Aims
A powerful Monte Carlo variance reduction technique introduced in (Cao, M. Y. Hussaini, and Zhang, 2004) uses local derivatives to accelerate Monte Carlo estimation. This work aims to:
- Develop a new derivative-driven estimator that works for SPDEs with uncertain data modelled as Gaussian random fields with Matérn covariance functions (infinite/high-dimensional problems) (Lindgren, Rue, and Lindström, 2011),
- Use second-order derivative (Hessian) information for improved variance reduction over our approach (in Hauseux, Hale, and Bordas, 2017),
- Demonstrate a software framework using FEhICS (Logg and Wells, 2010), dolfin-adjoint (Farrell et al., 2013) and PETSc (Balay et al., 2016) for automatic acceleration of MC estimation for a wide variety of PDEs on HPC architectures.

Setting
A non-linear parametric (ω) PDE:
\[ F(u, \omega) = 0. \]
We solve using the finite element method and pre-conditioned Newton-Krylov methods. Uncertain parameter modelled by a Gaussian random field:
\[ \omega \sim N(\mu, C). \]

Monte Carlo
We want to find:
\[ \mathbb{E}[\omega | \omega]. \]

Variance Reduction
Standard error estimate for Monte Carlo:
\[ \text{Standard error estimate for Monte Carlo:} \]
\[ \text{Variance of Estimator:} \]
\[ \text{VARIANCE REDUCTION} \]

Monte Carlo derivative
The second order sensitivity derivative Monte Carlo estimator can be written:
\[ \text{Second order sensitivity derivative Monte Carlo estimator can be written:} \]

Automatic Differentiation
We use dolfin-adjoint (Farrell et al., 2013) to automatically derive the adjoint and second-order adjoint equations and their finite element discretisation from their Unified Form Language description. This gives us access to routines for calculating the gradient and Hessian-vector action of ψ with respect to ω.

Correlation Term
The term:
\[ E\left[D^2_e \psi (\omega - \bar{\omega}) | \omega - \bar{\omega} \right], \]
can be written in a finite-dimensional Euclidean \( E^M \) setting as:
\[ E\left[\psi (\omega - \bar{\omega}) | H \right]. \]

Results
We solve a stochastic non-linear Burgers equation with uncertain viscosity coefficient \( \nu \) modelled as a random field. We can achieve variance reduction of three orders of magnitude over a standard MC method with only a few dozen extra PDE solves. We need far fewer realisations Z of the non-linear PDE to achieve convergence.

Find \( u \in H^2(\Omega) \) such that:
\[ F(u, \nu, \bar{\omega}) := \int_{\Omega} \nu \nabla u \cdot \nabla u - \frac{1}{2} \nu (u^2 - \bar{u}) \, dx = 0 \quad \forall \nu \in H^1(\Omega), \]
\[ \nu \sim N(1, C), \quad C := A^{-1}, \quad A := \nu - \nu^T. \]

Table 1

<table>
<thead>
<tr>
<th>Method</th>
<th>Normalised Variance of Estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Monte Carlo</td>
<td>1.0</td>
</tr>
<tr>
<td>Sensitivity derivative Monte Carlo N = 1</td>
<td>1.80 \times 10^{-2}</td>
</tr>
<tr>
<td>Sensitivity derivative Monte Carlo N = 2</td>
<td>1.13 \times 10^{-3}</td>
</tr>
</tbody>
</table>

Figure 1

Normalised spectrum of operator \( C^{1/2}H^{1/2} \) for non-linear Burgers problem with stochastic viscosity.

References