Derivations and differential operators on rings and fields

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Abstract

Let $R$ be an integral domain of characteristic zero. We prove that a function $D : R \to R$ is a derivation of order $n$ if and only if $D$ belongs to the closure of the set of differential operators of degree $n$ in the product topology of $R^R$, where the image space is endowed with the discrete topology. In other words, $f$ is a derivation of order $n$ if and only if, for every finite set $F \subset R$, there is a differential operator $D$ of degree $n$ such that $f = D$ on $F$. We also prove that if $d_1, \ldots, d_n$ are nonzero derivations on $R$, then $d_1 \circ \ldots \circ d_n$ is a derivation of exact order $n$.

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1 Introduction and main results

By a ring we mean a commutative ring with unit. An integral domain is a ring with no zero-divisors other than 0. The ring $R$ has characteristic zero if $n \cdot x \neq 0$ for every $x \in R \setminus \{0\}$ and for every positive integer $n$.

A derivation on a ring $R$ is a map $d : R \to R$ such that

$$d(x + y) = d(x) + d(y) \quad \text{and} \quad d(xy) = d(x)y + d(y)x$$

(1)

for every $x, y \in R$. Derivations of higher order are defined by induction as follows (cf. [5]).

Let $R$ be a ring. The identically 0 function defined on $R$ is called the derivation of order 0. Let $n > 0$, and suppose we have defined the derivations of order at most $n - 1$. A function $D : R \to R$ is called a derivation of order at most $n$, if $D$ is additive and satisfies

$$D(xy) - D(x)y - D(y)x = B(x, y)$$

(2)
for every $x, y \in R$, where $B(x, y)$ is a derivation of order at most $n - 1$ in each of its variables. We denote by $D^n(R)$ the set of derivations of order at most $n$ defined on $R$. We may write $D^n$ instead of $D^n(R)$ if the ring $R$ is clear from the context. We say that the order of a derivation $D$ is $n$ if $D \in D^n \setminus D^{n-1}$. (We have $D^{-1} = \emptyset$ by definition).

Clearly, a function $d : R \to R$ is a derivation if and only if $d \in D_1$.

Now we define differential operators on a ring $R$. We say that the map $D : R \to R$ is a differential operator of degree at most $n$ if $D$ is the linear combination, with coefficients from $R$, of finitely many maps of the form $d_1 \circ \ldots \circ d_k$, where $d_1, \ldots, d_k$ are derivations on $R$ and $k \leq n$. If $k = 0$ then we interpret $d_1 \circ \ldots \circ d_k$ as the identity function on $R$. We denote by $O^n(R)$ the set of differential operators of degree at most $n$ defined on $R$. We may write $O^n$ instead of $O^n(R)$ if the ring $R$ is clear from the context. We say that the degree of a differential operator $D$ is $n$ if $D \in O^n \setminus O^{n-1}$ (where $O^{-1} = \emptyset$ by definition).

The term “differential operator” is justified by the following fact. Let $K = \mathbb{Q}(t_1, \ldots, t_k)$, where $t_1, \ldots, t_k$ are algebraically independent over $\mathbb{Q}$. Then $K$ is the field of all rational functions of $t_1, \ldots, t_k$ with rational coefficients. It is clear that $d_i = \frac{\partial}{\partial t_i}$ is a derivation on $K$ for every $i = 1, \ldots, k$. Therefore, every differential operator

$$D = \sum_{i_1 + \ldots + i_k \leq n} c_{i_1, \ldots, i_k} \cdot \frac{\partial^{i_1 + \ldots + i_k}}{\partial t_1^{i_1} \ldots \partial t_k^{i_k}},$$

(3)

where the coefficients $c_{i_1, \ldots, i_k}$ belong to $K$, is a differential operator of degree at most $n$. The converse is also true: if $D$ is a differential operator of degree at most $n$ on the field $K = \mathbb{Q}(t_1, \ldots, t_k)$, then $D$ is of the form (3) (see [3, Proposition 3.2] and the proof of Lemma 2.6 below).

**Remark 1.1.** If $d$ is a derivation on $R$, then $c \cdot d$ is also a derivation for every $c \in R$. Thus every differential operator is the sum of terms of the form $d_1 \circ \ldots \circ d_k$, where $k \geq 1$ and $d_1, \ldots, d_k$ are derivations, and of a term $c \cdot j$, where $c \in R$ and $j$ is the identity function. Since $d(1) = 0$ for every derivation $d$, it follows that a differential operator $D$ satisfies $D(1) = 0$ if and only if the term $c \cdot j$ is missing; that is, if $D$ is the sum of terms of the form $d_1 \circ \ldots \circ d_k$, where $k \geq 1$ and $d_1, \ldots, d_k$ are derivations. We denote by $O^n_0$ the set of all differential operators $D$ of degree at most $n$ satisfying $D(1) = 0$.

Let $G$ be an Abelian semigroup, and let $H$ be an Abelian group. The *difference operator* $\Delta_g \ (g \in G)$ is defined by $\Delta_g f(x) = f(x + g) - f(x)$ for every $f : G \to H$ and $x \in G$. A function $f : G \to H$ is a generalized polynomial, if there is a $k$ such that $\Delta_{g_1} \ldots \Delta_{g_{k+1}} f = 0$ for every $g_1, \ldots, g_{k+1} \in G$. The smallest $k$ for which this holds for every $g_1, \ldots, g_{k+1} \in G$ is the degree of the generalized
polynomial \( f \), denoted by \( \text{deg} \ f \). The degree of the identically zero function is \(-1\) by definition. It is clear that the nonzero constant functions are generalized polynomials of degree 0, and the nonconstant additive functions; that is, the nonzero homomorphism from \( G \) to \( H \), are generalized polynomials of degree 1.

If \( X, Y \) are nonempty sets, then \( Y^X \) denotes the set of all maps \( f: X \to Y \). We endow the space \( Y \) with the discrete topology, and \( Y^X \) with the product topology. The closure of a set \( A \subset Y^X \) with respect to the product topology is denoted by \( \text{cl} \ A \). Clearly, a function \( f: X \to Y \) belongs to \( \text{cl} \ A \) if and only if, for every finite set \( F \subset X \) there is a function \( g \in A \) such that \( f(x) = g(x) \) for every \( x \in F \).

It is clear that a function \( f: G \to H \) is a generalized polynomial of degree at most \( n \) if and only if, for every finite set \( F \subset G \), there is a generalized polynomial \( h \) of degree at most \( n \) such that \( f = h \) on \( F \). This means that the set of generalized polynomials of degree at most \( n \) is closed in \( H^G \).

If \( R \) is a ring, then we denote by \( R^\ast \) the Abelian semigroup \( R \setminus \{0\} \) under multiplication. We denote by \( j \) the identity function on \( R \).

In this note our aim is to prove that, for every integral domain of characteristic zero and for every positive integer \( n \), we have \( D^n = \text{cl} \ O^n_0 \). That is, a map \( D: R \to R \) is a derivation of order at most \( n \) if and only if \( D \) belongs to the closure of the set of all differential operators of degree at most \( n \) satisfying \( D(1) = 0 \). More precisely, we prove the following result.

**Theorem 1.1.** Let \( R \) be an integral domain of characteristic zero, \( K \) its field of fractions, and let \( n \) be a positive integer. Then, for every function \( D: R \to R \), the following are equivalent.

(i) \( D \in D^n(R) \).

(ii) \( D \in \text{cl} (O^n_0(R)) \).

(iii) \( D \) is additive on \( R \), \( D(1) = 0 \), and \( D/j \), as a map from the semigroup \( R^\ast \) to \( K \), is a generalized polynomial of degree at most \( n \).

As an immediate consequence of the theorem above we find the following corollary.

**Corollary 1.1.** Let \( R \) be an integral domain of characteristic zero, \( K \) its field of fractions, and let \( n \) be a positive integer. Then, for every function \( D: R \to R \), the following are equivalent.

(i) \( D \in D^n(R) \setminus D^{n-1}(R) \).

(ii) \( D \in (\text{cl} O^n_0(R)) \setminus \text{cl} (O^{n-1}_0(R)) \).
(iii) $D$ is additive on $R$, $D(1) = 0$, and $D/j$, as a map from the semigroup $R^*$ to $K$, is a generalized polynomial of degree $n$.

Indeed, suppose $D \in D^n \setminus D^{n-1}$. Then, by Theorem 1.1, we have $D \in \text{cl} \mathcal{O}^n_0$. If $D \notin \text{cl} (\mathcal{O}^n_0) \setminus \text{cl} (\mathcal{O}^{n-1}_0)$, then $D \in \text{cl} \mathcal{O}^{n-1}_0$. This implies $D \in D^{n-1}$, which is impossible. Therefore, (i) of Corollary 1.1 implies (ii) of Corollary 1.1. The other implications can be shown similarly.

**Remark 1.2.** Theorem 1.1 and Corollary 1.1 do not hold without assuming that $R$ is of characteristic zero. Consider the following example.

Let $F_2$ denote the field having two elements, and let $R = F_2[x]$ be the ring of polynomials with coefficients from $F_2$. We put

$$D \left( \sum_{i=0}^{n} a_i \cdot x^i \right) = \sum_{i=2}^{n} \frac{i(i-1)}{2} \cdot a_i \cdot x^{i-2}$$

for every $n \geq 0$ and $a_0, \ldots, a_n \in F_2$. It is easy to check that $D$ is a derivation of order at most two on $R$. Since $D(x) = 0$ and $D(x^2) = 1$, it follows that $D$ is not a derivation, and thus $D \in D^2 \setminus D^1$.

On the other hand, if $d_1$ and $d_2$ are arbitrary derivations on $R$, then $d_1 \circ d_2$ is also a derivation. Indeed,

$$d_1(d_2(x^k)) = d_1(k \cdot x^{k-1} \cdot d_2(x)) = k(k-1) \cdot x^{k-2} \cdot d_1(x) \cdot d_2(x) + k \cdot x^{k-1} \cdot d_1(d_2(x))$$

for every $k \geq 2$. Since $k(k-1)$ is even, we find that

$$(d_1 \circ d_2)(x^k) = k \cdot x^{k-1} \cdot a$$

for every $k \geq 2$, where $a = d_1(d_2(x)) \in R$. It is easy to check that (4) is true for $k = 0$ and $k = 1$ as well. Since derivations are additive, (4) gives $d_1(d_2(p)) = a \cdot \frac{\partial p}{\partial x}$ for every $p \in R$, and thus $d_1 \circ d_2 \in \mathcal{O}^1_0$. This implies that $\mathcal{O}^2_0 = \mathcal{O}^1_0$, and thus $D^2$ is strictly larger than $\mathcal{O}^2_0$.

**Remark 1.3.** In the proof of Theorem 1.1 the crucial step is to show that if $R$ is of characteristic zero and the transcendence degree of the field of fractions $K$ of $R$ over $\mathbb{Q}$ is finite, then $D^n = \mathcal{O}^n_0$ (see Lemma 2.7). Comparing to Theorem 1.1 we find that under these conditions, for every function $f : R \to R$ we have

$$(f \in D^n \setminus D^{n-1}) \iff (f \in \mathcal{O}^n_0 \setminus \mathcal{O}^{n-1}_0) \iff D \text{ is additive on } K, D(1) = 0, \text{ and } D/j, \text{ defined on the group } K^*, \text{ is a generalized polynomial of degree } n.$$
Lemma 2.2. For every ring \( D \) to Remark 1.1, the set \( D \) is closed in \( \mathbb{R} \).

Proof. Since \( g \) and \( d \) are nonzero derivations on \( D \), it is enough to show that \( g \) and \( d \) are derivations of order 1.

The statement of the theorem above does not hold without assuming that \( R \) is of characteristic zero. Consider the example described in Remark 1.2, \( d \) is a derivation of order 1.

The statement of the theorem is not true for rings in general; not even for rings of characteristic zero. Let \( R = \mathbb{Q}[x] \times \mathbb{Q}[x] \), and put \( d_1(p, q) = (\frac{\partial p}{\partial x}, 0) \) and \( d_2(p, q) = (0, \frac{\partial q}{\partial x}) \) for every \((p, q) \in R\). Then \( d_1 \) and \( d_2 \) are nonzero derivations on \( R \), but \( d_1 \circ d_2 = 0 \).

2 Lemmas

Lemma 2.1. For every ring \( R \) and for every nonnegative integer \( n \), the set \( D^n \) is closed in \( \mathbb{R}^n \).

Proof. We prove by induction on \( n \). If \( n = 0 \), then \( D^0 = \{0\} \) is closed. Let \( n > 0 \), and suppose that \( D^{n-1} \) is closed. Let \( f \in \text{cl} \ D^n \) be arbitrary. We have to prove that \( f \in \text{cl} \ D^n \); that is, for every fixed \( y \in R \), the map \( x \mapsto g(x) = f(xy) - yf(x) - xf(y) \) belongs to \( D^{n-1} \). By the induction hypothesis, it is enough to show that \( g \in \text{cl} \ D^{n-1} \); that is, for every finite set \( \{ p, q \} \in \mathbb{R} \), there is a function \( h \in D^{n-1} \) such that \( g(x) = h(x) \) for every \( x \in F \).

If \( F \) is finite, then so is \( A = F \cup \{ x : x \in F \} \cup \{ y \} \). Since \( f \in \text{cl} \ D^n \), there is a function \( D \in D^n \) such that \( f(z) = D(z) \) for every \( z \in A \). If \( x \in F \), then \( x, y, xy, A \), and thus

\[
g(x) = f(xy) - yf(x) - xf(y) = D(xy) - yD(x) - xD(y).
\]

The function \( x \mapsto h(x) = D(xy) - yD(x) - xD(y) \) belongs to \( D^{n-1} \), as \( D \in D^n \). Since \( g(x) = h(x) \) for every \( x \in F \), the lemma is proved.

Lemma 2.2. For every ring \( R \) we have \( \text{cl} \ O_0^n \subset D^n \).

Proof. Since \( D^n \) is closed by Lemma 2.1, it is enough to show that \( O_0^n \subset D^n \). Let \( D \) be a differential operator of degree at most \( n \) satisfying \( D(1) = 0 \). According to Remark 1.1, \( D \) is the sum of terms of the form \( d_1 \circ \ldots \circ d_k \), where \( 1 \leq k \leq n \) and \( d_1, \ldots, d_k \) are derivations. Since \( D^n \) is a linear space, it is enough to show
that \( d_1 \circ \ldots \circ d_k \in D^k \) whenever \( k \geq 1 \) and \( d_1, \ldots, d_k \) are derivations. This, in turn, is easy to prove by induction on \( k \).

The statement of the following lemma is probably known. In order to make these notes as self-contained as possible, we provide the proof.

**Lemma 2.3.** Let \( G \) be an Abelian semigroup, and let \( K \) be a field. If \( p : G \to K \) is a generalized polynomial of degree \( n \geq 0 \) and \( a : G \to K \) is a nonzero additive function, then \( p \cdot a \) is a generalized polynomial of degree at most \( n + 1 \).

If \( K \) is of characteristic zero, then \( \deg (p \cdot a) = n + 1 \).

**Proof.** We prove by induction on \( n \). If \( n = 0 \), then \( p \) is a nonzero constant, and \( p \cdot a \) is a nonzero additive function, hence a generalized polynomial of degree 1.

Let \( n > 0 \), and suppose that the statement is true for \( n - 1 \). Let \( p \) be a generalized polynomial of degree \( n \). We have

\[
\Delta_g(p \cdot a)(x) = a(x) \cdot \Delta_g p(x) + a(g) \cdot p(x + g)
\]

for every \( x, g \in G \). Since \( \deg \Delta_g p(x) \leq n - 1 \), it follows from the induction hypothesis that \( \deg (a(x) \cdot \Delta_g p(x)) \leq n \). Therefore, by (5), we have \( \deg \Delta_g (p \cdot a) \leq n \) for every \( g \in G \), and thus \( \deg (p \cdot a) \leq n + 1 \). We have to prove that if \( K \) is characteristic zero, then \( \deg (p \cdot a) \geq n + 1 \).

Since the image space \( K \) is a torsion free and divisible Abelian group, it follows from Djoković’s theorem [1] that \( p = P_n + \ldots + P_1 + P_0 \), where \( P_i \) is a monomial of degree \( i \) for every \( i = 1, \ldots, n \), and \( P_0 \) is constant. Then there is a symmetric function \( A(x_1, \ldots, x_n) \), additive in each of its variables, such that \( P_n(x) = A(x_1, \ldots, x) \in G \). Since \( q = p - P_n \) is a generalized polynomial of degree \( \leq n - 1 \), it follows from the induction hypothesis that \( \deg (q \cdot a) \leq n \).

Therefore, in order to prove \( \deg (p \cdot a) \geq n + 1 \), it is enough to show that \( \deg (P_n \cdot a) = n + 1 \).

First we show that there exists an element \( g \in G \) such that \( P_n(g) \neq 0 \) and \( a(g) \neq 0 \). By assumption, there is an \( x \in G \) such that \( a(x) \neq 0 \). Since \( \deg P_n = n \geq 0 \), it follows that \( P_n \) is nonzero. Let \( y \in G \) be such that \( P_n(y) \neq 0 \). Now \( a(kx + y) = k \cdot a(x) + a(y) \) for every positive integer \( k \). Since \( a(x), a(y) \in K \) and \( a(x) \neq 0 \), we have \( a(kx + y) \neq 0 \) for every \( k \) with at most one exception.

Using the fact that \( A(x_1, \ldots, x_n) \) is symmetric and additive in each of its variables, we find

\[
P_n(kx + y) = \sum_{i=0}^{n} \binom{n}{i} A_i(kx, y)
\]

for every positive integer \( k \), where

\[
A_i(kx, y) = A\underbrace{(kx, \ldots, kx, y, \ldots, y)}_{i} = k^i \cdot A\underbrace{(x, \ldots, x, y, \ldots, y)}_{k-i}.
\]
Therefore, by (6), $Q(kx + y)$ is a polynomial of $k$ with coefficients from $K$. Since the constant term of this polynomial is $A(y, \ldots, y) \neq 0$, $Q(kx + y)$ is not the identically zero polynomial, and thus $P_n(kx + y) \neq 0$ for all but finitely many $k$. Therefore, we may choose a $k$ such that $P_n(g) \neq 0$ and $a(g) \neq 0$, where $g = kx + y$.

Let $Q = P_n \cdot a$, and suppose that $\deg Q \leq n$. Then $Q = Q_n + \ldots + Q_1 + Q_0$, where $Q_i$ is a monomial of degree $i$ for every $i = 1, \ldots, n$, and $Q_0$ is constant. For every $i = 1, \ldots, n$, there is there is a symmetric function $B_i(x_1, \ldots, x_i)$, additive in each of its variables, such that $Q_i(x) = B_i(x_1, \ldots, x_i)$ ($x \in G$). Then

$$Q(k \cdot g) = Q_0 + \sum_{i=1}^{n} B_i(kg, \ldots, kg) = Q_0 + \sum_{i=1}^{n} k^i \cdot B_i(g, \ldots, kg)$$

for every positive integer $k$. Therefore, the map $k \mapsto Q(k \cdot g)$ is a polynomial of degree $\leq n$ with coefficients from $K$. However,

$$Q(k \cdot g) = k^n \cdot A(g, \ldots, g) \cdot k \cdot a(g) = k^{n+1} \cdot A(g, \ldots, g) \cdot a(g)$$

is a polynomial of degree $n + 1$. This is a contradiction, proving $\deg Q = n + 1$.

\[\square\]

**Lemma 2.4.** Let $R$ be an integral domain, and let $K$ be its field of fractions. If $d_1, \ldots, d_n$ are nonzero derivations on $R$ and $D = d_1 \circ \ldots \circ d_n$, then $D/j$, as a map from the semigroup $R^*$ to $K$, is a generalized polynomial of degree at most $n$.

If $R$ is of characteristic zero, then $\deg D/j = n$.

**Proof.** We prove by induction on $n$. If $n = 1$, then $D$ is a nonzero derivation. It is clear that in this case $D/j$ is additive, hence a generalized polynomial of degree at most 1 on the semigroup $R^*$. Suppose $\deg D/j \leq 0$. Then $D/j$ is constant on $R^*$, and thus $D = c \cdot j$ on $R$, where $c \in R$ is a constant. Since $D$ is a derivation, we have $c = D(1) = 0$ and $d = 0$, a contradiction. Thus $\deg D/j = 1$.

Suppose that $n > 1$, and the statement is true for $n - 1$. Let $d_1, \ldots, d_n$ be nonzero derivations on $R$. By the induction hypothesis, $(d_2 \circ \ldots \circ d_n)/j = p$ is a generalized polynomial of degree at most $n - 1$. Since $d_1$ is a derivation, we have

$$D(x) = (d_1 \circ \ldots \circ d_n)(x) = d_1(p(x) \cdot x) = d_1(p(x)) \cdot x + p(x) \cdot d_1(x)$$

for every $x \in R^*$. Thus

$$D/j = (d_1 \circ p) + p \cdot (d_1/j) \quad (7)$$

on $R^*$. Since $p : R^* \to K$ is a generalized polynomial of degree $\leq n - 1$ and $d_1 : R \to R$ is additive, it follows that $d_1 \circ p$ is a generalized polynomial of degree
\[ \leq n - 1 \text{ on } R^*. \] (This is because, if \( G \) is an Abelian semigroup, \( H \) is an Abelian group, \( p: G \rightarrow H \) is a generalized polynomial of degree \( k \), and \( d: H \rightarrow H \) is additive, then \( d \circ p \) is a generalized polynomial of degree at most \( k \).)

If \( R \) is of characteristic zero, then so is \( K \). In this case \( p \cdot (d_1/j) \) is a generalized polynomial of degree \( n \) by Lemma 2.3, since \( d_1/j \) is nonzero and additive on \( R^* \). Therefore, \( D/j \) is a generalized polynomial of degree \( n \).

\begin{lemma}
Let \( R \) be an integral domain, and let \( K \) be its field of fractions. If \( D \in \mathcal{O}_0^n(R) \), then \( D/j \), as a map from the semigroup \( R^* \) to \( K \), is a generalized polynomial of degree at most \( n \).
\end{lemma}

\textit{Proof.} Let \( D \in \mathcal{O}_0^n \) be given. As the set of generalized polynomials of degree \( \leq n \) is closed, it is enough to show that for every finite set \( F \subseteq R^* \) there is a generalized polynomial \( p: R^* \rightarrow K \) such that \( \deg p \leq n \) and \( D/j = p \) on \( F \). Since \( D \in \mathcal{O}_0^n \), there is an \( f \in \mathcal{O}_0^n \) such that \( D = f \) on \( F \). It is clear from Remark 1.1 and Lemma 2.4 that \( f/j \) is a generalized polynomial of degree at most \( n \). Now we have \( D/j = f/j \) on \( F \), completing the proof.

The statement of the following lemma is proved, in a different context, in Lemma 3.3 of [3]. We give the proof adjusted to our purposes.

\begin{lemma}
Let \( R \) be a subring of \( \mathbb{C} \), let \( K \subseteq \mathbb{C} \) be its field of fractions, and suppose that the transcendence degree of \( K \) over \( \mathbb{Q} \) is finite. Let the map \( D: R \rightarrow R \) be additive. If \( D/j \), as a map from the semigroup \( R^* \) to \( \mathbb{C} \) is a generalized polynomial of degree at most \( n \), then \( D \in \mathcal{O}^n \).
\end{lemma}

\textit{Proof.} Let \( k \) be the transcendence degree of \( K \) over \( \mathbb{Q} \), and let the elements \( u_1, \ldots, u_k \in K \) be algebraically independent over \( \mathbb{Q} \). Let \( a_i, b_i \in R \) for every \( i = 1, \ldots, k \). Then the field \( \mathbb{Q}(a_1, b_1, \ldots, a_k, b_k) \) has transcendence degree \( k \) over \( \mathbb{Q} \), and thus we can chose elements \( t_1, \ldots, t_k \in \{a_1, b_1, \ldots, a_k, b_k\} \subseteq R^* \) such that \( t_1, \ldots, t_k \) are algebraically independent over \( \mathbb{Q} \).

By assumption, the function \( p = D/j \) is a generalized polynomial of degree \( \leq n \) on \( R^* \). By Djoković's theorem, we have \( p = P_n + \ldots + P_1 + P_0 \), where \( P_j \) is a monomial of degree \( j \) for every \( j = 1, \ldots, n \), and \( P_0 \) is constant. Using the fact that \( P_j(x) = A_j(x, \ldots, x) \), where \( A_j(x_1, \ldots, x_j) \) is symmetric and additive in each of its variables, it is easy to see that for every \( j = 1, \ldots, n \) there is a homogeneous polynomial \( \overline{p}_j \in K[x_1, \ldots, x_k] \) of degree \( j \) such that

\[ P_j(t_1^{i_1} \cdots t_k^{i_k}) = \overline{p}_j(i_1, \ldots, i_k) \]

whenever \( i_1, \ldots, i_k \) are nonnegative integers. (Note that the semigroup operation in \( R^* \) is multiplication.) Putting \( \overline{p} = P_0 + \sum_{j=1}^n \overline{p}_j \) we find that \( \overline{p} \in K[x_1, \ldots, x_k] \), and

\[ \overline{p}(t_1^{i_1} \cdots t_k^{i_k}) = q(i_1, \ldots, i_k) \]
for every \( i_1, \ldots, i_k \geq 0 \). We shall use the notation \( x^{[0]} = 1 \) and \( x^{[j]} = x(x - 1) \cdots (x - j + 1) \) for every \( j = 1, 2, \ldots \) and \( x \in \mathbb{Z} \). It is easy to see that every polynomial belonging to \( K[x_1, \ldots, x_k] \) and of degree \( \leq n \) can be written in the form \( \sum c_j \cdot x_1^{[j_1]} \cdots x_k^{[j_k]} \), where \( j = (j_1, \ldots, j_k) \) runs through the set of \( k \)-tuples of nonnegative integers with \( j_1 + \ldots + j_k \leq n \), and in each term the coefficient \( c_j \) belongs to \( K \). Therefore, the polynomial \( \bar{p} \) also has such a representation. Then we have

\[
D \left( t_1^{i_1} \cdots t_k^{i_k} \right) = p \left( t_1^{i_1} \cdots t_k^{i_k} \right) \cdot t_1^{i_1} \cdots t_k^{i_k} = \sum c_j \cdot t_1^{j_1} \cdots t_k^{j_k} \cdot t_1^{i_1} \cdots t_k^{i_k} = \sum c_j \cdot t_1^{i_1} \cdots t_k^{i_k} \cdot t_1^{j_1} \cdots t_k^{j_k} = E \left( t_1^{i_1} \cdots t_k^{i_k} \right)
\]

for every \( i_1, \ldots, i_k \geq 0 \), where \( E \) is the differential operator

\[
\sum c_j \cdot t_1^{j_1} \cdots t_k^{j_k} \cdot \frac{\partial^{j_1 + \cdots + j_k}}{\partial t_1^{j_1} \cdots \partial t_k^{j_k}}.
\]

By extending the derivations \( \partial / \partial t_i \) to \( K \), we can extend \( E \) to \( K \) as a differential operator \( \overline{E} \) of degree at most \( n \). Then \( \overline{E} \) is additive on \( K \), and \( \overline{E} / j \) is a generalized polynomial on \( K^* \) by Lemma 2.4. Let \( q(0) = 0 \), and let \( q(x) = p(x) - \overline{E}(x) / x \) for every \( x \in R^* \). Then \( q = j = D - \overline{E} \) is additive on \( R \), and \( q \) is a generalized polynomial on \( R^* \). Let \( G \) denote the semigroup generated by the elements \( t_1, \ldots, t_k \). Then \( q \) vanishes on \( G \) by (8). From these conditions it follows that \( q = 0 \) on \( R \). This is proved in [3, Lemma 3.6] under the stronger condition that \( G \) is the group (and not the semigroup) generated by \( t_1, \ldots, t_k \). One can see that the same argument works in our more general case as well; however, for the sake of completeness we give the proof in the appendix. Thus we have \( q = 0 \); that is, \( D = \overline{E} \) on \( R \), which completes the proof.

**Lemma 2.7.** Let \( R \) be a subring of \( \mathbb{C} \), let \( K \subset \mathbb{C} \) be its field of fractions, and suppose that the transcendence degree of \( K \) over \( \mathbb{Q} \) is finite. Then \( D^n(R) = \mathcal{O}_0^n(R) \).

**Proof.** By Lemma 2.2, we only have to show that \( D^n \subset \mathcal{O}_0^n \). It is easy to prove, by induction on \( n \) that if \( D \in D^n \), then \( D(1) = 0 \). Therefore, it is enough to show that if \( D \in D^n \), then \( D \) is a differential operator of degree at most \( n \). We prove by induction on \( n \).

The statement is obvious if \( n = 0 \). Let \( n > 0 \), and suppose that the statement is true for \( n - 1 \). Let \( D \) be a derivation of order at most \( n \). By Lemma 2.6, it is enough to show that \( p = D / j \), defined on the semigroup \( R^* \), is a generalized
polynomial of degree at most \( n \). Let \( y \in R^* \) be fixed. Dividing (2) by \( xy \) we obtain

\[
\frac{D(xy)}{xy} - \frac{D(x)}{x} - \frac{D(y)}{y} = \frac{B(x,y)}{xy},
\]

and thus \( p(xy) - p(x) - p(y) = \frac{B(x,y)}{xy} \) for every \( x \in K^* \). Therefore we have

\[
\Delta_y p(x) = \frac{1}{y} \cdot \frac{B(x,y)}{x} \tag{9}
\]
on \( R^* \). The map \( x \mapsto B(x,y) \) is a derivation of order at most \( n - 1 \). We also have \( B(1,y) = 0 \) by \( D(1) = 0 \). Therefore, by Lemma 2.4, the map \( x \mapsto B(x,y)/x \) is a generalized polynomial of degree at most \( n \). Then so is \( \Delta_y p \) by (9). Since this is true for every \( y \in K^* \), it follows that \( p \) is a generalized polynomial of degree at most \( n \).

\[\square\]

3 Proof of Theorems 1.1 and 1.2.

First we prove Theorem 1.1. The implication (ii)\( \implies \) (iii) is proved in Lemma 2.5.

(iii)\( \implies \) (ii): Suppose that \( D \) is additive, \( D(1) = 0 \), and \( D/j \) is a generalized polynomial of degree at most \( n \). In order to prove \( D \in \text{cl} \mathcal{O}^n_0 \), we have to show that for every finite set \( F \subset K \) there is a function \( f \in \mathcal{O}^n_0 \) such that \( D = f \) on \( F \). Let \( F \subset K \) be finite, and let \( L \) denote the subfield of \( K \) generated by \( F \). Obviously, the transcendence degree of \( L \) over \( \mathbb{Q} \) is finite. It is well-known that every field of characteristic zero and having finite transcendence degree over \( \mathbb{Q} \) is isomorphic to a subfield of \( \mathbb{C} \). Therefore, we may assume that \( L \subset \mathbb{C} \). Thus, by Lemma 2.6, the restriction \( D|_L \) of \( D \) to the field \( L \) is a derivation of order at most \( n \). Since \( D(1) = 0 \), we also have \( D|_L \in \mathcal{O}^n_0(L) \). It is well-known that every derivation on \( L \) can be extended to \( K \) as a derivation (see [4, pp. 351-352]). This implies that every differential operator on \( L \) of degree at most \( n \) can be extended to \( K \) as a differential operator of degree at most \( n \). If \( f \) is such an extension of \( D|_L \), then, obviously, \( D(x) = f(x) \) for every \( x \in F \). This proves (iii)\( \implies \) (ii).

(ii)\( \implies \) (i): This is Lemma 2.2.

(i)\( \implies \) (ii): Let \( D \in D^n \). In order to prove \( f \in \text{cl} \mathcal{O}^n_0 \) we have to show that for every finite set \( F \subset K \) there is a function \( f \in \mathcal{O}^n_0 \) such that \( D = f \) on \( F \). Let \( L \) denote the field generated by \( F \). Obviously, the transcendence degree of \( L \) over \( \mathbb{Q} \) is finite. Thus, by Lemma 2.7, the restriction \( D|_L \) of \( D \) to the field \( L \) is a derivation of order at most \( n \), vanishing at 1. Let \( f \) be an extension of \( D|_L \) to \( K \) as a function \( f \in \mathcal{O}^n_0 \). Then, obviously, \( D(x) = f(x) \) for every \( x \in F \). This proves (i)\( \implies \) (ii).

\[\square\]

The statement of Theorem 1.2 is an immediate consequence of Corollary 1.1 and Lemma 2.4.
4 Appendix

Lemma 4.1. Let $R$ be a subring of $\mathbb{C}$, and let $K \subset \mathbb{C}$ be its field of fractions. Suppose that the transcendence degree of $K$ over $\mathbb{Q}$ is $k < \infty$, and let the elements $t_1, \ldots, t_k \in R$ be algebraically independent over $\mathbb{Q}$. Let $f : R \to \mathbb{C}$ be additive on $R$ (with respect to addition) and such that $q = f/j$, as a map from the semigroup $R^*$ to $\mathbb{C}$ is a generalized polynomial. If $f = 0$ on the semigroup $G$ generated by $t_1, \ldots, t_k$, then $f = 0$ on $R$.

Proof. We prove by induction on $\deg q$. If $\deg q = 0$, then $q$ is constant. Since $f = 0$ on $G$, we have $q = 0$ on $G$, and thus $q = 0$ on $R$.

Suppose $m = \deg q > 0$, and that the statement is true for degrees less than $m$. Let $g \in G$ be fixed, and put $f_i(x) = g^{-1}f(gx) - f(x)$ $(x \in R)$. Then $f_i$ is additive on $R$. Also, $f_i/j$ is a generalized polynomial on $R^*$, since

$$\frac{f_i(x)}{x} = \frac{g^{-1}f(gx) - f(x)}{x} = \frac{f(gx)}{gx} - \frac{f(x)}{x} = q(gx) - q(x) = \Delta g q(x)$$

for every $x \in R^*$. Since $\deg (f_i/j) = \deg \Delta g \leq m - 1$ and $f_1 = 0$ on $G$, it follows from the induction hypothesis that $f_1 = 0$ on $R$. Thus $f(gx) = g \cdot f(x)$ for every $g \in G$ and $x \in R$. By the additivity of $f$ we obtain

$$f(cx) = c \cdot f(x) \quad (c \in \mathbb{Q}[t_1, \ldots, t_k], \ x \in R). \quad (10)$$

Since the transcendence degree of $K$ over $\mathbb{Q}$ is $k$ and $t_1, \ldots, t_k$ are algebraically independent over $\mathbb{Q}$, it follows that every element of $K$ is algebraic over $\mathbb{Q}(t_1, \ldots, t_k)$. Let $\alpha \in R$ be arbitrary. Then $\alpha$ is algebraic over the field $\mathbb{Q}(t_1, \ldots, t_k)$, and there are elements $c_0, \ldots, c_N \in \mathbb{Q}[t_1, \ldots, t_k]$ such that

$$c_N \alpha^N + \ldots + c_1 \alpha + c_0 = 0, \quad (11)$$

where $c_N \neq 0$ and $N$ is minimal. Let $f(\alpha^i) = a_i$ $(i = 0, 1, \ldots)$. Multiplying (11) by $\alpha^{n-N}$ for every $n \geq N$ we obtain

$$c_N \alpha^n + \ldots + c_1 \alpha^{n-N+1} + c_0 \alpha^{n-N} = 0.$$  

By (10) and by the additivity of $f$, this implies

$$c_N a_n + \ldots + c_1 a_{n-N+1} + c_0 a_{n-N} = 0$$

for every $n \geq N$. Therefore, the sequence $(a_n)$ satisfies a linear recurrence relation. It is well-known that $a_n$ can be uniquely represented in the form $a_n = \sum_{\lambda \in \Lambda} p_\lambda(n) \cdot \lambda^n$, where $\lambda$ runs through $\Lambda$, the set of roots of the characteristic
polynomial \( \chi(x) = c_N x^N + \ldots + c_0 \), and for every root \( \lambda \in \Lambda \), \( p_\lambda \in \mathbb{C}[x] \) is a polynomial of the degree less than the multiplicity of \( \lambda \).

Since \( N \) is minimal, the polynomial \( \chi \) is irreducible over \( \mathbb{Q}(t_1, \ldots, t_k) \). Therefore, every \( \lambda \) is a simple root of \( \chi \), and thus

\[
 a_n = \sum_{\lambda \in \Lambda} d_\lambda \cdot \lambda^n
\]  

for every \( n \), where \( d_\lambda \) is a constant for every \( \lambda \in \Lambda \).

Since \( q \) is a generalized polynomial on \( R^* \) it follows that the map \( n \mapsto q(\alpha^n) \) is a polynomial on \( \{0, 1, \ldots\} \). Now, we have \( a_n = f(\alpha^n) = q(\alpha^n) \cdot \alpha^n \) for every \( n \). The uniqueness of the representation (12) implies that \( \alpha \in \Lambda \), and the function \( n \mapsto q(\alpha^n) \ (n = 0, 1, \ldots) \) is constant. Since \( q(1) = f(1) = 0 \) by \( 1 \in G \), it follows that \( q(\alpha^n) = 0 \) for every \( n \). In particular, \( q(\alpha) = 0 \) and \( f(\alpha) = 0 \). Since this is true for every \( \alpha \in R \), we obtain \( f = 0 \) on \( R \). □

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