $0, 1 \in A$

Let $\Pi : A^3 \to A$ an operation that satisfies the equation

$$\Pi(x, y, y) = y.$$ 

Such a $\Pi$ is called a pivotal operation. In this talk, all $\Pi$ are pivotal.

Definition. An operation $f : A^n \to A$ is $\Pi$-decomposable if

$$f(x) = \Pi(x_k, f(x^1_k), f(x^0_k))$$

for all $x \in A^n$ and all $k \leq n$. 
Examples

\[ f(x) = \prod(x_k, f(x_k^1), f(x_k^0)) \]

Shannon decomposition: \( \prod(x, y, z) = xy + (1 - x)z \)

Median decomposition: \( \prod(x, y, z) = \text{med}(x, y, z) \)

Benefits:

- uniformly isolate the marginal contribution of a factor
- repeated applications lead to normal form representations
- lead to characterization of operation classes

\[ \Lambda_\Pi := \{ f \mid f \text{ is } \Pi\text{-decomposable} \} \]
\[ \Lambda_{\Pi} = \{ f \mid f \text{ is } \Pi\text{-decomposable} \} \]

**Problem.**

Characterize those \( \Lambda_{\Pi} \) which are clones.
\[ \Lambda_\Pi = \{ f \mid f \text{ is } \Pi\text{-decomposable} \} \]

**Problem.**

Characterize those \( \Lambda_\Pi \) which are clones.

\[ \Pi(x, 1, 0) = x \quad \text{(P)} \]
\[ \Pi(\Pi(x, y, z), u, v) = \Pi(x, \Pi(y, u, v), \Pi(z, u, v)) \quad \text{(AD)} \]
\[ \Lambda_\Pi = \{ f \mid f \text{ is } \Pi\text{-decomposable} \} \]

**Problem.**

Characterize those \( \Lambda_\Pi \) which are clones.

\[
\Pi(x, 1, 0) = x \quad (P)
\]
\[
\Pi(\Pi(x, y, z), u, v) = \Pi(x, \Pi(y, u, v), \Pi(z, u, v)) \quad (AD)
\]

**Proposition.** If \( \Pi \models (AD) \), the following are equivalent

(i) \( \Lambda_\Pi \) is a clone

(ii) \( \Lambda_\Pi \models (P) \)
Clones of pivotally decomposable Boolean operations

\[(P) + (AD) \iff \Lambda_\Pi \text{ is a clone} \quad (\star)\]

**Example.** For a Boolean clone \( C \), the following are equivalent

(i) There is \( \Pi \) such that \( C = \Lambda_\Pi \)
Clones of pivotally decomposable Boolean operations

\[(P) + (AD) \iff \Lambda_\Pi \text{ is a clone} \quad (\star)\]

**Example.** For a Boolean clone \(C\), the following are equivalent

(i) There is \(\Pi\) such that \(C = \Lambda_\Pi\)

(ii) \(C\) is the clone of (monotone) Boolean functions

What about the converse of \((\star)\)?
The case of $\Pi$-decomposable $\Pi$

\[
\begin{align*}
\Pi(\Pi(1, 0, 1), 0, 1) &= \Pi(1, \Pi(0, 0, 1), \Pi(1, 0, 1)) \\
\Pi(\Pi(0, 0, 1), 0, 1) &= \Pi(0, \Pi(0, 0, 1), \Pi(1, 0, 1))
\end{align*}
\] (WAD)
The case of \( \Pi \)-decomposable \( \Pi \)

\[
\begin{align*}
\Pi(\Pi(1, 0, 1), 0, 1) &= \Pi(1, \Pi(0, 0, 1), \Pi(1, 0, 1)) \\
\Pi(\Pi(0, 0, 1), 0, 1) &= \Pi(0, \Pi(0, 0, 1), \Pi(1, 0, 1))
\end{align*}
\]  \quad \text{(WAD)}

**Theorem.** If \( \Pi \in \Lambda_{\Pi} \) and \( \Pi \models (WAD) \), then

\((P) + (AD) \iff \Lambda_{\Pi} \text{ is a clone,} \)

and \( \Lambda_{\Pi} \) is the clone generated by \( \Pi \) and the constant maps.
**What happens if Π is not Π-decomposable?**

We have seen that if ΛΠ is a Boolean clone then Π ∈ ΛΠ.

There are some Π such that ΛΠ is a clone but Π ∉ ΛΠ.
What happens if $\Pi$ is not $\Pi$-decomposable?

We have seen that if $\Lambda_\Pi$ is a Boolean clone then $\Pi \in \Lambda_\Pi$.

There are some $\Pi$ such that $\Lambda_\Pi$ is a clone but $\Pi \notin \Lambda_\Pi$.

**Example.** Let $A = \{0, 1/2, 1\}$ and $\Pi$ be the pivotal operation s.t.

\[
\begin{align*}
\Pi(x, 1, 0) &= x & \Pi(x, 0, 1/2) &= 1 \\
\Pi(x, 0, 1) &= 1 - x & \Pi(x, 1/2, 1) &= 0 \\
\Pi(x, 1, 1/2) &= 1 & \Pi(x, 1/2, 0) &= 0
\end{align*}
\]
What happens if \( \Pi \) is not \( \Pi \)-decomposable?

We have seen that if \( \Lambda_\Pi \) is a Boolean clone then \( \Pi \in \Lambda_\Pi \).

There are some \( \Pi \) such that \( \Lambda_\Pi \) is a clone but \( \Pi \notin \Lambda_\Pi \).

**Example.** Let \( A = \{0, 1/2, 1\} \) and \( \Pi \) be the pivotal operation s.t.

\[
\begin{align*}
\Pi(x, 1, 0) &= x & \Pi(x, 0, 1/2) &= 1 \\
\Pi(x, 0, 1) &= 1 - x & \Pi(x, 1/2, 1) &= 0 \\
\Pi(x, 1, 1/2) &= 1 & \Pi(x, 1/2, 0) &= 0
\end{align*}
\]

\( \Pi \models (P), (AD) \) but \( \Pi \notin \Lambda_\Pi \)

since

\( \Pi(x, 1/2, 1/2) = 1/2 \) and

\( \Pi(1/2, \Pi(x, 1, 1/2), \Pi(x, 0, 1/2)) = 1 \)
Theorem. If $\Pi \in \Lambda_\Pi$ and $\Pi \models (P)$, then the following are equivalent

(i) $\Pi$ is symmetric

(ii) $\Pi(0, 0, 1) = \Pi(0, 1, 0)$ and $\Pi(1, 0, 1) = \Pi(1, 1, 0)$
Summary

- If $\Pi \in \Lambda_\Pi$ and $\Pi \models (\text{WAD})$, then
  \[(P) + (\text{AD}) \iff \Lambda_\Pi \text{ is a clone}\]

- There is a clone $\Lambda_\Pi$ such that $\Pi \not\in \Lambda_\Pi$
Summary

- If $\Pi \in \Lambda_\Pi$ and $\Pi \models (WAD)$, then
  \[(P) + (AD) \iff \Lambda_\Pi \text{ is a clone}\]

- There is a clone $\Lambda_\Pi$ such that $\Pi \not\in \Lambda_\Pi$

Problems.

Find a characterization of those $\Lambda_\Pi$ which are clones when $\Pi \not\in \Lambda_\Pi$.

Structure of the family of decomposable classes of operations?