Portfolio Optimisation with Conditioning Information

Jang SCHILTZ

based on joint work with Marc BOISSAUX

Luxembourg School of Finance
Faculty of Law, Economics and Finance
University of Luxembourg

3rd QFQP Symposium in Trier
April 4, 2017
Outline

1 The portfolio theory of Markowitz
Outline

1. The portfolio theory of Markowitz
2. Conditioned mean-variance portfolio optimisation with unconstrained portfolio weights
Outline

1. The portfolio theory of Markowitz
2. Conditioned mean-variance portfolio optimisation with unconstrained portfolio weights
3. Conditioned mean-variance portfolio optimisation with constrained portfolio weights
   - General formulation of the problem
   - Empirical study
4. Conditioned portfolio optimisation using higher moments of return
   - Analysis
   - Empirical results
5. Conditioned portfolio optimisation using multiple signals
   - Analysis
   - Empirical results
Outline

1. The portfolio theory of Markowitz
2. Conditioned mean-variance portfolio optimisation with unconstrained portfolio weights
3. Conditioned mean-variance portfolio optimisation with constrained portfolio weights
   - General formulation of the problem
   - Empirical study
4. Conditioned portfolio optimisation using higher moments of return
   - Analysis
   - Empirical results
Outline

1. The portfolio theory of Markowitz

2. Conditioned mean-variance portfolio optimisation with unconstrained portfolio weights
   - General formulation of the problem
   - Empirical study

3. Conditioned mean-variance portfolio optimisation with constrained portfolio weights
   - General formulation of the problem
   - Empirical study

4. Conditioned portfolio optimisation using higher moments of return
   - Analysis
   - Empirical results

5. Conditioned portfolio optimisation using multiple signals
   - Analysis
   - Empirical results
Outline

1. The portfolio theory of Markowitz
2. Conditioned mean-variance portfolio optimisation with unconstrained portfolio weights
3. Conditioned mean-variance portfolio optimisation with constrained portfolio weights
   - General formulation of the problem
   - Empirical study
4. Conditioned portfolio optimisation using higher moments of return
   - Analysis
   - Empirical results
5. Conditioned portfolio optimisation using multiple signals
   - Analysis
   - Empirical results
The notion of Risk

Suppose you can choose between 2 games:

Game 1: 5% chance of winning 1000 €; 95% chance of not winning anything.

Game 2: 5% chance of winning 5000 €; 95% chance of losing 200 €.

Expected return for game 1: 0.05 \times 1000 = 50 €

Expected return for game 2: 0.05 \times 5000 - 0.95 \times 200 = 60 €

Conclusion: The expected return of game 2 is higher.

Nevertheless, most people choose game 1, because they are risk averse.
The notion of Risk

Suppose you can choose between 2 games:

- **Game 1**: 5% chance of winning 1000€; 95% chance of not winning anything.

- **Game 2**: 5% chance of winning 5000€; 95% chance of losing 200€.

Expected return for game 1: \(0.05 \times 1000 = 50\) €

Expected return for game 2: \(0.05 \times 5000 - 0.95 \times 200 = 60\) €

Conclusion: The expected return of game 2 is higher. Nevertheless, most people choose game 1, because they are risk averse.
The notion of Risk

Suppose you can choose between 2 games:

- Game 1: 5% chance of winning 1000€; 95% chance of not winning anything.
- Game 2: 5% chance of winning 5000€; 95% chance of loosing 200€.

Conclusion: The expected return of game 2 is higher. Nevertheless, most people choose game 1, because they are risk averse.
The notion of Risk

Suppose you can choose between 2 games:

- **Game 1**: 5% chance of winning 1000€; 95% chance of not winning anything.
- **Game 2**: 5% chance of winning 5000€; 95% chance of losing 200€.

Which game should you choose?

Expected return for game 1: $0.05 \times 1000 = 50€$

Expected return for game 2: $0.05 \times 5000 - 0.95 \times 200 = 60€$

Conclusion: The expected return of game 2 is higher.

Nevertheless, most people choose game 1, because they are risk averse.
The notion of Risk

Suppose you can choose between 2 games:

- Game 1: 5% chance of winning 1000€; 95% chance of not winning anything.
- Game 2: 5% chance of winning 5000€; 95% chance of losing 200€.

Which game should you choose?

- Expected return for game 1: $0.05 \times 1000 = 50€$

Conclusion: The expected return of game 2 is higher. Nevertheless, most people choose game 1, because they are risk averse.
The notion of Risk

Suppose you can choose between 2 games:

- Game 1: 5% chance of winning 1000€; 95% chance of not winning anything.
- Game 2: 5% chance of winning 5000€; 95% chance of loosing 200€.

Which game should you choose?

- Expected return for game 1: \(0.05 \times 1000 = 50€\)
- Expected return for game 2: \(0.05 \times 5000 - 0.95 \times 200 = 60€\)

Conclusion: The expected return of game 2 is higher. Nevertheless, most people choose game 1, because they are risk averse.
The notion of Risk

Suppose you can choose between 2 games:

- Game 1: 5% chance of winning 1000€; 95% chance of not winning anything.
- Game 2: 5% chance of winning 5000€; 95% chance of loosing 200€.

Which game should you choose?

- Expected return for game 1: $0.05 \times 1000 = 50€$
- Expected return for game 2: $0.05 \times 5000 - 0.95 \times 200 = 60€$

Conclusion: The expected return of game 2 is higher.
The notion of Risk

Suppose you can choose between 2 games:

- Game 1: 5% chance of winning 1000€; 95% chance of not winning anything.
- Game 2: 5% chance of winning 5000€; 95% chance of losing 200€.

Which game should you choose?

- Expected return for game 1: \(0.05 \times 1000 = 50€\)
- Expected return for game 2: \(0.05 \times 5000 - 0.95 \times 200 = 60€\)

Conclusion: The expected return of game 2 is higher.

Nevertheless, most people choose game 1, because they are risk averse.
The notion of Risk

A risk averse investor doesn’t like to take risk. If he can choose between two investments with the same expected return, he will choose the less risky one.
The notion of Risk

A risk averse investor doesn’t like to take risk. If he can choose between two investments with the same expected return, he will choose the less risky one.

But usually investments with higher expected return also have higher risk.
The notion of Risk

A risk averse investor doesn’t like to take risk. If he can choose between two investments with the same expected return, he will choose the less risky one.

But usually investments with higher expected return also have higher risk.

So, there is a trade-off between expected return and risk.
The notion of Risk

A risk averse investor doesn’t like to take risk. If he can choose between two investments with the same expected return, he will choose the less risky one.

But usually investments with higher expected return also have higher risk.

So, there is a trade-off between expected return and risk.

In portfolio theory, expected return is measured by the average of the portfolio return, risk by its dispersion.
Prior to Markowitz’s work, investors focused on assessing the risks and rewards of individual securities in constructing their portfolios. Key insight of Markowitz: An asset’s risk and return should not be assessed by itself, but by how it contributes to a portfolio’s overall risk and return. Markowitz assumes that investors are risk averse, meaning that given two portfolios that offer the same expected return, investors will prefer the less risky one. Thus, an investor will take on increased risk only if compensated by higher expected returns. Since portfolio volatility depends on the correlations between the component assets, an investor can reduce portfolio risk simply by holding combinations of instruments that are not perfectly positively correlated.
Harry Markowitz (JF 1952)

Prior to Markowitz’s work, investors focused on assessing the risks and rewards of individual securities in constructing their portfolios.

Key insight of Markowitz:

An asset’s risk and return should not be assessed by itself, but by how it contributes to a portfolio’s overall risk and return.

Markowitz assumes that investors are risk averse, meaning that given two portfolios that offer the same expected return, investors will prefer the less risky one. Thus, an investor will take on increased risk only if compensated by higher expected returns.

Since portfolio volatility depends on the correlations between the component assets, an investor can reduce portfolio risk simply by holding combinations of instruments that are not perfectly positively correlated.
Prior to Markowitz’s work, investors focused on assessing the risks and rewards of individual securities in constructing their portfolios.

**Key insight of Markowitz:** An asset’s risk and return should not be assessed by itself, but by how it contributes to a portfolio’s overall risk and return.
Harry Markowitz (JF 1952)

Prior to Markowitz’s work, investors focused on assessing the risks and rewards of individual securities in constructing their portfolios.

**Key insight of Markowitz:** An asset’s risk and return should not be assessed by itself, but by how it contributes to a portfolio’s overall risk and return.

Markowitz assumes that investors are risk averse, meaning that given two portfolios that offer the same expected return, investors will prefer the less risky one. Thus, an investor will take on increased risk only if compensated by higher expected returns.
Prior to Markowitz’s work, investors focused on assessing the risks and rewards of individual securities in constructing their portfolios.

**Key insight of Markowitz:** An asset’s risk and return should not be assessed by itself, but by how it contributes to a portfolio’s overall risk and return.

Markowitz assumes that investors are risk averse, meaning that given two portfolios that offer the same expected return, investors will prefer the less risky one. Thus, an investor will take on increased risk only if compensated by higher expected returns.

Since portfolio volatility depends on the correlations between the component assets, an investor can reduce portfolio risk simply by holding combinations of instruments that are not perfectly positively correlated.
Efficient portfolio

An efficient portfolio is defined as the portfolio that maximizes the expected return for a given amount of risk, or the portfolio that minimizes the risk for a given expected return.
An efficient portfolio is defined as the portfolio that maximizes the expected return for a given amount of risk, or the portfolio that minimizes the risk for a given expected return.
Efficient frontier without risk-free assets
Efficient frontier without risk-free assets

The efficient frontier is the curve that shows all efficient portfolios in a risk-return framework.
Minimum variance portfolio

Suppose an investor wants to invest all his money with the lowest possible amount of risk. Because he will always invest in an efficient portfolio, he will choose the portfolio on the efficient frontier with minimum standard deviation.
Minimum variance portfolio

Suppose an investor wants to invest all his money with the lowest possible amount of risk. Because he will always invest in an efficient portfolio, he will choose the portfolio on the efficient frontier with minimum standard deviation.
Minimum variance portfolio

Suppose an investor wants to invest all his money with the lowest possible amount of risk. Because he will always invest in an efficient portfolio, he will choose the portfolio on the efficient frontier with minimum standard deviation.
Optimal portfolio in the Markowitz sense

The theory of Markowitz stipulates that investors try to maximize their utility function $u$ that depends on his risk aversion parameter $\gamma$ and is given by $u(\gamma) = E[P] - \frac{1}{2}\gamma \sigma^2 P$, where $E[P]$ denotes the (unconditional) expected portfolio return. The higher value of $\gamma$ the more risk averse is the investor. The optimal portfolio for an investor is the portfolio with maximum utility.
Optimal portfolio in the Markowitz sense

The theory of Markowitz stipulates that investors try to maximize their utility function $u$ that depends on his risk aversion parameter $\gamma$ and is given by

$$u(\gamma) = E[P] - \frac{1}{2} \gamma \sigma_P^2,$$

where $E[P]$ denotes the (unconditional) expected portfolio return.
The theory of Markowitz stipulates that investors try to maximize their utility function $u$ that depends on his risk aversion parameter $\gamma$ and is given by

$$u(\gamma) = E[P] - \frac{1}{2} \gamma \sigma_P^2,$$

where $E[P]$ denotes the (unconditional) expected portfolio return.

The higher value of $\gamma$ the more risk averse is the investor.
Optimal portfolio in the Markowitz sense

The theory of Markowitz stipulates that investors try to maximize their utility function $u$ that depends on his risk aversion parameter $\gamma$ and is given by

$$u(\gamma) = E[P] - \frac{1}{2}\gamma\sigma_P^2,$$

where $E[P]$ denotes the (unconditional) expected portfolio return.

The higher value of $\gamma$ the more risk averse is the investor.

The optimal portfolio for an investor is the portfolio with maximum utility.
Optimal portfolio in the Markowitz sense

Graphically, the portfolio with maximum utility is gained by moving the utility curve as high as possible. The utility curve is the curve that shows the possible combinations of mean and standard deviation that result in the same utility.
Optimal portfolio in the Markowitz sense

Graphically, the portfolio with maximum utility is gained by moving the utility curve as high as possible. The utility curve is the curve that shows the possible combinations of mean and standard deviation that result in the same utility.
Optimal portfolio in the Markowitz sense

Graphically, the portfolio with maximum utility is gained by moving the utility curve as high as possible. The utility curve is the curve that shows the possible combinations of mean and standard deviation that result in the same utility.
Suppose now, that there exists a risk-free asset guaranteeing a risk-free rate $r_f$. The Sharpe ratio (JF 1964) examines the performance of an investment by adjusting for its risk. The ratio measures the excess return (or risk premium) per unit of deviation in an investment asset. For a portfolio $P$, of volatility $\sigma_P$, the Sharpe ratio $SR_P$ is then defined as

$$SR_P = \frac{E[P] - r_f}{\sigma_P}.$$
Sharpe ratio

Suppose now, that there exists a risk-free asset guaranteeing a risk-free rate of return $r_f$. 
Sharpe ratio

Suppose now, that there exists a risk-free asset guaranteeing a risk-free rate of return $r_f$.

The Sharpe ratio (JF 1964) examines the performance of an investment by adjusting for its risk.
Sharpe ratio

Suppose now, that there exists a risk-free asset guaranteeing a risk-free rate of return $r_f$.

The Sharpe ratio (JF 1964) examines the performance of an investment by adjusting for its risk.

The ratio measures the excess return (or risk premium) per unit of deviation in an investment asset.
Sharpe ratio

Suppose now, that there exists a risk-free asset guaranteeing a risk-free rate of return $r_f$.

The Sharpe ratio (JF 1964) examines the performance of an investment by adjusting for its risk.

The ratio measures the excess return (or risk premium) per unit of deviation in an investment asset.

For a portfolio $P$, of volatility $\sigma_P$, the Sharp ratio $SR_P$ is then defined as

$$SR_P = \frac{E[P] - r_f}{\sigma_P}.$$
Tangency portfolio without risk-free asset

If an investor wants to invest in a portfolio with maximum Sharpe ratio, getting thus the highest expected return per unit of risk, he chooses the tangency portfolio, which is the most "risk-efficient portfolio". Graphically, this portfolio is the point where a line through the origin is tangent to the efficient frontier, because this point has the property that is has the highest possible mean-standard deviation ratio.
If an investor wants to invest in a portfolio with maximum Sharpe ratio, getting thus the highest expected return per unit of risk, he chooses the tangency portfolio, which is the most "risk-efficient portfolio".

Graphically, this portfolio is the point where a line through the origin is tangent to the efficient frontier, because this point has the property that is has the highest possible mean-standard deviation ratio.
Tangency portfolio without risk-free asset

Graphically, this portfolio is the point where a line through the origin is tangent to the efficient frontier, because this point has the property that is has the highest possible mean-standard deviation ratio.
The theory of Markowitz implies that the new efficient frontier is a straight line, starting at the risk-free point and tangent to the old efficient frontier. This line, called also the capital allocation line is the efficient frontier in the presence of a risk-free asset.
The theory of Markowitz implies that the new efficient frontier is a straight line, starting at the risk-free point and tangent to the old efficient frontier.
Tangency portfolio with risk-free assets

The theory of Markowitz implies that the new efficient frontier is a straight line, starting at the risk-free point and tangent to the old efficient frontier. This line, called also the capital allocation line is the efficient frontier in the presence of a risk-free asset.
Optimal portfolio with risk-free assets
Optimal portfolio with risk-free assets

Maximizing the utility function of an investor in the presence of a risk-free asset gives the following optimal portfolio.
Optimal portfolio with risk-free assets

Maximizing the utility function of an investor in the presence of a risk-free asset gives the following optimal portfolio.
Outline

1. The portfolio theory of Markowitz

2. Conditioned mean-variance portfolio optimisation with unconstrained portfolio weights

3. Conditioned mean-variance portfolio optimisation with constrained portfolio weights
   - General formulation of the problem
   - Empirical study

4. Conditioned portfolio optimisation using higher moments of return
   - Analysis
   - Empirical results

5. Conditioned portfolio optimisation using multiple signals
   - Analysis
   - Empirical results
Ferson and Siegel postulate that there exists some relationship $\mu(s)$ between a signal $s$ and each asset return $r$ observed at the end of the investment interval:

$$r_t = \mu(s_t - 1) + \epsilon_t,$$

with $E[\epsilon_t | s_{t-1}] = 0$.

How do we optimally use this information in an otherwise classical portfolio optimization process?

Ferson and Siegel provided an analytical solution for the case where the weights are unconstrained.
Ferson and Siegel postulate that there exists some relationship $\mu(s)$ between a signal $s$ and each asset return $r$ observed at the end of the investment interval:

$$r_t = \mu(s_{t-1}) + \epsilon_t,$$

with $E[\epsilon_t | s_{t-1}] = 0.$
Ferson and Siegel postulate that there exists some relationship $\mu(s)$ between a signal $s$ and each asset return $r$ observed at the end of the investment interval:

$$r_t = \mu(s_{t-1}) + \epsilon_t,$$

with $E[\epsilon_t|s_{t-1}] = 0$.

How do we optimally use this information in an otherwise classical portfolio optimization process?
Ferson and Siegel postulate that there exists some relationship $\mu(s)$ between a signal $s$ and each asset return $r$ observed at the end of the investment interval:

$$r_t = \mu(s_{t-1}) + \epsilon_t,$$

with $E[\epsilon_t|s_{t-1}] = 0$.

How do we optimally use this information in an otherwise classical portfolio optimization process?

Ferson and Siegel provided an analytical solution for the case where the weights are unconstrained.
The Ferson-Siegel paper (JF 2001)

Theorem (Ferson and Siegel, 1991)

Given unconditional expected return $\mu_P$, $n$ risky assets, and no risk-free asset, the unique portfolio having minimum unconditional variance is determined by the weights

$$u'(s) = \frac{e'\Lambda(s)}{e'\Lambda(s)e} + \frac{\mu_P - E\left[\frac{e'\Lambda(s)\mu(s)}{e'\Lambda(s)e}\right]}{E\left[\mu'(s)\left(\Lambda(s) - \frac{\Lambda(s)ee'\Lambda(s)}{e'\Lambda(s)e}\right)\mu(s)\right]} \mu'(s) \left(\Lambda(s) - \frac{\Lambda(s)ee'\Lambda(s)}{e'\Lambda(s)e}\right)$$

where $e$ is an $n$–vector of ones and $\Lambda(s) = [\mu(s)\mu(s) + \Sigma^2_{\epsilon}]^{-1}$; $\Sigma^2_{\epsilon}$ is the conditional covariance matrix $E[\epsilon\epsilon'|s]$. 

The Ferson-Siegel paper (JF 2001)

Theorem (Ferson and Siegel, 1991)

Given unconditional expected return $\mu_P$, $n$ risky assets, and a risk-free asset, with return rate $r_f$, the unique portfolio having minimum unconditional variance is determined by the weights

$$u'(s) = \frac{\mu_P - r_f}{E[(\mu(s) - r_f e)'\Lambda(s)(\mu(s) - r_f e)]} (\mu(s) - r_f e)'\Lambda(s),$$

where $e$ is an $n$–vector of ones and

$$\Lambda(s) = [((\mu(s) - r_f e)(\mu(s) - r_f e)' + \Sigma_\epsilon)^{-1}; \Sigma_\epsilon$$

is the conditional covariance matrix $E[\epsilon\epsilon'|s]$. 
Problems with the Ferson-Siegel results

In practice, portfolio weights are often not unconstrained.
Problems with the Ferson-Siegel results

In practice, portfolio weights are often not unconstrained.

In particular, negative portfolio weights correspond to short sales of the underlying assets. It is generally unrealistic to assume that large short positions can be entered at no extra cost.
Problems with the Ferson-Siegel results

In practice, portfolio weights are often not unconstrained.

In particular, negative portfolio weights correspond to short sales of the underlying assets. It is generally unrealistic to assume that large short positions can be entered at no extra cost.

This may even be prohibited for legal reasons, as in the case of pension funds.
Problems with the Ferson-Siegel results

In practice, portfolio weights are often not unconstrained.

In particular, negative portfolio weights correspond to short sales of the underlying assets. It is generally unrealistic to assume that large short positions can be entered at no extra cost.

This may even be prohibited for legal reasons, as in the case of pension funds.

Clearly, ignoring these restrictions is even less realistic when dealing with less liquid assets such as investment funds.
Problems with the Ferson-Siegel results

In practice, portfolio weights are often not unconstrained.

In particular, negative portfolio weights correspond to short sales of the underlying assets. It is generally unrealistic to assume that large short positions can be entered at no extra cost.

This may even be prohibited for legal reasons, as in the case of pension funds.

Clearly, ignoring these restrictions is even less realistic when dealing with less liquid assets such as investment funds.

Another plausible constraint on feasible investment weights would restrict them to a certain interval usually centred on zero, so as to avoid entering excessively large positions in particular assets.
Problems with the Ferson-Siegel results

In practice, portfolio weights are often not unconstrained.

In particular, negative portfolio weights correspond to short sales of the underlying assets. It is generally unrealistic to assume that large short positions can be entered at no extra cost.

This may even be prohibited for legal reasons, as in the case of pension funds.

Clearly, ignoring these restrictions is even less realistic when dealing with less liquid assets such as investment funds.

Another plausible constraint on feasible investment weights would restrict them to a certain interval usually centred on zero, so as to avoid entering excessively large positions in particular assets.

For the traditional Markowitz problem, the introduction of portfolio weights constraints of any kind means a closed-form solution is no longer available, and a numerical algorithm has to be used.
Outline

1. The portfolio theory of Markowitz
2. Conditioned mean-variance portfolio optimisation with unconstrained portfolio weights
3. Conditioned mean-variance portfolio optimisation with constrained portfolio weights
   - General formulation of the problem
   - Empirical study
4. Conditioned portfolio optimisation using higher moments of return
   - Analysis
   - Empirical results
5. Conditioned portfolio optimisation using multiple signals
   - Analysis
   - Empirical results
Possible signals

Taken from a continuous scale ranging from purely macroeconomic indices to investor sentiment indicators. Indicators taking into account investor attitude may be based on some model or calculated in an ad-hoc fashion.

Examples include

- short-term treasury bill rates (Fama and Schwert 1977);
- CBOE Market Volatility Index (VIX) (Whaley 1993) or its European equivalents (VDAX etc.);
- risk aversion indices using averaging and normalisation (UBS Investor Sentiment Index 2003) or PCA reduction (Coudert and Gex 2007) of several macroeconomic indicators;
- global risk aversion indices (GRAI) (Kumar and Persaud 2004) based on a measure of rank correlation between current returns and previous risks;
- option-based risk aversion indices (Tarashev et al. 2003);
- sentiment indicators directly obtained from surveys (e.g. University of Michigan Consumer Sentiment Index)
Unconditioned expected return and variance given conditioning information

These are obtained as expectation integrals over the signal domain. If a risk-free asset with return $r_t$ is available,

$$E(P) = E \left[ u'(s)(\mu(s) - r_f1) \right] = E[l_1(u, s)]$$

and

$$\sigma^2(P) = E \left[ u'(s) \left( (\mu(s) - r_t1)(\mu(s) - r_t1)' + \sigma^2_\epsilon u(s) \right) \right] - \mu_P^2$$

$$= E[l_2(u, s)] - \mu_P^2$$

for an expected unconditional return of $\mu_P$ and a conditional covariance matrix $\sigma^2_\epsilon$. 
Optimal control formulation

Minimize \[ J_{[s^-, s^+]}(x, u) = \int_{s^-}^{s^+} l_2(u, s)p_s(s)ds \] as \( s^- \to -\infty, s^+ \to +\infty \)

subject to \[ \dot{x}(s) = l_1(u, s)p_s(s), \quad \forall s \in [s^-, s^+] \]
with
\[ \lim_{s \to -\infty} x(s) = x_-, \quad \lim_{s \to +\infty} x(s) = x_+ , \]
and
\[ u(s) \in U, \quad \forall s \in [s^-, s^+] \]

where \( U \subseteq \mathbb{R}^n \), \( x(s) \in \mathbb{R}^m \) and \( L \) as well as \( f \) are continuous and differentiable in both \( x \) and \( u \).

Since the signal \( s \) is not necessarily bounded, the resulting control problem involves expectation integrals with infinite boundaries in the general case.
Necessity and sufficiency results generalised

- The Pontryagin Minimum Principle (PMP) and Mangasarian sufficiency theorem are shown to continue holding if the control problem domain corresponds to the full real axis: the corresponding optimal control problems are well-posed.

- The PMP is then used to show that the given optimal control formulation of the conditioned mean-variance problem generalizes classical (Ferson and Siegel; Markowitz) problem expressions.
Aim of the study

Carry out backtests executing constrained-weight conditioned optimization strategies with different settings.
Data set

- 11 years of daily data, from January 1999 to February 2010 (2891 samples)
- Risky assets: 10 different EUR-based funds commercialized in Luxembourg chosen across asset categories (equity, fixed income) and across Morningstar style criteria
- Risk-free proxy: EURIBOR with 1 week tenor
- Signals: VDAX, volatility of bond index, PCA-based indices built using both 2 and 4 factors and estimation window sizes of 50, 100 and 200 points, Kumar and Persaud currency-based GRAI obtained using 1 month and 3 month forward rates
Benchmark problem

- Take VDAX index as signal, with 60 point estimation window and weights constrained to allow for long investment only
- Rebalance Markowitz-optimal portfolio alongside conditioned optimal portfolio, both with and without the availability of a risk-free proxy asset, over the 11-year period
- Assume lagged relationship $\mu(s)$ between signal and return can be represented by a linear regression
- Use kernel density estimates for signal densities
- Estimate the above using a given rolling window size (15 to 120 points)
- Use direct collocation discretisation method for numerical problem solutions
Benchmark problem (2)

- Vary the parameters to check both for robustness of strategy results and whether results can be further improved while staying with a linear regression model for the relationship between signals and returns.
- Obtain efficient frontier for every date and choose portfolio based on quadratic utility functions with risk aversion coefficients between 0 and 10.
- Compare sharp ratios (ex ante), additive observed returns (ex post), observed standard deviations (ex post) of both strategies.
- Try different window sizes, different signal lags, weight averages over different signal points, different signals.
As would be expected, the constrained optimal weights are not simply a truncated version of the unconstrained optimal (Ferson-Siegel) weights.

Kernel density estimate for GRAI 3M signal, 26 Mar 99

Optimal weight functionals using GRAI 3M signal, positive weights only, 26 Mar 99
Typical kernel density estimate for signal and resulting optimal weight functionals

As would be expected, the constrained optimal weights are not simply a truncated version of the unconstrained optimal (Ferson-Siegel) weights.
Weights in the constrained and unconstrained case

Figure 4.6: Optimal weight functionals for conditioned problem, no short constraint (left) / weights constrained to be positive (right), 26 March 1999, 60 day estimation windows, VDAX index.
With risk-free asset

Ex post observed relative excess additive returns, standard deviation ratios

Observed excess returns and std ratios for base case with risk-free asset (conditioned wrt Markowitz)
With risk-free asset

Time path of additive strategy returns for $\lambda = 2$

Cumulative strategy returns (risk aversion = 2, positive weights only)

- With signalling
- Markowitz
With risky assets only

Ex post observed relative excess additive returns, standard deviation ratios

Observed excess returns and std ratios for risky asset only base case (conditioned wrt Markowitz)
With risky assets only

Time path of additive strategy returns for $\lambda = 2$
Ex post results for different estimation window sizes

- Excess returns (and standard deviations) larger as window sizes increase
- Trade-off between statistical quality of estimates and impact of conditional nonstationarities
Ex post results for different signal lags

- Excess returns larger and standard deviations smaller as lag size increases
- Trade-off between statistical quality of estimators and easier modelling
Ex post results for weight averages over different number of signal points

**Observed Sharpe ratios for risky asset only case, VDAX signal, averaging weights over different intervals (conditioned wrt Markowitz)**

- Negligible changes in excess returns, slight changes in standard deviations: little risk attached to signal observations
Ex post results for different signals

Best results seen for baseline VDAX signal, averaging seems to distract from signal power.
Outline

1. The portfolio theory of Markowitz
2. Conditioned mean-variance portfolio optimisation with unconstrained portfolio weights
3. Conditioned mean-variance portfolio optimisation with constrained portfolio weights
   - General formulation of the problem
   - Empirical study
4. Conditioned portfolio optimisation using higher moments of return
   - Analysis
   - Empirical results
5. Conditioned portfolio optimisation using multiple signals
   - Analysis
   - Empirical results
Higher-moment optimisation

In the present, we obtain and exercise optimal control formulations of conditioned problems involving higher moments of returns to evaluate the impact of conditioning information on higher moment optimisation for the first time.

Model user preferences with respect to the third and fourth moments of returns (skewness (S) and kurtosis (K)) as well as mean (M) and variance (V).

Can either work as in MV case, replacing expected return or variance by, respectively, skewness or kurtosis (MK efficient frontier) or use (polynomial) utility functions to capture investor preferences with respect to more than two moments at the same time (MVK and MVSK optimisation).
Higher-moment optimisation

- In the present, we obtain and exercise optimal control formulations of conditioned problems involving higher moments of returns to evaluate the impact of conditioning information on higher moment optimisation for the first time.
Higher-moment optimisation

- In the present, we obtain and exercise optimal control formulations of conditioned problems involving higher moments of returns to evaluate the impact of conditioning information on higher moment optimisation for the first time.

- Model user preferences with respect to the third and fourth moments of returns (skewness (S) and kurtosis (K)) as well as mean (M) and variance (V).
Higher-moment optimisation

- In the present, we obtain and exercise optimal control formulations of conditioned problems involving higher moments of returns to evaluate the impact of conditioning information on higher moment optimisation for the first time.

- Model user preferences with respect to the third and fourth moments of returns (skewness (S) and kurtosis (K)) as well as mean (M) and variance (V).

- Can either work as in MV case, replacing expected return or variance by, respectively, skewness or kurtosis (MK efficient frontier)
Higher-moment optimisation

- In the present, we obtain and exercise optimal control formulations of conditioned problems involving higher moments of returns to evaluate the impact of conditioning information on higher moment optimisation for the first time.

- Model user preferences with respect to the third and fourth moments of returns (skewness \( S \) and kurtosis \( K \)) as well as mean \( M \) and variance \( V \).

- Can either work as in MV case, replacing expected return or variance by, respectively, skewness or kurtosis (MK efficient frontier)

- or use (polynomial) utility functions to capture investor preferences with respect to more than two moments at the same time (MVK and MVSK optimisation).
MK optimisation as an optimal control problem

Minimize \( J_{[s^-, s^+]}(x, u) = \int_{s^-}^{s^+} l_4(u, s)p_s(s)\,ds \) as \( s^- \to -\infty, s^+ \to +\infty \)

subject to \( \dot{x}(s) = l_1(u, s)p_s(s) \quad \forall s \in [s^-, s^+] \), with \( \lim_{s \to -\infty} x(s) = x_-, \lim_{s \to +\infty} x(s) = x_+ \),

and \( u(s) \in U, \quad \forall s \in [s^-, s^+] \)

where \( l_1(u; s) \) and \( l_4(u; s) \) are integrands chosen such that the signal domain integral of \( l_i p_s(s) \) corresponds in either case to unconditional ith moment metrics of expected portfolio returns, \( \mu_P \) is the expected unconditional portfolio return and \( p_s(s) \) is the signal density function.
MVK/MVSK optimisation as an optimal control problem

minimise \( J(s)(x(s), u(s)) = -\int_{l_S} \left( a_1 \frac{dx_1}{ds} + a_2 \frac{dx_2}{ds} + a_3 \frac{dx_3}{ds} + a_4 \frac{dx_4}{ds} \right) ds \)

subject to \( \frac{dx_i}{ds} = l_i(u, s)p_s(s), i \in \{1, 2, 3, 4\} \),
\( x_1(s^-) = x_2(s^-) = x_3(s^-) = x_4(s^-) = 0 \)
and \( u(s) \in U \ \forall s \in l_S \)

where the \( l_i(u, s) \) are integrands chosen such that the \( x_i(s^+) \) correspond to unconditional \( i \)th moment metrics of expected portfolio returns.

Optimisation involving the third moment always entails a nonconvex cost function: this is problematic from the numerical point of view.
Ex ante and ex post kurtoses for MK optimisation

Large reduction in kurtosis both ex ante and ex post seen when conditioning information is used.

Ex ante / ex post returns kurtosis for different points on the efficient frontier, VDAX, 60D window

Returns kurtosis (absolute)

Point index on efficient frontier

Ex post MK without signalling
Ex post MK with signalling
Ex post MV without signalling
Ex ante MK without signalling
Ex ante MK with signalling
Ex ante MV without signalling
Large reduction in kurtosis both ex ante and ex post seen when conditioning information is used.
Ex ante and ex post utility values for MVK optimisation

Average expected / realised utilities for different utility functions, MVK problem, VDAX, 60D window

Returns utility vs Utility function#

- Realised with signalling
- Realised without signalling
- Expected with signalling
- Expected without signalling
Ex ante and ex post improvements in ASR for MVS K optimisation

Average expected / realised improvements in ASR for different utility functions, VDAX, 60D window

- Realised
- Expected
The adjusted Sharpe ratio (ASR, Pzier and White (2006)) takes into account third and fourth moments of expected returns.
Ex ante and ex post improvements in ASR for MVSK optimisation

- The adjusted Sharpe ratio (ASR, Pzier and White (2006)) takes into account third and fourth moments of expected returns.
- MVSK improvements are consistent with the MVK case: some evidence that skewness preferences may be taken into account in practice.
Analysis gives an example of how the optimal control formulation of conditioned problems may be applied to different problem variations not previously accessible.
Summary

- Analysis gives an example of how the optimal control formulation of conditioned problems may be applied to different problem variations not previously accessible.

- Results provide further evidence (in addition to the existing empirical two-moment literature) to suggest that conditioned optimisation increases strategy performance in a universal and robust manner.
Outline

1. The portfolio theory of Markowitz
2. Conditioned mean-variance portfolio optimisation with unconstrained portfolio weights
3. Conditioned mean-variance portfolio optimisation with constrained portfolio weights
   - General formulation of the problem
   - Empirical study
4. Conditioned portfolio optimisation using higher moments of return
   - Analysis
   - Empirical results
5. Conditioned portfolio optimisation using multiple signals
   - Analysis
   - Empirical results
Optimal control translation

Two signals \( s^{(1)} \) and \( s^{(2)} \) with \( s = (s^{(1)} s^{(2)}) \), investor utility function \( U(x) = a_1 x + a_2 x^2 \), joint signal density \( p_s \) give

\[
\text{minimise} \quad J_{ls}(x(s), u(s)) = \int_{I_S} \left( a_1 \frac{\partial^2 x_1}{\partial s^{(1)} \partial s^{(2)}} + a_2 \frac{\partial^2 x_2}{\partial s^{(1)} \partial s^{(2)}} \right) ds
\]

subject to

\[
\frac{\partial^2 x_1}{\partial s^{(1)} \partial s^{(2)}} = u'(s) \mu(s) p_s(s),
\]

\[
\frac{\partial^2 x_2}{\partial s^{(1)} \partial s^{(2)}} = \left( \left( u'(s) \mu(s) \right)^2 + u'(s) \Sigma^2 u(s) \right) p_s(s),
\]

\[
x_1(s^-) = x_2(s^-) = 0
\]

and

\[
u(s) \in U \ \forall s \in I_S
\]

as the resulting mean-variance equivalent optimisation problem formulation.
Multidimensional results

- Optimal control problems involving a higher-dimensional objective function integration variable and first-order state PDEs are called *Dieudonné-Rashevsky* problems.

- Multidimensional analogues of PMP have been established (Cesari 1969) for problems of the Dieudonné-Rashevsky type.

- The problem with cross-derivatives just given represents a form equivalent to Dieudonné-Rashevsky (Udriste 2010).
2-D discretisation scheme

- Use a 2-D direct collocation scheme: *direct* means both control and state variables are discretised, *collocation* means PDE and other constraints have to be met exactly at prespecified (collocation) points on the grid.

- Use control values constant on each surface element and state values on vertices to which bilinear interpolation is applied.

- Provide analytical expressions for the (sparse) gradient and Hessian matrices to the numerical solver so convergence rate and computational cost remain manageable.
2-D discretisation scheme (2)

(a) Control discretisation constant over surface elements.

(b) Bilinear state discretisation.
2-D discretisation scheme convergence result

**Theorem**

At the collocation points \( s_{i+1/2,j+1/2} \), the Pontryagin costate equations are verified to order the chosen grid mesh \( h \):

\[
\nabla_s \cdot \lambda = - \sum_{\alpha=1}^{2} \lambda^{(\alpha)}_{i+1/2,j+1/2} \frac{\partial f^{(\alpha)}_{i+1/2,j+1/2}}{\partial x} + O(h).
\]

Also, for any optimal control interior to the admissible set \( U \), the proposed scheme is consistent with the first-order condition on the Hamiltonian \( \mathcal{H} \):

\[
\frac{\partial \mathcal{H}}{\partial u(s)} = 0 \ \forall s \in I_s.
\]
2-D discretisation gradient and Hessian matrix sparsity patterns

- Gradient dimensions for $N \times N$-point grid and $n$ assets are
  \[
  \left[ (N - 1)^2 n + 2N^2 \right] \times \left[ 3(N - 1)(N - 2) + 3(N - 2) + 5 \right]
  \]
- Hessian dimensions in that case are
  \[
  \left[ (N - 1)^2 n + 2N^2 \right] \times \left[ (N - 1)^2 n + 2N^2 \right]
  \]
Typical optimal weight functional

Optimal weights are found as vector functions of the two signals
2-signal backtest

- Simultaneously use VDAX (pure equity risk) and BONDIDX (volatility of Barclays Aggregate Euro Bond Index, pure interest rate risk) as signals.
- Obtain optimal portfolio weights for daily rebalancing by optimising unconditional expected utilities for quadratic investor utility functions $U(x) = a_1x + a_2x^2$ and three different levels of risk aversion: $a_2 = -0.2$, $a_2 = -0.5$ and $a_2 = -0.7$.
- Compare utilities and Sharpe ratios (ex ante and ex post), maximum drawdowns / drawdown durations (MD/MDD) and observed returns time paths for Markowitz, 1 signal and 2 signal strategies.
Ex ante/ex post utilities obtained, VDAX / BONDIDX, 60D window

Utility vs. |a2|
Backtest average Sharpe ratios

Ex ante/ex post Sharpe ratios, VDAX / BONDIDX, 60D window

SR vs. |a2|
Backtest average maximum drawdown (durations)

Maximum drawdowns (MD), VDAX / BONDIDX, 60D window

Maximum drawdown durations (MDD), VDAX / BONDIDX, 60D window
Backtest cumulative return time paths, $a_2 = -0.5$

Cumulative strategy returns ($a_2 = -0.5$, VDAX / BONDI DX, positive weights only)
Summary

- Improvement with a second signal is substantial ex ante, but very marginal ex post: estimation risk larger than for a single signal.
- The suggested numerical solution scheme can be generalised to even more signals, but a curse of dimensionality applies:
  - computational cost: will diminish in impact over time
  - statistical (kernel density estimate): fundamentally prevents the use of more than three signals unless simplifications are made.
- Marginal ex post improvements, however, suggest an averaging effect (as seen for single PCA indices in earlier single signal study) takes place for more signals, such that this limitation is not seen as that restrictive.