VISUAL CHARACTERIZATION OF ASSOCIATIVE QUASITRIVIAL NONDECREASING OPERATIONS ON FINITE CHAINS

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Abstract. In this paper we provide visual characterization of associative quasitrivial nondecreasing operations on finite chains. We also provide a characterization of bisymmetric quasitrivial nondecreasing binary operations on finite chains. Finally, we estimate the number of operations belonging to the previous classes.

1. Introduction

The study of aggregation operations defined on finite ordinal scales (i.e, finite chains) have been in the center of interest in the last decades, e.g., [3,6,11,17,19–24,26,30,31]. Among these operations, discrete uninorms has an important role in fuzzy logic and decision making.

In this paper we investigate associative quasitrivial nondecreasing operations on finite chains. In [4,7,27] idempotent discrete uninorms (i.e. idempotent symmetric nondecreasing associative operations with neutral elements defined on finite chains) have been characterized. Since every idempotent uninorm is quasitrivial (see e.g. [2]), in some sense this paper is a continuation of these works where we eliminate the assumption of symmetry of the operations.

We briefly recall the most relevant results on the unit interval $[0,1]$. Czogała-Drewniak [2] proved the following. The properties and notation used below are defined precisely in Section 2.

Theorem 1.1. *For every associative nondecreasing idempotent binary operation $F\colon [0,1]^2 \to [0,1]$ that has neutral element $s \in [0,1]$ there exists a nonincreasing function $g\colon [0,1] \to [0,1]$ with a fixed point s, such that the operation F can be described by the following formula*

$$F(x,y) = \begin{cases} x \wedge y & \text{if } x \in [0,1], y \in [0, g(x)), \\ x \vee y & \text{if } x \in [0,1], y \in (g(x), 1], \\ x \wedge y \text{ or } x \vee y & \text{if } x \in [0,1], y = g(x). \end{cases}$$

By Theorem 1.1, an associative monotonic idempotent operation with neutral element is quasitrivial. Martin, Mayor and Torrens in [18] gave a complete characterization of associative quasitrivial nondecreasing operations on $[0,1]$. Since their result contains many subcases, we omit the details. The interested reader

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are referred to see [18, Proposition 5 and Theorem 4] and [28]. (For the multivariable generalization of these results see [15].) We note that in [27] the analogue of the result of Czogała-Drewniak for finite chains has been provided assuming the symmetry of such operations.

The study of n-ary operations $F: X^n \to X$ satisfying the associativity property (see Definition 2.1) stemmed from the work of Dörnte [8] and Post [25]. In [9, 10] the reducibility (see Definition 2.2) of associative n-ary operations have been studied by adjoining neutral elements. In [1] a complete characterization of quasitrivial associative n-ary operations has been presented. In [7] the quasitrivial symmetric nondecreasing associative n-ary operations defined on chains have been characterized. Recently, in [16] it was proved that associative idempotent nondecreasing n-ary operations defined on any chain are reducible. Using reducibility (see Theorem 3.1) a characterization of associative quasitrivial nondecreasing n-ary operations for any $2 \le n \in \mathbb{N}$ can be obtained automatically by a characterization of associative quasitrivial nondecreasing binary operations.

The paper is organized as follows. In Section 2 we present the most important definitions. In Section 3, we recall ([16, Theorem 4.8]) the reducibility of associative idempotent nondecreasing n-ary operations and, hence, in the sequel we mainly focus on the binary case. We introduce the basic concept of visualization for quasitrivial monotone binary operations and present some preliminary results due to this concept. Here we discuss an important visual test of non-associativity (Lemma 3.5). Section 4 is devoted to the visual characterization of associative quasitrivial nondecreasing operations with so-called 'downward-right paths' (Theorems 4.12 and 4.13). We also present an Algorithm which provides the contour plot of any associative quasitrivial nondecreasing operation. In Section 5 we characterize the bisymmetric quasitrivial nondecreasing binary operations (Theorem 5.3). In Section 6 we calculate the number of associative quasitrivial nondecreasing operations defined on a finite chain of given size with and also without the assumption of the existence of neutral elements (Theorem 6.1). We get similar estimations for the number of bisymmetric quasitrivial nondecreasing binary operations defined on a finite chain of given size (Proposition 6.5). In Section 7 we present some problems for further investigation. Finally, using a slight modification of the proof of [16, Theorem 3.2], in the Appendix we show that every associative quasitrivial monotonic n-ary operations are nondecreasing.

2. Definition

Here we present the basic definitions and some preliminary results. First we introduce the following notation. For any integer $l \ge 0$ and any $x \in X$, we set $l \cdot x = x, \dots, x$ (l times). For instance, we have $F(3 \cdot x_1, 2 \cdot x_2) = F(x_1, x_1, x_1, x_2, x_2)$.

Definition 2.1. Let X be an arbitrary nonempty set. A operation $F : X^n \to X$ is called

- *idempotent* if $F(n \cdot x) = x$ for all $x \in X$;
- *quasitrivial* (or *conservative*) if

$$F(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$$

for all $x_1, \dots, x_n \in X$;

- *(n-ary) associative* if
$$F(x_1,\dots,x_{i-1},F(x_i,\dots,x_{i+n-1}),x_{i+n},\dots,x_{2n-1}) = F(x_1,\dots,x_i,F(x_{i+1},\dots,x_{i+n}),x_{i+n+1},\dots,x_{2n-1})$$
for all $x_1,\dots,x_{2n-1}\in X$ and all $i\in\{1,\dots,n-1\}$;
- *(n-ary) bisymmetric* if
$$F(F(\mathbf{r}_1),\dots,F(\mathbf{r}_n)) = F(F(\mathbf{c}_1),\dots,F(\mathbf{c}_n))$$
for all $n\times n$ matrices $[\mathbf{r}_1 \cdots \mathbf{r}_n] = [\mathbf{c}_1 \cdots \mathbf{c}_n]^T \in X^{n\times n}$.

Remark 1. Quasitrivial operations were introduced in universal algebra (see [1,14]) and are known as conservative operations in aggregation theory (see [13]).

We say that $F : X^n \to X$ has a *neutral element* $e\in X$ if for all $x\in X$ and all $i\in\{1,\dots,n\}$
$$F((i-1)\cdot e, x, (n-i)\cdot e) = x.$$

Hereinafter we simply write that an n-ary operation is associative or bisymmetric if the context clarifies the number of its variables. We also note that if $n=2$ we get the binary definition of associativity, quasitriviality, idempotency, and neutral element property.

Let $(X,\leq)$ be a nonempty chain (i.e, a totally ordered set). An operation $F\colon X^n\to X$ is said to be

- *nondecreasing* (resp. *nonincreasing*) if
$$F(x_1,\dots,x_n)\leq F(x_1',\dots,x_n') \quad (\text{resp. } F(x_1,\dots,x_n)\geq F(x_1',\dots,x_n'))$$
whenever $x_i\leq x_i'$ for all $i\in\{1,\dots,n\}$,
- *monotone in the i-th variable* if for all fixed elements $a_1,\dots a_{i-1},a_{i+1},\dots,a_n$ of X, the 1-ary function defined as
$$f_i(x) := F(a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_n)$$
is nondecreasing or nonincreasing.
- *monotone* if it is monotone in each of its variables.

Definition 2.2. We say that $F : X^n \to X$ *is derived from* a binary operation $G : X^2 \to X$ if F can be written of the form

$$F(x_1,\dots,x_n) = x_1\circ\cdots\circ x_n, \tag{1}$$

where $x\circ y = G(x,y)$. It is easy to see that G is associative (and F is n-ary associative) if and only if (1) is well-defined. If such a G exists, then we say that F is *reducible.*

Remark 2. Definition 2.2 stems from the work of Dudek and Mukhin [9,10].

We denote the *diagonal* of X^2 by $\Delta_X = \{(x,x) : x\in X\}$. Let L_k denote $\{1,\dots,k\}$ endowed with the natural ordering $(\leq)$. Then L_k is a finite chain. Moreover, every finite chain with k element can be identified with L_k and the domain of an n-variable operation defined on a finite chain can be identified with $\underbrace{L_k\times\cdots\times L_k}_{n} = (L_k)^n$ for some $k\in\mathbb{N}$.

For an arbitrary poset $(X,\leq)$ and $a\leq b\in X$ we denote the elements between a and b by $[a,b]\subseteq X$. In particular, for L_k
$$[a,b] = \{m\in L_k : a\leq m\leq b\}.$$

We also introduce the lattice notion of the minimum ($\wedge$) and the maximum ($\vee$) as follows

$$x_1 \wedge \cdots \wedge x_n = \wedge_{i=1}^n x_i = \min\{x_1, \dots, x_n\},$$
$$x_1 \vee \cdots \vee x_n = \vee_{i=1}^n x_i = \max\{x_1, \dots, x_n\}.$$

The binary operations Proj_x and Proj_y denote the projection on the first and the second coordinate, respectively. Namely, $\mathrm{Proj}_x(x,y) = x$ and $\mathrm{Proj}_y(x,y) = y$ for all $x, y \in X$.

3. Basic concept and preliminary results

The following general result was published as [16, Theorem 4.8] recently.

Theorem 3.1. *Let X be a nonempty chain and $F : X^n \to X$ $(n \geq 2)$ be an associative idempotent nondecreasing operation. Then there exists uniquely an associative idempotent nondecreasing binary operation $G : X^2 \to X$ such that F is derived from G. Moreover, G can be defined by*

$$G(a,b) = F(a,(n-1)\cdot b) = F((n-1)\cdot a, b) \;\; (a,b \in X). \tag{2}$$

Remark 3. By the definition (2) of G, it is clear that if F is quasitrivial, then G is also.

According to Theorem 3.1 and Remark 3, a characterization of associative quasitrivial nondecreasing binary operations automatically implies a characterization for the n-ary case. Therefore, from now on we deal with the binary case ($n = 2$).

3.1. **Visualization of binary operations.** In this section we prove and reprove basic properties of quasitrivial associative nondecreasing binary operations in the spirit of visualization.

Observation 3.2. *Let X be a nonempty chain and let $F : X^2 \to X$ be a quasitrivial monotone operation. If $F(x,t) = x$, then $F(x,s) = x$ for every $s \in [x \wedge t, x \vee t]$. Similarly, if $F(x,t) = t$, then $F(s,t) = t$ for every $s \in [x \wedge t, x \vee t]$.*

For an operation $F : X^2 \to X$ we say that the points (x,y) and (u,v) of X^2 are *connected* if $F(x,y) = F(u,v)$. Otherwise, they are said to be *disconnected*. The *level-set* $s_t = \{(a,b) \in X^2 : F(a,b) = t \in X\}$ consists of connected points that have value t. The *contour line* l_t of F is shortest curve that contains s_t. The *contour plot* contains the contour lines l_t on X^2 for all $t \in X$. It is a graphical interpretation of F on X^2. According to Observation 3.2, if X is a chain and F is a quasitrivial monotone operation, then this contour plot can be drawn using only horizontal and vertical line segments starting from the diagonal (as in Figure 1.). It is clear that these lines do not cross each other by the monotonicity of F.

As a consequence we get the following.

Corollary 3.3. *Let X be a nonempty chain and $F : X^2 \to X$ be a quasitrivial operation.*

$$F \text{ is monotone} \iff F \text{ is nondecreasing.}$$

Proof. We only need to prove that every monotone quasitrivial operation is nondecreasing.

As an easy consequence of Observation 3.2 and the quasitriviality of F, we have $F(s,x) \leq F(t,x)$ and $F(x,t) \leq F(s,t)$ for any $x, s, t \in X$ that satisfies $s \in [x,t]$.

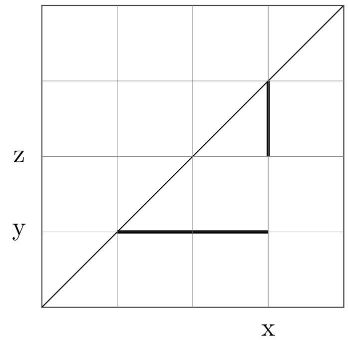

FIGURE 1. $F(x,y) = y$ and $F(x,z) = x$

This implies that F is nondecreasing in the first variable. Similar argument shows the statement for the second variable.

□

Remark 4. The analogue of Corollary 3.3 holds whenever $n > 2$. The proof is essentially the same as the proof of [16, Theorem 3.10]. Thus we present it in Appendix A.

In the sequel we are dealing with associative, quasitrivial and nondecreasing operations.

There are several known forms of the following proposition. This type of result was first proved in [18]. The form as stated here is [4, Proposition 18].

Proposition 3.4. *Let X be an arbitrary nonempty set and let $F : X^2 \to X$ be a quasitrivial operation. Then the following assertions are equivalent.*

(i) *F is* **not** *associative.*
(ii) *There exist pairwise distinct $x, y, z \in X$ such that $F(x,y), F(x,z), F(y,z)$ are pairwise distinct.*
(iii) *There exists a rectangle in X^2 such that one of the vertices is on Δ_X and the three remaining vertices are in $X^2 \setminus \Delta_X$ and pairwise disconnected.*

Now we present an equivalent form of the previous statement if F is nondecreasing that will be useful in the sequel.

Lemma 3.5. *Let X be chain and $F : X^2 \to X$ a quasitrivial, nondecreasing operation. Then F is* **not** *associative if and only if there are pairwise distinct elements $x, y, z \in X$ that give one of the following pictures.*

Proof. By Proposition 3.4, F is not associative if and only if there exist distinct $x, y, z \in X$ satisfying one of the following cases:

$$F(x,y) = x, F(x,z) = z, F(y,z) = y \text{ (Case 1)}, \tag{3}$$

or

$$F(x,y) = y, F(y,z) = z, F(x,z) = x \text{ (Case 2)}. \tag{4}$$

Since $x, y, z \in X$ pairwise distinct elements, they can be ordered in 6 possible configuration of type $x < y < z$. For each case either (3) or (4) holds. Therefore we have 12 configurations as possible realizations of Case 1 or Case 2.

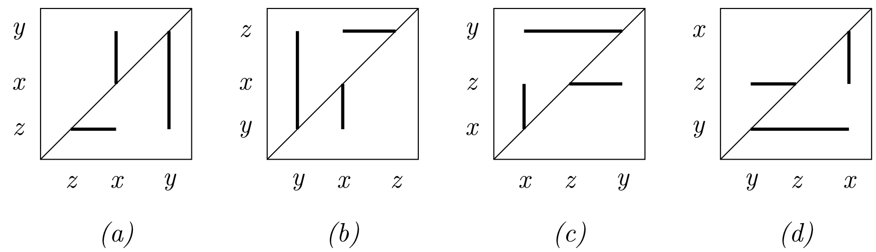

FIGURE 2. Four pictures that guarantee the non-associativity of F

Let us consider Case 1 (when equation (3) holds) and assume $x < y < z$. This implies the situation of Figure 3.

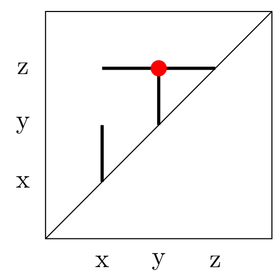

FIGURE 3. Case 1 and $x < y < z$ ('Fake' example)

The red point signs the problem of this configuration, since two lines with different values cross each other. There is no such a quasitrivial monotone operation.

Thus this subcase provides 'fake' example to study associativity. From the total, 8 cases are 'fake' in this sense.

The remaining 4 cases are presented in the statement. Figure 2 (a) and (b) represent the cases when equation (3) holds, and Figure 2 (c) and (d) represent the cases when (4) holds. □

Since for a 2-element set none of the cases of Figure 2 can be realized, as an immediate consequence of Lemma 3.5 we get the following.

Corollary 3.6. *Every quasitrivial nondecreasing operation* $F : L_2^2 \to L_2$ *is associative.*

As a byproduct of this visualization we obtain a simple alternative proof for the following fact. This was proved first in [18, Proposition 2].

Corollary 3.7. *Let* X *be nonempty chain and* $F : X^2 \to X$ *be a quasitrivial symmetric nondecreasing operation then* F *is associative.*

Proof. If we add the assumption of symmetry of F, each cases presented in Figure 2 have crossing lines (as in Figure 4), which is not possible. Thus F is automatically associative. □

For finite chains more can be stated.

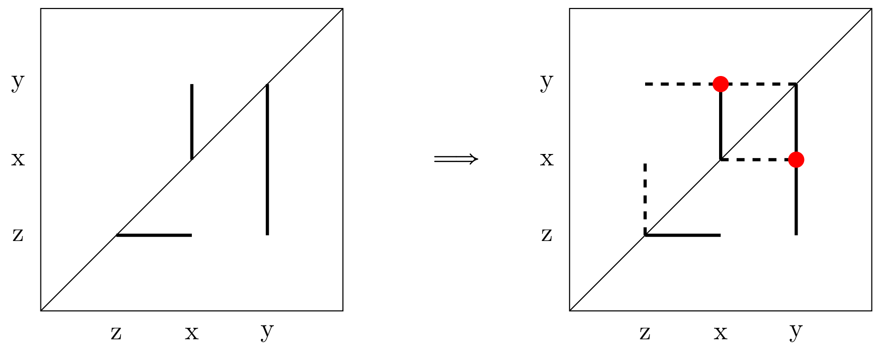

FIGURE 4. The symmetric case

Proposition 3.8 ([4, Proposition 11.]). *If* $F : L_k^2 \to L_k$ *is quasitrivial symmetric nondecreasing then it is associative and has a neutral element.*

Remark 5. The conclusion that F has a neutral element is not necessarily true when $X = [0,1]$ (see [18]). This fact is one of the main difference between the cases $X = L_k$ and $X = [0,1]$.

If we assume that F has a neutral element (as it follows by Proposition 3.8 for finite chains), then as a consequence of Observation 3.2 we get the following pictures (Figure 5) for quasitrivial monotone operations having neutral elements. In Figure 5 the neutral element is denoted by e.

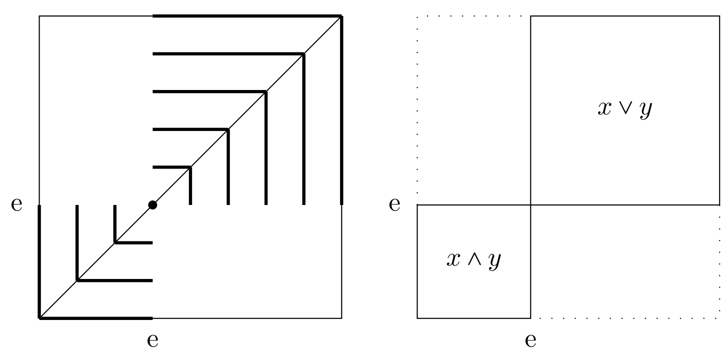

FIGURE 5. Partial description of a quasitrivial monotone operations having neutral elements

4. VISUAL CHARACTERIZATION OF ASSOCIATIVE QUASITRIVIAL NONDECREASING OPERATIONS DEFINED ON L_k

From now on we denote the upper and the lower 'triangle' by

$$T_1 = \{(x,y) : x,y \in L_k, x \le y\}, \quad T_2 = \{(x,y) : x,y \in L_k, x \ge y\},$$

respectively, as in Figure 6. We note that $T_1 \cap T_2$ is the diagonal Δ_{L_k}.

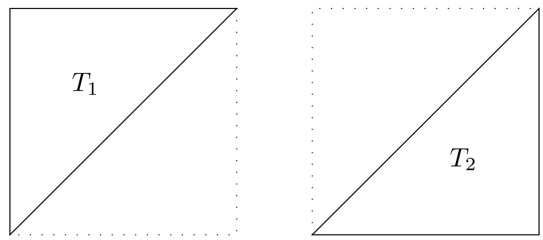

FIGURE 6. The upper and lower 'triangles' T_1 and T_2

Definition 4.1. For a operation $F : L_k^2 \to L_k$ there can be defined the *upper symmetrization* F_1 *and lower symmetrization* F_2 of F as

$$F_1(x,y) = \begin{cases} F(x,y) & \text{if } (x,y) \in T_1 \\ F(y,x) & \text{if } (y,x) \in T_1 \end{cases} \quad \text{and} \quad F_2(x,y) = \begin{cases} F(x,y) & \text{if } (x,y) \in T_2 \\ F(y,x) & \text{if } (y,x) \in T_2, \end{cases}$$

Briefly, $F_1(x,y) = F(x \wedge y, x \vee y)$, $F_2(x,y) = F(x \vee y, x \wedge y)$ $\forall x,y \in L_k$.

Fodor [12] (see also [29, Theorem 2.6]) shown the following statement.

Proposition 4.2. *Let X be a nonempty chain and $F : X^2 \to X$ be an associative operation. Then F_1 and F_2, the upper and the lower symmetrization of F, are also associative.*

This idea makes it possible to investigate the two 'parts' of a non-symmetric associative operation as one-one half of two symmetric associative operations.

By Proposition 3.8, both symmetrization of a nondecreasing quasitrivial operation $F : L_k^2 \to L_k$ has a neutral element.

Definition 4.3. We call an element *upper (or lower) half-neutral element* of F if it is the neutral element of the upper (or the lower) symmetrization. For simplicity we always denote the upper and lower half-neutral element of F by e and f, respectively.

Summarizing the previous results we get following partial description.

Proposition 4.4. *Let $F : L_k^2 \to L_k$ be an associative quasitrivial nondecreasing operation. Then it has an upper and an lower half-neutral element denoted by e and f. Moreover, if $e \le f$ then*

$$F(x,y) = \begin{cases} x \wedge y & \text{if } x \vee y \le e \\ y & \text{if } e \le x \le f \\ x \vee y & \text{if } f \le x \wedge y \end{cases}$$

Analogously, if $f \le e$ then

$$F(x,y) = \begin{cases} x \wedge y & \text{if } x \vee y \le f \\ x & \text{if } f \le x \le e \\ x \vee y & \text{if } e \le x \wedge y \end{cases}$$

We note that $e = f$ iff F has a neutral element.

The following lemma is essential for the visual characterization.

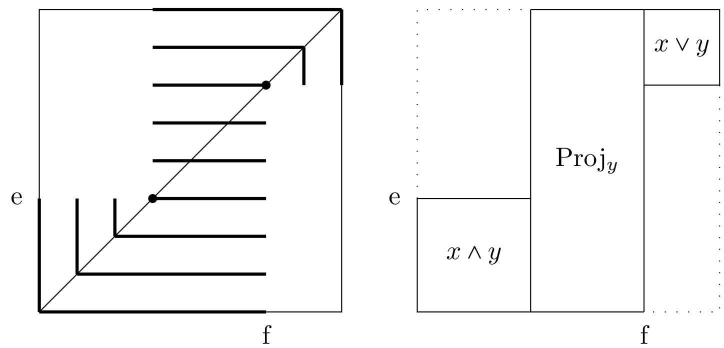

FIGURE 7. Partial description of associative quasitrivial monotone operations when $e \leq f$

Lemma 4.5. *Let $F : L_k^2 \to L_k$ be an associative quasitrivial nondecreasing operation. Assume that there exists $a < b \in L_k$ such that $F(a,b) = a$ and $F(b,a) = b$. Then one of the following holds:*

(a) If $F(a+1,a) = a$, then

$$F(x,b) = b \text{ and } F(y,a) = a$$

for every $x \in [a+1,b]$ and $y \in [a,b-1]$.

(b) If $F(a+1,a) = a+1$, then $F(x,y) = x$ $(= Proj_x(x,y))$ for all $x,y \in [a,b]$.

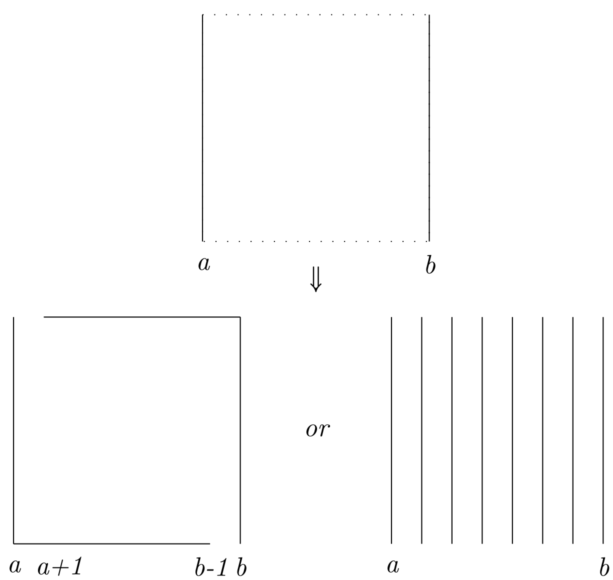

FIGURE 8. Graphical interpretation of Lemma 4.5

Proof. Assume first that $F(a+1,a) = a$. Then it follows that $F(a+1,b) = b$, otherwise we get Figure 2 (a). Using Observation 3.2 we have that $F(x,b) = b$ for every $x \in [a+1,b]$. The equation $F(b-1,b) = b$ implies that $F(b-1,a) = a$, otherwise we are in the situation of Figure 2 (b). Similarly, as above we get that $F(y,a) = a$ for every $y \in [a, b-1]$. Here we note that an analogue argument gives the same result if we assume originally that $F(b-1,b) = b$.

Now assume that $F(a+1,a) = a+1$. This immediately implies that $F(x,a) = x$ for every $x \in [a,b]$ by quasitriviality, since it cannot be a by the nondecreasingness of F. Using Observation 3.2 again, it follows that $F(x,y) = x$ for all $y \in [a,x]$. Since $F(b-1,b) = b$ also implies the previous case, the assumption $F(a+1,a) = a+1$ implies $F(b-1,b) = b-1$. Similarly as above, this condition implies that $F(x,b) = x$ for all $x \in [a,b]$ and, by Observation 3.2, it follows that $F(x,y) = x$ for every $y \in [x,b]$. Altogether we get that $F(x,y) = x = \mathrm{Proj}_x(x,y)$ as we stated. □

Remark 6. Analogue of Lemma 4.5 can be formalized as follows.

Let $F : L_k^2 \to L_k$ be an associative quasitrivial nondecreasing operation. Assume that there exists $a < b \in L_k$ such that $F(b,a) = a$ and $F(a,b) = b$. Then one of the following holds:

(a) *If $F(a,a+1) = a$, then*

$$F(b,x) = b \text{ and } F(a,y) = a$$

for every $x \in [a+1,b]$ and $y \in [a,b-1]$.

(b) *If $F(a,a+1) = a+1$, then $F(x,y) = y (= Proj_y(x,y))$ for all $x,y \in [a,b]$.*

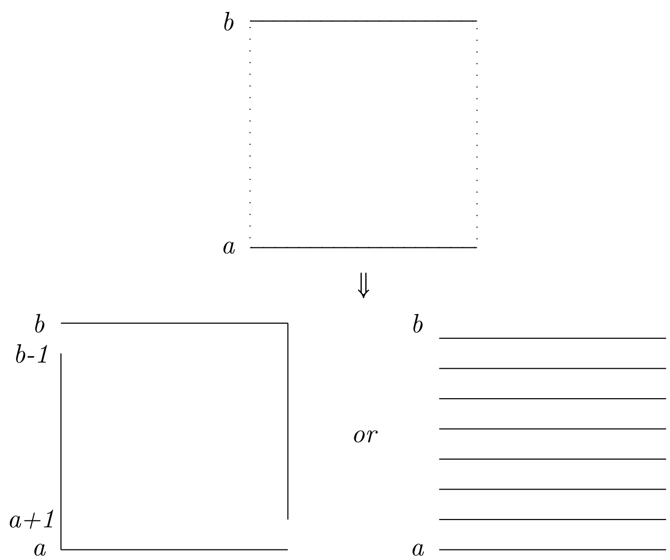

Figure 9. Graphical interpretation of Remark 6

The proof of this statement is analogue to Lemma 4.5 using Figure 2(c) and (d) instead of Figure 2(a) and (b), respectively.

From the previous results we conclude the following.

Lemma 4.6. *Let $F : L_k^2 \to L_k$ be an associative quasitrivial and nondecreasing operation and e and f the upper and the lower half-neutral elements, respectively, and let $a, b \in L_k$ $(a < b)$ be given. If $F(x,y) = x$ for every $x, y \in [a,b]$ (i.e, Lemma 4.5 (b) holds), then $f < e$ and $[a,b] \subseteq [f,e]$. Similarly, if $F(x,y) = y$ for every $x, y \in [a,b]$ (i.e, Remark 6 (b) holds), then $e < f$ and $[a,b] \subseteq [e,f]$.*

Proof. This is a direct consequence of Proposition 4.4. If a or b is not in $[e \wedge f, e \vee f]$ then $\tilde{F} = F|_{[a,b]^2}$ contains a part where $\tilde{F}$ is a minimum or a maximum. Moreover, it is also easily follows that if $F(x,y) = x$ for every $x, y \in [a,b]$, then $f < e$ must hold. Similarly, $F(x,y) = y$ for every $x, y \in [a,b]$ implies $e < f$. □

Corollary 4.7. *Let F, e, f be as in Lemma 4.6 and assume that $a, b \in X$ such that $a < b$ and $F(a,b) \neq F(b,a)$. Then*

(i) Lemma 4.5(b) holds iff $f < e$ and $a, b \in [f,e]$,
(ii) Remark 6(b) holds iff $e < f$ and $a, b \in [e,f]$.
(iii) Lemma 4.5(a) or Remark 6(a) holds iff $a, b \notin [e \wedge f, e \vee f]$.

With other words we have:

Corollary 4.8. *Let F, e, f be as in Lemma 4.6. Then $F(a,b) = F(b,a)$, if $a \notin [e \wedge f, e \vee f]$ and $b \in [e \wedge f, e \vee f]$, or $b \notin [e \wedge f, e \vee f]$ and $a \in [e \wedge f, e \vee f]$.*

This form makes it possible to extend the partial description. (See Figure 10 for the case $e < f$.)

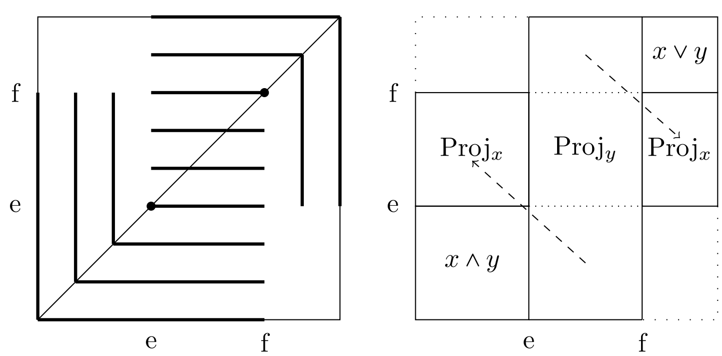

FIGURE 10. Extended partial description of associative quasitrivial monotone operations when $e < f$

Using Lemma 4.5 and Remark 6 we can provide a visual characterization of associative quasitrivial nondecreasing operations. The characterization based on the following algorithm which outputs the contour plot of F.

Before we present the algorithm we note that the letters indicated in the following figures represent the value of operation F in the corresponding points or lines (not a coordinate of the points itself as usual).

Algorithm

Initial setting: Let $Q_1 = L_k^2$ and $F : L_k^2 \to L_k$ be an associative quasitrivial nondecreasing operation.

Step i. For $Q_i = [a,b]^2$ ($a \le b$) we distinguish cases according to the values of $F(a,b)$ and $F(b,a)$. Whenever Q_i contains only 1 element ($a = b$) for some i, then we are done.

I. (a) If $F(a,b) = F(b,a) = a$, then draw straight lines between the points (b,a) and (a,a) and between (a,b) and (a,a). Let $Q_{i+1} = [a+1,b]^2$. (See Figure 11.)

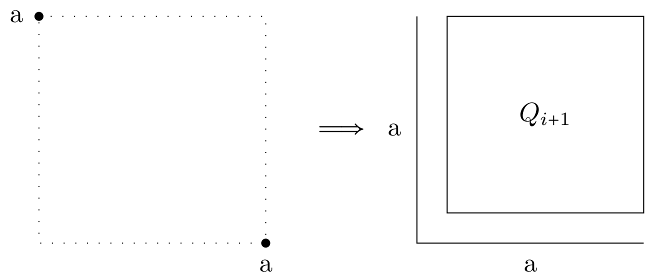

FIGURE 11. Case I.(a)

(b) If $F(a,b) = F(b,a) = b$, then draw straight lines between the points (a,b) and (b,b) and between (b,a) and (b,b). Let $Q_{i+1} = [a,b-1]^2$.

II. (a) If $F(a,b) = a, F(b,a) = b$ and $F(a+1,a) = a+1$, then $F(x,y) = x$ for all $x,y \in [a,b]$ and we are done. (See Figure 12)

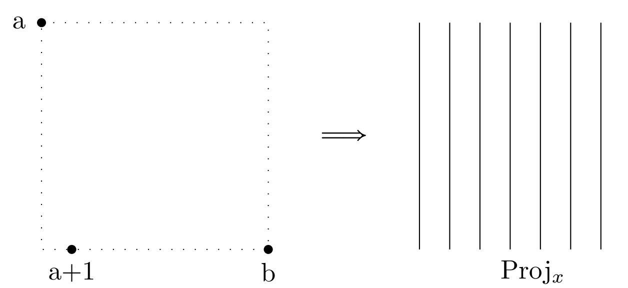

FIGURE 12. Case II.(a)

(b) If $F(a,b) = b, F(b,a) = a$ and $F(a,a+1) = a+1$, then $F(x,y) = y$ for all $x,y \in [a,b]$ and we are also done.

III. (a) If $F(a,b) = a, F(b,a) = b$ and $F(a+1,a) = a$, then Lemma 4.5 (a) holds and we have Figure 13. Let $Q_{i+1} = [a+1,b-1]^2$.

(b) If $F(a,b) = b, F(b,a) = a$ and $F(a,a+1) = a$, then Remark 6 (a) holds. Let $Q_{i+1} = [a+1,b-1]^2$.

It is clear that the algorithm is finished after finitely many steps. Let us denote this number of steps by $l \in \mathbb{N}$.

We also denote the top-left and the bottom-right corner of Q_i by p_i and q_i ($i = 1,\dots,l$), respectively.

Let $\mathcal{P}$ (and $\mathcal{Q}$) denote the path containing p_i (and q_i) for $i \in \{1,\dots,l\}$ and line segments between consecutive p_i's (and q_i's). Let us denote the line segment between p_i and p_{i+1} by $\overline{p_i,p_{i+1}}$. We set the notation $\mathcal{P} = (p_j)_{j=1}^l$ and $\mathcal{Q} = (q_j)_{j=1}^l$.

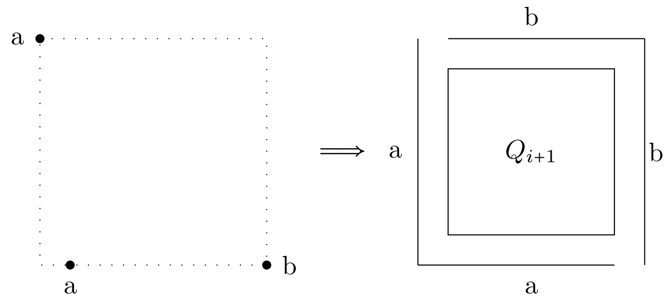

FIGURE 13. Case III.(a)

Clearly, we get the path $\mathcal{P}$ if we start at the top-left corner of L_k^2 and in each step we move either one place to the right or one place downward or one place diagonally downward-right.

Definition 4.9. We say that a path is a *downward-right path* of L_k if in each step it moves to the nearest point of L_k^2 either one place to the right or one place downward or one place diagonally downward-right.

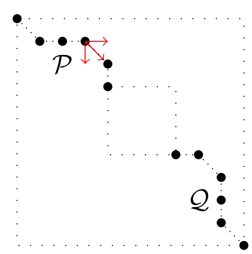

FIGURE 14. The path $\mathcal{P}$ is a downward-right path

If $\overline{p_i, p_{i+1}}$ is horizontal or vertical, then the reduction from Q_i to Q_{i+1} is uniquely determined. Moreover, if $\overline{p_i, p_{i+1}}$ is horizontal, then $F(x,y) = F(y,x) = x \wedge y$, where $p_i = (x,y)$ and $q_i = (y,x)$. Similarly, if $\overline{p_i, p_{i+1}}$ is vertical, then $F(x,y) = F(y,x) = x \vee y$, where $p_i = (x,y)$ and $q_i = (y,x)$. On the other hand if $\overline{p_i, p_{i+1}}$ is diagonal, then we have a free choice for the value of F in p_i. This is determined by either Lemma 4.5 (a) or Remark 6 (a). Since in this case the value of F in q_i is different from p_i, the value in q_i is automatically defined. It is also clear from the algorithm that the path $\mathcal{Q}$ is the reflection of $\mathcal{P}$ to the diagonal Δ_{L_k}.

Using the previous paragraph and Observation 3.2 it is possible to reconstruct operations from a given downward-right path $\mathcal{P}$ which starts at $p_1 = (1,k)$.

Example 4.10. We illustrate the reconstruction on $L_6 \times L_6$. The paths $\mathcal{P} = (p_j)_{j=1}^5$ and $\mathcal{Q} = (q_j)_{j=1}^5$ denoted by red and blue, respectively. According to the previous observations we get the following pictures (see Figure 15). It can be clearly seen that $\mathcal{Q}$ is the reflection of $\mathcal{P}$ to the diagonal Δ_{L_6}, and 4 is the neutral element of the reconstructing operation, where $\mathcal{P}$ and $\mathcal{Q}$ touch each other and reach the diagonal Δ_{L_6}. For the precise statement and proof see Theorem 4.13.

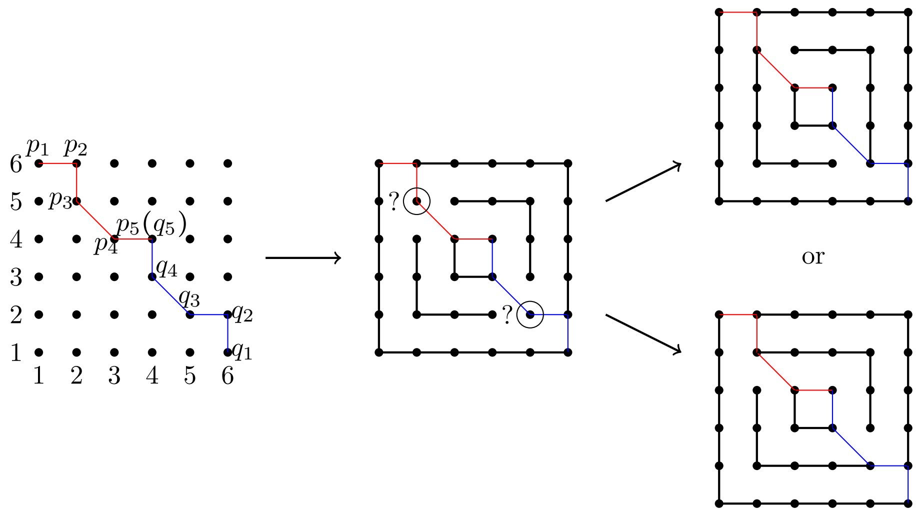

FIGURE 15. Reconstruction of F from the path $\mathcal{P}$

Definition 4.11. Let $\mathcal{P} \subset L_k^2$ be the downward-right path from $(1,k)$ to (a,b) $(a<b)$ and let $\mathcal{Q}$ be the reflection of $\mathcal{P}$ to the diagonal Δ_{L_k}.

We say that $(x,y) \in L_k^2 \setminus (\mathcal{P} \cup \mathcal{Q} \cup [a,b]^2)$ is *above* $\mathcal{P} \cup \mathcal{Q}$ if there exists $p=(x,w) \in \mathcal{P}$ such that $y>w$ or $q=(w,y) \in \mathcal{Q}$ such that $x>w$.

Similarly, we say that $(x,y) \in L_k^2 \setminus (\mathcal{P} \cup \mathcal{Q} \cup [a,b]^2)$ is *below* $\mathcal{P} \cup \mathcal{Q}$ if there exists a $p=(x,w) \in \mathcal{P}$ such that $y<w$ or a $q=(w,y) \in \mathcal{Q}$ such that $x<w$.

Using this terminology we can summarize the previous observations and we get the following characterization. The next statement can be seen as the analogue of Theorem 1.1 for finite chains. Moreover, it is generalized to the case when F has no neutral element.

Theorem 4.12. *For every associative quasitrivial nondecreasing operation $F: L_k^2 \to L_k$ there exist half-neutral elements $a,b \in L_k$ $(a \le b)$ and a downward-right path $\mathcal{P}=(p_j)_{j=1}^l$ (for some $l \in \mathbb{N}, l<k$) from $(1,k)$ to (a,b). We denote the reflection of $\mathcal{P}$ to the diagonal Δ_{L_k} by $\mathcal{Q}=(q_j)_{j=1}^l$. Then for every $(x,y) \notin \mathcal{P} \cup \mathcal{Q}$*

$$F(x,y)=\begin{cases} x \vee y, & \text{if } (x,y) \text{ is above } \mathcal{P} \cup \mathcal{Q} \\ x \wedge y, & \text{if } (x,y) \text{ is below } \mathcal{P} \cup \mathcal{Q} \\ Proj_x(x,y) \text{ or } Proj_y(x,y), & \text{if } (x,y) \in [a,b]^2, \end{cases}$$

and for every $(x,y) \in \mathcal{P} \cup \mathcal{Q}$

$$F(x,y)=\begin{cases} x \wedge y & \text{if } (x,y)=p_i \text{ or } q_i \text{ and } \overline{p_i,p_{i+1}} \text{ is horizontal,} \\ x \vee y, & \text{if } (x,y)=p_i \text{ or } q_i \text{ and } \overline{p_i,p_{i+1}} \text{ is vertical,} \\ x \text{ or } y, & \text{if } (x,y)=p_i \text{ and } \overline{p_i,p_{i+1}} \text{ is diagonal,} \\ x \text{ or } y, & \text{if } (x,y)=q_i \text{ and } \overline{q_i,q_{i+1}} \text{ is diagonal.} \end{cases}$$

If a is the lower half-neutral element f and b is the upper half-neutral element e, then F is $Proj_x$ on $[a,b]^2$, otherwise it is $Proj_y$.

Moreover F is symmetric expect on $[a,b]^2$ and at the points $p_i \in \mathcal{P}$ and $q_i \in \mathcal{Q}$ where $\overline{p_i,p_{i+1}}$ is diagonal $(i \in \{1,\dots,l-1\})$.

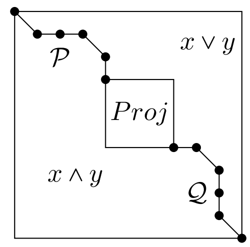

FIGURE 16. Characterization of associative quasitrivial nondecreasing operations on finite chains

Proof. The statement is clearly follows from the Algorithm and the definition of paths $\mathcal{P}$ and $\mathcal{Q}$. ☐

The converse statement can be formalized as follows. The next statement and Theorem 4.12 together provide a characterization of associative quasitrivial nondecreasing operations on finite chains. This characterization can be seen as the analogue of Martin-Mayor-Torrens's result [18, Theorem 4.] for finite chains.

Theorem 4.13. *Let $\mathcal{P} = (p_j)_{j=1}^{l}$ be a downward-right path in $T_1 \subset L_k^2$ from $(1,k)$ to (a,b) $(a \leq b)$ and let $\mathcal{Q} = (q_j)_{j=1}^{l}$ be its reflection to the diagonal Δ_{L_k}. Let $F : L_k^2 \to L_k$ be defined for every $(x,y) \notin \mathcal{P} \cup \mathcal{Q}$ as*

$$F(x,y) = \begin{cases} x \vee y, & \text{if } (x,y) \text{ is above } \mathcal{P} \cup \mathcal{Q}, \\ x \wedge y, & \text{if } (x,y) \text{ is below } \mathcal{P} \cup \mathcal{Q}, \\ Proj_x(x,y) \text{ or } Proj_y(x,y) \text{ (uniformly)}, & \text{for every } (x,y) \in [a,b]^2. \end{cases}$$

and for every $(x,y) \in \mathcal{P} \cup \mathcal{Q}$

$$F(x,y) = \begin{cases} x \wedge y & \text{if } (x,y) = p_i \text{ or } q_i \text{ and } \overline{p_i, p_{i+1}} \text{ is horizontal}, \\ x \vee y, & \text{if } (x,y) = p_i \text{ or } q_i \text{ and } \overline{p_i, p_{i+1}} \text{ is vertical}, \\ x \text{ or } y \text{ (arbitrarily)}, & \text{if } (x,y) = p_i \text{ and } \overline{p_i, p_{i+1}} \text{ is diagonal}. \end{cases}$$

If $(x,y) = q_i$ and $\overline{q_i, q_{i+1}}$ (or equivalently $\overline{p_i, p_{i+1}}$) is diagonal, then $F(x,y) \in \{x,y\}$ and $F(x,y) \neq F(y,x)$ uniquely define $F(x,y)$. Then F is associative quasitrivial and nondecreasing.

Proof. It is clear that F is defined for every $(x,y) \in L_k^2$ and F is quasitrivial and nondecreasing. Now we show that F is associative. If it is not the case, then by Lemma 3.5, one of the cases of Figure 2 is realized. Let $u,v,w \in L_k$ $(u < v < w)$ denote the elements where its realized. Clearly $F(u,w) \neq F(w,u)$ and F is not a projection on $[u,w]^2$. Thus, by the definition of F, it follows that $(u,w) \in \mathcal{P}$ and $(w,u) \in \mathcal{Q}$. Hence $p_i = (u,w)$ for some $i = \{1,\ldots,l-1\}$ and $\overline{p_i, p_{i+1}}$ is diagonal. Thus we have one of the following situation (Figure 17).

Therefore, since $u < v < w$, it follows that $F(u,v) \neq v, F(v,u) \neq v, F(w,v) \neq v, F(v,w) \neq v$. Hence, none of the cases of Figure 2 can be realized. Thus F is associative. ☐

Remark 7. According to Theorems 4.12 and 4.13 it is clear that there is a surjection from the set of associative quasitrivial nondecreasing operations defined on L_k to the downward-right paths defined on T_1 and started at $(1,k)$ (and ended somewhere

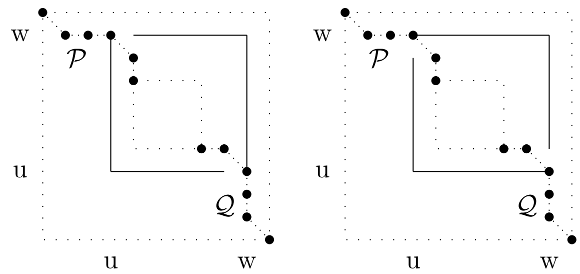

FIGURE 17. Two remaining cases

in T_1). This surjection is a bijection if and only if the path $\mathcal{P}$ does not contain a diagonal move and $a = b$. This condition is equivalent that F is symmetric (and has a neutral element).

Corollary 4.14. *Let $F : L_k^2 \to L_k$ be an associative quasitrivial nondecreasing operation. If F is symmetric, then it is uniquely determined by a downward-right path $\mathcal{P}$ containing only horizontal and vertical line segments and it starts at $(1, k)$ and reaches the diagonal Δ_{L_k}.*

As a consequence of the previous corollary we obtain the result of [27, Theorem 4.] (see also [4, Theorem 14.]).

Corollary 4.15. *The number of associative quasitrivial nondecreasing symmetric operation defined on L_k is 2^{k-1}.*

Proof. Every path from $(1, k)$ to the diagonal Δ_{L_k} using right or downward moves contains k points. According to Corollary 4.14, in each point of the path, except the last one, we have two options which direction we move further. This immediately implies that the number of associative quasitrivial nondecreasing symmetric operation defined on L_k is 2^{k-1}. □

In Theorem 6.1, as an application of the results of this section, we calculate the number of associative quasitrivial nondecreasing operations defined on L_k and also the number of associative quasitrivial nondecreasing operations on L_k that have neutral elements.

Remark 8. (a) We note that from the proof of Lemma 4.5 throughout this section we essentially use that F is defined on a finite chain.
(b) In the continuous case [2, 18] and also in the symmetric case [4, 27] it is always possible to define a one variable function g, such that the extended graph of g separates the points of the domain of the binary operation F into two parts where F is a minimum and a maximum, respectively. Now the paths $\mathcal{P}$ and $\mathcal{Q}$ play the role of the extended graph of g. Because of the diagonal moves of the path $\mathcal{P}$, it does not seems so clear how such a 'separating' function can be defined in the non-symmetric discrete case.

5. Bisymmetric operations

In this section we show a characterization of bisymmetric quasitrivial nondecreasing binary operations based on the previous section.The following statement was proved as [4, Lemma 22.].

Lemma 5.1. *Let X be an arbitrary set and $F : X^2 \to X$ be an operation. Then the following assertions hold.*

(a) If F is bisymmetric and has a neutral element, then it is associative and symmetric.
(b) If F is bisymmetric and quasitrivial, then F is associative.
(c) If F is associative and symmetric, then it is bisymmetric.

Using also the results of Section 4 we get the following statement.

Theorem 5.2. *Let $F : L_k^2 \to L_k$ be a bisymmetric quasitrivial nondecreasing operation. Then there exists the upper half-neutral element e and the lower half-neutral element f and F is symmetric on $(L_k \setminus [e \wedge f, e \vee f])^2$.*

Proof. According to Lemma 5.1(b), every quasitrivial bisymmetric operations are associative. Thus, by Proposition 4.4 it has an upper and lower half-neutral element (e and f, respectively).

Let us assume that $e \leq f$ (the case when $f \leq e$ can be handled similarly).

If there exists $u, v \in L_k$ such that $u < v$, $F(u,v) \neq F(v,u)$, then by Corollary 4.7, either $u, v \in [e, f]$ (then we do not need to prove anything) or $u, v \notin [e, f]$. Moreover, if $u, v \notin [e, f]$, then Lemma 4.5(a) or Remark 6(a) holds. The existence of e implies that $v - u \geq 2$.

If

$$u = F(u,v) \neq F(v,u) = v$$

is satisfied, then Lemma 4.5 (a) holds (i.e, $F(x,v) = v$ if $x \in [u+1, v]$ and $F(y,u) = u$ if $y \in [u, v-1]$). Since $v - u \geq 2$, $u + 1 \leq v - 1$, hence $F(u+1, u) = u$. On the other hand, F is monotone and idempotent, thus by Observation 3.2, $F(v,t) = v$ and $F(u,t) = u$ for all $t \in [u,v]$. Using bisymmetric equation we get the following

$$u = F(u,v) = F(F(u+1,u), F(v,v-1)) = F(F(u+1,v), F(u,v-1)) = F(v,u) = v,$$

which is a contradiction.

Similarly, if

$$v = F(u,v) \neq F(v,u) = u$$

is satisfied, then Remark 6 (a) holds (i.e, $F(v,x) = v$ if $x \in [u+1, v]$ and $F(u,y) = u$ if $y \in [u, v-1]$). Since $v - u \geq 2$, $u + 1 \leq v - 1$, hence $F(v-1, v) = v$. Applying Observation 3.2 again, we have $F(t,v) = v$ and $F(t,u) = u$ for all $t \in [u,v]$. Using bisymmetric equation we get a contradiction as

$$u = F(v,u) = F(F(v-1,v), F(u,u+1)) = F(F(v-1,u), F(v,u+1)) = F(u,v) = v.$$

□

Applying Theorem 5.2 we get the following characterization.

Theorem 5.3. *Let $F : L_k^2 \to L_k$ be a quasitrivial nondecreasing operation. Then F is bisymmetric if and only if there exists $a, b \in L_k$ ($a \leq b$) and a downward-right path $\mathcal{P} = (p_j)_{j=1}^{l}$ (for some $l \in \mathbb{N}$) from $(1,k)$ to (a,b) containing only horizontal and vertical line segments such that for every $(x,y) \notin \mathcal{P} \cup \mathcal{Q}$*

(5)

$$F(x,y) = \begin{cases} x \vee y, & \text{if } (x,y) \text{ is above } \mathcal{P} \cup \mathcal{Q}, \\ x \wedge y, & \text{if } (x,y) \text{ is below } \mathcal{P} \cup \mathcal{Q}, \\ Proj_x(x,y) \text{ or } Proj_y(x,y) \text{ (uniformly)}, & \text{for every } (x,y) \in [a,b]^2. \end{cases}$$

and for every $(x,y)\in\mathcal{P}\cup\mathcal{Q}$

$$F(x,y)=\begin{cases} x\wedge y & \text{if } (x,y)=p_i \text{ or } q_i \text{ and } \overline{p_i,p_{i+1}} \text{ is horizontal,}\\ x\vee y, & \text{if } (x,y)=p_i \text{ or } q_i \text{ and } \overline{p_i,p_{i+1}} \text{ is vertical,}\end{cases} \tag{6}$$

where $\mathcal{Q}=(q_j)_{j=1}^{l}$ *is the reflection of* $\mathcal{P}$ *to the diagonal* Δ_{L_k}*.*

In particular, F *is symmetric on* $L_k^2\smallsetminus[a,b]^2$ *and one of the projections on* $[a,b]^2$*.*

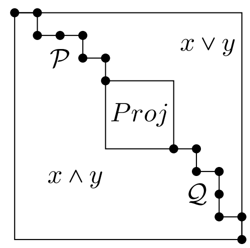

Figure 18. Characterization of bisymmetric quasitrivial nondecreasing operations on finite chains

Proof. (Necessity) Since F is bisymmetric and quasitrivial, by Lemma 5.1(b), F is associative. By Theorem 4.12, there exist half-neutral elements $a,b\in L_k$ $(a<b)$ and a downward-right path $\mathcal{P}$ from $(1,k)$ to (a,b). By Theorem 5.3, F is symmetric on $L_k^2\smallsetminus[a,b]^2$. Thus $\mathcal{P}$ does not contain a diagonal line segment. Hence, applying again Theorem 4.12 we get that F satisfies (5) and (6).

(Sufficiency) The operation F defined by (5) and (6) satisfies the conditions of Theorem 4.13, thus F is quasitrivial nondecreasing and associative. Now we show that F is bisymmetric (i.e, $\forall u,v,w,z\in L_k$

$$F(F(u,v),F(w,z))=F(F(u,w),F(v,z)).) \tag{7}$$

Let us assume that $F(x,y)=Proj_x$ on $[a,b]^2$ (for $F(x,y)=Proj_y$ on $[a,b]^2$ the proof is analogue). By Corollary 4.7, this implies that $a=f$ and $b=e$ $(f<e)$ and, by Proposition 4.4, it is clear that

$$F(x,y)=x\ \ \forall x\in L_k, \forall y\in[a,b]. \tag{8}$$

Since F is associative, we have

$$F(F(u,v),F(w,z))=F(F(F(u,v),w),z)=F(F(u,F(v,w)),z)$$

and

$$F(F(u,w),F(v,z))=F(F(F(u,w),v),z)=F(F(u,F(w,v)),z).$$

If $F(v,w)=F(w,v)$, then (7) follows and we are done.

If $F(v,w)\neq F(w,v)$, then $v,w\in[a,b]^2$ and, since $F(x,y)=Proj_x$ on $[a,b]^2$, $F(v,w)=v$ and $F(w,v)=w$. Then, by (8),

$$F(F(u,F(v,w)),z)=F(F(u,v),z)=F(u,z),$$
$$F(F(u,F(v,w)),z)=F(F(u,w),z)=F(u,z).$$

Thus F is bisymmetric. □

Remark 9. (a) There is a one-to-one correspondence between downward-right paths containing only vertical and horizontal line segments and the quasitrivial nondecreasing bisymmetric operations if we fix that the operation is $Proj_x$ on $[a,b]^2$ (a and b are the half neutral-elements of the operation). The same is true, if the operation is $Proj_y$ on $[a,b]^2$.

(b) The nondecreasing assumption can be substituted by monotonicity. Indeed, by Corollary 3.3, monotonicity is equivalent with nondecreasingness for quasitrivial operations.

6. The number of operations of given class

This section is devoted to calculate the number of associative quasitrivial nondecreasing operations. Byproduct of the following argument we also consider the number of associative quasitrivial nondecreasing operations having neutral elements. With the same technique one can easily deduce the number of bisymmetric quasitrivial nondecreasing binary operations (see Proposition 6.5).

Theorem 6.1. *Let A_k denote the number of associative quasitrivial nondecreasing operations defined on L_k and B_k denote the number of associative quasitrivial nondecreasing operations defined on L_k and having neutral elements. Then*

$$A_k = \frac{1}{6}\big((2+\sqrt{3})(1+\sqrt{3})^k + (2-\sqrt{3})(1-\sqrt{3})^k - 4\big),$$

$$B_k = \frac{1}{2\cdot\sqrt{3}}\big((1+\sqrt{3})^k - (1-\sqrt{3})^k\big).$$

The following observations show that these numbers are related to the downward-right path $\mathcal{P} = (p_j)_{j=1}^{l}$ (for some $l \le k$) in T_1 starting from $(1,k)$. Let $m_{\mathcal{P}}$ be the number of diagonal line segments $\overline{p_i, p_{i+1}} \in \mathcal{P}$ ($i \in \{1,\dots,l-1\}$). We say that the downward-right path $\mathcal{P}$ is *weighted* with weight $2^{m_{\mathcal{P}}}$.

Lemma 6.2. *(a) B_k is the sum of the weights of weighted paths that starts at $(1,k)$ and reaches Δ_{L_k}.*

(b) $A_k + B_k$ is twice the sum of the weights of weighted paths in T_1 that starts at $(1,k)$ and ends at any point of T_1.

Proof. (a) Applying Theorem 4.12, it is clear that if an associative quasitrivial nondecreasing binary operation F has a neutral element, then the downward-right path $\mathcal{P}$ defined for F reaches the diagonal Δ_{L_k}. By Theorem 4.13, there can be defined $2^{m_{\mathcal{P}}}$ different operations for a given path $\mathcal{P}$ that reaches the diagonal, since we have a choice in each case when the path contains a diagonal line segment. This show the first part of the statement.

(b) This statement follows from the fact that for any associative quasitrivial nondecreasing operation F one can define a downward-right path which starts at $(1,k)$ and ends somewhere in T_1. If its end in (a,b) where $a < b$ (not on Δ_{L_k}), then F is one of the projections in $[a,b]^2$, and a and b are the half-neutral elements of F. This makes the extra 2 factor in the statement.

Let Π_1 denote set of the weighted paths in T_1 that starts at $(1,k)$ and ends at (a,b) where $a < b$. Similary, Π_2 denote the set of weighted paths that starts at $(1,k)$ and reaches Δ_{L_k}. Hence,

$$A_k = 2\cdot\sum_{\mathcal{P}\in\Pi_1} 2^{m_{\mathcal{P}}} + \sum_{\mathcal{P}\in\Pi_2} 2^{m_{\mathcal{P}}}$$

According to the (a) part

$$B_k = \sum_{\mathcal{P}\in\Pi_2} 2^{m_{\mathcal{P}}}.$$

Adding these equations, we get the statement for $A_k + B_k$.

□

Now we present a recursive formula for A_k and B_k.

Lemma 6.3. *(a) $B_1 = 1$, $B_2 = 2$ and $B_k = 2\cdot B_{k-1} + 2\cdot B_{k-2}$ for every $k \geq 3$.*
(b) $A_k = 2\sum_{i=1}^{k} B_i - B_k$ for every $k \in \mathbb{N}$.

Proof. (a) $B_1 = 1$, $B_2 = 2$ are clear. The recursive formula follows from the Algorithm presented in Section 4 and the definition of downward-right path $\mathcal{P} = (p_j)_{j=1}^{l}$. Now we assume that $k \geq 3$. If $\overline{p_1, p_2}$ is horizontal or vertical, then Case I. (a) or (b) of the Algorithm holds (see also Figure 11). Thus we reduce the square Q_1 of size k to a square Q_2 of size $k-1$. If $\overline{p_1, p_2}$ is diagonal, then Case III (a) or (b) holds (see also Figure 13). Thus we reduce the square Q_1 of size k to a square Q_2 of size $k-2$. By definition, the number of associative quasitrivial nondecreasing operations having neutral elements defined on a square of size k is B_k. Thus we get that $B_k = 2\cdot B_{k-1} + 2\cdot B_{k-2}$.

(b) This follows from Lemma 6.2 (b) and the fact that 'sum of the weights of weighted paths from $(1,k)$ to any point of T_1' is exactly $\sum_{i=1}^{k} B_i$. Indeed, let $s \in \{1,\dots,k\}$ be fixed. Then B_s is equal to the sum of the weights of weighted paths $\mathcal{P}$ that starts at $(1,k)$ and ends at (a,b) where $b - a = s$.

□

Proof of Theorem 6.1. We use a standard method of second-order linear recurrence equations for the formula of Lemma 6.3 (a). Therefore,

$$B_k = c_1\cdot(\alpha_1)^k + c_2(\alpha_2)^k,$$

where α_1, α_2 ($\alpha_1 < \alpha_2$) are the solutions of the equation $x^2 - 2x - 2 = 0$. Thus, $\alpha_1 = 1-\sqrt{3}, \alpha_2 = 1+\sqrt{3}$. By the initial condition $B_1 = 1$ and $B_2 = 2$, we get that $c_1 = -c_2 = \frac{1}{2\sqrt{3}}$. Thus,

$$B_k = \frac{1}{2\cdot\sqrt{3}}\big((1+\sqrt{3})^k - (1-\sqrt{3})^k\big).$$

According to Lemma 6.3 (b), A_k can be calculated as $2\cdot\sum_{i=1}^{k} B_i - B_k$.
This provides that

$$A_k = \frac{1}{6}\big((2+\sqrt{3})(1+\sqrt{3})^k + (2-\sqrt{3})(1-\sqrt{3})^k - 4\big)$$

□

Here we present a list of the first 10 value of A_k: $A_1 = 1$, $A_2 = 4$, $A_3 = 12, A_4 = 34, A_5 = 94, A_6 = 258, A_7 = 706, A_8 = 1930, A_9 = 5274, A_{10} = 14410$.

By Theorem 3.1, we get the similar results for the n-ary case.

Corollary 6.4. *(a) The number of associative quasitrivial nondecreasing operations $F : L_k^n \to L_k$ ($k \in \mathbb{N}$) having neutral elements is*

$$\frac{1}{2\cdot\sqrt{3}}\big((1+\sqrt{3})^k - (1-\sqrt{3})^k\big),$$

(b) *The number of associative quasitrivial nondecreasing operations* $F : L_k^n \to L_k$ $(k \in \mathbb{N})$ *is*

$$\frac{1}{6}\Big((2+\sqrt{3})(1+\sqrt{3})^k + (2-\sqrt{3})(1-\sqrt{3})^k - 4\Big).$$

Proposition 6.5. *Let* C_k *denote number of bisymmetric quasitrivial nondecreasing binary operations defined in* L_k *and* D_k *denote the number of bisymmetric quasitrivial nondecreasing binary operations having neutral elements. Then*

$$D_k = 2^{k-1},$$
$$C_k = 3 \cdot 2^{k-1} - 2.$$

Proof. (a) By Lemma 5.1 and Proposition 3.8, bisymmetric quasitrivial nondecreasing binary operations having neutral elements defined on L_k are exactly the associative quasitrivial symmetric nondecreasing binary operations. Thus by Corollary 4.15, we get that $D_k = 2^{k-1}$.

(b) Same argument as in Lemma 6.3(b) shows that $C_k = 2\sum_{i=1}^{k} D_i - D_k$. Using this we get that $C_k = 2 \cdot (2^k - 1) - 2^{k-1} = 3 \cdot 2^{k-1} - 2$.

□

Remark 10. During the finalization of this paper the author have been informed that Miguel Couceiro, Jimmy Devillet and Jean-Luc Marichal found an alternative and independent approach for similar estimations in their upcoming paper [5].

7. Open problems and further perspectives

First we summarize the most important results of our paper. In this article we introduced a geometric interpretation of quasitrivial nondecreasing associative binary operations. We gave a characterization of such operations on finite chains using downward-right paths. Combining this with a reducibility argument we provided characterization for the n-ary analogue of the problem. As a remarkable application of our visualization method we gave characterization of bisymmetric quasitrivial nondecreasing binary operation on finite chains. As a byproduct of our argument we estimated the number of operations belonging to these classes.

These results initiate the following open problems.

(1) Characterize the n-ary bisymmetric quasitrivial nondecreasing operations. If these operations are also associative, then we can apply reducibility to deduce a characterization for them. On the other hand if $n \geq 3$, then not all of such operations are associative as the following example shows. Let $F: X^n \to X$ $(n \geq 3)$ be the projection on the i^{th} coordinate where i is neither 1 or n. Then it is easy to show that it is bisymmetric quasitrivial nondecreasing but not associative.

(2) Find a visual characterization of associative idempotent nondecreasing operations. Quasitrivial operations are automatically idempotent. Since idempotent operations are essentially important in fuzzy logic, this problem has its own interest.

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Appendix

This section is devoted to prove the analogue of Corollary 3.3. As it was already mentioned in Remark 4, the proof is just a slight modification of the proof of [16, Theorem 3.2]. The difference is based on the following easy lemma.

Lemma 7.1. *Let X be a chain and $F : X^n \to X$ be an associative monotone operation. Then F is non-decreasing in the first and the last variable.*

Proof. The argument for the first and for the last variable is similar. We just consider it for the first variable. From the definition of associativity it is clear that an associative operation $F\colon X^n \to X$ is satisfies

$$F(F(x_1,\dots,x_n),x_{n+1},\dots,x_{2n-1}) = F(x_1,F(x_2,\dots,x_{n+1}),x_{n+2},\dots,x_{2n-1}). \tag{9}$$

for every $x_1,\dots,x_{2n-1} \in X$. Now let us fix $x_2,\dots,x_{2n-1} \in X$ and define

$$h(x) = F(F(x,x_2,\dots,x_n),x_{n+1},\dots,x_{2n-1}).$$

The operation F is monotonic in the first variable thus it is clear that $h(x)$ is nondecreasing, since we apply F twice when x is in the first variable. Then using (9) we get that F must be nondecreasing in the first variable. □

As it was also mentioned in [16] the following condition is an easy application of [1, Theorem 1.4] using the statement therein for $A_2 = \varnothing$.

Theorem 7.2. *Let X be an arbitrary set. Suppose $F : X^n \to X$ be a quasitrivial associative operation. If F is not derived from a binary operation G, then n is odd and there exist b_1, b_2 $(b_1 \neq b_2)$ such that for any $a_1,\dots,a_n \in \{b_1,b_2\}$*

$$F(a_1,\dots,a_n) = b_i \ (i = \{1,2\}), \tag{10}$$

where b_i occurs odd number of times.

Proposition 7.3. *Let X be a totally ordered set and let $F\colon X^n \to X$ be an associative, quasitrivial, monotone operation. Then F is reducible.*

Proof. According to Theorem 7.2, if F is not reducible, then n is odd. Hence $n \geq 3$ and there exist b_1, b_2 satisfying equation (10). Since $b_1 \neq b_2$, we may assume that $b_1 < b_2$ (the case $b_2 < b_1$ can be handled similarly). By the assumption (10) for b_1 and b_2 we have

$$F(n\cdot b_1) = b_1,\ F(b_2,(n-1)\cdot b_1) = b_2,\ F(b_2,(n-2)\cdot b_1,b_2) = b_1. \tag{11}$$

By Lemma 7.1, F is nondecreasing in the first and the last variable. Thus we have

$$F(n\cdot b_1) \leq F(b_2,(n-1)\cdot b_1) \leq F(b_2,(n-2)\cdot b_1,b_2).$$

This implies $b_1 = b_2$, a contradiction. □

The following was proved as [16, Corollary 4.9].

Corollary 7.4. *Let X be a nonempty chain and $n \geq 2$ be an integer. An associative, idempotent, monotone operation $F : X^n \to X$ is reducible if and only if F is nondecreasing.*

Using Proposition 7.3 and Corollary 7.4 we get the statement.

Corollary 7.5. *Let $n \geq 2 \in \mathbb{N}$ be given, X be a nonempty chain and $F : X^n \to X$ be an associative quasitrivial operation.*

$$F \text{ is monotone } \Longleftrightarrow F \text{ is nondecreasing.}$$

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