Extending Typicality for Description Logics

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Abstract. Recent extensions of description logics for dealing with different forms of non-monotonic reasoning don't take us beyond the case of *defeasible subsumption*. In this paper we enrich the DL $\mathcal{EL}_\bot$ with a (constrained version of) a typicality operator $\bullet$, the intuition of which is to capture the most typical members of a class, providing us with the DL $\mathcal{EL}_\bot^\bullet$. We argue that $\mathcal{EL}_\bot^\bullet$ is the smallest step one can take to increase the expressivity beyond the case of defeasible subsumption for DLs, while still retaining all the rationality properties an appropriate notion of defeasible subsumption is required to satisfy, and investigate what an appropriate notion of non-monotonic entailment for $\mathcal{EL}_\bot^\bullet$ should look like.

1 Introduction

Given the success story of description logics (DLs) [1] as knowledge representation formalisms for modern AI applications, the past decades have witnessed many attempts to endow them with non-monotonic reasoning features. These essentially range from preferential approaches [2–9] to circumscription [10–12], amongst others [13–17].

Up until now, most of the effort in this direction has been put into the study of defeasible inheritance via non-monotonic versions of the subsumption relation. To witness, Britz et al. [4, 5] have introduced *defeasible subsumption statements* of the form $C \mathrel{\sqsubseteq\!\!\!\!\!\sim} D$, whereas Giordano et al. [7, 18] have enriched the concept language with a *typicality operator* $\mathbf{T}(\cdot)$ allowing us to state subsumptions of the form $\mathbf{T}(C) \sqsubseteq D$. Both approaches aim at formalising the intuition according to which "Cs are usually Ds" or "typical Cs are Ds" and build on the well-established KLM approach (after Kraus, Lehmann and Magidor [19]) to defeasible consequence of the form $\varphi \mathrel{|\!\!\sim} \psi$ in propositional logic.

Recently, Bonatti et al. [15] also introduce a typicality-like operator $\mathbf{N}(\cdot)$ in the concept language. In their framework, it becomes possible to define concepts of the form $\exists r.\mathbf{N}(C)$, referring to those objects that are related to typical Cs. This allows us to state subsumptions of the form $\mathbf{N}(C) \sqsubseteq \exists r.\mathbf{N}(D)$, for which properties beyond the Boolean ones would be required in order to characterise the behaviour of defeasibility. However, Bonatti et al.'s construction is based on a particular semantic framework and does not satisfy all the rationality properties associated with preferential approaches.

In previous work, Booth et al. [20, 21] investigated the addition of a typicality operator $\bullet$ to propositional logic, of which the semantics is given in terms of KLM ranked models [22]. It turns out the logic thus obtained is more expressive than that of KLM conditional statements, allowing us to move beyond the defeasible conditionals of the propositional case. Following up on that, Booth et al. [23] investigated two semantic versions of entailment in the presence of $\bullet$, constructed using two different forms of minimality. Both are based on the notion of Rational Closure (RC) as defined by Lehmann and Magidor [22] for KLM-style propositional conditionals. It was shown that (*i*) these notions of entailment can be viewed as generalised definitions of Rational Closure, (*ii*) that they are equivalent w.r.t. the conditional language originally proposed by Kraus et al., but (*iii*) they are different in the language enriched with $\bullet$.

In this paper we show that these results provide us with the springboard necessary to truly move beyond the case of defeasible subsumption in a preferential DL setting. In doing so we take the following route. After presenting the background required for the rest of the work (Section 2), we introduce an *unrestricted* form of typicality for the DL $\mathcal{EL}_\bot$ (Section 3). In Section 3 we also point out the issues brought about by the typicality operator in $\mathcal{EL}_\bot$, which takes us to the study of typicality for $\mathcal{EL}_\bot^\bullet$, a language extending the DL $\mathcal{EL}_\bot$ (Section 4). In Section 5, we show that it is possible to define a notion of non-monotonic entailment for $\mathcal{EL}_\bot^\bullet$ which satisfies properties seen as important in a defeasible-reasoning context. We conclude the paper with a discussion and directions for future investigation. A document with the proofs of all the propositions can be downloaded at http://tinyurl.com/zley8ab.

2 Background

2.1 KLM-Style Defeasible Description Logics

Description Logics (DLs) [1] are a family of decidable fragments of first-order logic exhibiting interesting computational properties. In this paper we focus on one of the least expressive DLs, referred to as $\mathcal{EL}_\bot$. We will see that $\mathcal{EL}_\bot$ is already sufficient to express the problem we have in moving from the propositional case to the DL case, and to present significant language constraints that can enforce some desired properties.

The (concept) language of $\mathcal{EL}_\bot$ is built upon a finite set of atomic *concept names* $\mathsf{N}_\mathscr{C}$ and a finite set of *role names* $\mathsf{N}_\mathscr{R}$ such that $\mathsf{N}_\mathscr{C} \cap \mathsf{N}_\mathscr{R} = \emptyset$. We shall use $A, B, \ldots$as 'meta-variables' for the atomic concepts, and $r, s, \ldots$to denote role names. With $C, D, \ldots$ we denote the complex $\mathcal{EL}_\bot$-concepts, built according to the following rule:

$$C ::= A \mid \top \mid \bot \mid (C \sqcap C) \mid \exists r.C$$

With $\mathcal{L}_{\mathcal{EL}_\bot}$ we denote the *language* of all $\mathcal{EL}_\bot$ concepts. The semantics of $\mathcal{EL}_\bot$ is the standard set-theoretic Tarskian semantics. An *interpretation* is a structure $\mathcal{I} := \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$, where $\Delta^\mathcal{I}$ is a non-empty set called the *domain*, and $\cdot^\mathcal{I}$ is an *interpretation function* mapping concept names A to subsets $A^\mathcal{I}$ of $\Delta^\mathcal{I}$ and role names r to binary relations $r^\mathcal{I}$ over $\Delta^\mathcal{I}$: $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$.

Given an interpretation $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$, $\cdot^\mathcal{I}$ is extended to interpret complex concepts of $\mathcal{L}_{\mathcal{EL}_\bot}$ in the following way:

$$\top^\mathcal{I} = \Delta^\mathcal{I}, \;\; \bot^\mathcal{I} = \emptyset, \;\; (C \sqcap D)^\mathcal{I} := C^\mathcal{I} \cap D^\mathcal{I},$$
$$(\exists r.C)^\mathcal{I} := \{x \in \Delta^\mathcal{I} \mid \text{for some } y, (x, y) \in r^\mathcal{I} \text{ and } y \in C^\mathcal{I}\}$$

Given $C, D \in \mathcal{L}_{\mathcal{EL}_\bot}$, $C \sqsubseteq D$ is a *subsumption statement*, read "C is subsumed by D". $C \equiv D$ is an abbreviation for $C \sqsubseteq D$ and $D \sqsubseteq C$. We shall denote subsumption statements with $\alpha, \beta, \ldots$. An $\mathcal{EL}_\bot$ *TBox* (alias *terminology*) $\mathcal{T}$ is a finite set of subsumption statements. An interpretation $\mathcal{I}$ *satisfies* a subsumption statement $C \sqsubseteq D$ (denoted $\mathcal{I} \Vdash C \sqsubseteq D$) if and only if $C^\mathcal{I} \subseteq D^\mathcal{I}$. (And then $\mathcal{I} \Vdash C \equiv D$ if and only if $C^\mathcal{I} = D^\mathcal{I}$.) We say that an interpretation $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$ (denoted $\mathcal{I} \Vdash \mathcal{T}$) if and only if $\mathcal{I} \Vdash \alpha$ for every $\alpha \in \mathcal{T}$. A statement α is (classically) *entailed* by a TBox $\mathcal{T}$, denoted $\mathcal{T} \models \alpha$, if and only if every model of $\mathcal{T}$ satisfies α. If $\mathcal{T} = \emptyset$, then we have that $\mathcal{I} \Vdash \alpha$ for all interpretations $\mathcal{I}$, in which case we say α is a validity and denote with $\models \alpha$. The notion of *defeasible subsumption* [4] is captured by statements of the form $C \precsim D$, read "an element of C is usually an element of D", where C and D are complex concepts of the underlying classical DL. The semantics of defeasible subsumption is defined in terms of ordered structures called *ranked interpretations* [2]:

Definition 1 (Ranked Interpretation). *A ranked interpretation is a structure $\mathcal{R} := \langle \Delta^\mathcal{R}, \cdot^\mathcal{R}, \prec^\mathcal{R} \rangle$, where $\langle \Delta^\mathcal{R}, \cdot^\mathcal{R} \rangle$ is a standard DL interpretation and $\prec^\mathcal{R}$ is a modular order over $\Delta^\mathcal{R}$ satisfying the smoothness condition (given $C \in \mathcal{L}_{\mathcal{EL}_\bot}$, for every $x \in C^\mathcal{R}$, either $x \in \min_{\prec^\mathcal{R}} C^\mathcal{R}$ or there is $y \in C^\mathcal{R}$ such that $y \prec^\mathcal{R} x$ and $y \in \min_{\prec^\mathcal{R}} C^\mathcal{R}$.*[5]

Given a set X, $\prec \subseteq X \times X$ is *modular* if and only if there is a ranking function $rk : X \longrightarrow \mathbb{N}$ ($\mathbb{N}$ being the set of the natural numbers) such that for every $x, y \in X$, $x \prec y$ if and only if $rk(x) < rk(y)$.[6]

In a ranked interpretation $\mathcal{R}$, the intuition is that those objects that are lower down in the ordering are deemed more normal (or typical) than those higher up. A ranked interpretation $\mathcal{R}$ satisfies the defeasible subsumption $C \precsim D$ (noted $\mathcal{R} \Vdash C \precsim D$) if and only if $\min_\prec C^\mathcal{R} \subseteq D^\mathcal{R}$, in other words, if the most typical Cs fall under D. $\mathcal{R}$ satisfies a classical subsumption statement $C \sqsubseteq D$ (noted $\mathcal{R} \Vdash C \sqsubseteq D$) if and only if $C^\mathcal{R} \subseteq D^\mathcal{R}$. A ranked interpretation $\mathcal{R}$ is a *ranked model* of a set of statements (defeasible or classical) $\mathcal{T}$ if and only if $\mathcal{R} \Vdash \alpha$ for every $\alpha \in \mathcal{T}$.

The domain $\Delta^\mathcal{R}$ of a ranked model $\mathcal{R}$ can be partitioned into a sequence of layers $\{L_0^\mathcal{R}, L_1^\mathcal{R}, L_2^\mathcal{R}, \ldots\}$, where $x \in L_0^\mathcal{R}$ iff $x \in \Delta^\mathcal{R}$ and there is no $y \in \Delta^\mathcal{R}$ s.t. $y \prec^\mathcal{R} x$; $x \in L_{i+1}^\mathcal{R}$ iff $x \in \Delta^\mathcal{R} \setminus \bigcup_{0 \leq j \leq i} L_j^\mathcal{R}$ and and there is no $y \in \Delta^\mathcal{R} \setminus \bigcup_{0 \leq j \leq i} L_j^\mathcal{R}$ s.t. $y \prec^\mathcal{R} x$.

Using the layers we can also introduce the function $h_\mathcal{R}$, indicating the height of an object in a model $\mathcal{R}$: for every $x \in \Delta^\mathcal{R}$, $h_\mathcal{R}(x) = i$ iff $x \in L_i^\mathcal{R}$. Such a notion of height can be extended to concepts, where the height of a concept C is the lowest layer in which some object falls under the concept C: $h_\mathcal{R}(C) = i$ iff $\min_{\prec^\mathcal{R}}(C^\mathcal{R}) \cap L_i^\mathcal{R} \neq \emptyset$.

[5] Given $X \subseteq \Delta^\mathcal{R}$, with $\min_{\prec^\mathcal{R}} X$ we denote the *minimal elements* of X w.r.t. $\prec^\mathcal{R}$, i.e., the set $\{x \in X \mid \text{for every } y \in X, y \not\prec^\mathcal{R} x\}$.

[6] The standard definition of modularity does not refer to the set $\mathbb{N}$, but to a random (potentially uncountable) totally ordered set. Since the DLs logics used here, enriched with defeasible subsumption, satisfy the FMP, it is provable that to study relevant notion of entailment we can restrict our attention to countable domains, and in particular to constraint the notion of modularity as we have done here (see [2, Section 5] for details). Since the FMP holds also w.r.t. the language we will use from Section 4 on (see Theorem 1 and Corollary 1), we are justified in using this contrained definition of modularity. We have chosen to use such a definition for the sake of exposition.

Given a TBox $\mathcal{T}$, possibly containing defeasible subsumption statements, we are interested in which other subsumption statements (defeasible or classical), follow from $\mathcal{T}$. Among the possible notions of entailment that can be defined into this formal framework, *Rational Closure* (RC), originally introduced in the propositional framework [22] has a particular relevance, because of the following reasons: (*i*) it satisfies desirable logical properties [22]; (*ii*) it satisfies the principle of *presumption of typicality* [24, p. 63], which, informally, specifies that a situation (in the DL case, the objects in the domain) should be assumed to be as typical as possible (w.r.t. the background information in a knowledge base), and (*iii*) it can easily be implemented on top of classical reasoners.

RC is commonly viewed as the *basic* (although certainly not the only acceptable) form of nonmonotonic entailment, on which other, more venturous forms of entailment can be constructed. It is for this reason that various adaptations to DLs have been proposed [2, 25]. Informally, from a semantic point of view, in order to define a model characterising RC we have to *minimise* the position of the individuals in the model, that is, the RC of $\mathcal{T}$ is characterised by the model(s) of $\mathcal{T}$ in which the objects are positioned in the lowest possible layer: the lower the position in the model, the more typicality is enforced in the interpretation of individuals. A particularly desirable property of RC is that, both in the propositional and in the DL cases, there is a unique minimal configuration among the models of the KB, a property that guarantees the satisfaction of desirable logical properties [6], and ease the definition of a model characterising RC and the definition of a decision procedure. For more details on RC for DLs, and how to introduce minimal configurations in the semantics, the reader is referred to the work by Britz et al. [2] and by Giordano et al. [25].

2.2 Propositional Typicality Logic

Propositional Typicality Logic (PTL) [20, 21] is a recently proposed logic allowing for the representation of and reasoning with an explicit notion of *typicality*. It is obtained by enriching classical propositional logic with a *typicality operator* $\bullet$, the intuition of which is to capture the most typical (or normal) situations in which a given sentence holds. Here we briefly present the main results about PTL relevant for our purposes.

Let $\mathcal{P}$ be a finite set of propositional *atoms*, denoted by $p, q, \ldots$, possibly with subscripts. The language of PTL, denoted by $\mathcal{L}^{\bullet}$, is recursively defined by:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \bullet\varphi$$

All the other truth-functional connectives ($\vee$, $\rightarrow$, $\leftrightarrow$, $\ldots$) are defined in terms of $\neg$ and $\wedge$ in the usual way. Moreover, $\top$ and $\bot$ are seen as abbreviations for, respectively, $p \vee \neg p$ and $p \wedge \neg p$, for some $p \in \mathcal{P}$.

Intuitively, a sentence of the form $\bullet\varphi$ is understood to refer to the typical situations in which φ holds. Note that φ can itself be a $\bullet$-sentence. The semantics of PTL is in terms of the propositional counterpart of our ranked interpretations (Definition 1). Let $\mathcal{U}$ denote the set of all propositional *valuations* $v : \mathcal{P} \longrightarrow \{0, 1\}$.

Definition 2. *A (propositional) ranked interpretation is a pair* $\mathcal{R} := \langle \mathcal{V}, \prec \rangle$, *where* $\mathcal{V} \subseteq \mathcal{U}$ *and* $\prec \subseteq \mathcal{V} \times \mathcal{V}$ *is a modular order on* $\mathcal{V}$.

Let $\mathcal{R} = \langle \mathcal{V}, \prec \rangle$ and let $v \in \mathcal{V}$. Classical sentences are evaluated as usual, whereas $v \Vdash \bullet\varphi$ (the valuation v satisfies $\bullet\varphi$) if and only if $v \Vdash \varphi$ (the valuation v satisfies φ) and there is no $v' \prec v$ such that $v' \Vdash \varphi$. A PTL Knowledge Base is a finite set of formulas, that is satisfiable iff there is a (propositional) ranked model $\mathcal{R}$ s.t. $\mathcal{R}$ satisfies all the formulas in it.

A useful property of the typicality operator $\bullet$ is that it allows us to express KLM-style conditionals. That is, for every ranked interpretation $\mathcal{R}$ and all propositional sentences φ, ψ, $\mathcal{R} \Vdash \varphi \mid\!\sim \psi$ if and only if $\mathcal{R} \Vdash \bullet\varphi \rightarrow \psi$. The converse does not hold since it can be shown that there are $\mathcal{L}^\bullet$-sentences that cannot be expressed as a set of KLM-style $\mid\!\sim$-statements [21]. In other words, $\bullet$ delivers a more expressive framework than the original KLM conditional language.

Booth et al. [23] have recently investigated different (semantic) versions of entailment for PTL. These are based on the notion of RC as defined in the propositional case [22], and are constructed using minimality. An important realisation from these results is that in augmenting the expressivity beyond defeasible conditionals, the notion of minimality we want to introduce becomes problematic: notions that were equivalent in the propositional case now give back different results.

3 Typicality Operators in DLs

$\mathcal{EL}_\bot$ can be extended with a typicality operator on concepts as follows:

$$C ::= A \mid (C \sqcap C) \mid \exists r.C \mid \top \mid \bot \mid \bullet C$$

The intuition is that a concept $\bullet C$ represents the set of the most typical elements of the concept C. As in the propositional case, we use ranked interpretations to interpret the language, and use the ranking to interpret the typicality operator. The classical DL interpretation function is extended to include the concepts containing the typicality operator with the condition $(\bullet C)^{\mathcal{R}} := \min_{\prec^{\mathcal{R}}}(C^{\mathcal{R}})$.

A TBox is still a finite set of subsumption axioms $C \sqsubseteq D$, where C and D are concepts in this extended language. As in the classical case, a ranked interpretation $\mathcal{R}$ (Defintion 1) satisfies a subsumption axiom $C \sqsubseteq D$ ($\mathcal{R} \Vdash C \sqsubseteq D$) iff $C^{\mathcal{R}} \subseteq D^{\mathcal{R}}$. A TBox $\mathcal{T}$ is *satisfiable* iff there is a ranked interpretation $\mathcal{R}$ s.t. $\mathcal{R} \Vdash \alpha$ for every $\alpha \in \mathcal{T}$. Up to now the proposed extensions of DLs to model defeasible reasoning about what typically holds have taken into account the introduction of defeasible subsumption (see Section 2.1). Analogous to the PTL case, a subsumption $C \precsim D$ can be translated in our framework as $\bullet C \sqsubseteq D$. Giordano et al. [26, 8, 25] have introduced a typicality operator $\mathbf{T}$ in the language, but only axioms of the form $\mathbf{T}(C) \sqsubseteq D$ are allowed. Hence it does not go beyond the expressivity of defeasible subsumption. Bonatti et al. [15] introduce a typicality operator $\mathbf{N}$ ('Normally') and they use it to model, beyond the defeasible subsumption, concepts such as $\exists r.\mathbf{N}(C)$. This is a useful construct to have in modelling defeasibility in DLs. However, their work is done in a semantical framework that differs from the preferential approach. As it was seen in the case of PTL, enriching the language with a typicality operator allows us to express useful information beyond the defeasible conditionals. In the DL framework we are allowed to give information about the typicality of the objects connected through roles. That is, we can express subsumptions such as $C \sqsubseteq \exists r. \bullet C$ or $C \sqsubseteq \forall r. \bullet C$. Such axioms turn out to be very interesting in the DL framework, since they allow us to model useful inferences.

Example 1. Assume we have an ontology modelling the information that, typically, husbands are married to wives ($\bullet H \sqsubseteq \exists m.W$), and that typical wives are female ($\bullet W \sqsubseteq F$). From such an ontology it would be reasonable to conclude that, typically, husbands are married to women ($\bullet H \sqsubseteq \exists m.F$): since we have no information forcing us to conclude that the wife of a typical husband is atypical in any way, we would like to reason assuming that she is typical, and hence a woman, if not informed of the contrary. Therefore, we want to be able to conclude that presumably typical husbands are married to women ($\bullet H \sqsubseteq \exists m.F$).

To the best of our knowledge, none of the proposals in the preferential approaches are able to enforce the maximisation of typicality for the concepts appearing after roles (as W in $\exists r.W$).[7] An alternative solution is to follow the same approach that Bonatti et al. [15] have applied in their own framework. That is, to enforce the typicality of the concepts after a role by means of a typicality operator. In our example it would correspond to introducing into the ontology the axiom $\bullet H \sqsubseteq \exists m. \bullet W$, that, combined with $\bullet W \sqsubseteq F$, would allow us to derive the desired conclusion ($\bullet H \sqsubseteq \exists m.F$).

Here we extend our work [23] to the DL case and investigate how the minimisation techniques behave when moving beyond the expressivity of defeasible subsumption axioms. The role connections in DLs make the comparison of distinct interpretations slightly trickier than the propositional case. In order to compare the objects appearing in the distinct models, that could be characterised by distinct domains, we introduce a correspondence relation among objects, a step analogous to the one taken by Giordano et al. [25]. Given a ranked interpretation $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$, we indicate with $\mathcal{I}^{\mathcal{R}} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}} \rangle$ its *classical part*. Consider the set $\mathcal{U}$ composed of all the pairs $(\mathcal{I}, x)$ where $\mathcal{I}$ is a classical $\mathcal{EL}_\bot$ interpretation and $x \in \Delta^{\mathcal{I}}$.

Definition 3 (Function f). *Let $(\mathcal{I}, x), (\mathcal{J}, y) \in \mathcal{U}$. $(\mathcal{I}, x) \sim_c (\mathcal{J}, y)$ iff, for every $C \in \mathcal{L}_{\mathcal{EL}_\bot}$,*

$$x \in C^{\mathcal{I}} \text{ iff } y \in C^{\mathcal{J}}.$$

Given a pair $(\mathcal{I}, x)$, we indicate with $f((\mathcal{I}, x))$ the set of its correspondent pairs:

$$f((\mathcal{I}, x)) = \{(\mathcal{J}, y) \in \mathcal{U} \mid (\mathcal{I}, x) \sim_c (\mathcal{J}, y)\}.$$

In what follows, we will use the expression $(\mathcal{R}, x)$, with $\mathcal{R}$ a ranked interpretation, as referring to the pair $(\mathcal{I}^{\mathcal{R}}, x)$. Let $\Delta^{\mathcal{R}}$ be the domain of a ranked model $\mathcal{R}$ partitioned into $n+1$ layers $\{L_0^{\mathcal{R}}, \dots, L_n^{\mathcal{R}}\}$; let $f(L_i^{\mathcal{R}})$ indicate the set containing the equivalence classes that correspond to each $(\mathcal{R}, x)$, with $x \in L_i^{\mathcal{R}}$, that is, $f(L_i^{\mathcal{R}}) = \{f((\mathcal{R}, x)) \mid x \in L_i^{\mathcal{R}}\}$.

In order to implement the principle of *presumption of typicality* in our framework, we introduce into the present framework the kind of investigation that was started for

[7] At least, this is the case for TBox reasoning. For the case of ABox reasoning, namely in deriving typicality assertions about individuals participating in a role, the issue has been addressed by Giordano et al. [7, 8].

PTL. Among the possible options identified by Booth et al. [23], one notion of minimality turned out to be particularly interesting, *LM-minimality*, since, for every satisfiable PTL Knowledge Base, there is only one minimal configuration and one minimal characteristic model. Such uniqueness of the minimal model makes this notion of minimality particularly desirable, since it guarantees the satisfaction of formal desiderata and makes the definition of a decision procedure simpler [23]. First of all, we will check here whether such a notion of minimality preserves its desirable properties in the DL framework.

The notion of LM-minimality, that in the conditional case gives back a notion of entailment corresponding to RC, is defined referring to the maximisation of the size of lower (more typical) layers. The bigger a lower layer in an interpretation, the higher is the level of typicality that such an interpretation enforces. This notion of minimality is modelled through the relation $\trianglelefteq_{\mathrm{LM}}$, that can be described as follows.

Take any pair of ranked interpretations $\mathcal{R}_1$ and $\mathcal{R}_2$. $\Delta^{\mathcal{R}_1}$ is partitioned into $\{L_1, \dots, L_n\}$ and $\Delta^{\mathcal{R}_2}$ into $\{M_1, \dots, M_n\}$ (we can assume they have the same number of layers; if it is not the case, we can fill up the tail of the shorter sequence of layers with $\emptyset$'s).

Definition 4 (Relation $\trianglelefteq_{\mathrm{LM}}$).

$$\begin{aligned} \mathcal{R}_1 \trianglelefteq_{\mathrm{LM}} \mathcal{R}_2 \textit{ iff either } & f(L_i) = f(M_i) \textit{ for all } i \\ \textit{or} \quad & \textit{for the first } j \textit{ s.t. } f(L_j) \neq f(M_j) \\ & \textit{we have } f(L_j) \supseteq f(M_j) \end{aligned}$$

$\mathcal{R} \triangleleft_{LM} \mathcal{R}'$ iff $\mathcal{R} \trianglelefteq_{\mathrm{LM}} \mathcal{R}'$ and not $\mathcal{R}' \trianglelefteq_{\mathrm{LM}} \mathcal{R}$. Given a TBox $\mathcal{T}$, a model $\mathcal{R}$ of $\mathcal{T}$ is LM-minimal *iff there is no model $\mathcal{R}'$ of $\mathcal{T}$ s.t. $\mathcal{R}' \triangleleft_{\mathrm{LM}} \mathcal{R}$.*

The related notion of entailment is defined as follows.

Definition 5 (LM-entailment $\models_{LM}$). *Given a TBox $\mathcal{T}$ and an axiom $C \sqsubseteq D$, $\mathcal{T} \models_{LM} C \sqsubseteq D$ iff*

$$\mathcal{R} \Vdash C \sqsubseteq D \textit{ for every LM-minimal model } \mathcal{R} \textit{ of } \mathcal{T}$$

In the conditional case we end up with a unique ranked model characterising an entailment relation corresponding to RC. In PTL this uniqueness property is still preserved [23] since every satisfiable PTL knowledge base has a single LM-minimal model. In the case of DL TBoxes, however, the uniqueness of the LM-minimal model does not hold: we can have models of the same TBox that are not $\trianglelefteq_{\mathrm{LM}}$-comparable and for which there is no model that is $\trianglelefteq_{\mathrm{LM}}$-preferred to both of them.

Example 2. Consider a Tbox $\mathcal{T} = \{\bullet B \sqsubseteq F, P \sqsubseteq B, P \sqsubseteq E, E \sqcap F \sqsubseteq \bot, C \sqsubseteq \exists r.(\bullet D \sqcap \bullet P)\}$. We try to build minimal models of TBox $\mathcal{T}$. The concept P cannot be satisfied by any object in Layer 0 of the model, since every P is a B, but typical Bs are F, that is incompatible with P. Hence every object satisfying $\bullet P$ must be in Layer 1 or higher. We also require that every object satisfying C is connected through a role r to some object satisfying both $\bullet D$ and $\bullet P$. Hence, because of what we just deduced about $\bullet P$, no object in C can be connected through r to any object in Layer 0. We have two possibilities: either there are no objects satisfying D in Layer 0, allowing the existence of some object satisfying $\bullet D$ and $\bullet P$, or we introduce some objects satisfying D in

Layer 0, in which case we have to conclude that the interpretation of C is the empty set, since there are no objects satisfying $\bullet D \sqcap \bullet P$. Both the options allow for the definition of LM-minimal models of $\mathcal{T}$ that are not $\trianglelefteq_{\mathrm{LM}}$-comparable.

4 The Logic $\mathcal{EL}_\perp^\bullet$

To recap, on the one hand we would like to extend the language of defeasible subsumption in DLs. In particular, the introduction of concepts such as $\exists r. \bullet C$ would be very useful. On the other hand, Example 2 shows that going beyond the expressivity of defeasible subsumption in DLs results in the loss of unique minimal configurations, even for LM-minimality, the most 'reliable' form of minimisation in PTL.

There are two possible ways to try and resolve this matter. We can either constrain the expressivity of the language, or we can investigate possible changes in the definition of minimisation. We shall see below that it is necessary to consider both. With respect to expressivity, we consider here the smallest extension of $\mathcal{EL}_\perp$ that is interesting from a modelling point of view. The concepts are defined as:

$$C ::= A \mid C \sqcap C \mid \exists r.C \mid \top \mid \bot$$

$$D_1 ::= C \mid \bullet C$$

$$D_2 ::= C \mid \exists r. \bullet C$$

Axioms are of the form $D_1 \sqsubseteq D_2$. We will refer to this language as $\mathcal{EL}_\perp^\bullet$. $\mathcal{EL}_\perp^\bullet$ minimally extends defeasible subsumption, allowing only for the use of concepts of the form $\exists r. \bullet C$ in order to model representation problems such as the one in Example 1. However, constraining the language to $\mathcal{EL}_\perp^\bullet$ is not sufficient, as Example 3 shows.

Example 3. Consider a Tbox $\mathcal{T} = \{\bullet B \sqsubseteq F, P \sqsubseteq B, P \sqsubseteq E, E \sqcap F \sqsubseteq \bot, \bullet\top \sqsubseteq A, C \sqsubseteq \exists r. \bullet D\}$. $\mathcal{R}_1$ and $\mathcal{R}_2$ are two models of such a TBox. $\mathcal{R}_1$ is composed of a single object x, and we have $C^{\mathcal{R}_1} = B^{\mathcal{R}_1} = F^{\mathcal{R}_1} = A^{\mathcal{R}_1} = D^{\mathcal{R}_1} = \{x\}$, $P^{\mathcal{R}_1} = E^{\mathcal{R}_1} = \emptyset$, and $r^{\mathcal{R}_1} = \{(x,x)\}$. $\mathcal{R}_2$ is composed of two objects z, y s.t. $z \prec^{\mathcal{R}_2} y$, and we set $C^{\mathcal{R}_2} = F^{\mathcal{R}_2} = A^{\mathcal{R}_2} = \{z\}$, $P^{\mathcal{R}_2} = E^{\mathcal{R}_2} = D^{\mathcal{R}_2} = \{y\}$, $B^{\mathcal{R}_2} = \{z, y\}$, and $r^{\mathcal{R}_1} = \{(z,y)\}$. It is easy to check that they are both models of $\mathcal{T}$; note that $z \notin (\exists r.A)^{\mathcal{R}_2}$, and that $y \in \bullet D^{\mathcal{R}_2}$, since we do not have any object satisfying D in the lower layer.

We shall now try to define a model $\mathcal{R}_3$ of $\mathcal{T}$ that is $\trianglelefteq_{\mathrm{LM}}$-preferred to both of them. Such a model should contain in Layer 0 two objects x', z' s.t. $x' \in f(\mathcal{R}_1, x)$ and $z' \in f(\mathcal{R}_2, z)$. Now, in order to satisfy the axiom $C \sqsubseteq \exists r. \bullet D$, z' must be related through r to some object $u \in \bullet D^{\mathcal{R}_3}$; since we have at least $x' \in D^{\mathcal{R}_3}$ in Layer 0, z' must be related through r to some object $u \in D^{\mathcal{R}_3}$ in Layer 0 (it could be x' itself, for example). However, all the objects in Layer 0 must be in $A^{\mathcal{R}_3}$ because of the axiom $\bullet\top \sqsubseteq A$, and therefore $z' \in (\exists r. \bullet D)^{\mathcal{R}_3}$ enforces the conclusion $z' \in (\exists r.A)^{\mathcal{R}_3}$. However, since $z \notin (\exists r.A)^{\mathcal{R}_2}$, $z' \notin f(\mathcal{R}_2, z)$.

Constraining of the language is not enough, hence we consider possible changes in the definition of minimality. In the example above, the main issue in preventing the definition of a model that is $\trianglelefteq_{\mathrm{LM}}$-preferred to both $\mathcal{R}_1$ and $\mathcal{R}_2$ is the following: in $\mathcal{R}_2$ the object y satisfies $\bullet D$ only in the particular context defined by the models, since y must be in Layer 1 or higher (since it satisfies P) and there are no objects satisfying D in the lower layer of $\mathcal{R}_2$. But it is logically possible to have models of $\mathcal{T}$ with objects satisfying D in Layer 0, as it is the case with $\mathcal{R}_1$. Our (refined) proposal is that, in minimisation, we consider a notion of typicality (that is, an interpretation of the $\bullet$ operator) that is more *objective*. That is, it refers to the most typical objects satisfying a concept C w.r.t. all the models of a TBox $\mathcal{T}$, and not only w.r.t. the objects represented in a specific model.

We thus refine the notion of LM-preference w.r.t. a particular TBox. For every concept C, we indicate with $h_{\mathcal{T}}(C)$ the minimal height of the concept C w.r.t. all the models of $\mathcal{T}$.

- For each $\mathcal{EL}_{\bot}$ concept C and each ranked model $\mathcal{R}$, $h_{\mathcal{R}}(C) = i$ iff $\min_{\prec^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq L_i^{\mathcal{R}}$.
- $h_{\mathcal{T}}(C) = \min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \Vdash \mathcal{T}\}$.

Using the function $h_{\mathcal{T}}$, we can formally define our idea about the interpretation of typicality, and define a refinement of LM-minimality.

Definition 6 (LM*-minimality). *Given a TBox $\mathcal{T}$, a model $\mathcal{R}$ of $\mathcal{T}$ is LM*-minimal iff:*

1. *$\mathcal{R}$ is a LM-minimal model of $\mathcal{T}$;*
2. *For every $C \in \ell_{\mathcal{T}}^{\mathcal{EL}^{\bot}}$, either $h_{\mathcal{R}}(C) = \infty$ or $h_{\mathcal{R}}(C) = h_{\mathcal{T}}(C)$.*

The definition of the related notion of entailment is as in the previous cases.

Definition 7 (LM*-entailment $\models_{\mathbf{LM}^*}$). *Given a TBox $\mathcal{T}$ and an axiom $C \sqsubseteq D$, $\mathcal{T} \models_{LM^*} C \sqsubseteq D$ iff*

$$\mathcal{R} \Vdash C \sqsubseteq D \textit{ for every LM}^*\textit{-minimal model } \mathcal{R} \textit{ of } \mathcal{T}.$$

From the constructions of Britz et al. [2] and Giordano et al. [25] it can be shown that, in case of TBoxes containing only defeasible subsumption axioms of the form $C \precsim D$ (and hence being equivalent to $\bullet C \sqsubseteq D$), LM*-minimality corresponds to LM-minimality, since for every TBox $\mathcal{T}$, in every LM-minimal model $\mathcal{R}$ of $\mathcal{T}$, and for every $C \in \ell_{\mathcal{T}}^{\mathcal{EL}^{\bot}}$, $h_{\mathcal{R}}(C) = h_{\mathcal{T}}(C)$. In what follows we are going to describe a procedure to identify, given a TBox $\mathcal{T}$, the model characterising the entailment relation based on LM*-minimality.

5 Construction of Characteristic Schema

In order to prove what follows it is relevant to know that $\mathcal{EL}_{\bot}^{\bullet}$ satisfies the Finite Model Property (FMP) and a related Finite Countermodel Property (FCMP).

Theorem 1 (FMP). *Let $\mathcal{T}$ be a finite $\mathcal{EL}_{\bot}^{\bullet}$ TBox. If $\mathcal{T}$ has a ranked model $\mathcal{R}$, then $\mathcal{T}$ has a* finite *ranked model $\mathcal{R}'$ s.t., for every $C \in \ell_{\mathcal{T}}^{\mathcal{EL}_{\bot}}$, $h_{\mathcal{R}}(C) = h_{\mathcal{R}'}(C)$.*

Corollary 1 (FCMP). *Let $\mathcal{T}$ be a finite $\mathcal{EL}_{\perp}^{\bullet}$ TBox and $C \sqsubseteq D$ an $\mathcal{EL}_{\perp}^{\bullet}$ axiom. If $\mathcal{T}$ has a ranked model $\mathcal{R}$ s.t. $\mathcal{R}$ is not a model of $C \sqsubseteq D$, then $\mathcal{T}$ has also a* finite *ranked model $\mathcal{R}'$ s.t. it is a counter-model of $C \sqsubseteq D$ and, for every $C \in \ell_{\mathcal{T}}^{\mathcal{EL}_{\perp}}$, $h_{\mathcal{R}}(C) = h_{\mathcal{R}'}(C)$.*

The first step in order to define a characteristic model for LM*-entailment in $\mathcal{EL}_{\perp}^{\bullet}$ is a transformation of the TBox. Given a TBox $\mathcal{T}$, let $\mathcal{T}'$ be the TBox obtained by substituting in $\mathcal{T}$ every concept $\exists r. \bullet C$ with $\exists r.C$. That is, we transform every axiom $D_1 \sqsubseteq \exists r. \bullet C$ in $\mathcal{T}$ in the following way:

- $D_1 \sqsubseteq \exists r. \bullet C \rightsquigarrow D_1 \sqsubseteq \exists r.C$.

Such a transformation preserves satisfiability.

Proposition 1. *For every $\mathcal{EL}_{\perp}^{\bullet}$ TBox $\mathcal{T}$, $\mathcal{T}$ is satisfiable iff $\mathcal{T}'$ is.*

For a TBox $\mathcal{T}$, let $\ell_{\mathcal{T}}$ be the set of all the concepts appearing as subconcepts in the axioms in $\mathcal{T}$, and in particular let $\ell_{\mathcal{T}}^{\mathcal{EL}^{\perp}} \subseteq \ell_{\mathcal{T}}$ be the set of the $\mathcal{EL}_{\perp}$-concepts appearing as subconcepts in $\mathcal{T}$. Given a Tbox $\mathcal{T}$, a ranked model $\mathcal{R}$ of $\mathcal{T}$ and an object $x \in \Delta^{\mathcal{R}}$:

$$g(\mathcal{R}, x) = \{(\mathcal{R}', y) \mid \forall C \in \ell_{\mathcal{T}}^{\mathcal{EL}^{\perp}}, y \in C^{\mathcal{R}'} \text{ iff } x \in C^{\mathcal{R}}\}$$

That is, g is a function that defines equivalence classes of pairs $(\mathcal{R}, x)$ w.r.t. the satisfaction of the $\mathcal{EL}_{\perp}$-concepts appearing in $\mathcal{T}$. Each $g(\mathcal{R}, x)$ can be also seen as an object with an associated *label*, that is, the set of concepts in $\ell_{\mathcal{T}}^{\mathcal{EL}^{\perp}}$ that it satisfies. We will abuse notation by writing $C \in g(\mathcal{R}, x)$ to indicate that $C \in \ell_{\mathcal{T}}^{\mathcal{EL}^{\perp}}$ and $x \in C^{\mathcal{R}}$. We will also use the letters $u, v, \ldots$ to indicate the sets $g(\mathcal{R}, x)$. Let $\Delta^{\mathcal{T}}$ be the set of all the $g(\mathcal{R}, x)$ defined from a TBox $\mathcal{T}$ (modulo equivalence). $\Delta^{\mathcal{T}}$ will be a finite set, since $\ell_{\mathcal{T}}^{\mathcal{EL}^{\perp}}$ is. We will use the domain $\Delta^{\mathcal{T}}$ to define a different kind of ranked interpretation we refer to as a *schema*, where the objects in $\Delta^{\mathcal{T}}$ are not connected through roles between themselves. For example, if $\exists r.C \in g(\mathcal{R}, x)$, there is no role r connecting $g(\mathcal{R}, x)$ to some $g(\mathcal{R}', y)$ s.t. $C \in g(\mathcal{R}', y)$. We will denote the schemas for a TBox $\mathcal{T}$ with the letters $M, N, \ldots$, where a schema $N = \langle \Delta^N, \prec^N \rangle$ is composed of a set $\Delta^N \subseteq \Delta^{\mathcal{T}}$ and $\prec^N$ is a modular order over Δ^N (the finitess of Δ^N guarantees the satisfaction of the smoothness condition). Formally, schemas can be seen as propositional valuations where every atomic concept $A \in \ell_{\mathcal{T}}^{\mathcal{EL}^{\perp}}$ and every concept of the form $\exists r.C \in \ell_{\mathcal{T}}^{\mathcal{EL}^{\perp}}$ are treated like atomic letters in propositional logic. Still, it is constrained in such a way to respect the $\mathcal{EL}_{\perp}$ logical constraints: for example, if $\exists r.A_1 \sqcap A_2 \in u$, it must be the case that $\exists r.A_1 \in u$. Given a TBox $\mathcal{T}'$, the notion of interpretation in a model $N = \langle \Delta^N, \prec^N \rangle$ is defined as:

- for every concept $C \in \ell_{\mathcal{T}}^{\mathcal{EL}^{\perp}}$, $C^N = \{g(\mathcal{R}, x) \in \Delta^N \mid C \in g(\mathcal{R}, x)\}$;
- $(\bullet C)^N = \min_{\prec^N}(C^N)$;
- $N \Vdash D_1 \sqsubseteq D_2$ iff $D_1^N \subseteq D_2^N$.

A schema N is a model of a TBox $\mathcal{T}$ iff it satisfies all the axioms in $\mathcal{T}$. We also introduce the following notation:

- $(D_1 \sqsubseteq D_2)^N = \{g(\mathcal{R}, x) \in \Delta^N \mid \text{if } g(\mathcal{R}, x) \in D_1^N, \text{ then } g(\mathcal{R}, x) \in D_2^N\}$;

- $\mathcal{T}^N = \{g(\mathcal{R}, x) \in \Delta^N \mid g(\mathcal{R}, x) \in (D_1 \sqsubseteq D_2)^N$ for every $D_1 \sqsubseteq D_2 \in \mathcal{T}\}$;

The procedure to create a model for LM*-minimisation is divided into two sub-procedures. For any ranked interpretation $N = \langle \Delta^N, \prec^N \rangle$ ($\Delta^N \subseteq \Delta^{\mathcal{T}}$) and $S \subseteq \Delta^N$, we define $N \downarrow S$ (the restriction of N to S) as $\langle \Delta^N \cap S, \prec^N \cap (S \times S) \rangle$.

The overall procedure is defined as follows:

- Given a TBox $\mathcal{T}$, transform it into a TBox $\mathcal{T}'$, as described above. In $\mathcal{T}'$ there can be axioms of only two types, $C_1 \sqsubseteq C_2$ and $\bullet C_1 \sqsubseteq C_2$, with C_1, C_2 being classical $\mathcal{EL}_\perp$ concepts. Note that $\ell^{\mathcal{EL}^\perp}_{\mathcal{T}'} = \ell^{\mathcal{EL}^\perp}_{\mathcal{T}}$, and consequently $\Delta^{\mathcal{T}'} = \Delta^{\mathcal{T}}$.
- **Procedure 1:**

  We will build a sequence of interpretations $N_0, N_1, \ldots$, with $N_i = \langle S_0, \prec^i \rangle$. The first time we execute this procedure we start with $S_0 := \Delta^{\mathcal{T}}$.

  Step 1 Initialise $\prec_0 := \emptyset$ (*start with an initial ranked interpretation in which all objects are equally preferred*).
  Step 2 $S_{i+1} := \mathcal{T}'^{\mathsf{N}_i}$ (*separate the objects which satisfy $\mathcal{T}'$ w.r.t. the current ranked interpretation N_i from those that do not*).
  Step 3 If $S_{i+1} = S_i$ then return $N^*(\mathcal{T}') := N_i \downarrow S_{i+1}$ (*if the division is the same as in the previous round then eliminate completely from the current ranked interpretation those objects that do not satisfy $\mathcal{T}'$ w.r.t. N_i and return the interpretation that remains*) and go to Procedure 2.
  Step 4 Otherwise $\prec_{i+1} := \prec_i \cup (S_{i+1} \times S^c_{i+1})$, $i := i + 1$ and go to Step 2 (*otherwise create a new ranked interpretation N_{i+1} by making every object not in S_{i+1} less plausible than every object* in S_{i+1}. *Note that S^c here denotes $\Delta^{\mathcal{T}} \setminus S$*).

  So, we end up with a schema $N^*(\mathcal{T}') = \langle \Delta^*, \prec^* \rangle$.
- **Procedure 2:**

  Step 1 Set $\mathcal{A}_{\{\bullet C \sqsubseteq D\}} := \{C \mid \bullet C \sqsubseteq D \in \mathcal{T}'\}$ and check $h_{N^*(\mathcal{T}')}(C)$ for every $C \in \mathcal{A}_{\bullet C \sqsubseteq D}$.
  Step 2 Impose a linear order of all the elements of Δ^*: $t_0 = \langle v_1, \ldots, v_n \rangle$.
  Step 3 $i := 0$.
  Step 4 $j := 0$.
  Step 5 $t_{i+1} := t_i$.
  Step 6 Let $v_j = g(\mathcal{R}, x)$. If $v_j \notin t_{i+1}$, do nothing. Else, for all the concepts of form $\exists r.C \in v_j$, check if there is a $w \in t_{i+1}$ s.t. $C \in w$. If there is one, do nothing; if there is not, eliminate v_j from t_{i+1} ($t_{i+1} := t_{i+1} \setminus \{v_j\}$).
  Step 7 $j := j + 1$.
  Step 8 If $j \leq n$, go back to step 6. If $j > n$, the if $t_{i+1} = t_i$, set $M^*(\mathcal{T}') := N^*(\mathcal{T}') \downarrow T_i$ (where T_i is the set of the objects in t_i). Else, set $i := i + 1$ and go back to Step 4.
  Step 9 If $h_{M^*(\mathcal{T}')}(C) = h_{N^*(\mathcal{T}')}(C)$ for every $C \in \mathcal{A}_{\bullet C \sqsubseteq D}$, then return $M_{\mathcal{T}'} := M^*(\mathcal{T}')$. Else set $S_0 := T_i$ and go back to Procedure 1.

The output of Procedure 1 is a schema $N^*(\mathcal{T}')$ that does not consider the role connections. The output of Procedure 2 is a schema $M^*(\mathcal{T}')$ that eliminates the objects that imply the presence of unsatisfiable role connections. For example, the output of Procedure 1 could contain some object v s.t. $\exists r.C \in v$, but $C \notin u$ for any u in the domain. Such objects must be eliminated, and Procedure 2 takes care of that. Nonetheless, Procedure 2 could erase the lower layer of some concept $C \in \ell_{\mathcal{T}}^{\mathcal{EL}_\perp}$, and consequently there could be some axiom $\bullet C \sqsubseteq D_2 \in \mathcal{T}'$ that is not satisfied anymore. We go back to Procedure 1 to take care of that. We go on until we do not find a stable configuration $M_{\mathcal{T}'}$. Since $\Delta^{\mathcal{T}}$ is a finite set, the overall procedure must terminate.

From $M_{\mathcal{T}'}$ we can obtain a model $M_{\mathcal{T}}$ by modifying the labels of the objects: for every axiom $D_1 \sqsubseteq \exists r. \bullet C \in \mathcal{T}$, for every object in $M_{\mathcal{T}'}$ satisfying D_1 add to its label also the concept $\exists r. \bullet C$ (note that $\exists r.C$ was already in its label), and $(\exists r. \bullet C)^{M_{\mathcal{T}}} = \{g(\mathcal{R}, x) \in \Delta^{M_{\mathcal{T}}} \mid \exists r. \bullet C \in g(\mathcal{R}, x)\}$.

Proposition 2. *The schema $M_{\mathcal{T}}$ is a model of $\mathcal{T}$.*

Now that we have $M_{\mathcal{T}}$, we can use it to build DL ranked models.

Definition 8. *[DL-completion] Given a TBox $\mathcal{T}$ and the model $M_{\mathcal{T}}$, partitioned into the layers $\{L_0^M, \dots, L_n^M\}$, a ranked DL model $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$ is a* DL-completion *of $M_{\mathcal{T}}$ iff*

- *$\mathcal{R}$ is partitioned into a sequence of layers $\{L_0^{\mathcal{R}}, \dots, L_m^{\mathcal{R}}\}$ ($m \geq n$).*
- *If $x \in L_i^{\mathcal{R}}$, then $(\mathcal{R}, x) \in g(\mathcal{R}', y)$ for some $g(\mathcal{R}', y) \in L_j^M$, $j \leq i$.*
- *For every $g(\mathcal{R}', y) \in L_i^M$, there is at least one $x \in L_i^{\mathcal{R}}$ s.t. $(\mathcal{R}, x) \in g(\mathcal{R}', y)$.*

If $\Delta^{\mathcal{R}}$ is finite, $\mathcal{R}$ is a finite *DL-completion of $M_{\mathcal{T}}$.*

Let $\mathscr{R}_{M_{\mathcal{T}}}$ be the set of all the DL interpretations that DL-complete $M_{\mathcal{T}}$, and $\mathscr{R}_{M_{\mathcal{T}}}^{fin} \subseteq \mathscr{R}_{M_{\mathcal{T}}}$ the set of the finite ones. We can prove that DL-completion are always definable from a schema $M_{\mathcal{T}}$ and are models of $\mathcal{T}$.

Proposition 3. *Given a schema $M_{\mathcal{T}}$, $\mathscr{R}_{M_{\mathcal{T}}}^{fin}$ is non-empty.*

Proposition 4. *If $\mathcal{R}$ is a DL-completion of $M_{\mathcal{T}}$, then $\mathcal{R}$ is a model of $\mathcal{T}$.*

Now we define a model $\mathcal{R}^{\mathcal{T}} = \langle \Delta^{\mathcal{R}^{\mathcal{T}}}, \cdot^{\mathcal{R}^{\mathcal{T}}}, \prec^{\mathcal{R}^{\mathcal{T}}} \rangle$ that in practice unifies in a single ranked model all the finite models in $\mathscr{R}_{M_{\mathcal{T}}}^{fin}$. $\mathcal{R}^{\mathcal{T}}$ is defined in the following way:

- $\Delta^{\mathcal{R}^{\mathcal{T}}} = \{x_{\mathcal{R}} \mid x \in \Delta^{\mathcal{R}}$ and $\mathcal{R} \in \mathscr{R}_{M_{\mathcal{T}}}^{fin}\}$ (we unify all the domains of the models in $\mathscr{R}_{M_{\mathcal{T}}}^{fin}$ in an unique domain).
- For every atomic concept A $x_{\mathcal{R}} \in A^{\mathcal{R}^{\mathcal{T}}}$ iff $x \in A^{\mathcal{R}}$.
- For every role r, $(x_{\mathcal{R}}, y'_{\mathcal{R}}) \in r^{\mathcal{R}^{\mathcal{T}}}$ iff $\mathcal{R} = \mathcal{R}'$ and $(x, y) \in r^{\mathcal{R}}$.
- $h_{\mathcal{R}^{\mathcal{T}}}(x^{\mathcal{R}}) = h_{\mathcal{R}}(x)$.

$\mathcal{R}^{\mathcal{T}}$ is still a DL-completion of $M_{\mathcal{T}}$, and it is the model we are looking for. That is, it characterises LM*-entailment.

Proposition 5 (Uniqueness of the LM*-minimal schema). *Given an $\mathcal{EL}_\perp^\bullet$ TBox $\mathcal{T}$, all its LM*-minimal models are DL-completions of the schema $M_{\mathcal{T}}$.*

Theorem 2. *Given a TBox $\mathcal{T}$, $\mathcal{R}^{\mathcal{T}}$ characterises the LM*-entailment. That is,*

$$\mathcal{T} \models_{LM^*} C \sqsubseteq D \textit{ iff } \mathcal{R}^{\mathcal{T}} \Vdash C \sqsubseteq D$$

Example 4. Let $\mathcal{T}$ be the TBox of Example 3. The TBox $\mathcal{T}'$ is obtained substituting $C \sqsubseteq \exists r. \bullet D$ with $C \sqsubseteq \exists r.D$, and $\ell_{\mathcal{T}'}^{\mathcal{EL}_\bot} = \{B, F, P, E, A, C, \exists r.D, D\}$. In order to build $M_{\mathcal{T}'}$, first we fill $\Delta^{\mathcal{T}}$ with one object for each subset of $\ell_{\mathcal{T}'}^{\mathcal{EL}_\bot}$ (since $|\ell_{\mathcal{T}'}^{\mathcal{EL}_\bot}| = 8$, $|\Delta^{\mathcal{T}}| = 2^8$). Through Procedure 1 we end up with a schema $N_{\mathcal{T}'}^*$ in which *(i)* all the elements of $\Delta^{\mathcal{T}}$ satisfying P and not B, P and not E, $E \sqcap F$, C and not $\exists r.D$ are eliminated, and *(ii)* among the remaining elements of $\Delta^{\mathcal{T}}$, all the ones satisfying B and not F (hence also all the ones satisfying P) and the ones not satisfying A go to the upper Layer 1. Moving to Procedure 2, in the domain there are objects satisfying $\exists r.D$, hence the procedure checks that there are objects in the domain satisfying D, and the procedure terminates with $M_{\mathcal{T}'} := N_{\mathcal{T}'}^*$. The schema $M_{\mathcal{T}}$ is obtained just adding the concept $\exists r. \bullet D$ to all the objects satisfying C, and $M_{\mathcal{T}}$ is the schema we use to define LM*-minimality.

The fact that LM*-minimality can be characterised through a single model guarantees that some desirable logical properties are preserved, since the set of axioms satisfied by a single ranked model defines a rational subsumption [2].

6 Conclusion and Future Work

In this paper we started an investigation of what it means to reason about typicality in DLs once we go beyond the expressivity of defeasible subsumption. In the framework of DLs, the enforcement of the principle of *presumption of typicality* through minimisation procedures turns out to be even more complex than in the case of PTL, since even the notion of LM-entailment, that in the case of PTL can be decided with easy procedures and gives back reliable results, becomes difficult to manage in the DL framework. We considered an extension of the language $\mathcal{EL}_\bot$ that is strongly constrained, but still very interesting from the point of view of knowledge representation. We defined a notion of minimal entailment that can be characterised through a single model. In this way we guarantee that the subsumption relation behaves rationally. The result also opens the possibility of the definition of an effective decision procedure.

To the best of our knowledge the only proposals for extending the expressivity of reasoning about typicality in DLs beyond the expressivity of defeasible subsumption axioms are the ones by Bonatti et al. [15], that is developed in a different semantics, and by Giordano et al. [7, 8], that is defined for ABox reasoning. Hence our work is the first in which the investigation focuses on TBox reasoning in a language that extends that of defeasible subsumption in a preferential framework.

The next step is the definition of an effective procedure to decide the LM*-entailment of a $\mathcal{EL}_\bot^\bullet$ TBox $\mathcal{T}$: the most promising options are either the use of the schema $M_{\mathcal{T}}$ or, using as a starting point a tractable procedure to decide RC for $\mathcal{EL}_\bot$ enriched with defeasible subsumption [27], defining a decision procedure for $\mathcal{EL}_\bot^\bullet$ TBoxes that is correct w.r.t. the model $\mathcal{R}^{\mathcal{T}}$. This will be followed by an analysis of ABox reasoning and the investigation of other languages, either extending $\mathcal{EL}_\bot^\bullet$ or related to the DL-lite family [28].

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Appendix - Proofs

Theorem 1 (Finite Model Property - FMP). *Let $\mathcal{T}$ be a finite $\mathcal{EL}_{\bot}^{\bullet}$ TBox. If $\mathcal{T}$ has a ranked model $\mathcal{R}$, then $\mathcal{T}$ has a* finite *ranked model $\mathcal{R}'$.*

Proof. Given a model $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$ of $\mathcal{T}$, we can build a finite model $\mathcal{R}' = \langle \Delta^{\mathcal{R}'}, \cdot^{\mathcal{R}'}, \prec^{\mathcal{R}'} \rangle$ in the following way:

- $\Delta^{\mathcal{R}'} = \{g(\langle \mathcal{R}, x\rangle) \mid x \in \Delta^{\mathcal{R}}\}$ (g is the same function used in Section 5 of the paper);
- for every atomic concept $A \in \ell_{\mathcal{T}}^{\mathcal{EL}_{\bot}}$, $A^{\mathcal{R}'} = \{g(\langle \mathcal{R}, x\rangle) \in \Delta^{\mathcal{R}'} \mid x \in A^{\mathcal{R}}\}$;
- every atomic concept $A \notin \ell_{\mathcal{T}}^{\mathcal{EL}_{\bot}}$ can be interpreted freely;
- for every role r, $r^{\mathcal{R}'} = \{\langle g(\langle \mathcal{R}, x\rangle), g(\langle \mathcal{R}, y\rangle)\rangle \mid \langle x, y\rangle \in r^{\mathcal{R}}\}$;
- $g(\langle \mathcal{R}, x\rangle) \prec^{\mathcal{R}'} g(\langle \mathcal{R}, y\rangle)$ iff there is an object $x' \in g(\langle \mathcal{R}, x\rangle) \cap \Delta^{\mathcal{R}}$ s.t. for all the objects $y' \in g(\langle \mathcal{R}, y\rangle) \cap \Delta^{\mathcal{R}}$, $x' \prec^{\mathcal{R}} y'$.

It can be easily checked by induction on the construction of the concepts that for every concept C in $\ell_{\mathcal{T}}^{\mathcal{EL}_{\bot}}$, $x \in C^{\mathcal{R}}$ iff $g(\langle \mathcal{R}, x\rangle) \in C^{\mathcal{R}'}$.

We need to prove that $\prec^{\mathcal{R}'}$ is a modular relation (smoothness comes out for free, since it is a finite model). Let rk be a ranking function that associate every $g(\langle \mathcal{R}, x\rangle) \in \Delta^{\mathcal{R}'}$ to the number $h_{\mathcal{R}}(z)$, where $\langle \mathcal{R}, z\rangle \in g(\langle \mathcal{R}, x\rangle)$ and $h_{\mathcal{R}}(z) \leq h_{\mathcal{R}}(y)$ for every $\langle \mathcal{R}, y\rangle \in g(\langle \mathcal{R}, x\rangle)$. It is immediate to see from the definition of $\prec^{\mathcal{R}'}$ that for every $g(\langle \mathcal{R}, x\rangle), g(\langle \mathcal{R}, y\rangle) \in \Delta^{\mathcal{R}'}$, $g(\langle \mathcal{R}, x\rangle) \prec^{\mathcal{R}'} g(\langle \mathcal{R}, y\rangle)$ iff $rk(g(\langle \mathcal{R}, x\rangle)) < rk(g(\langle \mathcal{R}, y\rangle))$.

Finally, we have to prove that $\mathcal{R}'$ is a model of $\mathcal{T}$.

In order to obtain that, we first prove the following: for every $C \in \ell_{\mathcal{T}}^{\mathcal{EL}_{\bot}}$ and for every $g(\langle \mathcal{R}, x\rangle) \in \Delta^{\mathcal{R}'}$, if in $g(\langle \mathcal{R}, x\rangle)$ there is a z s.t. $z \in \bullet C^{\mathcal{R}}$, then $g(\langle \mathcal{R}, x\rangle) \in \bullet C^{\mathcal{R}'}$.

If $z \in \bullet C^{\mathcal{R}}$, then $h_{\mathcal{R}}(z) = h_{\mathcal{R}}(C)$. Assume that $g(\langle \mathcal{R}, x\rangle) \notin \bullet C^{\mathcal{R}'}$; since $g(\langle \mathcal{R}, x\rangle) \in C^{\mathcal{R}'}$, it means that there is a $g(\langle \mathcal{R}, y\rangle) \in \Delta^{\mathcal{R}'}$ s.t. $g(\langle \mathcal{R}, y\rangle) \in \bullet C^{\mathcal{R}'}$, that implies that $g(\langle \mathcal{R}, y\rangle) \prec^{\mathcal{R}'} g(\langle \mathcal{R}, x\rangle)$. However, that cannot be the case, since we would have that in $g(\langle \mathcal{R}, y\rangle)$ there is an object $z' \in \Delta^{\mathcal{R}}$ s.t. $z' \in C^{\mathcal{R}}$ and $z' \prec^{\mathcal{R}} z''$ for every $z'' \in g(\langle \mathcal{R}, x\rangle)$, against the assumption that $z \in g(\langle \mathcal{R}, x\rangle)$ and $z \in (\bullet C)^{\mathcal{R}}$.

So, now we know that:

- for every concept C in $\ell_{\mathcal{T}}^{\mathcal{EL}_{\bot}}$, $x \in C^{\mathcal{R}}$ iff $g(\langle \mathcal{R}, x\rangle) \in C^{\mathcal{R}'}$;
- for every concept $\bullet C$ in $\ell_{\mathcal{T}}$, $g(\langle \mathcal{R}, x\rangle) \in C^{\mathcal{R}'}$ iff there is a $z \in g(\langle \mathcal{R}, x\rangle)$ s.t. $z \in \bullet C^{\mathcal{R}}$;
- for every role r, $r^{\mathcal{R}'} = \{\langle g(\langle \mathcal{R}, x\rangle), g(\langle \mathcal{R}, y\rangle)\rangle \mid \langle x, y\rangle \in r^{\mathcal{R}}\}$;

From these three points it is easy to derive that the satisfaction of the axioms in $\mathcal{T}$ is preserved from $\mathcal{R}$ to $\mathcal{R}'$. It is sufficient to check all the four kinds of axioms allowed in our language:

1. $C_1 \sqsubseteq C_2$
2. $\bullet C_1 \sqsubseteq C_2$
3. $C_1 \sqsubseteq \exists r. \bullet C_2$

4. $\bullet C_1 \sqsubseteq \exists r. \bullet C_2$

□

Corollary 1 (Finite Countermodel Property - FCMP). *Let $\mathcal{T}$ be a finite $\mathcal{EL}_{\bot}^{\bullet}$ TBox and $C \sqsubseteq D$ an $\mathcal{EL}_{\bot}^{\bullet}$ axiom. If $\mathcal{T}$ has a ranked model $\mathcal{R}$ s.t. $\mathcal{R}$ is not a model of $C \sqsubseteq D$, then $\mathcal{T}$ has also a* finite *ranked model $\mathcal{R}'$ s.t. it is a counter-model of $C \sqsubseteq D$.*

Proof. It is sufficient to go through the proof of Theorem 1, just considering also the concepts C and D (and their sub-concepts) in the definition of the function g.
□

It is also easy to check from Definition 8 in the paper that if a model $\mathcal{R}$ is a DL-completion of a schema, the finite model defined using the procedures in the above proofs are finite DL-completion of the same schema.

Proposition 1. *For every $\mathcal{EL}_{\bot}^{\bullet}$ TBox $\mathcal{T}$, $\mathcal{T}$ is satisfiable iff $\mathcal{T}'$ is.*

Proof. If $\mathcal{R}$ is a model of $\mathcal{T}$, it is immediately also a model of $\mathcal{T}'$. For the other direction, let $\mathcal{R}' = \langle \Delta^{\mathcal{R}'}, \cdot^{\mathcal{R}'}, \prec^{\mathcal{R}'} \rangle$ be a model of $\mathcal{T}'$ and let $\mathcal{D} \subseteq \mathcal{T}$ be the set of the axioms in $\mathcal{T}$ with the form $D_1 \sqsubseteq \exists r. \bullet C$ that have been transformed into axioms $D_1 \sqsubseteq \exists r.C$ in $\mathcal{T}'$.

Define a model $\mathcal{R} = \langle \Delta^{\mathcal{R}'}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}'} \rangle$ as:

- for every atomic concept A, $A^R = A^{R'}$;
- for every role r and every pair $x, y \in \Delta^{R'}$, if $(x, y) \in r^{R'}$ then $(x, y) \in r^R$;
- for every axiom $D_1 \sqsubseteq \exists r. \bullet C \in \mathcal{D}$ and every $x \in D_1^{\mathcal{R}'}$, add to $r^{\mathcal{R}}$ (if not present already) a pair (x, y) s.t. $y \in (\bullet C)^{\mathcal{R}'}$.

It is easy to prove by induction on the construction of the concepts that for every $\mathcal{EL}_{\bot}$-concept C, $C^{\mathcal{R}} = C^{\mathcal{R}'}$. Also, since every $x \in \Delta^{\mathcal{R}'}$ preserves the same height in $\mathcal{R}$ ($\prec^{\mathcal{R}'}$ is unchanged), for every concept $\bullet C$, $(\bullet C)^{\mathcal{R}} = (\bullet C)^{\mathcal{R}'}$.

About the last condition in the definition of $\mathcal{R}$, note that $(\bullet C)^{\mathcal{R}'}$ must be non-empty: $\mathcal{R}'$ is a model of $\mathcal{T}'$, for every $D_1 \sqsubseteq \exists r. \bullet C \in \mathcal{T}$, $D_1 \sqsubseteq \exists r.C$ is in $\mathcal{T}'$ and $x \in D_1^{\mathcal{R}'}$, hence there must be an object $z \in C^{\mathcal{R}'}$ s.t. $(x, z) \in r^{\mathcal{R}'}$; by smoothness, $C^{\mathcal{R}'} \neq \emptyset$ imposes $(\bullet C)^{\mathcal{R}'} \neq \emptyset$.

We can have four kinds of axioms in $\mathcal{T}$:

1. $C_1 \sqsubseteq C_2$
2. $\bullet C_1 \sqsubseteq C_2$
3. $C_1 \sqsubseteq \exists r. \bullet C_2$
4. $\bullet C_1 \sqsubseteq \exists r. \bullet C_2$

For each axiom of kind 1 or 2 we immediately obtain that $\mathcal{R}$ satisfies it. The last condition in the definition of $\mathcal{R}$ guarantees also the satisfaction of the axioms of kind 3 and 4 (the ones in $\mathcal{D}$).
□

Proposition 2. *The schema $M_{\mathcal{T}}$ is a model of $\mathcal{T}$.*

Proof. We need to prove that the outcome of out overall procedure, $M_{\mathcal{T}'}$, is a model of $\mathcal{T}'$. It is sufficient to prove that every iteration of Procedure 1 gives back a schema that satisfies $\mathcal{T}'$, and the last iteration of Procedure 2 preserves the satisfaction of $\mathcal{T}'$.

Every iteration of Procedure 1 gives necessarily back a schema that satisfies $\mathcal{T}'$, since the construction is such that $\mathcal{T}'^{N^*(\mathcal{T}')} = \Delta^{N^*(\mathcal{T}')}$.

Then, assume that after the last application of Procedure 2 we have an object $u \in \Delta^{M}_{\mathcal{T}'}$ s.t. $u \notin \mathcal{T}'^{M_{\mathcal{T}'}}$. In $\mathcal{T}'$ we have only axioms of the kinds $C_1 \sqsubseteq C_2$ or $\bullet C_1 \sqsubseteq C_2$, hence such a situation could arise only for axioms of the form $\bullet C_1 \sqsubseteq C_2$, that is, there is an object that is in $(\bullet C_1)^{M_{\mathcal{T}'}}$ but not in $C_2^{M_{\mathcal{T}'}}$. At the end of the last iteration of Procedure 1 $u \in \mathcal{T}'^{N^*(\mathcal{T}')}$ (otherwise it would have been eliminated), then we would have $u \notin \mathcal{T}'^{M_{\mathcal{T}'}}$ only if after the last iteration of Procedure 1 $u \notin (\bullet C_1)^{N^*(\mathcal{T}')}$ but after the last iteration of Procedure 2 $u \in (\bullet C_1)^{M_{\mathcal{T}'}}$; such a situation cannot occur because of the condition on the last application of Procedure 2 ($h_{M^*(\mathcal{T}')}(C) = h_{N^*(\mathcal{T}')}(C)$ for every $C \in \mathcal{A}_{\bullet C \sqsubseteq D}$).

$M_{\mathcal{T}'}$ being a model of $\mathcal{T}'$ implies that $M_{\mathcal{T}}$ is a model of $\mathcal{T}$.

□

Proposition 3. *Given a schema $M_{\mathcal{T}}$, $\mathscr{R}^{fin}_{M_{\mathcal{T}}}$ is non-empty.*

Proof. We give an example of how to build a DL ranked interpretation R from $M_{\mathcal{T}}$.

Populate $\Delta^{\mathcal{R}}$ with one object for each object in $\Delta^{M_{\mathcal{T}}}$: $\Delta^{\mathcal{R}} = \{x^{g(\mathcal{R}',y)} \mid g(\mathcal{R}',y) \in \Delta^{M_{\mathcal{T}}}\}$.

1. for every $x^{g(\mathcal{R}',y)} \in \Delta^{\mathcal{R}}$, $h_{\mathcal{R}}(x^{g(\mathcal{R}',y)}) = h_{M_{\mathcal{T}}}(g(\mathcal{R}',y)))$;
2. for every atomic concept A, $A^{\mathcal{R}} = \{x^{g(\mathcal{R}',y)} \mid y \in A^{\mathcal{R}'}\}$;
3. for every concept $\exists r.C$ s.t. $y \in \exists r.C^{R'}$, add a role connection from $x^{g(\mathcal{R}',y)}$ to some object in $z^{g(\mathcal{R}^*,u)}$ in $\Delta^{\mathcal{R}}$ s.t. $u \in C^{R^*}$;
4. for every concept $\exists r. \bullet D$ s.t. it has been added to the label of $g(\mathcal{R}',y)$, add a role connection from $x^{g(\mathcal{R}',y)}$ to some object in $z^{g(\mathcal{R}^*,u)}$ in $\Delta^{\mathcal{R}}$ s.t. $u \in C^{R^*}$ and $z^{g(\mathcal{R}^*,u)}$ is minimal in R w.r.t. objects associated to C.

The procedure creating $M_{\mathcal{T}'}$ from $N^*(\mathcal{T}')$ guarantees that we can satisfy point 3; the definition of $M_{\mathcal{T}}$ from $M_{\mathcal{T}'}$ guarantees that we can satisfy also point 4 (see the proof of Proposition 1).

□

Proposition 4. *If $\mathcal{R}$ is a DL-completion of $M_{\mathcal{T}}$, then $\mathcal{R}$ is a model of $\mathcal{T}$.*

Proof. We can have in $\mathcal{T}$ axioms of four possible forms: $C_1 \sqsubseteq C_2$, $\bullet C_1 \sqsubseteq C_2$, $C_1 \sqsubseteq \exists r. \bullet C_2$, $\bullet C_1 \sqsubseteq \exists r. \bullet C_2$ (C_1, C_2 being $\mathcal{EL}_\bot$ axioms). The definition of DL-completion guarantees that the satisfaction of an axiom is preserved for each kind of axiom.

□

Proposition 5. *Given an $\mathcal{EL}_\bot^\bullet$ TBox $\mathcal{T}$, all its LM*-minimal models are DL-completions of the schema $M_{\mathcal{T}}$.*

Proof. Assume there is an LM*-minimal model $\mathcal{R}$ of $\mathcal{T}$ that is not a DL-completion of $M_{\mathcal{T}}$. Then it is a DL-completion of another schema M'. We can define M' building a DL model as in the proof of the Finite Model Property, Theorem 1, then changing it into a schema just eliminating all the role connections, but preserving the interpretation of all the concepts in $\ell_{\mathcal{T}}^{\mathcal{EL}_\perp}$.

This means that there is an i s.t. $L_j^{M_{\mathcal{T}}} = L_j^{M'}$ for every $j < i$ (if $i > 0$), and there is a $g(\langle \mathcal{R}, x \rangle) \in L_i^{M'}$ that is not in $\bigcup_{k \leq i} L_k^{M_{\mathcal{T}}}$. We need to prove that cannot be the case. $g(\langle \mathcal{R}, x \rangle)$ must have been eliminated from $\bigcup_{k \leq i} L_k^{M_{\mathcal{T}}}$ either in one iteration of Procedure 1 or in one iteration of Procedure 2.

Assume it happened in the first application of Procedure 1. Assume that $i = 0$, that is, $g(\langle \mathcal{R}, x \rangle) \in L_0^{M'}$ but $g(\langle \mathcal{R}, x \rangle) \notin L_0^{M_{\mathcal{T}}}$. That cannot be the case, since the elimination of $g(\langle \mathcal{R}, x \rangle)$ from Layer 0 by Procedure 1 implies that there is no schema satisfying $\mathcal{T}$ with $g(\langle \mathcal{R}, x \rangle)$ in the lower layer. Assume that $i > 0$, and, by induction hypothesis, that there is no schema satisfying $\mathcal{T}'$ with $g(\langle \mathcal{R}, x \rangle)$ in a layer j, $j < i$, all the lower layers being equal to the correspondent ones in $N^*(\mathcal{T}')$; the same argument applies, that is, Procedure 1 implies that there is no schema satisfying $\mathcal{T}'$ with $g(\langle \mathcal{R}, x \rangle)$ in the layer i. This also implies that there is no LM*-minimal schema of $\mathcal{T}$ in which there are objects eliminated in the first application of Procedure 1.

Now, assume $g(\langle \mathcal{R}, x \rangle)$ has been eliminated from $\bigcup_{k \leq i} L_k^{M_{\mathcal{T}}}$ in the first application of Procedure 2. Since the objects eliminated from $\Delta^{\mathcal{T}}$ in Procedure 1 cannot be in any LM*-minimal schema of $\mathcal{T}$, also the objects eliminated by Procedure 2 cannot be in any schema that can be DL-completed into a DL-model, since they are all objects that should be role-connected to objects that cannot be in the schema.

Let $g(\langle \mathcal{R}, x \rangle)$ be eliminated from $\bigcup_{k \leq i} L_k^{M_{\mathcal{T}}}$ either in the k^{th} application of Procedure 1 or Procedure 2. By induction hypothesis $g(\langle \mathcal{R}, x \rangle)$ has not yet been eliminated from $\bigcup_{k \leq i} L_k^{M_{\mathcal{T}}}$, and all the objects eliminated from $\Delta^{\mathcal{T}}$ in the previous applications of Procedures 1 and 2 are correct.

Assume $g(\langle \mathcal{R}, x \rangle)$ has been eliminated from $\bigcup_{k \leq i} L_k^{M_{\mathcal{T}}}$ in the k^{th} application of Procedure 1. If $i = 0$, then there cannot be a schema of $\mathcal{T}'$ that has $g(\langle \mathcal{R}, x \rangle)$ in the layer 0. If $i > 0$, analogously to the first case, by induction hypothesis there is no schema satisfying $\mathcal{T}'$ with $g(\langle \mathcal{R}, x \rangle)$ in a layer j, $j < i$; the same argument applies, that is, Procedure 1 implies that there is no schema satisfying $\mathcal{T}'$ with $g(\langle \mathcal{R}, x \rangle)$ in the layer i, all the lower layers being equal to the ones in $N^*(\mathcal{T}')$.

Again, Procedure 2 in every iteration destroys only objects that in the DL-completion would be role-connected to objects that cannot be in any LM*-minimal model, since they are eliminated by Procedure 1.

Hence, it is not possible to have an LM*-minimal schema that differs from $M_{\mathcal{T}}$.
□

Theorem 2. *Given a TBox $\mathcal{T}$, $\mathcal{R}^{\mathcal{T}}$ characterises the LM*-entailment. That is,*

$$\mathcal{T} \models_{LM^*} C \sqsubseteq D \textit{ iff } \mathcal{R}^{\mathcal{T}} \Vdash C \sqsubseteq D$$

Proof. Let $\mathcal{R}$ be a LM*-minimal model of $\mathcal{T}$. Then it is a DL-completion of $M_{\mathcal{T}}$. If $\mathcal{R} \not\Vdash D_1 \sqsubseteq D_2$, then by the FCMP (Corollary 1) there is a finite model $\mathcal{R}^{fin}$ of $\mathcal{T}$ that does not satisfy $D_1 \sqsubseteq D_2$, that is in $\mathscr{R}^{fin}_{M_{\mathcal{T}}}$. Hence $\mathcal{R}^{fin}$ takes part in the definition of $\mathcal{R}^{\mathcal{T}}$. It is easy to check that, in such a case, whatever form $D_1 \sqsubseteq D_2$ has, $\mathcal{R}^{\mathcal{T}}$ would not satisfy it.