Recent results on conservative and symmetric $n$-ary semigroups

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Joint work with Jimmy Devillet and Jean-Luc Marichal

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Let $X$ be an arbitrary set. An operation $F : X^n \to X$ is said to be $(n\text{-})$associative if

$$F(x_1, \ldots, x_{i-1}, F(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{2n-1}) = F(x_1, \ldots, x_i, F(x_{i+1}, \ldots, x_{i+n}), x_{i+n+1}, \ldots, x_{2n-1})$$

for all $x_1, \ldots, x_{2n-1} \in X$ and all $i \in \{1, \ldots, n-1\}$. 
$n$-ary semigroups

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Natural generalization: For $n = 2$ we get

$$F(F(x, y), z) = F(x, F(y, z))$$

holds for every $x, y, z \in X$. 
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Other important definitions

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An operation $F: X^n \rightarrow X$ is said to be

- reflexive (or idempotent) if $F(x, \ldots, x) = x$ for all $x \in X$;
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An operation $F: X^n \to X$ is said to be

- **reflexive** (or idempotent) if $F(x, \ldots, x) = x$ for all $x \in X$;
- **conservative** (or quasitrivial, selective) if $F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$ for all $x_1, \ldots, x_n \in X$;
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- **symmetric** if $F(x_1, \ldots, x_n)$ is invariant under any permutation of $x_1, \ldots, x_n$;

Assuming that $(X, \leq)$ is a chain, an operation $F: X^n \to X$ is said to be **nondecreasing** (w.r.t. $\leq$) if $F(x_1, \ldots, x_n) \leq F(x'_1, \ldots, x'_n)$ whenever $x_i \leq x'_i$ for all $i \in \{1, \ldots, n\}$.

We also introduce the notation $\overline{x, \ldots, x}_{n} = n \cdot x$. 

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Neutral element

**Definition**

Let $F: X^n \to X$ be an operation.

- An element $e \in X$ is said to be a **neutral element** of $F$ if

$$F((i - 1) \cdot e, x, (n - i) \cdot e) = x$$

for all $x \in X$ and all $i \in \{1, \ldots, n\}$. 

Example

$F(x_1, x_2, x_3) \equiv x_1 + x_2 + x_3 \pmod{2}$ on $X = \mathbb{Z}_2$. 

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  $F(x) = F(y)$.

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A neutral element is unique if $n = 2$ and not necessarily unique if $n \geq 3$.

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**Example**

$$F(x_1, x_2, x_3) \equiv x_1 + x_2 + x_3 \pmod{2} \text{ on } X = \mathbb{Z}_2.$$
Connectivity and neutral element

Example (More generally)

\[ F(x_1, \ldots, x_n) \equiv x_1 + \cdots + x_n \pmod{(n - 1)} \text{ on } X = \mathbb{Z}_{n-1} \ (n \geq 3). \]
Connectivity and neutral element

Example (More generally)

\[ F(x_1, \ldots, x_n) \equiv x_1 + \cdots + x_n \pmod{(n - 1)} \] on \( X = \mathbb{Z}_{n-1} \) \((n \geq 3)\).

Proposition

Let \( X \) be a chain. If \( F : X^n \to X \) is a nondecreasing operation, then \( F \) has at most one neutral element.
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Let \( F : X^n \to X \) be a reflexive operation. If \( \underline{x} = (x_1, \ldots, x_n) \in X^n \) is isolated for \( F \), then necessarily \( x_1 = \cdots = x_n \).
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Corollary
Any conservative operation \( F : X^n \to X \) has at most one isolated point.
Proposition

Let $F: X^n \to X$ be a conservative operation and let $e \in X$. If $(n \cdot e)$ is isolated for $F$, then $e$ is a neutral element.

The converse holds if and only if $n = 2$.

Counter example

Let $X = \{a, b, e\}$ and let $F: X^3 \to X$ be defined as

$$F(x, y, z) = \begin{cases} a, & \text{if there are more } a's \text{ than } b's \text{ among } x, y, z, \\ b, & \text{if there are more } b's \text{ than } a's \text{ among } x, y, z, \\ e, & \text{otherwise} \end{cases}$$

The operation $F$ is conservative and has $e$ as the neutral element. However, we have $F(e, e, e) = F(a, b, e)$ and hence the point $(e, e, e)$ is not isolated for $F$. 

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Why are neutral elements so important?

Definition

Let \( F: X^n \to X \) and \( H: X^2 \to X \) be associative operations. \( F \) is said to be derived from (or reducible to) \( H \) if \( F(x_1, \ldots, x_n) = x_1 \circ \cdots \circ x_n \) for all \( x_1, \ldots, x_n \in X \), where \( x \circ y = H(x, y) \).
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**Theorem (Dudek-Mukhin)**

Let \( X \) be a nonempty set. A function \( F : X^n \to X \) can be derived from an associative function \( H : X^2 \to X \) if and only if \( F \) has a neutral element or there can be adjoin a neutral element to \( X \) for \( F \).
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Corollary

If $F: X^n \to X$ is associative and has a neutral element $e \in X$, then $F$ is derived from the associative operation $H: X^2 \to X$ defined by $H(x, y) = F(x, (n - 2) \cdot e, y)$. 

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The case when $F : X^n \rightarrow X$ is an associative, monotone, reflexive function that has a neutral element was well understood. In this presentation we extend the investigation:

**Proposition**

Let $X$ be a chain and $F : X^n \rightarrow X$ be an associative, reflexive, nondecreasing function that has a neutral element. Then $F$ is conservative.
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Proposition (Martin-Mayor-Torrens, Couceiro-Devillet-Marichal)

Let $X$ be a chain. If $G: X^2 \rightarrow X$ is conservative, symmetric, and nondecreasing, then it is associative.
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Theorem (Main theorem)

Let $X$ be a chain and let $F: X^n \to X$ ($n \geq 3$) be a conservative, symmetric, and nondecreasing function. The following assertions are equivalent.

(i) $F$ is associative (i.e.: $(X, F)$ is a $n$-ary semigroup).

(ii) $F((n-1) \cdot x, y) = F(x, (n-1) \cdot y)$ for all $x, y \in X$.

(iii) There exists a conservative and nondecreasing operation $G: X^2 \to X$ such that $F(x_1, \ldots, x_n) = G(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i)$, $x_1, \ldots, x_n \in X$.

Moreover, the operation $G$ is unique, symmetric, and associative in assertion (iii).
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(iii) There exists a conservative and nondecreasing operation $G: X^2 \to X$ such that

$$F(x_1, \ldots, x_n) = G(\land_{i=1}^n x_i, \lor_{i=1}^n x_i), \quad x_1, \ldots, x_n \in X. \quad (1)$$
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Moreover, the operation $G$ is unique, symmetric, and associative in assertion (iii).
Consequences

Corollary

Let $X$ be a chain. If $F: X^n \to X$ is of the form (1), where $G: X^2 \to X$ is conservative, symmetric and nondecreasing, then $F$ is associative and derived from $G$. 

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**Corollary**

Let $X$ be a chain. If $F : X^n \to X$ is a conservative, symmetric, nondecreasing and associative, then $F$ has a neutral element or we can adjoin one.
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Proposition

Let $X$ be a chain and let $e \in X$. Assume that $F: X^n \to X$ is of the form (1), where $G: X^2 \to X$ is conservative, nondecreasing and symmetric. Then the following assertions are equivalent.

(i) $e$ is a neutral element of $F$.
(ii) $e$ is a neutral element of $G$.
(iii) The point $(e, e)$ is isolated for $G$.
(iv) The point $(n \cdot e)$ is isolated for $F$.
Back to the neutral element

Proposition

Let $X$ be a chain and let $e \in X$. Assume that $F : X^n \to X$ is of the form (1), where $G : X^2 \to X$ is conservative, nondecreasing and symmetric. Then the following assertions are equivalent.

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Corollary

Let $X$ and $F$ as above. Then $F$ has a neutral element iff there exists an isolated point for $F$.
The single-peaked ordering

Proposition (Ackerman)

Let $X$ be a set and $H : X^2 \to X$ be an associative, conservative, symmetric function, then there exists a linear ordering $\leq$ on $X$ such that $F$ is the maximum operation on $(X, \leq)$.

Corollary

An operation $F : X^n \to X$ is conservative, symmetric, associative, and derived from a conservative and associative operation $H : X^2 \to X$ iff there exists a linear ordering $\leq$ on $X$ such that $F$ is the maximum operation on $(X, \leq)$, i.e.,

$$F(x_1, \ldots, x_n) = x_1 \lor \cdots \lor x_n, \quad x_1, \ldots, x_n \in X.$$
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(2)
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Definition

In this case if $(X, \leq)$ is a chain, then we say that new ordering $\leq$ is single-peaked w.r.t. $\leq$.
Example

Consider the real operation $F: [0, 1]^2 \rightarrow [0, 1]$ defined as

$$F(x, y) = \begin{cases} x \lor y, & \text{if } x + y \geq 1, \\ x \land y, & \text{otherwise} \end{cases}$$  \hspace{1cm} (3)
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Denoting the single-peaked linear ordering on $[0, 1]$ by $\leq$, then

$$x \leq y \iff (y \leq x < 1 - y \text{ or } 1 - y \leq x \leq y), \quad x, y \in [0, 1].$$
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So for every $x \in [0, 1]$, there is no $y \in [0, 1]$ such that $x < y < 1 - x$ or $1 - x < y < x$. From this observation one can show that the chain $([0, 1], \leq)$ cannot be embedded into the reals $(\mathbb{R}, \leq)$. 
Thank you for your kind attention.
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