FACULTY OF LAW, ECONOMICS AND FINANCE

CREA Discussion Paper 2016-18

Economics

Centre for Research in Economics and Management
University of Luxembourg

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November, 2016

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Factor Income Distribution and Endogenous Economic Growth - When Piketty meets Romer -

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Abstract: We scrutinize Thomas Piketty's (2014) theory concerning the relationship between an economy's long-run growth rate, its capital-income ratio, and its factor income distribution put forth in his recent book *Capital in the Twenty-First Century*. We find that a smaller long-run growth rate may be associated with a smaller capital-income ratio. Hence, the key implication of Piketty's *Second Fundamental Law of Capitalism* does not hold. In line with Piketty's theory a smaller long-run growth rate may go together with a greater capital share. However, the mechanics behind this result are the opposite of what Piketty suggests. Our findings obtain in variants of Romer's (1990) seminal model of endogenous technological change. Here, both the economy's savings rate and its growth rate are *endogenous* variables whereas in Piketty's theory they are both *exogenous* parameters. Including demographic growth in the spirit of Jones (1995) shows that a smaller growth rate of the economy may imply a lower capital share contradicting a central claim in Piketty's book.

Keywords: Endogenous Technological Change, Capital Accumulation, Aggregate Factor Income Distribution

JEL-Classification: E10, E21, E25, O33, O41

This Version: November 17, 2016

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For valuable comments we would like to thank Christos Koulovatianos, Gautam Tripathi and audiences at Luxembourg, Mainz, the annual congresses 2016 of the EEA and of the German Economic Association. Both authors gratefully acknowledge financial support from the University of Luxembourg under the program "Agecon C - Population Aging: An Exploration of its Effect on Economic Performance and Culture."

1 Introduction

What is the relationship between economic inequality and economic growth? Since the 19th century - if not earlier - up to the times of *Capital in the Twenty-First Century* by Thomas Piketty (2014) this question has been at the heart of many policy debates. Does economic growth cause more inequality or vice versa? Who benefits, who suffers from economic growth, capitalists or workers?

We argue that a plausible answer to these questions has to acknowledge that both, economic inequality and economic growth, are simultaneously determined. Hence, a satisfactory understanding of the determinants of economic inequality and economic growth requires an analytical framework that treats both phenomena as endogenous. This is the central methodological perspective of the present paper.

From this standpoint we scrutinize Piketty (2014)'s theory concerning the relationship between an economy's long-run growth rate, its capital-income ratio, and its factor income distribution. We accomplish this in variants of Paul Romer (1990)'s seminal model of endogenous technological change (with demographic growth as suggested by Jones (1995) and without it). Here, long-run growth is driven by endogenous R&D efforts of profit-maximizing research firms, and the economy's factor income distribution is endogenous, too. Hence, changes in the economic environment are bound to affect the distribution of factor incomes and economic growth simultaneously. We establish the qualitative impact of these changes on the long-run growth rate, the capital-income ratio, and the factor income distribution. More importantly, we contrast these findings with the predictions of Piketty's theory. This is why and where Piketty meets Romer.

We show that several key implications of Piketty's two fundamental laws of capitalism are violated. This discrepancy arises since Piketty treats the economy's growth rate and its savings rate as exogenous parameters whereas in our analysis both are endogenous variables. As a consequence of this change in perspective, there is no longer a direct causal effect of the steady-state growth rate of an economy on the steady-state capital-income ratio. Rather, "deep" parameters that describe the economy's technology, preferences, policy, demographics, or market structure cause both variables. This leads to the conclusion that changes in these parameters induce adjustments that are often inconsistent with Piketty's fundamental laws. To provide more intuition and two fixed points for the discussion that follows it proves useful to quickly summarize Piketty's theory and the predictions he derives from it.

Piketty's Theory and Predictions. Piketty (2014) asserts two so-called fundamental laws of capitalism that are used to explain and predict long-run trends in the capital-income ratio, β, and in the capital share, α. The *first fundamental law of capitalism*, henceforth "first law," is the definition of the capital share of national income, $\alpha = r \times K/Y$ (Piketty (2014), p. 52 ff.). Here, r is "the average annual rate of return on capital, including profits, dividends, interest, rents, and other income from capital, expressed as a percentage of its total value" (ibidem, p. 25), K is "the total market value of everything owned by the residents and the government of a given country at a given point in time, provided that it can be traded on some market" (ibidem, p. 48), and Y is national income, i. e., "the sum of all income available to the residents of a given country in a given year, regardless of the legal classification of that income" (ibidem, p. 43).

Let us use a "*" to denote steady-state values. Then, the statement of Piketty's *second fundamental law of capitalism*, henceforth "second law," is $\beta^* = s/g$ where s is the exogenous savings rate defined, following Solow (1956), as $s \equiv \dot{K}/Y$ and g is the exogenous growth rate of the economy reflecting demographic and technical change (ibidem, p. 166 ff.). The second law is a variant of the Harrod-Domar-Solow condition for the steady state of an economy with capital accumulation. It should be seen as a relationship holding in the long run when the capital stock grows at the same rate as the economy, i. e., $\dot{K}/K = g$. Then, it follows from the definition of the savings rate as

$$s \equiv \frac{\dot{K}}{Y} = \frac{\dot{K}}{K} \times \frac{K}{Y} = g \times \frac{K}{Y} \quad \Leftrightarrow \quad \beta^* = \frac{s}{g}.$$

Piketty uses the second law to assert that a decline in the growth rate, g, explains a higher capital-income ratio (see, e. g., ibidem, p. 175, p. 183, or, with a special emphasis on population growth, p. 166). Taking the second law to extremes he argues that a society with zero population and productivity growth will see its capital-income ratio rise indefinitely (see, e. g., Piketty and Saez (2014), p. 480). We summarize these assertions as Prediction 1.

Prediction 1 *(Predictions for the Steady-State Capital-Income Ratio)*

1. *The smaller the economy's growth rate the greater is the capital-income ratio, i. e.,*

$$\frac{\partial \beta^*}{\partial g} < 0.$$

2. *As $g \to 0$ the capital-income ratio becomes unbounded, i. e.,*

$$\lim_{g \to 0} \beta^* = \infty.$$

The second law gives rise to a steady-state rate of return on capital $r^* = r(\beta^*)$ with

$r'(\beta^*) < 0$.[1] Accordingly, the steady-state capital share is $\alpha^* = r(\beta^*) \times \beta^*$. This expression links the economy's growth rate to its capital share, i. e.,

$$\frac{d\alpha^*}{dg} = \frac{\partial \alpha^*}{\partial \beta} \times \frac{\partial \beta^*}{\partial g} = \left[\underbrace{r'(\beta^*) \times \beta^* + r(\beta^*)}_{(+)} \right] \times \underbrace{\frac{\partial \beta^*}{\partial g}}_{(-)} .$$

Piketty argues that a hike in β is unlikely to induce a strong decline in the rate of return on capital since the elasticity of substitution between capital and labor is greater than unity. Hence, the term in brackets is positive since the "the volume effect will outweigh the price effect" (Piketty (2014), p. 221). Accordingly, the capital share is predicted to increase if g falls (ibidem, p. 216 or p. 220). We merge these assertions into Prediction 2.

Prediction 2 *(Predictions for the Steady-State Capital Share)*

1. *The smaller the economy's growth rate the greater is the capital share in national income, i. e.,*

$$\frac{d\alpha^*}{dg} < 0.$$

2. *The latter holds since "the volume effect will outweigh the price effect."*

Our Contribution. Are Prediction 1 and Prediction 2 justified or justifiable? The main results of our analysis show that both predictions are problematic.

Main Results. Let us develop the flavor for why these predictions are questionable by way of a simple heuristic example. In contradiction to Claim 1 of Prediction 1, we establish in the main part of the paper that a greater instantaneous discount rate, ρ, reduces the economy's steady-state growth rate *and* its capital-income ratio (see Proposition 3 in conjunction with Proposition 7).[2] The intuition comes in two steps.

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[1]This follows from basic neoclassical growth theory. Let national income be $Y = F(K, L) - \delta K$ where F is a neoclassical production function, L is labor, and δK is capital depreciation. In per worker terms, we may write $y = f(k) - \delta k$, where $y = Y/L$, $k = K/L$, and $f(k) \equiv F(k, 1)$ with $f'(k) > 0 > f''(k)$. Then, $\beta = k/y$ and total differentiation of the latter delivers $dk/d\beta > 0$. Due to competitive marginal cost pricing the rate of return on capital satisfies $r = f'(k(\beta)) - \delta$. Hence, $r'(\beta^*) \equiv dr/d\beta = f''(k(\beta)) \times (dk/d\beta) < 0$.

[2]The literature traces differences across countries in the discount rate back to various sources. They include differences in religious believe systems (see, e. g., Weber (1930)), in the level of income per capita (see, e. g., Das (2003)), or in the geographical variation of the natural return to agricultural investments (Galor and Ömer Özak (2016)).

First, we show that in steady state the capital-income ratio can be expressed as (see Proposition 8)

$$\beta^* = \frac{s^*(\rho)}{g^*(\rho)};$$

here, $s^*(\rho)$ is the savings rate as defined by Piketty, computed for the steady state of Romer's model, and $g^*(\rho)$ is Romer's endogenous growth rate.[3] Hence, the second law holds but now both rates are endogenous and depend on ρ. This implies, in particular, that there is no direct causal effect of the steady-state growth rate of the economy on the steady-state capital-income ratio.

Second, we establish that increasing ρ lowers the steady-state capital-income ratio, i. e.,

$$\frac{d\beta^*}{d\rho} = \beta^* \left[\underbrace{\frac{(s^*)'(\rho)}{s^*(\rho)}}_{(-)} - \underbrace{\frac{(g^*)'(\rho)}{g^*(\rho)}}_{(-)} \right] < 0, \quad \text{since} \quad \left| \frac{(s^*)'(\rho)}{s^*(\rho)} \right| > \left| \frac{(g^*)'(\rho)}{g^*(\rho)} \right|.$$

This finding comes about since $(s^*)'(\rho) < 0$, $(g^*)'(\rho) < 0$, and the proportionate decline in $s^*(\rho)$ dominates the proportionate decline in $g^*(\rho)$. Hence, a greater ρ means lower growth and a smaller capital-income ratio.

We also show that contrary to Claim 2 of Prediction 1 the long-run capital-income ratio remains finite for an economy without growth. Intuitively, such an economy will close down its research sector and behaves very much like the textbook Ramsey model. In particular, the stationary steady state exhibits a finite capital-income ratio.[4]

As to Prediction 2, two findings are remarkable. First, Claim 1 of Prediction 2 may hold even though Claim 1 of Prediction 1 does not. For instance, a lower growth rate may lead to a higher capital share in the long run. However, the way this result comes about must contradict Claim 2 of Prediction 2. To see this, consider again the case of an increase in ρ. As argued above, this leads to a decline in the growth rate and to a lower capital-income ratio. Moreover, it also increases the rate of return on capital, i. e., $r = r(\rho)$ with

[3]Proposition 3 and Proposition 8 show respectively that the savings rate and the growth rate will not only depend on ρ but on a whole vector of parameters that characterize the economy. For simplicity, we suppress these other parameters in the notation here.

[4]More formally, one can show that there is $\rho_c > 0$ such that $g^*(\rho) > 0$ for $\rho < \rho_c$ and $\lim_{\rho \to \rho_c} g^*(\rho) = 0$. Since $s(\rho) = \dot{K}/Y$, it also holds that $\lim_{\rho \to \rho_c} s^*(\rho) = 0$ since $\dot{K} \to 0$ while Y remains strictly positive. Then, an application of l'Hôpital's rule to the steady-state capital-output ratio reveals that

$$\lim_{\rho \to \rho_c} \beta^* = \frac{\lim_{\rho \to \rho_c} (s^*(\rho))'}{\lim_{\rho \to \rho_c} (g^*(\rho))'} = \bar{\beta}^* < \infty.$$

$r'(\rho) > 0$.[5] Then, the first law says that

$$\alpha^* = r^*(\rho) \times \beta^*(\rho).$$

Moreover, the proportionate increase in $r^*(\rho)$ dominates the proportionate decline in $\beta^*(\rho)$ (see Proposition 12), i. e.,

$$\frac{d\alpha^*}{d\rho} = \alpha^* \left[\underbrace{\frac{(r^*)'(\rho)}{r^*(\rho)}}_{(+)} + \underbrace{\frac{(\beta^*)'(\rho)}{\beta^*(\rho)}}_{(-)} \right] > 0 \quad \text{since} \quad \frac{(r^*)'(\rho)}{r^*(\rho)} > \left| \frac{(\beta^*)'(\rho)}{\beta^*(\rho)} \right|.$$

In other words, the price effect outweighs the volume effect and not vice versa.

Second, the validity of Claim 1 of Prediction 2 depends crucially on how the economy's long-run growth rate is determined. In Romer's economy without demographic growth we find that this claim holds for almost all model parameters (see Proposition 3 in conjunction with Proposition 12). In Romer's economy with demographic growth Claim 1 fails to hold. Not even the effect of a declining demographic growth rate yields unequivocal predictions consistent with Prediction 2 and Prediction 1 (see Proposition 13 in conjunction with Proposition 15 and Proposition 14).

Additional Results. A second set of results transcends Piketty's theory and predictions and highlights the "deep" parameters that affect the capital-income ratio and the capital share in the long run in a consistent way across all model variants that we consider. Somewhat surprisingly, we find only three parameters that are shared by all variants and have consistent predictive power: the tax rate on capital earnings, the instantaneous discount rate, and the depreciation rate of physical capital (see, Table 1 and Table 2).

These tables also reveal that the predictions of the remaining parameters are specific to the particular model variant. Consider the degree of product differentiation of intermediates. If this degree falls (and μ increases) then price competition gets fiercer, and intermediate-good prices will be lower. This implies a greater (smaller) capital-income ratio and a greater (smaller) capital share in the model without (with) demographic growth.

Finally, our analysis suggests that the effect of some parameters depends on whether the economy is stationary or growing. For instance, in a stationary economy a lower degree of product differentiation reduces the capital-income ratio and the capital share

[5] The intuition comes from the Euler condition. In steady state the rate of return on capital must adjust so that the infinitely-lived household embarks on a path along which consumption grows at rate $g(\rho)$. Hence, in the simplest case with log-utility the steady-state Euler condition reads $g^*(\rho) = r - \rho$ and $r'(\rho) \equiv dr/d\rho = 1 + g'(\rho) > 0$ as $|g'(\rho)| < 1$.

whereas for the growing economy without demographic growth the opposite is predicted (compare Proposition 5 and Proposition 10 to Table 1).

Related Literature. This paper relates and contributes to the broad and growing literature on the link between the factor income distribution of an economy and its growth rate. First, it contributes to the discussion surrounding the validity of Piketty's two fundamental laws of capitalism. Acemoglu and Robinson (2014), Blume and Durlauf (2015), or Ray (2015) hint at the "endogeneity problem" of Piketty's theory. However, these authors do not provide a formal analysis which is a central issue of the present paper. Homburg (2015) and Krusell and Smith (2015) question the plausibility of the second law from the perspective of Piketty's savings hypothesis. For instance, Krusell and Smith (2015) emphasize that Claim 2 of Prediction 1 requires the economy's gross savings rate to approach unity. However, contrary to the present paper these authors follow Piketty and treat the growth rate of the economy as an exogenous parameter.

Second, our analysis contributes to the recent literature documenting and explaining contemporaneous trends in the evolution of the factor income distribution (see, e. g., Elsby, Hobijn, and Sahin (2013), Bridgman (2014), Karabarbounis and Neiman (2014a), Karabarbounis and Neiman (2014b), Growiec, McAdam, and Muck (2015), or Rognlie (2015)). We provide a broad set of testable predictions about the long-run relationship between "deep" parameters of an economy, its speed of economic growth and its factor income distribution that are relevant for a comprehensive understanding of the real facts. Our analysis supports the view that the distinction between gross and net shares is quite relevant (Bridgman (2014)). Allowing for capital depreciation brings our analytical findings closer to reality but sometimes at a cost of some cumbersome extra algebra. More importantly, we show via simulation exercises that some qualitative results change their sign once we switch from a world with realistic levels of capital depreciation to a world void of depreciation.

Third, the present paper extends and complements previous contributions that study the income distribution-growth nexus in the AK models of Frankel (1962), Romer (1986), or Rebelo (1991). Here, parameters characterizing the economic environment that are associated with slower growth are also associated with a smaller share of capital in aggregate output (see, e. g., Bertola (1993), or Bertola, Foellmi, and Zweimüller (2006), pp. 81-87). The same qualitative result obtains in a first-generation endogenous growth model without capital accumulation sketched in Bertola, Foellmi, and Zweimüller (2006), Chapter 10. However, to the best of our knowledge the present paper is the first that conducts a comprehensive analysis of the relationship between the factor income distribution, the capital-income ratio, and economic growth in an R&D-based endogenous growth model with capital accumulation and provides the detailed link to Piketty's theory. Moreover, unlike the predictions derived from the AK model or the variety-expansion model without capital accumulation our analysis suggests that changes in some parameters of the economy give rise to a negative correlation between the capital share and the economy's growth rate, a finding consistent with Claim 1 of Prediction 2.

Structure of the Paper. The remainder of this paper is organized as follows. Section 2 presents the details of the model. Section 2.1 describes the economic sectors, and Section 2.2 has the definition of the dynamic general equilibrium. Our main results are established in Section 3. Here, we first characterize the equilibrium distribution of factor incomes (Section 3.1). Then, we switch to the analysis of the long-run. Section 3.2 establishes the steady-state equilibrium and derives the determinants of the steady-state growth rate. The following two sections provide the link to Prediction 1 and Prediction 2. Section 3.3 studies the determinants of the capital-income ratio, and Section 3.4 has the analysis of the capital share. Section 4 adds demographic growth to the picture. The focus of Section 4.1 is on the capital-income ratio and Prediction 1, whereas the focus of Section 4.2 is on the capital share and Prediction 2. Section 5 concludes. If not indicated otherwise, proofs are relegated to the Appendix, Section 6.

2 The Model

We study the factor income distribution in a variant of Romer's model of endogenous technological change extended to allow for a variable degree of monopoly power of intermediate-good firms, depreciation of the physical capital stock, and for an active government. As will become clear below, these features will have an effect on the factor income distribution.

Time is continuous, i. e., $t \in [0, \infty)$. At all t, the economy has a unique final good that may be consumed or accumulated as physical capital. Besides the market for the final good, there are markets for bonds, stocks, physical capital, labor, and for intermediate goods. Moreover, there is a government that levies a tax on capital incomes and pays subsidies to innovators. In all periods its budget is balanced. This may necessitate lump-sum taxes or lump-sum transfers to the household sector. All prices are expressed in units of the contemporaneous final good.

We denote $g_x(t) = \dot{x}(t)/x(t)$ the instantaneous growth rate of some variable x at t. For notational simplicity we shall suppress the time argument whenever this is not confusing. Observe also that our notation below slightly differs from Piketty's. In particular, we follow standard textbook notation and denote by r the real interest rate on bonds, K is the stock of physical capital, and Y is the total output of the final good.

2.1 Economic Sectors

2.1.1 The Final-Good Sector

The final good, Y, is produced by a single competitive representative firm. Labor, $L_Y > 0$, and a continuum of $M > 0$ different intermediate goods serve as inputs in the production function

$$Y = L_Y^{1-\gamma} \left(\int_0^M x\left(j\right)^{\mu} dj \right)^{\frac{\gamma}{\mu}}, \quad 0 < \gamma \leq \mu < 1, \tag{2.1}$$

where $x\left(j\right), j \in \left[0, M\right]$, denotes the quantity of intermediate good j and M is the "number" of available intermediate goods at t. The parameter μ determines the elasticity of substitution, $1/(1-\mu) > 1$, between intermediates. The output elasticity of the intermediate-good aggregate is given by γ. If $\mu \geq \gamma$, then all first-order conditions derived below are also sufficient for a profit maximum. Let w denote the real wage per (homogeneous) worker paid in the economy and $p\left(j\right)$ the price of intermediate good j. Then, the demand for labor and the demands for each intermediate good are the solution to

$$\max_{L_Y, \{x(j)\}_{j=0}^{j=M}} Y - wL_Y - \int_0^M p\left(j\right) x\left(j\right) dj \tag{2.2}$$

and given by

$$L_Y = \left(1-\gamma\right) \frac{Y}{w}, \tag{2.3}$$

$$\gamma x\left(j\right)^{\mu-1} L_Y^{1-\gamma} \left(\int_0^M x\left(j\right)^{\mu} dj \right)^{\frac{\gamma}{\mu}-1} = p(j) \quad \text{for all } j \in [0, M]. \tag{2.4}$$

2.1.2 The Intermediate-Good Sector

The intermediate-good sector comprises M monopolists with a perpetual patent for a single variety $j \in [0, M]$. The production function of all monopolists is the same and linear with one unit of capital services producing one unit of an intermediate, i. e.,

$$x\left(j\right) = k\left(j\right), \tag{2.5}$$

where $k\left(j\right)$ is the amount of capital employed by monopolist j. Each monopolist's revenue is

$$\pi\left(j\right) = p\left(j\right) x\left(j\right) - Rk(j) \tag{2.6}$$

where $x\left(j\right)$ and $k\left(j\right)$ are linked via (2.5), and R is the rental rate of capital. Revenue maximization delivers

$$p\left(j\right) = p = \frac{R}{\mu}. \tag{2.7}$$

Hence, μ measures the monopoly power of intermediate-good firms. Then, (2.4) in conjunction with (2.1) gives

$$x(j) = x = \frac{\gamma\mu}{R}\frac{Y}{M}. \tag{2.8}$$

Moreover, with (2.5) - (2.8) we obtain

$$\pi(j) = \pi = (1-\mu)\, px. \tag{2.9}$$

2.1.3 The Research Sector

Previous to its marketing an intermediate good must be invented through research. To capture this consider a single competitive research firm with access to a technology for the creation of new intermediate-good varieties given by

$$\dot{M} = \frac{L_M}{a} M. \tag{2.10}$$

Here, $L_M \geq 0$ measures the workforce employed in the research sector, and $a > 0$ determines its productivity. The research output depends also on the current stock of technological knowledge represented by M to which access is for free.

Following its invention the blueprint of the new variety in conjunction with a perpetual patent is sold at the price v. Then, free entry into the research sector implies a zero-profit condition, so that the price received per invention cannot be greater than the cost of creating it, i. e.,

$$v \leq \frac{a}{M}(1-\sigma)\, w \quad \text{with "=", if } \dot{M} > 0, \tag{2.11}$$

where $\sigma \in (0,1)$ is the subsidy rate of the labor cost associated with each innovation.

Since new patents are auctioned off the highest bidder pays a price per variety equal to the net present value of all future after-tax revenues. Hence, the value of a patent, v, in units of the current final good is equal to the present value of all future profits. According to (2.9), these profits are the same for all varieties. Let $\tau \in (0,1)$ denote the tax rate on capital earnings. Then, the market capitalization of each variety j at t is

$$v(t) = (1-\tau)\int_t^\infty \pi(t) e^{-(s-t)(1-\tau)\bar{r}(s)} ds, \tag{2.12}$$

where $\bar{r}(s)$ is the average interest rate over the interval $[t,s]$, i. e.,

$$\bar{r}(s) = \frac{1}{s-t}\int_t^s r(\nu) d\nu, \tag{2.13}$$

and $r(t)$ is the real interest rate on bonds at t.

Since bonds, capital, and stocks are perfect substitutes as stores of value, for all t the no-arbitrage condition of the capital market must equate the after-tax returns associated with each of these assets, i. e., $(1-\tau)r = (1-\tau)(R-\delta) = (1-\tau)\pi/v + g_v$. Here, δ is the instantaneous depreciation rate of physical capital, $R-\delta$ is the pre-tax rate of return on holding one unit of final-good output in capital, and $\pi/v + g_v$ is pre-tax rate of return on holding one unit of final-good output in shares. As stated, the capital income tax is levied on the rental rate of capital net of depreciation, on paid dividends and not on the accounting profit, $\dot{v}$. Therefore, it proves useful to introduce $\delta_v \equiv -g_v/(1-\tau)$ as the depreciation rate of share prices so that the no-arbitrage condition can be written as

$$r = R - \delta = \frac{\pi}{v} - \delta_v. \tag{2.14}$$

2.1.4 The Household Sector

There is a single representative household comprising $L(t) = L$ members. Each household member supplies one unit of homogeneous labor inelastically to the labor market. Besides its labor endowment, the household owns the capital stock, K, and all firms in the economy. The household values streams of consumption per household member, $c(t) = C(t)/L$ according to

$$U = L\int_0^\infty \frac{c(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt, \quad \rho > 0, \quad \theta > 0. \tag{2.15}$$

Here, ρ is the instantaneous discount rate, and θ is the inverse of the inter-temporal elasticity of substitution in consumption.

The flow budget constraint may be expressed as

$$Lc + \dot{K} + v\dot{M} = (1-\tau)(R-\delta)K + wL_Y + wL_M + (1-\tau)\pi M + T. \tag{2.16}$$

The right-hand side states the household's income flows net of capital depreciation. They comprise the after-tax income from physical capital, labor income earned in manufacturing and research, the after-tax dividend income from the ownership of M intermediate-good monopolists, and the lump-sum payment necessary to balance the government's budget, T. This income is spent on consumption or savings. The latter may involve net investments in the accumulation of the capital stock, $\dot{K}$, and/or the purchase of newly emitted shares, $v\dot{M}$.

Let $A \equiv K + vM$ denote the household's total assets. Then, standard arguments deliver the optimal solution to the household's optimization problem. The latter satisfies the Euler condition

$$g_c = \frac{\dot{c}}{c} = \frac{(1-\tau)r - \rho}{\theta} \tag{2.17}$$

and the transversality condition

$$\lim_{t\to\infty} A(t)\exp\left(-(1-\tau)\int_0^t r(v)dv\right) = 0 \tag{2.18}$$

as necessary and sufficient conditions.

2.2 Dynamic General Equilibrium - Definition

Given $M(0) > 0$, $K(0) > 0$, $L > 0$, and a time-invariant policy (τ, σ), a dynamic general equilibrium comprises an allocation,

$$\{Y(t), L_Y(t), x(j,t), k(j,t), M(t), K(t), L_M(t), c(t)\}_{t=0}^{\infty},$$

a price system,

$$\{w(t), r(t), p(j,t), R(t), \pi(j,t), v(t)\}_{t=0}^{\infty},$$

where $j \in [0, M(t)]$, and a path of lump-sum transfers $\{T(t)\}_{t=0}^{\infty}$ such that i) producers of the final good choose labor and the quantity of all available intermediates taking prices as given, ii) intermediate-good monopolists maximize profits taking their respective demand curves and the rental rate of capital as given, iii) research firms enter the market, take prices for new blueprints, the wage for researchers, and the subsidy rate as given, and earn zero profits, iv) firms contemplating entry into the business of producing a novel intermediate take the price of blueprints as given, finance the purchase of the blueprint through the issue of new shares, pay the promised dividends to the household sector, and earn zero inter-temporal profits, v) the household sector makes consumption and savings decisions taking prices, the tax on capital incomes and the lump-sum transfer as given, vi) all markets clear, vii) the government budget is balanced so that $T = \tau\left(\left(R-\delta\right)K + \pi M\right) - \sigma w L_M$.

3 Factor Income Distribution and Economic Growth - Piketty meets Romer

This section studies the long-run relationship between the factor income distribution and the economy's growth rate. To accomplish this it proves useful to highlight first some important and intuitive results for equilibrium factor incomes and factor shares. This is the purpose of Section 3.1. Then, we turn to the long-run and characterize the steady-state equilibrium in Section 3.2. Piketty meets Romer in Section 3.3 and 3.4. These sections contain central findings of this paper. Section 3.3 studies the steady-state capital-income ratio and relates it to Prediction 1. The focus of Section 3.4 is on the steady-state capital share and Prediction 2.

3.1 Equilibrium Factor Incomes and Factor Shares

At any t, the economy is endowed with three factors of production, labor (in two uses), $L = L_Y + L_M$, technological knowledge, M, and physical capital, K. Their respective factor incomes are as follows.

Proposition 1 *(Equilibrium Factor Incomes)*

Equilibrium factor incomes satisfy

$$wL_M = \max\left\{0, \frac{v\dot{M}}{1-\sigma}\right\}, \tag{3.1}$$

and

$$wL_Y + \pi M + RK = Y, \tag{3.2}$$

where

$$\frac{wL_Y}{Y} = 1-\gamma, \quad \frac{\pi M}{Y} = (1-\mu)\,\gamma, \quad \textit{and} \quad \frac{RK}{Y} = \mu\gamma. \tag{3.3}$$

Proposition 1 makes three important observations about the income flows in the economy. First, equation (3.1) says that the wage income of labor in research is equal to the value of the new shares that the household sector buys from new intermediate-good monopolists discounted by the fraction of the wage bill borne by research firms, $1-\sigma$. The intuition is the following. At all t the household sector buys $\dot{M}$ new shares, each at a price v. These shares are emitted by new intermediate-good monopolists that use the revenue raised from the sale of these shares to pay research firms in exchange for the blueprint. In addition, research firms receive the subsidy that covers the fraction σ of their total labor costs. In equilibrium research firms just break even. Hence, whenever $L_M > 0$ we have $v\dot{M} + \sigma wL_M = wL_M$. Trivially, if $L_M = 0$ then $wL_M = 0$ and (3.1) follows.

To grasp the remaining results of Proposition 1 observe that in equilibrium the market for the capital input clears so that $x = K/M$. Hence, aggregate output of the final good becomes

$$Y = L_Y^{1-\gamma} M^{\gamma(1/\mu-1)} K^{\gamma}. \tag{3.4}$$

Then, equation (3.2) has the second observation: the remuneration of the three factors of production that show up in (3.4) adds up to the total output of the final good. More interestingly, the price for the service of each "unit" of technological knowledge corresponds to the dividend that the household sector receives from the respective intermediate-good firm that uses this unit.

Finally, equation (3.3) informs us about the shares of income in final-good output that accrue to industrial labor, technological knowledge, and physical capital. Intuitively, the equilibrium remuneration of industrial labor is obtained from the first-order condition (2.3). Hence, the aggregate wage bill of the final-good sector is $wL_Y = (1-\gamma)\,Y$. As expected for a Cobb-Douglas production function, the fraction of total output that goes to L_Y workers is equal to the output elasticity of these workers.

Since final-good production exhibits constant returns to scale in all rival inputs it must be that

$$\int_0^M p\,(j)\,x\,(j)\,dj = pxM = \gamma Y. \tag{3.5}$$

Consequently, the remuneration of technological knowledge and physical capital sums up to γY. But what is the split? Invoking (2.9) and (3.5) one finds that the remuneration of the total stock of technological knowledge amounts to $\pi M = (1-\mu)\,\gamma Y$. Ceteris paribus, a greater μ means a greater elasticity of substitution between intermediates which reduces the mark-up charged by intermediate-good monopolists. Accordingly, the share of dividends in γY falls. Finally, with (3.5) we obtain the remuneration of physical capital as $RK = \gamma Y - \pi M = \mu\gamma Y$. Hence, μ determines the breakup of γY into income for technological knowledge and physical capital.[6]

Next, we turn to the equilibrium factor shares. Let GDP denote the economy's gross domestic product, i. e., its total value added. In equilibrium GDP satisfies

$$GDP = Y + v\dot{M}. \tag{3.6}$$

Intuitively, for any symmetric configuration of the production sector we have $GDP = (Y - Mpx) + Mpx + v\dot{M}$ where the expression in parenthesis is the value added in the final-good sector, the second and the third term show the value added in the intermediate-good sector and in the research sector. As a result, GDP is the sum of the total output of the final good and the value created by research firms.

By definition, net domestic product is $NDP \equiv GDP - \delta K$. Since the economy is closed NDP coincides with national income.[7]

[6] Observe that the equilibrium remuneration of L_Y, M, and K is such that labor earns its marginal product whereas technological knowledge and capital earn less than their respective marginal product. Indeed, with (3.4) we derive $\partial Y/\partial M = \gamma\,(1/\mu - 1)\,Y/M > \pi = (1-\mu)\,\gamma Y/M$ and $\partial Y/\partial K = \gamma Y/K > R = \mu\gamma Y/K$. Both inequalities hold since $\mu < 1$, i. e., because there is some degree of product differentiation that allows intermediate-good monopolists to charge a strictly positive mark-up.

[7] Indeed, from the right-hand side of the household budget constraint (2.16) and equilibrium transfers $T = \tau\,((R-\delta)K + \pi M) - \sigma w L_M$ one finds that total income net of capital depreciation may be expressed as

$$RK + wL_Y + \pi M + \frac{v\dot{M}}{1-\sigma} - \sigma wL - \delta K = Y + v\dot{M} - \delta K = GDP - \delta K = NDP,$$

where use is made of (3.1) and (3.2).

The capital share, α, and the labor share, $1-\alpha$, relate, respectively, the economy's income from asset holdings (net of capital depreciation) and its total wage bill (net of wage subsidies) to its NDP, i. e.,[8]

$$\alpha \equiv \frac{(R-\delta)K + \pi M}{NDP} \quad \text{and} \quad 1-\alpha \equiv \frac{wL - \sigma w L_M}{NDP}. \tag{3.7}$$

These definitions are the counterparts to the (pre-tax) factor shares that Piketty considers (see, e. g., Piketty (2014), pp. 200-203). To study the determinants of the factor shares we now turn to the long run.

3.2 Steady-State Equilibrium

A steady-state equilibrium is a path along which all variables of the model grow at constant, possibly different exponential rates. We denote steady-state values by a "*" and define

$$\zeta \equiv \frac{1-\sigma}{1-\tau} > 0, \quad \eta \equiv \frac{\gamma(1-\mu)}{\mu(1-\gamma)} \in (0,1], \quad \text{and} \quad \vartheta \equiv \frac{\zeta}{\eta\mu} > 0.$$

Roughly speaking, ζ states the effect of policy measures, η accounts for the fact that market power, μ, is independent of the technology parameter, γ, and ϑ captures the interaction between the two. Let g^* denote the steady-state growth rate of the economy.

Proposition 2 *(Steady-State Equilibrium)*

There exists a unique steady-state equilibrium if

$$\rho > (1-\theta)\, g^*. \tag{3.8}$$

The steady-state growth rate of technological knowledge is

$$g_M^* = \max\left\{0, \frac{\mu\left(\frac{L}{a} - \vartheta\rho\right)}{\mu + \zeta\left(\theta + \eta^{-1} - 1\right)}\right\}. \tag{3.9}$$

The steady-state growth rate of the economy is

$$g^* = g_Y^* = g_K^* = g_c^* = \eta g_M^*. \tag{3.10}$$

Moreover, it holds that

$$g_v^* = g_\pi^* = -\left(\eta^{-1} - 1\right) g^* \leq 0. \tag{3.11}$$

[8] If $\sigma > 0$ then wL_M exceeds the value added of research labor by the subsidy $\sigma w L_M$ (see, equation (3.1)). The definition of $1-\alpha$ takes this into account. Hence, its numerator corresponds to the value added by labor in both sectors.

Proposition 2 states key properties of the steady-state equilibrium. Condition (3.8) assures that $(1-\tau)r^* > g^*$, so that the household's transversality condition holds. The steady-state growth rate of technological knowledge of (3.9) may be zero which will allow us to compare factor shares of a stationary to those of a growing economy.

For a growing economy the steady state involves $g_M^* > g^*$ whenever $\mu > \gamma$ (and $\eta < 1$). To see why, recall that the equilibrium output of the final good is given by (3.4). In steady state L_Y is time-invariant, and the ratio of physical capital to final-good output must be constant, i. e., $g_K = g_Y$. Moreover, with $\mu > \gamma$ the output elasticity of M becomes strictly smaller than $1-\gamma$. This drives a wedge between g_M and g_K. Accordingly, M must grow faster than K and Y.[9]

Finally, equation (3.11) states that the steady state of a growing economy has $g_\pi^* = g_v^* < 0$ if $\mu > \gamma$. Intuitively, this reflects a declining turnover of existing intermediate-good firms as $g_x^* = g_K^* - g_M^* < 0$. Hence, in steady state dividends and share prices fall at a constant rate.

From (3.10) we may write $g^* = g(\boldsymbol{\omega})$ where g is a function that maps the vector of parameters upon which g^* depends, $\boldsymbol{\omega} = (L, a, \tau, \sigma, \rho, \theta, \gamma, \mu)$, into $\mathbb{R}_+$. For further reference the following proposition summarizes the derivative properties of $g(\cdot)$ for a growing economy.

Proposition 3 *(Comparative Statics of g^*)*

Suppose $L/a > \vartheta\rho$. Then, it holds that

$$\frac{\partial g^*}{\partial L} > 0, \quad \frac{\partial g^*}{\partial a} < 0, \quad \frac{\partial g^*}{\partial \tau} < 0, \quad \frac{\partial g^*}{\partial \sigma} > 0, \quad \frac{\partial g^*}{\partial \rho} < 0, \quad \frac{\partial g^*}{\partial \theta} < 0, \quad \frac{\partial g^*}{\partial \gamma} > 0.$$

Moreover, there is ε such that $0 < g^* < \varepsilon$ and

$$\frac{\partial g^*}{\partial \mu} < 0.$$

According to Proposition 3 the steady-state growth rate responds in an intuitive way to the considered parameter changes: an economy with more workers grows faster (the

[9]For a complementary way to see this write (3.4) as $Y = (L_Y M^\eta)^{1-\gamma} K^\gamma$ so that technological knowledge appears as labor-augmenting. To have both sides of this equation grow at the same rate while $g_K = g_Y$ it is necessary that $\eta g_M = g_K$.

scale effect), a lower labor productivity in the research sector slows down growth, taxing capital reduces the growth rate while subsidizing research increases it, more patience and a greater inter-temporal elasticity of substitution in consumption on the side of households spurs growth, and a larger output elasticity of intermediates leads to faster growth. Moreover, for sufficiently small growth rates smaller mark-ups in the intermediate-good sector reduce the long-run growth rate.

Finally, the steady-state real interest rate is given by the Euler equation of (2.17)

$$r^* = \frac{\theta g^* + \rho}{1 - \tau}. \tag{3.12}$$

For further reference, we may also write $r^* = r\left(\tau, \rho, \theta, g\left(\boldsymbol{\omega}\right)\right)$.

3.3 The Capital-Income Ratio in the Long Run

This section studies the long-run determinants of the capital-income ratio (or wealth-to-NDP ratio). We derive two main results, each of them contradicts Prediction 1. First, we show in Section 3.3.1 that a stationary economy has a finite capital-income ratio. Second, we establish in Section 3.3.2 that the long-run capital-income ratio may increase even though g^* increases. Throughout, we follow Piketty and let β denote the capital-income ratio defined as

$$\beta \equiv \frac{A}{NDP}. \tag{3.13}$$

3.3.1 The Stationary Economy

Consider the stationary steady state where $g^* = 0$, $L_Y = L$, $\dot{M} = 0$, and $GDP = Y$. Since the economy has two assets in positive net supply, claims on physical capital and shares in intermediate-good firms, the capital-income ratio is

$$\bar{\beta} = \frac{K + vM}{Y - \delta K}, \tag{3.14}$$

where the bar indicates that the economy is stationary. The following proposition gives $\bar{\beta}^*$.

Proposition 4 *(Capital-Income Ratio in a Stationary Steady State)*

Suppose that $L/a \leq \vartheta\rho$. Then, the steady-state capital-income ratio is

$$\bar{\beta}^* = \frac{\dfrac{\mu\gamma}{r^* + \delta} + \dfrac{(1-\mu)\gamma}{r^*}}{1 - \delta\dfrac{\mu\gamma}{r^* + \delta}} < \infty, \tag{3.15}$$

where r^* is given by the Euler equation (3.12).

Hence, unlike Piketty's assertion the capital-income ratio of a stationary economy is finite. The numerator of $\bar{\beta}^*$ features the present discounted value of a permanent income stream that accrues to physical capital, $R^*K^*/Y^* = \mu\gamma$, discounted at rate $r^* + \delta$ and the present discounted value of a permanent income stream that accrues to technological knowledge, $\pi^* M^*/Y^* = \mu(1-\gamma)$, discounted at rate r^*. The different discount rates reflect that capital depreciates whereas the stock market value, v^*, of all M^* intermediate-good firms does not. The denominator shows NDP^*/Y^*. The Euler condition delivers $r^* = \rho/(1-\tau)$ so that

$$\bar{\beta}^* = \left(\frac{\gamma\,(1-\tau)}{\rho}\right) \times \left(\frac{\rho + \delta(1-\tau)(1-\mu)}{\rho + \delta(1-\tau)(1-\gamma\mu)}\right). \tag{3.16}$$

To build intuition, two interesting special cases are worth mentioning. The first arises when $\delta \to 0$. In the limit, the permanent income stream of either asset is discounted at rate r^*. Moreover, void of physical capital depreciation $NDP = Y$. Hence, the capital-income ratio boils down to the present discounted value of the permanent income stream that accrues to both assets, γ, discounted at rate r^*. In other words, $\lim_{\delta \to 0} \bar{\beta}^* = \gamma(1-\tau)/\rho$. It is independent of μ since the distinction between income streams accruing to the owners of physical capital and of shares no longer matters. Hence, neglecting capital depreciation will not only alter the value of $\bar{\beta}^*$ but may also hide potentially important determinants.

The second special case arises for $\mu \to 1$. In the limit, all intermediate goods are perfect substitutes, $\pi = v = 0$, and (3.4) gives $Y = L^{1-\gamma}K^{\gamma}$, i. e., technological knowledge disappears as a factor of production. Intuitively, when all intermediate goods are identical then there are no gains from specialization and the way the capital stock is allocated to intermediates no longer matters. In fact, we are back in the standard neoclassical growth model with a Cobb-Douglas production function, and $\lim_{\mu \to 1} \bar{\beta}^* = \gamma(1-\tau)/\left(\rho + \delta(1-\tau)(1-\gamma)\right)$.

Overall, the steady-state capital-income ratio of the stationary economy reflects the interaction between technology, preferences, policy, and market structure. The following proposition shows how these features affect $\bar{\beta}^*$.

Proposition 5 *(Determinants of the Capital-Income Ratio in a Stationary Steady State)*

Suppose that $L/a < \vartheta\rho$. Then, it holds that

$$\frac{d\bar{\beta}^*}{d\gamma} > 0, \quad \frac{d\bar{\beta}^*}{d\rho} < 0, \quad \frac{d\bar{\beta}^*}{d\tau} < 0, \tag{3.17}$$

$$\frac{d\bar{\beta}^*}{d\delta} < 0, \quad \frac{d\bar{\beta}^*}{d\mu} < 0. \tag{3.18}$$

To grasp the intuition behind Proposition 5 note that the sign of the comparative statics for γ, ρ, and τ is determined by the respective effect of these parameters on the first term in (3.16).[10] As a consequence, $\bar{\beta}^*$ increases in the share of the remuneration of physical capital and technological knowledge in final-good output, γ. Moreover, $\bar{\beta}^*$ is greater the more patient households are (the smaller ρ) and the smaller the tax on asset returns. Both, a smaller ρ and a smaller τ call for a lower interest rate to make a flat consumption profile consistent with household optimization. As a consequence, all discount rates in (3.16) fall and shift $\bar{\beta}^*$ upwards.

A greater rate of capital depreciation reduces $\bar{\beta}^*$. From (3.15) we see that a higher δ means a greater discount rate for the income stream of physical capital and a smaller ratio NDP/Y. As it turns out, the former of the two opposing channels dominates. Finally, $\bar{\beta}^*$ is smaller if intermediates become better substitutes and monopoly mark-ups fall. Intuitively, as μ increases income is shifted away from shareholders to the owners of physical capital. Accordingly, the numerator of (3.15) falls since the income associated with physical capital is more heavily discounted. At the same time, the denominator falls since NDP/Y declines. Again, the former channel dominates the latter.

3.3.2 The Growing Economy

If the economy grows then $g^* > 0$, $L_M > 0$, $\dot{M} > 0$, and $GDP = Y + v\dot{M}$. Accordingly, the capital-income ratio is

$$\beta = \frac{K + vM}{Y + v\dot{M} - \delta K}. \tag{3.19}$$

Proposition 6 *(Capital-Income Ratio in a Steady State with Growth)*

Suppose that $L/a > \vartheta\rho$. Then, the steady-state capital-income ratio is

$$\beta^* = \frac{\dfrac{\mu\gamma}{r^* + \delta} + \dfrac{(1-\mu)\gamma}{r^* + \delta_v^*}}{1 + \dfrac{\mu(1-\gamma)}{r^* + \delta_v^*} g^* - \delta \dfrac{\mu\gamma}{r^* + \delta}}. \tag{3.20}$$

Moreover,

$$\lim_{L/a - \vartheta\rho \downarrow 0} \beta^* = \bar{\beta}^*. \tag{3.21}$$

[10] In fact, for all three parameters one can show that the effect on the second term in (3.16) is of opposite sign.

Proposition 6 extends the expression of the steady-state capital-income ratio derived for the stationary economy in Proposition 4 to a growing economy. The comparison between (3.15) and (3.20) highlights two new features. First, since the value of shares may decline over time the permanent income stream that accrues to shareholders is now discounted at rate $r^* + \delta_v^*$ where δ_v^* is endogenous. Second, the denominator shows the value added in the research sector.

Proposition 6 also establishes that in the limit $L/a - \vartheta\rho \downarrow 0$, i. e., when $g_M^* \to 0$, we have $\beta^* \to \bar{\beta}^*$. Hence, in contrast to Prediction 1, as $g^* \to 0$ the capital-income ratio remains finite.

To highlight the significance of (3.21) consider changing the level of μ. From Proposition 2 we readily derive a critical degree of product differentiation, $\mu_c < 1$, such that for $\gamma < \mu < \mu_c < 1$ and $L/a - \vartheta\rho > 0$ it holds that $\lim_{\mu\to\mu_c} g_M^* = \lim_{\mu\to\mu_c} g^* = 0$. Then,[11]

$$\lim_{\mu\to\mu_c} \beta^* = \frac{\gamma(1-\tau)}{\rho} \times \frac{\rho + \delta(1-\tau)(1-\mu_c)}{\rho + \delta(1-\tau)(1-\mu_c\gamma)} = \bar{\beta}^*,$$

i. e., the capital-income ratio converges to the one of a stationary economy with a degree of product differentiation between intermediate goods equal to $\mu_c < 1$. Hence, β^* may be equal to $\bar{\beta}^*$ even though intermediates are far from being perfect substitutes.

The following proposition has the determinants of β^*.

Proposition 7 *(Comparative Statics of β^*)*

Suppose that $L/a > \vartheta\rho$. Then, it holds that

$$\frac{d\beta^*}{d\rho} < 0, \quad \frac{d\beta^*}{d\theta} < 0, \quad \frac{d\beta^*}{d\tau} < 0, \tag{3.22}$$

$$\frac{d\beta^*}{dL} < 0, \quad \frac{d\beta^*}{da} > 0, \quad \frac{d\beta^*}{d\sigma} < 0, \tag{3.23}$$

where all derivatives are evaluated at $\mu = \gamma$ and $\delta = 0$.

Moreover, it holds that

$$\frac{d\beta^*}{d\mu} > 0, \quad and \quad \frac{d\beta^*}{d\gamma} \gtreqless 0 \quad \Leftrightarrow \quad \theta \lesseqgtr \frac{\sigma(1-\mu)(1-\tau)}{1-\sigma}, \tag{3.24}$$

[11] In fact, $\mu_c = 1 - a(1-\gamma)\zeta\rho/(L\gamma) < 1$. Qualitatively similar results can be derived for the policy parameters τ and σ. Indeed, one readily verifies that there is a critical tax on capital, τ_c, such that for $0 < \tau < \tau_c < 1$ and $L/a - \vartheta\rho > 0$ we have $\lim_{\tau\to\tau_c} g_M^* = 0$. In the same vein, there is a critical subsidy rate, σ_c, such that for $0 < \sigma_c < \sigma < 1$ and $L/a - \vartheta\rho > 0$ we have $\lim_{\sigma\to\sigma_c} g_M^* = 0$.

where the latter derivatives are evaluated at $\mu = \gamma$, $\delta = 0$, *and* $g^* = 0$.

Finally, it holds for sufficiently small values of $g^* > 0$ *that*

$$\frac{d\beta^*}{d\delta} < 0. \tag{3.25}$$

Proposition 7 gives the long-run responses of the capital-income ratio to changes in all parameters of the model.[12] To link these findings to Piketty's Prediction 1 consider the first and the second line of Table 1 which shows the signa of the comparative statics for g^* of Proposition 3 and those for β^* of Proposition 7. Hence, Prediction 1 does not hold for ρ, θ, and τ. Changes in these parameters move g^* and β^* in the same direction. The same contradiction may hold for changes in γ. Prediction 1 does not hold for δ either. Increasing the rate of physical capital depreciation leaves g^* unaffected but reduces β^*. However, for σ, μ, γ, L, and a Prediction 1 holds true: changes that reduce the economy's growth rate will at the same time increase the capital-income ratio.

For further reference, note that Proposition 7 allows us to express β^* as

$$\beta^* = \beta\left(\boldsymbol{\psi}, g\left(\boldsymbol{\omega}\right)\right), \tag{3.26}$$

where $\boldsymbol{\psi} = (\tau, \rho, \theta, \gamma, \mu, \delta)$ is the vector of parameters that exercise a "direct" effect on β^*, i. e., an effect that does not materialize through $g\left(\boldsymbol{\omega}\right)$.[13]

How come that some parameter changes are in line with Prediction 1 while others are not? To address this question we must establish the link between Proposition 7 and Piketty's second law. This requires an appropriate definition of the following net savings rates:

$$s_K \equiv \frac{Y - Lc - \delta K}{NDP} \quad \text{and} \quad s_M \equiv \frac{v\dot{M} + \dot{v}M}{NDP}. \tag{3.27}$$

Here, $s_K \geq 0$ states net investment in the accumulation of physical capital as a fraction of NDP whereas $s_M \geq 0$ is net investment in the accumulation of shares as a fraction of NDP. Then, $s \equiv s_K + s_M$ is the net savings rate of the economy as defined by Piketty.

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[12]Unfortunately, analytic results become cumbersome once we move away from $\mu = \gamma$ or allow for $\delta > 0$. However, numerical exercises reveal that the qualitative results stated in (3.22) - (3.23) hold true for a wide range of parameter values once we allow for $\mu > \gamma$ and $\delta > 0$. The same is true for the sign of $d\beta^*/d\mu$ if, in addition, we allow for sufficiently small values of $g^* > 0$. The Mathematica file with these exercises is available upon request.

[13]This follows since $r^* = r\left(\tau, \rho, \theta, g\left(\boldsymbol{\omega}\right)\right)$ and Proposition 2 allow us to write $\delta_v^* = \left(\eta^{-1} - 1\right) g\left(\boldsymbol{\omega}\right)/(1 - \tau)$. To obtain (3.26) replace these terms in (3.20).

Table 1: Comparative Statics of g^*, β^*, and α^* (Evaluations as in Proposition 12, 7 and 3).

Variables \ Parameters	ρ	θ	τ	σ	μ	γ	δ	L	a
g^*	$-$	$-$	$-$	$+$	$-$	$+$	/	$+$	$-$
β^*	$-$	$-$	$-$	$-$	$+$	$+/-$	$-$	$-$	$+$
α^*	$+$	$+$	$+$	$-$	$+$	$+$	$-$	$-$	$+$

Proposition 8 *(Piketty's Second Law)*

At all t, the evolution of the stock of assets $A = K + vM$ may be written as

$$\dot{A} = sNDP \tag{3.28}$$

so that in steady state Piketty's second law holds, i. e.,

$$\beta^* = \frac{s^*}{g^*}. \tag{3.29}$$

Moreover, $s^* = s\left(\boldsymbol{\psi}, g\left(\boldsymbol{\omega}\right)\right)$ so that β^* may be written as

$$\beta^* = \frac{s\left(\boldsymbol{\psi}, g\left(\boldsymbol{\omega}\right)\right)}{g\left(\boldsymbol{\omega}\right)}. \tag{3.30}$$

Proposition 8 establishes that s is appropriately defined so that the second law holds. Indeed, noting that in steady state A grows at rate g^* we just need to divide (3.28) by A and rearrange things to get the desired relationship (3.29). Finally, (3.30) shows how the second law translates into a framework that views s^* and g^* as endogenous variables. We can use this expression to make sense of the comparative statics given in Proposition 7. For instance, consider a parameter out of $(\rho, \theta, \tau, \mu, \gamma) \in \{\boldsymbol{\psi}, \boldsymbol{\omega}\}$, say ρ. Then, the total effect of changing ρ on β^* may be decomposed as follows:

$$\frac{d\beta^*}{d\rho} = \frac{1}{g^*}\left[\underbrace{\frac{\partial s(\boldsymbol{\psi}, g(\boldsymbol{\omega}))}{\partial \rho}}_{(-)} + \left(\underbrace{\underbrace{\frac{\partial s(\boldsymbol{\psi}, g(\boldsymbol{\omega}))}{\partial g^*}}_{(+)} - \beta^*}_{(-)}\right)\underbrace{\frac{\partial g(\boldsymbol{\omega})}{\partial \rho}}_{(-)}\right] < 0.$$

From Proposition 3 we know that $\partial g(\boldsymbol{\omega})/\partial\rho < 0$. Moreover, the sign of the term in parenthesis is determined by the partial effect of g^* on β^*. It is negative since the negative direct effect of g^* on β^* outweighs the positive indirect effect that comes about through an induced increase in s^*.[14] Hence, it follows that the direct (negative) effect of ρ on s^*, i. e., $\partial s(\boldsymbol{\psi}, g(\boldsymbol{\omega}))/\partial\rho < 0$, determines the sign of $d\beta^*/d\rho$.

So, why does Prediction 1 not hold for changes in ρ? An increase in ρ lowers g^* which increases β^*. However, there are two more channels which operate on s^*. First, a lower g^* reduces s^* and, second, a greater ρ reduces s^*. In the words used in the Introduction, the proportionate decline in s^* is stronger than the proportionate decline in g^*. Hence, β^* falls in respond to a greater ρ.

For parameters like L, a, σ which are elements of $\boldsymbol{\omega}$ but not of $\boldsymbol{\psi}$ the decomposition analysis simplifies since there is no direct effect through $s(\boldsymbol{\psi}, g(\boldsymbol{\omega}))$. For instance, consider σ. Then,

$$\frac{d\beta^*}{d\sigma} = \frac{1}{g^*}\left(\underbrace{\underbrace{\frac{\partial s(\boldsymbol{\psi}, g(\boldsymbol{\omega}))}{\partial g^*}}_{(+)} - \beta^*}_{(-)}\right)\underbrace{\frac{\partial g(\boldsymbol{\omega})}{\partial \sigma}}_{(+)} < 0.$$

Hence, a higher research subsidy increases the growth rate of the economy but reduces β^* since the partial effect of g^* on β^* is negative. Accordingly, Prediction 1 holds.

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[14]Indeed, writing (3.30) as $\beta^* = s(\boldsymbol{\psi}, g^*)/g^*$ it follows that $\partial\beta^*/\partial g^* = 1/g^*(\partial s^*/\partial g^* - \beta^*)$. Evaluated at $\delta = 0$ and $\mu = \gamma$ the latter is

$$\left.\frac{\partial\beta^*}{\partial g^*}\right|_{\mu=\gamma,\,\delta=0} = -\frac{\mu(1-\tau)(\theta + (1-\mu)\mu(1-\tau))}{\left(g_M^*(\theta + (1-\mu)\mu(1-\tau)) + \rho\right)^2} < 0,$$

where, however, $\partial s^*/\partial g^*|_{\mu=\gamma,\,\delta=0} > 0$. The negative sign of $\partial\beta^*/\partial g^*$ is consistent with an elasticity $(\partial s^*/\partial g^*)(g^*/s^*) < 1$ which is usually borne out by the data (see, e. g., Krusell and Smith (2015), pp. 738-739, for some supporting evidence).

Finally, for the depreciation rate of physical capital the comparative statics simplify even further since δ is an element of $\boldsymbol{\psi}$ but not of $\boldsymbol{\omega}$. Here, we have

$$\frac{d\beta^*}{d\delta} = \frac{1}{g^*} \underbrace{\frac{\partial s\left(\boldsymbol{\psi}, g\left(\boldsymbol{\omega}\right)\right)}{\partial \delta}}_{(-)} < 0$$

since a higher depreciation rate reduces the economy's net savings rate. Prediction 1 does not hold since changes in δ leave g^* unchanged.

3.4 The Capital Share in the Long Run

This section studies the determinants of the steady-state capital share in a stationary and a growing economy and relates them to Prediction 2 of Piketty's theory.

Recall that the capital share is defined as the fraction of total income from asset holdings net of capital depreciation in NDP. Let $\tilde{r} \equiv ((R-\delta)K + \pi M) \,/\, (K + vM)$ denote the average rate of return on assets as a percentage of their total value. Then, the capital share of equation (3.7) may be expressed as

$$\begin{aligned} \alpha &= \left(\frac{(R-\delta)K + \pi M}{K + vM}\right) \times \left(\frac{K + vM}{NDP}\right) \\ &= \tilde{r} \times \beta, \end{aligned} \tag{3.31}$$

which is a restatement of Piketty's first law.

3.4.1 The Stationary Economy

Let $\bar{\alpha}$ denote the capital share of a stationary economy.

Proposition 9 *(The Capital Share in a Stationary Steady State)*

Suppose that $L/a < \vartheta\rho$. Then, the steady-state capital-share is

$$\bar{\alpha}^* = \gamma \times \frac{1 - \dfrac{\delta\mu}{r^* + \delta}}{1 - \delta\dfrac{\mu\gamma}{r^* + \delta}} < \gamma, \tag{3.32}$$

where r^* is given by the Euler equation (3.12).

Intuitively, in the limit $\delta \to 0$ the capital share is equal to the share of final-good output that accrues to physical capital and technological knowledge, i. e., $\bar{\alpha}^* = \gamma$. Hence, γ is the gross capital share. For $\delta > 0$, $\bar{\alpha}^* < \gamma$. Since capital income is smaller than the total output of the final good, i. e., $RK + \pi M < Y$, the proportional reduction of the numerator of $\bar{\alpha}^*$ is stronger than the reduction of the denominator.

Using the Euler equation (3.12) we obtain[15]

$$\bar{\alpha}^* = \gamma \times \left(\frac{\rho + \delta(1-\tau)(1-\mu)}{\rho + \delta(1-\tau)(1-\gamma\mu)} \right). \tag{3.33}$$

This expression highlights a shortcoming of an approach that focusses on the gross capital share by setting $\delta = 0$: the net capital share may depend on a different set of parameters.

Proposition 10 *(Determinants of the Capital Share in a Stationary Steady State)*

Suppose that $L/a < \vartheta\rho$. Then, it holds that

$$\frac{d\bar{\alpha}^*}{d\gamma} > 0, \quad \frac{d\bar{\alpha}^*}{d\rho} > 0, \quad \frac{d\bar{\alpha}^*}{d\tau} > 0, \tag{3.34}$$

$$\frac{d\bar{\alpha}^*}{d\delta} < 0, \quad \frac{d\bar{\alpha}^*}{d\mu} < 0. \tag{3.35}$$

The comparative statics of Proposition 10 may be decomposed into a price and a volume effect. Consider a change in ρ,

$$\frac{d\bar{\alpha}^*}{d\rho} = \underbrace{\frac{\partial r^*}{\partial \rho}}_{(+)} \bar{\beta}^* + \frac{d\bar{\beta}^*}{d\rho} r^* > 0.$$

Since $r^* = \rho/(1-\tau)$ we have $\partial r^*/\partial\rho > 0$. Thus, the price effect dominates the volume effect, $d\bar{\beta}^*/d\rho < 0$ (see Proposition 5). An analogous argument applies to changes in τ. The remaining parameters, γ, δ, and μ leave r^* unchanged. Hence, the sign of their effect on $\bar{\beta}^*$ determines the sign of the effect on $\bar{\alpha}^*$.

Finally, observe that α is the pre-tax capital share. We define the after-tax capital share as $(1-\tau)\alpha$. Then, one readily verifies that $(1-\tau)\bar{\alpha}^*$ falls in response to an increase in τ.

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[15]Hence, in the limit $\mu \to 1$, i. e., when the economy coincides with the textbook neoclassical growth model featuring a Cobb-Douglas production function, we have $\lim_{\mu\to 1} \bar{\alpha}^* = \gamma \times \rho/\left(\rho + \delta(1-\tau)(1-\gamma)\right)$.

3.4.2 The Growing Economy

The following proposition has the steady-state capital share of a growing economy.

Proposition 11 *(Capital Share in a Steady State with Growth)*

Suppose that $L/a > \vartheta\rho$. Then, the steady-state capital-share is

$$\begin{aligned} \alpha^* &= \tilde{r}^* \times \beta^* \\ &= (r^* + \delta_v^*)\left(\frac{r^* + \delta(1-\mu)}{\mu\delta_v^* + r^* + \delta(1-\mu)}\right) \times \beta^*, \end{aligned} \tag{3.36}$$

where β^* is given by (3.20).

Moreover,

$$\lim_{L/a - \vartheta\rho \downarrow 0} \alpha^* = \bar{\alpha}^*. \tag{3.37}$$

Proposition 11 generalizes Proposition 9 to a growing economy. The first term in (3.36) is the steady-state average rate of return on assets. It exceeds r^* whenever $\mu > \gamma$ (and $\delta_v^* > 0$) since the no-arbitrage condition requires $\pi/v > r^*$ to make investors willing to hold shares that loose value over time.

Proposition 11 also establishes that the capital share of the stationary economy will be reached in the limit $L/a - \vartheta\rho \downarrow 0$, i. e., when $g^* \to 0$ and, hence, $\delta_v^* \to 0$.

The following proposition has the comparative statics of the steady-state capital share.

Proposition 12 *(Determinants of the Steady-State Capital Share with Growth)*

It holds that

$$\frac{d\alpha^*}{d\rho} > 0, \quad \frac{d\alpha^*}{d\theta} > 0, \quad \frac{d\alpha^*}{d\tau} > 0, \quad \frac{d\alpha^*}{d\delta} < 0, \tag{3.38}$$

$$\frac{d\alpha^*}{dL} < 0, \quad \frac{d\alpha^*}{da} > 0, \quad \frac{d\alpha^*}{d\sigma} < 0, \tag{3.39}$$

where all derivatives are evaluated at $\mu = \gamma$ and $\delta = 0$. Moreover, it holds that

$$\frac{d\alpha^*}{d\mu} > 0 \quad and \quad \frac{d\alpha^*}{d\gamma} > 0, \tag{3.40}$$

where the latter are evaluated at $\mu = \gamma$, $\delta = 0$, and $g^* = 0$.

Proposition 12 provides the comparative statics of α^* with respect to all parameters of the model.[16] To interpret these findings in light of Piketty's Prediction 2 compare the first and the third line of Table 1. It follows that Prediction 2 holds true for almost all parameters, i. e., parameter changes that lead to a decline in g^* also imply an increase in α^*. The exceptions are γ and δ. A smaller γ reduces g^* and α^* whereas a smaller δ increases α^* while leaving g^* unaffected.

Does Proposition 12 reflect the dominance of the volume effect over the price effect? This depends on the parameter. For instance, consider the effect of ρ and σ on α^*.[17] From (3.36) the total effect of ρ on α^* may be decomposed in a price and a volume effect

$$\frac{d\alpha^*}{d\rho} = \underbrace{\frac{d\tilde{r}^*}{d\rho}}_{(+)} \beta^* + \underbrace{\frac{d\beta^*}{d\rho}}_{(-)} \tilde{r}^* > 0.$$

From Proposition 7 we know that $d\beta^*/d\rho < 0$. Therefore, $d\alpha^*/d\rho > 0$ must be due to $d\tilde{r}^*/d\rho > 0$. Hence, contrary to Prediction 2, it is the price effect that outweighs the volume effect (as already hinted at in the Introduction).

Similarly, the effect of a change in σ on α^* may be decomposed as follows

$$\frac{d\alpha^*}{d\sigma} = \underbrace{\frac{d\tilde{r}^*}{d\sigma}}_{(+)} \beta^* + \underbrace{\frac{d\beta^*}{d\sigma}}_{(-)} \tilde{r}^* < 0.$$

From Proposition 7 we know that $d\beta^*/d\sigma < 0$ while $d\tilde{r}^*/d\sigma > 0$. Hence, the negative sign of $d\alpha^*/d\sigma$ is brought about by a volume effect that dominates the price effect which is consistent with Prediction 2.

Finally, consider the effect of an increase in τ on the steady-state after-tax capital share, $(1-\tau)\alpha^*$. One readily verifies that this effect is negative since a tax hike reduces both the after-tax average rate of return, $(1-\tau)\tilde{r}^*$, and the capital-income ratio, β^*.[18] Hence, we arrive at the conclusion that a higher tax slows down growth and reduces the after-tax capital share.

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[16] Again, analytic results become cumbersome once we move away from the indicated evaluation. Nevertheless, numerical exercises reveal that the qualitative results in (3.38) remain valid when $\mu > \gamma$ and $\delta > 0$ is allowed for. The same holds true for changes in γ if, in addition, we allow for sufficiently small values of $g^* > 0$. However, all signs in (3.39) may change when we allow for $\mu > \gamma$ and $\delta > 0$. The same is true for the sign of $d\alpha^*/d\mu$. A Mathematica file is available upon request.

[17] To simplify the notation we suppress the information about where a particular derivative is evaluated. It is understood that all decompositions hold if the evaluation is as in Proposition 12, 7 and 3.

[18] Indeed, when evaluated at $\mu = \gamma$ one finds

$$\frac{d(1-\tau)\tilde{r}^*}{d\tau} = -\frac{(1-\sigma)\theta\mu(L\theta + a\rho)}{a\left(\theta(1-\sigma) + \mu(1-\tau)\right)^2} < 0.$$

4 Demographic Growth

Since g comprises demographic and technical change it is obvious that changes in demographic growth will affect the steady-state capital-income ratio and the capital share. The purpose of this section is to study this relationship in light of Prediction 1 and 2.

To incorporate population growth we extend the model of Section 2 along the lines suggested by Jones (1995). Accordingly, the representative household comprises $L(t) = L(0)\exp(tg_L) > 0$ members where $g_L \in \mathbb{R}_+$ is the instantaneous population growth rate. Moreover, the technology for the creation of new intermediate-good varieties is now given by

$$\dot{M} = \frac{L_M}{a} M^{\phi} l_M^{\lambda-1}, \quad \phi < 1, \quad 0 < \lambda \leq 1. \tag{4.1}$$

Here, l_M captures an externality due to duplication in the R&D process, and $l_M = L_M$ holds in equilibrium.[19]

Let g_J^* denote the steady-state growth rate of per-capita variables. The following proposition characterizes the steady state of this economy.

Proposition 13 *(Steady-State Equilibrium with Population Growth)*

Let $\phi < 1$, $0 < \lambda \leq 1$, and $g_L \geq 0$. Then, there exists a unique steady-state equilibrium if

$$\rho > (1-\theta)\left(g_J^* + g_L\right). \tag{4.2}$$

The steady-state growth rate of technological knowledge is

$$g_M^* = \frac{\lambda}{1-\phi} g_L. \tag{4.3}$$

The steady-state growth rate of per-capita variables is

$$g_J^* = \eta g_M^* \tag{4.4}$$

whereas economic aggregates grow at rate $g_J^* + g_L$. Moreover, it holds that

$$g_{vJ}^* = g_{\pi J}^* = -\left(\eta^{-1} - 1\right) g_J^* + g_L \gtreqless 0. \tag{4.5}$$

[19] The economy of this section is very close to the one for which Scrimgeour (2015) studies the effect of changes in tax rates on government revenue. While this author is not concerned with issues related to the factor income distribution his and our analytical settings differ only in the way how accounting profits and losses associated with changing share prices are treated for tax purposes. In Scrimgeour (2015) the tax on capital earnings applies also to accounting profits and losses whereas in our analysis it does not (compare the no-arbitrage condition of Scrimgeour (2015), p. 705, to our equation (2.14)).

Condition (4.2) assures the transversality condition. The steady-state growth rate of technological knowledge in (4.3) follows immediately from the research technology (4.1). For the reason discussed in the context of Proposition 2 the steady state has $g_M^* > g_J^*$ whenever $\mu > \gamma$. Finally, equation (4.5) reveals that in steady state the share price and the dividend of intermediate-good firms need not decline even if $\mu > \gamma$. Intuitively, in the presence of positive population growth, $g_L > 0$, the turnover of intermediate-good producers increases as the market size for intermediates grows. This may even offset the tendency of a declining turnover arising from $g_M^* > g_J^*$.

4.1 The Capital-Income Ratio in the Long Run

This section establishes that the effect of population growth on the steady-state capital-labor ratio is ambiguous. Hence, depending on the circumstances faster population growth may increase or decrease the long-run capital-income ratio. Since faster population growth unequivocally increases the growth rate of the economy this violates Prediction 1.

Let β_J^* denote the steady-state capital-income ratio. By construction, β_J^* is still given by the right-hand side of (3.20) with g^* and δ_v^* being respectively replaced by g_J^* and $\delta_{vJ}^* = -g_{vJ}^*/(1-\tau)$. From (4.3) and (4.4) we may express the steady-state growth rate of per-capita variables as $g_J^* = g_J(\boldsymbol{\omega}_J)$ where g_J is a function and $\boldsymbol{\omega}_J = (g_L, \lambda, \phi, \gamma, \mu)$ is the vector of parameters upon which g_J^* depends. Let $\boldsymbol{\psi}_J = (\tau, \rho, \theta, \gamma, \mu, \delta, g_L)$ denote the vector of parameters that have a direct effect on β_J^*. Then, $\beta_J^* = \beta_J\left(\boldsymbol{\psi}_J, g_J(\boldsymbol{\omega}_J)\right)$.

Since, $g_L \to 0$ implies $g_J^* \to 0$ and $g_v^* \to 0$ it becomes obvious from Proposition 6 and Proposition 4 that

$$\lim_{g_L \to 0} \beta_J^* = \bar{\beta}^*, \tag{4.6}$$

i. e., void of population growth the capital-income ratio is the one of the stationary economy. Hence, contrary to Prediction 1 without growth the steady-state capital-income ratio remains finite.

The following proposition has the determinants of the capital-income ratio.

Proposition 14 *(Comparative Statics of β_J^*)*

There is $\varepsilon > 0$ such that $0 < g_L < \varepsilon$ and

Table 2: Comparative Statics of $g_j^* + g_L$, β_J^*, and α_J^* (Evaluations as in Proposition 15, 7 and 3).

Variables \ Parameters	ρ	θ	τ	σ	μ	γ	δ	L	a	λ	ϕ	g_L
$g_J^* + g_L$	/	/	/	/	$-$	$+$	/	/	/	$+$	$+$	$+$
β_J^*	$-$	$-$	$-$	/	$-$	$+$	$-$	/	/	$-$	$-$	$+/-$
α_J^*	$+$	/	$+$	/	$-$	$+$	$-$	/	/	$+/-$	$+/-$	$+/-$

$$\frac{d\beta_J^*}{dg_L} \gtreqless 0, \quad \frac{d\beta_J^*}{d\lambda} < 0, \quad \frac{d\beta_J^*}{d\phi} < 0. \tag{4.7}$$

$$\frac{d\beta_J^*}{d\rho} < 0, \quad \frac{d\beta_J^*}{d\theta} < 0, \quad \frac{d\beta_J^*}{d\tau} < 0, \tag{4.8}$$

$$\frac{d\beta_J^*}{d\mu} < 0, \quad \frac{d\beta_J^*}{d\gamma} > 0, \quad \frac{d\beta_J^*}{d\delta} < 0, \tag{4.9}$$

Moreover, β_J^* does not depend on L, a, and σ.

To link these findings to Piketty's Prediction 1 consider the first and the second line of Table 2. Hence, only changes in λ and ϕ induce adjustments in $g_J^* + g_L$ and β_J^* of opposite sign. Changes in g_L may or may not have this property. For all other parameters, Prediction 1 fails. For instance, changes in ρ, θ, τ, or δ leave $g_J^* + g_L$ unchanged while affecting β_J^*. Changes in μ and γ shift $g_J^* + g_L$ and β_J^* in the same direction whereas L, a, and σ neither affect $g_J^* + g_L$ nor β_J^*.

To provide further intuition for why Prediction 1 may or may not hold consider Piketty's second law from Proposition 8. Since now A grows at rate $g_A^* = g_J^* + g_L$ we obtain from $\dot{A} = sNDP$ that

$$\beta_J^* = \frac{s^*}{g_J^* + g_L}. \tag{4.10}$$

In analogy to Proposition 8, this allows us to express β_J^* as[20]

$$\beta_J^* = \frac{s_J\left(\boldsymbol{\psi}_J, g_J(\boldsymbol{\omega}_J)\right)}{g_J(\boldsymbol{\omega}_J) + g_L}. \tag{4.11}$$

Then, the the total effect of changing g_L on β_J^* may be decomposed as follows:

$$\frac{d\beta^*}{dg_L} = \left(\frac{1}{g_J(\boldsymbol{\omega}_J) + g_L}\right) \times \left[\left(\underbrace{\frac{\partial s_J\left(\boldsymbol{\psi}_J, g_J(\boldsymbol{\omega}_J)\right)}{\partial g_L} - \beta^*}_{(+)}\right) + \left(\underbrace{\frac{\partial s\left(\boldsymbol{\psi}_J, g_J(\boldsymbol{\omega}_J)\right)}{\partial g_J^*} - \beta^*}_{(-)}\right)\underbrace{\frac{\partial g_J(\boldsymbol{\omega}_J)}{\partial g_L}}_{(+)}\right]$$

The first parenthesis in the bracket turns out to be positive. It shows the difference between the partial effect of g_L on s_J^* and its direct effect on β_J^*. The last product in brackets can be shown to be negative. It captures the effect of g_L on the growth rate of per-capita variables. As a consequence, the sign of $d\beta^*/dg_L$ is in general indeterminate which is inconsistent with Prediction 1.

Next, we turn to the effect of a higher intra-temporal externality in research, λ.[21] Since there is no direct effect of λ on s^* we have

$$\frac{d\beta^*}{d\lambda} = \left(\frac{1}{g_J(\boldsymbol{\omega}_J) + g_L}\right) \times \left[\left(\underbrace{\frac{\partial s\left(\boldsymbol{\psi}_J, g_J(\boldsymbol{\omega}_J)\right)}{\partial \lambda} - \beta^*}_{(-)}\right)\underbrace{\frac{\partial g_J(\boldsymbol{\omega}_J)}{\partial \lambda}}_{(+)}\right] < 0.$$

From Proposition 13 we deduce that $\partial g_J^*/\partial\lambda > 0$, whereas the term in parenthesis is negative. Hence, in line with Prediction 1 we have $d\beta^*/d\lambda < 0$.

Finally, consider the effect of ρ on β_J^*.[22] From Proposition 13 it becomes obvious that $\partial g_J^*/\partial\rho = 0$. Hence, (4.11) delivers

$$\frac{d\beta^*}{d\rho} = \left(\frac{1}{g_J(\boldsymbol{\omega}_J) + g_L}\right) \times \underbrace{\frac{\partial s_J\left(\boldsymbol{\psi}_J, g_J(\boldsymbol{\omega}_J)\right)}{\partial \rho}}_{(-)} < 0,$$

and the sign of $d\beta^*/d\rho$ reflects the direct effect of ρ on s^*. Hence, contrary to Prediction 1 a change in ρ induces a change in β_J^* even though it leaves g_J^* unaffected.

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[20]To see this, we proceed as in the proof of Proposition 8. Here, equation (4.10) allows us to write and define $s_J^* = \beta_J^* \times (g_J^* + g_L) = \beta_J\left(\boldsymbol{\psi}_J, g_J(\boldsymbol{\omega}_J)\right) \times (g_J(\boldsymbol{\omega}_J) + g_L) \equiv s_J\left(\boldsymbol{\psi}_J, g_J(\boldsymbol{\omega}_J)\right)$.

[21]Mutatis mutandis, the analysis carries over to the effect of a change in ϕ.

[22]Mutatis mutandis, the analysis for the change in ρ carries over to the remaining comparative statics of (4.8).

4.2 The Capital Share in the Long Run

Let α_J^* denote the steady-state capital share in the model with population growth. By construction α_J^* is given by the right-hand side of (3.36) in Proposition 11 where $\tilde{r}^*$ is replaced by $\tilde{r}_J^*$ and β^* by β_J^*. Hence, we may write

$$\alpha_J^* = \tilde{r}_J^* \times \beta_J^*.$$

Since, $g_L \to 0$ implies $\delta_v^* = -g_{vJ}/(1-\tau) \to 0$ it becomes obvious with (4.6) that

$$\lim_{g_L \to 0} \alpha_J^* = \bar{\alpha}^*, \tag{4.12}$$

i. e., void of population growth the factor income distribution is the one of the stationary economy.

The following proposition has the comparative statics for the long-run capital share with population growth.

Proposition 15 *(Comparative Statics of the Capital Share with Population Growth)*

There is $\varepsilon > 0$ *such that* $0 < g_L < \varepsilon$ *and*

$$\frac{d\alpha_J^*}{dg_L} \gtreqless 0, \quad \frac{d\alpha_J^*}{d\lambda} \gtreqless 0, \quad \frac{d\alpha_J^*}{d\phi} \gtreqless 0, \tag{4.13}$$

$$\frac{d\alpha_J^*}{d\mu} < 0, \quad \frac{d\alpha_J^*}{d\gamma} > 0, \quad \frac{d\alpha_J^*}{d\delta} < 0, \tag{4.14}$$

$$\frac{d\alpha_J^*}{d\rho} > 0, \quad \frac{d\alpha_J^*}{d\theta} = 0, \quad \frac{d\alpha_J^*}{d\tau} > 0. \tag{4.15}$$

Moreover, α_J^* *is independent of* L*,* a*, and* σ*.*

Hence, the effect of demographic growth on the steady-state capital share is not unequivocal.

Does Piketty's Prediction 2 hold? In general, the answer is no. The first and the third line of Table 2 show that Claim 1 of Prediction 2 is problematic since a smaller growth rate $g_J^* + g_L$ may not be associated with a greater α_J^*. A lower level of $g_J^* + g_L$ may be due to changes in g_L, λ, ϕ, μ, or γ. Changes in g_L, λ, and ϕ that reduce the economy's growth rate may or may not increase its capital share.[23]

Changes in μ or γ, that reduce $g_J^* + g_L$ will also reduce α_J^*.[24] Finally, α_J^* increases without affecting the economy's growth rate if ρ or τ increase, and if δ falls. Furthermore, the size of the labor force, L, labor productivity in research, a, and research subsidies, σ, have no effect on the steady-state capital share. Proposition 14 establishes that they do not affect the capital-income ratio.

Do the findings of Proposition 15 arise since the "volume effect outweighs the price effect" as asserted by Claim 2 of Prediction 2? In general, the answer is again no. To see why we focus on the effect of demographic growth, g_L, where

$$\frac{d\alpha^*}{dg_L} = \frac{d\tilde{r}_J^*}{dg_L}\beta_J^* + \frac{d\beta_J^*}{dg_L}\tilde{r}_J^* \gtreqless 0. \tag{4.16}$$

We know from Proposition 14 that g_L has an ambiguous effect on β^*. Similarly, one can show that the effect of g_L on $\tilde{r}_J^*$ is ambiguous. Hence, in general the hypothesis of a volume effect outweighing the price effect receives little support.

5 Concluding Remarks

According to David Ricardo the principal problem in Political Economy is to discover the laws which regulate the distribution of income (see Ricardo (1821), preface). Thomas Piketty's *Capital in the Twenty-First Century* (2014) is a forceful reminder of this assessment. More so, the author comprehensively documents the relevant empirical phenomena and presents two "fundamental laws of capitalism" that are meant to explain a large part of these stylized facts. Yet, are these laws what Ricardo had hoped for? Should we use these laws to formulate predictions about the future?

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[23]By example, we show in the Proof of Proposition 15 that μ, the degree of product differentiation of intermediates, plays an important role for the sign of $d\alpha_J^*/dg_L$. If μ is large then this sign is positive and Claim 1 of Prediction 2 is violated. The opposite holds for small values of μ. Similar findings are obtained for the signs of $d\alpha_J^*/d\lambda$ and $d\alpha_J^*/d\phi$.

[24]Observe that a higher μ also implies a lower price of intermediates. Indeed, from (2.7), (2.14), and (3.12) one finds $p^* = \left(\theta g_J^* + \rho + \delta\right)/(\mu(1-\tau))$ so that $dp^*/d\mu < 0$. Hence, the argument proposed by Karabarbounis and Neiman (2014a) according to which falling prices of investment goods lead to more capital accumulation and more capital income does not hold. This argument is, however, consistent with the model of Section 2 that dispenses with demographic growth. Here, the effect of μ on α^* is positive (see, Proposition 12).

The present paper argues that a central weakness of Piketty's laws and the conclusions he draws from them is an "endogeneity problem." The variables that explain the factor income distribution in the long run, namely, the real rate of return on assets, the economy's savings rate and its growth rate, are *all* endogenous variables. Therefore, in contrast to Piketty's predictions, what matters for the steady-state capital share is not that the economy's growth rate falls, what matters is why it falls. The cause of the decline in the economy's growth rate will affect the equilibrium of the economy as a whole, including its factor income distribution.

Our analysis identifies cases where the implications of Piketty's second law are violated. Due to an exogenous shock the steady-state capital-income ratio may well increase if the economy's growth rate increases. In spite of this violation, our analysis of Romer's model without population growth tends to confirm Piketty's assertion that slower long-run growth goes together with a greater capital share. However, the underlying intuition is quite different from Piketty's. In the model with population growth this assertion receives little support. In particular, slower demographic growth may be associated with a greater or a smaller capital share. On the whole, we conclude that neither Prediction 1 about the implications of the second law, nor Prediction 2 on the role of the growth rate for the capital share, should be uncritically used to forecast the capital-income ratio and the factor income distribution.

Clearly, there are important channels that our research does not touch upon even though they are likely to be relevant for the determination of the factor income distribution in the long run. For instance, our analysis is mute on the role of housing as an important determinant of the share of capital in net income (see, e. g., Bonnet, Bono, Chapelle, and Wasmer (2014), Rognlie (2015), Grossmann and Steger (2016)), and it neglects the role of wage bargaining as opposed to marginal product pricing or open economy issues. At a more technical level, our findings rely on a Cobb-Douglas production function of the final good sector. Therefore, the factor shares for capital, technological knowledge, and industrial labor in final-good production are constant (see Proposition 1). This raises the question of how our qualitative findings would change under a more general production function allowing for an elasticity of substitution between the composite of all intermediates and industrial labor different from unity.[25]

Overall, our results suggest that technology, preferences, policy, demographics, and market structure shape the factor income distribution. Yet, we concur with Blume and Durlauf

[25] A problem that a generalization along these lines faces is that a steady-state path with a positive rate of technical change can no longer exist since technical change is "capital-augmenting." In an economy where capital accumulates and the aggregate production function of the final-good has constant returns to scale in capital and labor, Uzawa's theorem applies so that there can be no capital-augmenting technical change in steady state (see, e. g., Uzawa (1961) or Irmen (2016)).

(2015) and others that these dimensions are themselves endogenous and determined, e. g., by advances in scientific and medical knowledge (Fogel (2004)) or by institutional changes (Acemoglu and Robinson (2014)). The development of a comprehensive understanding of the laws that govern the distribution of income must also take these features into account.

6 Appendix: Proofs

6.1 Proof of Proposition 1

To be found in the main text. ■

6.2 Proof of Proposition 2

The economy's resource constraint and the zero-profit condition of research firms imply $Lc + \dot{K} = Y - \delta K$. Therefore, in steady state Y, K, and c must grow at the same rate. Moreover, the equilibrium output of the final good is given by (3.4) with L_Y being constant in steady state (see below). Accordingly, $g_Y = \gamma\,(1/\mu - 1)\, g_M + \gamma g_K$. With $g_Y = g_K$ we have $g_Y = \eta g_M$ so that

$$g_Y = g_K = g_c = \eta g_M. \tag{6.1}$$

Next, we derive g_M^* of (3.9). We start with supply-side conditions that deliver a relationship between g_M and r. Upon combining the labor market equilibrium, $L_Y + L_M = L$, and the research technology (2.10) we obtain successively

$$\begin{aligned} g_M &= \frac{L - L_Y}{a} \\ &= \frac{L}{a} - (1-\gamma)\,\frac{Y}{wa} \\ &= \max\left\{0, \frac{L}{a} - (1-\sigma)\,(1-\gamma)\,\frac{Y}{Mv}\right\} \\ &= \max\left\{0, \frac{L}{a} - \vartheta\left((1-\tau)\,r - g_v\right)\right\} \\ &= \max\left\{0, \frac{\frac{L}{a} - \vartheta\,(1-\tau)\,r}{1+\vartheta\,(1-\eta)}\right\}. \end{aligned} \tag{6.2}$$

Since g_M is time-invariant the first line implies that L_Y must be a constant. The second line takes the final-good sector's demand for labor of (2.3) to substitute for L_Y. The third line uses the free entry condition of the research sector (2.11). To obtain the fourth line observe that rearranging the no-arbitrage condition (2.14) gives $v = (1-\tau)\,\pi / \left((1-\tau)\,r - g_v\right)$. Moreover, from (3.3) we know that $\pi = (1-\mu)\gamma Y/M$. Finally, the fifth line uses the fact that in steady state (2.14) implies $g_\pi = g_v$ where $g_\pi = g_Y - g_M$. In conjunction with (6.1) this gives

$$g_v = g_\pi = g_Y - g_M = -\,(1-\eta)\,g_M \leq 0. \tag{6.3}$$

Using the latter delivers (6.2) which is the desired supply-side relationship between g_M and r.

A second equation linking g_M and r obtains from the Euler equation (2.17) for a constant growth rate g_c. With (6.1) the latter becomes

$$r = \frac{\theta\eta g_M + \rho}{1-\tau}. \tag{6.4}$$

Combining (6.2) and (6.4) delivers (3.9). Obviously, the steady state also satisfies (3.10 and (3.11).

Finally, observe that the transversality condition requires $(1-\tau)r^* > g^*$ in steady state. Using the Euler condition (6.4) in conjunction with (3.9) one readily verifies that this inequality is satisfied under condition (3.8). ■

6.3 Proof of Proposition 3

From Proposition 2 we know that $L/a > \vartheta\rho$ implies $g_M^* > 0$. Then, it holds that

$$\frac{\partial g_M^*}{\partial L/a} = \frac{\mu}{\zeta\left(\theta + \eta^{-1} - 1\right) + \mu} > 0,$$

$$\frac{\partial g_M^*}{\partial \zeta} = -\frac{\eta\mu(L(1-\eta(1-\theta)) + a\rho)}{a(\zeta(1-\eta(1-\theta)) + \eta\mu)^2} < 0,$$

$$\frac{\partial g_M^*}{\partial \rho} = -\frac{\mu\vartheta}{\zeta\left(\theta + \eta^{-1} - 1\right) + \mu} < 0,$$

$$\frac{\partial g_M^*}{\partial \theta} = -\frac{\mu\zeta\left(\frac{L}{a} - \vartheta\rho\right)}{\left(\zeta\left(\theta + \eta^{-1} - 1\right) + \mu\right)^2} < 0,$$

$$\frac{\partial g_M^*}{\partial \gamma} = \frac{\zeta(1-\mu)\mu^2\left(\frac{L}{a} - \eta\vartheta\rho + \frac{\rho}{\mu}\left(\zeta\theta + \mu\right)\right)}{\left(\zeta(\mu-\gamma) + \gamma(1-\mu)\left(\zeta\theta + \mu\right)\right)^2} > 0, \quad \text{and} \quad \frac{\partial \eta}{\partial \gamma} = \frac{1-\mu}{(1-\gamma)^2\mu} > 0.$$

Since $g^* = \eta g_M^*$ these results prove the sign of the comparative statics of g^* for L, a, τ, σ, ρ, θ, and γ. For μ we find

$$\lim_{L/a - \vartheta\rho \downarrow 0} \frac{\partial g^*}{\partial \mu} = -\frac{\rho\zeta}{\eta(1-\mu)\left(\mu + \zeta(\theta + \eta^{-1} - 1)\right)} < 0.$$

■

6.4 Proof of Proposition 4

From Proposition 2 the inequality $L/a \leq \vartheta\rho$ implies $g_M^* = g^* = g_v^* = 0$. Then, the no-arbitrage condition (2.14) delivers $r^* = R^* - \delta = \pi^*/v^*$, and with (3.3) of Proposition 1 we obtain (3.15). Finiteness follows immediately from (3.16). ■

6.5 Proof of Proposition 5

One readily verifies that

$$\frac{\partial \bar{\beta}^*}{\partial \gamma} = \left(\frac{1-\tau}{\rho}\right) \times \left(\frac{(\rho + \delta(1-\tau)(1-\mu))\left(\rho + \delta(1-\tau)\right)}{(\rho + \delta(1-\tau)(1-\gamma\mu))^2}\right) > 0,$$

$$\frac{\partial \bar{\beta}^*}{\partial \rho} = -\left(\frac{(1-\tau)\gamma}{\rho}\right) \times \left(\frac{\rho^2 + 2\rho\delta(1-\tau)(1-\mu) + \delta^2(1-\tau)^2(1-\mu)(1-\gamma\mu)}{\rho\left(\rho + \delta(1-\tau)(1-\gamma\mu)\right)^2}\right) < 0,$$

$$\frac{\partial \bar{\beta}^*}{\partial \tau} = -\left(\frac{\gamma}{\rho}\right) \times \left(\frac{\rho^2 + 2\rho\delta(1-\tau)(1-\mu) + \delta^2(1-\tau)^2(1-\mu)(1-\gamma\mu)}{(\rho + \delta(1-\tau)(1-\gamma\mu))^2}\right) < 0,$$

$$\frac{\partial \bar{\beta}^*}{\partial \delta} = -\frac{\gamma(1-\gamma)\mu(1-\tau)^2}{(\rho + \delta(1-\tau)(1-\gamma\mu))^2} < 0,$$

$$\frac{\partial \bar{\beta}^*}{\partial \mu} = -\left(\frac{(1-\tau)\gamma}{\rho}\right) \times \left(\frac{(1-\gamma)\delta(1-\tau)\left(\delta(1-\tau) + \rho\right)}{(\rho + \delta(1-\tau)(1-\gamma\mu))^2}\right) < 0.$$

■

6.6 Proof of Proposition 6

From Proposition 2 we know that $L/a > \vartheta\rho$ implies $g_M^* \geq g^* > 0$ and $g_v^* \leq 0$ with strict inequality for $\mu > \gamma$. Hence, whenever the latter condition is satisfied the value of each share depreciates at rate $-g_v^* > 0$. Using δ_v^*, the no-arbitrage condition (2.14), (3.3) of Proposition 1, and $g^* = \eta g_M^*$ of Proposition 2 we obtain (3.20).

Equation (3.21) obtains as $\lim_{L/a-\vartheta\rho\downarrow 0} g_M^* = 0$. Since, $g^* = \eta g_M^*$ and $\eta > 0$ this implies $g^* \to 0$, $g_v^* \to 0$, and $\delta_v^* \to 0$. ■

6.7 Proof of Proposition 7

Starting from (3.20) some tedious but straightforward algebra using (2.17), (3.12), and Proposition 2 delivers

$$\begin{aligned} \beta^* &= \frac{\gamma(1-\tau)}{\rho + \theta g^* - g_v^*} \\ &\times \frac{\rho + \delta\,(1-\tau)\,(1-\mu) + \theta g^* - \mu g_v^*}{\rho + \theta g^* + \delta\,(1-\tau)\,(1-\mu\gamma) + (\rho + \theta g^* + (1-\tau)\,\delta)\,\frac{(1-\tau)(1-\gamma)\mu g^*}{\rho+\theta g^* - g_v^*}} \end{aligned} \tag{6.5}$$

With g^* the following comparative statics can be computed.

$$\begin{aligned}
\left.\frac{d\beta^*}{d\rho}\right|_{\mu=\gamma,\,\delta=0} &= -\frac{a^2(\mu(1-\sigma)+\sigma)(\mu(1-\tau)+\theta(1-\sigma))}{(a\rho(\mu(1-\sigma)+\sigma)+L\,(\mu(1-\tau)(1-\mu)+\theta))^2} < 0, \\
\left.\frac{d\beta^*}{d\theta}\right|_{\mu=\gamma,\,\delta=0} &= -\frac{a(\mu(1-\sigma)+\sigma)\,(L(\mu(1-\tau)) - a\rho(1-\sigma))}{(a\rho(\mu(1-\sigma)+\sigma)+L\,(\mu(1-\tau)(1-\mu)+\theta))^2} < 0, \\
\left.\frac{d\beta^*}{d\tau}\right|_{\mu=\gamma,\,\delta=0} &= -\frac{a\mu(\mu(1-\sigma)+\sigma)(a\rho+L\theta)}{(a\rho(\mu(1-\sigma)+\sigma)+L\,(\mu(1-\tau)(1-\mu)+\theta))^2} < 0, \\
\left.\frac{d\beta^*}{dL}\right|_{\mu=\gamma,\,\delta=0} &= -\frac{a\,(\mu(1-\tau)(1-\mu)+\theta)\,(\mu(1-\tau)+\theta(1-\sigma))}{(a\rho(\mu(1-\sigma)+\sigma)+L\,(\mu(1-\tau)(1-\mu)+\theta))^2} < 0, \\
\left.\frac{d\beta^*}{da}\right|_{\mu=\gamma,\,\delta=0} &= \frac{L\,(\mu(1-\tau)(1-\mu)+\theta)\,(\mu(1-\tau)+\theta(1-\sigma))}{(a\rho(\mu(1-\sigma)+\sigma)+L\,(\mu(1-\tau)(1-\mu)+\theta))^2} > 0, \\
\left.\frac{d\beta^*}{d\sigma}\right|_{\mu=\gamma,\,\delta=0} &= -\frac{a\,(\mu(1-\tau)(1-\tau)+\theta)\,(a\rho+L\theta)}{(a\rho(\mu(1-\sigma)+\sigma)+L\,(\mu(1-\tau)(1-\mu)+\theta))^2} < 0.
\end{aligned}$$

Moreover, one finds

$$\begin{aligned}
\left.\frac{d\beta^*}{d\mu}\right|_{\mu=\gamma,\,\delta=0,\,g^*=0} &= \frac{\mu(1-\tau)}{\rho} \times \frac{(1-\sigma)(\theta+(1-\mu)\mu(1-\tau))}{(1-\mu)(\theta(1-\sigma)+\mu(1-\tau))} > 0, \\
\left.\frac{d\beta^*}{d\gamma}\right|_{\mu=\gamma,\,\delta=0,\,g^*=0} &= \frac{\mu(1-\tau)}{\rho} \times \frac{(1-\mu)\mu(1-\tau)-(1-\sigma)\theta}{(1-\mu)\,(\theta(1-\sigma)+\mu(1-\tau))} \gtreqless 0.
\end{aligned}$$

As to $d\beta^*/d\delta$ one can show that

$$\lim_{L/a-\vartheta\rho\downarrow 0} \frac{d\beta^*}{d\delta} = \frac{d\bar{\beta}^*}{d\delta} < 0.$$

Hence, there is a neighborhood where $L/a - \vartheta\rho > 0$, $g^* > 0$, and $d\beta^*/d\delta < 0$. ■

6.8 Proof of Proposition 8

To prove (3.28) consider the flow budget constraint of the household sector (2.16). A balanced budget of the government means that $T = \tau((R-\delta)K + \pi M) - \sigma w L_M$. Then, with (3.2) of Proposition 1 we may write (2.16) as

$$\dot{K} + v\dot{M} = Y - Lc - \delta K + wL_M(1-\sigma). \tag{6.6}$$

Since final-good output is required for capital accumulation it must be that $\dot{K} = Y - Lc - \delta K$. Hence, we may define $s_K \equiv \dot{K}/NDP$. Moreover, denote by $\tilde{s}_M$ the value of newly emitted shares in NDP, i. e., $\tilde{s}_M \equiv v\dot{M}/NDP$. Then, with (3.1) of Proposition 1 the budget constraint (6.6) may be written as

$$\dot{K} + v\dot{M} = (s_K + \tilde{s}_M)\, NDP. \tag{6.7}$$

Next, observe that the evolution of $A \equiv K + vM$ satisfies

$$\dot{A} = \dot{K} + v\dot{M} + \dot{v}M = (s_K + s_M)\, NDP, \tag{6.8}$$

where use is made of the definitions stated in (3.27). Invoking $s \equiv s_K + s_M$ delivers (3.28).

As to the proof of (3.29) recall that in steady-state A grows at rate $g^* > 0$ so that dividing (6.8) by A gives $g^* = s^*\,(NDP/A)^*$. Rearranging delivers (3.29).

Next, observe that equation (3.29) allows us to write and define

$$s^* = \beta^* \times g^* = \beta(\boldsymbol{\psi}, g(\boldsymbol{\omega})) \times g(\boldsymbol{\omega}) \equiv s(\boldsymbol{\psi}, g(\boldsymbol{\omega})). \tag{6.9}$$

Hence, s^* can be expressed as a function of $\boldsymbol{\psi}$ and $g^* = g(\boldsymbol{\omega})$. It follows that β^* of (3.26) may be written as stated in (3.30). ■

6.9 Proof of Proposition 9

Observe that the no-arbitrage condition (2.14) delivers $\tilde{r}^* = r^*$. Hence, $\bar{\alpha}^* = r^* \times \bar{\beta}^*$. With β^* of (3.15) it follows that $r^* \times (\mu\gamma/(r^*+\delta) + (1-\mu)\gamma/r^*) = \gamma \times (1 - \delta\mu/(r^*+\delta))$. Moreover, $1 - \delta\mu\gamma/(r^*+\delta) = NDP^*/Y^*$. ■

6.10 Proof of Proposition 10

From (3.33) it is immediate that

$$\begin{aligned}
\frac{d\bar{\alpha}^*}{d\gamma} &= \frac{(\delta(1-\tau)+\rho)(\delta(1-\tau)(1-\mu)+\rho)}{(\delta(1-\tau)(1-\gamma\mu)+\rho)^2} > 0,\\
\frac{d\bar{\alpha}^*}{d\rho} &= \frac{(1-\gamma)\gamma\delta\mu(1-\tau)}{(\delta(1-\tau)(1-\gamma\mu)+\rho)^2} > 0,\\
\frac{d\bar{\alpha}^*}{d\tau} &= \frac{(1-\gamma)\gamma\delta\mu\rho}{(\delta(1-\tau)(1-\gamma\mu)+\rho)^2} > 0,\\
\frac{d\bar{\alpha}^*}{d\delta} &= -\frac{(1-\gamma)\gamma\mu\rho(1-\tau)}{(\delta(1-\tau)(1-\gamma\mu)+\rho)^2} < 0,\\
\frac{d\bar{\alpha}^*}{d\mu} &= -\frac{(1-\gamma)\gamma\delta(1-\tau)(\delta(1-\tau)+\rho)}{(\delta(1-\tau)(1-\gamma\mu)+\rho)^2} < 0.
\end{aligned}$$

■

6.11 Proof of Proposition 11

By definition, $\tilde{r}$ may be expressed as

$$\tilde{r} = \frac{\frac{(R-\delta)K}{Y} + \frac{\pi M}{Y}}{\frac{K}{Y} + \frac{vM}{Y}}$$

With (2.14), (3.3) and the definition of δ_v^*, appropriate rearranging delivers $\tilde{r}^*$ as the first two factors stated in (3.36).

Equation (3.37 holds since with Proposition 2 we have $\delta_v^* = (1-\eta^{-1})g^*/(1-\tau)$. Hence, $\delta_v^* = 0$ obtains $g^* = 0$. ■

6.12 Proof of Proposition 12

The following comparative statics are based on α^* of (3.36). It holds that

$$\begin{aligned}
\left.\frac{d\alpha^*}{d\rho}\right|_{\mu=\gamma,\,\delta=0} &= \frac{a(1-\mu)\mu L(\theta(1-\sigma)+\mu(1-\tau))}{(a\rho(\mu(1-\sigma)+\sigma)+L(\theta+(1-\mu)\mu(1-\tau)))^2} > 0,\\
\left.\frac{d\alpha^*}{d\theta}\right|_{\mu=\gamma,\,\delta=0} &= \frac{(1-\mu)\mu L(\mu L(1-\tau)-a\rho(1-\sigma))}{(a\rho(\mu(1-\sigma)+\sigma)+L(\theta+(1-\mu)\mu(1-\tau)))^2} > 0,\\
\left.\frac{d\alpha^*}{d\tau}\right|_{\mu=\gamma,\,\delta=0} &= \frac{(1-\mu)\mu^2 L(a\rho+\theta L)}{(a\rho(\mu(1-\sigma)+\sigma)+L(\theta+(1-\mu)\mu(1-\tau)))^2} > 0,\\
\left.\frac{d\alpha^*}{dL}\right|_{\mu=\gamma,\,\delta=0} &= -\frac{a(1-\mu)\mu\rho(\theta(1-\sigma)+\mu(1-\tau))}{(a\rho(\mu(1-\sigma)+\sigma)+L(\theta+(1-\mu)\mu(1-\tau)))^2} < 0,\\
\left.\frac{d\alpha^*}{da}\right|_{\mu=\gamma,\,\delta=0} &= \frac{(1-\mu)\mu L\rho(\theta(1-\sigma)+\mu(1-\tau))}{(a\rho(\mu(1-\sigma)+\sigma)+L(\theta+(1-\mu)\mu(1-\tau)))^2} > 0,\\
\left.\frac{d\alpha^*}{d\sigma}\right|_{\mu=\gamma,\,\delta=0} &= -\frac{a(1-\mu)\mu\rho(a\rho+\theta L)}{(a\rho(\mu(1-\sigma)+\sigma)+L(\theta+(1-\mu)\mu(1-\tau)))^2} < 0,\\
\left.\frac{d\alpha^*}{d\delta}\right|_{\mu=\gamma,\,\delta=0} &= -\frac{a(1-\mu)\mu(\theta(1-\sigma)+\mu(1-\tau))(a\rho\sigma+L(\theta+\mu(1-\tau)))}{(a\rho(\mu(1-\sigma)+\sigma)+L(\theta+(1-\mu)\mu(1-\tau)))^2} < 0,
\end{aligned}$$

$$\begin{aligned}
\left.\frac{d\alpha^*}{d\mu}\right|_{\mu=\gamma,\,\delta=0,\,g^*=0} &= \frac{\mu^2(1-\sigma)(1-\tau)}{\theta(1-\sigma)+\mu(1-\tau)} > 0,\\
\left.\frac{d\alpha^*}{d\gamma}\right|_{\mu=\gamma,\,\delta=0,\,g^*=0} &= \frac{\theta(1-\sigma)+\mu\sigma(1-\tau)}{\theta(1-\sigma)+\mu(1-\tau)} > 0.
\end{aligned}$$

■

6.13 Proof of Proposition 13

See Scrimgeour (2015). ■

6.14 Proof of Proposition 14

We obtain β_J^* from (6.5) in conjunction with Proposition 13. Hence,

$$\begin{aligned}\beta_J^* &= \frac{\gamma(1-\tau)}{\rho+\theta g_J^* - g_{vJ}^*} \\ &\times \frac{\rho+\delta(1-\tau)(1-\mu)+\theta g_J^* - \mu g_{vJ}^*}{\rho+\theta g_J^* + \delta(1-\tau)(1-\mu\gamma) + \left(\rho+\theta g_J^* + (1-\tau)\delta\right)\frac{(1-\tau)(1-\gamma)\mu g_J^*}{\rho+\theta g_J^* - g_{vJ}^*}},\end{aligned}$$

where

$$g_J^* = \frac{\eta\lambda}{1-\phi} g_L \quad \text{and} \quad g_{vJ}^* = -(\eta^{-1}-1)g_J^* + g_L.$$

Then, one readily verifies that

$$\begin{aligned}\left.\frac{d\beta_J^*}{d\rho}\right|_{g_L=0} &= \frac{d\bar{\beta}^*}{d\rho} < 0, \quad \left.\frac{d\beta_J^*}{d\tau}\right|_{g_L=0} = \frac{d\bar{\beta}^*}{d\tau} < 0 \\ \left.\frac{d\beta_J^*}{d\mu}\right|_{g_L=0} &= \frac{d\bar{\beta}^*}{d\mu} < 0, \quad \left.\frac{d\beta_J^*}{d\gamma}\right|_{g_L=0} = \frac{d\bar{\beta}^*}{d\gamma} > 0 \\ \left.\frac{d\beta_J^*}{d\delta}\right|_{g_L=0} &= \frac{d\bar{\beta}^*}{d\delta} < 0.\end{aligned}$$

Moreover, it holds that

$$\frac{d\beta^*}{dL} = \frac{d\beta^*}{da} = \frac{d\beta^*}{d\sigma} = 0.$$

As to the remaining comparative statics observe that

$$\begin{aligned}\left.\frac{d\beta^*}{dg_L}\right|_{g_L=0} &= \frac{-\frac{\gamma\eta\theta\lambda\mu(1-\tau)}{(1-\phi)(\delta(1-\tau)+\rho)^2} - \frac{\gamma(1-\mu)(1-\tau)\left(\frac{\lambda(\eta\theta-\eta+1)}{1-\phi}-1\right)}{\rho^2}}{1-\frac{\gamma\delta\mu(1-\tau)}{\delta(1-\tau)+\rho}} \\ &- \frac{\left(\frac{\gamma\mu(1-\tau)}{\delta(1-\tau)+\rho}+\frac{\gamma(1-\mu)(1-\tau)}{\rho}\right)\left(\frac{\gamma\delta\eta\theta\lambda\mu(1-\tau)}{(1-\phi)(\delta(1-\tau)+\rho)^2}+\frac{\gamma\lambda(1-\mu)(1-\tau)}{\rho(1-\phi)}\right)}{\left(1-\frac{\gamma\delta\mu(1-\tau)}{\delta(1-\tau)+\rho}\right)^2} \gtreqless 0,\end{aligned}$$

$$\begin{aligned}\frac{d\beta^*}{d\lambda} &\approx \left.\frac{\partial\beta^*}{d\lambda}\right|_{g_L=0} + \left.\frac{\partial^2\beta^*}{\partial\lambda\partial g_L}\right|_{g_L=0}(g_L-0) \\ &= 0 + \frac{-\frac{\gamma\eta\theta\mu(1-\tau)}{(1-\phi)(\delta(1-\tau)+\rho)^2} - \frac{\gamma(1-\mu)(1-\tau)(\eta\theta-\eta+1)}{\rho^2(1-\phi)}}{1-\frac{\gamma\delta\mu(1-\tau)}{\delta(1-\tau)+\rho}}(g_L-0) \\ &- \frac{\left(\frac{\gamma\mu(1-\tau)}{\delta(1-\tau)+\rho}+\frac{\gamma(1-\mu)(1-\tau)}{\rho}\right)\left(\frac{\gamma\delta\eta\theta\mu(1-\tau)}{(1-\phi)(\delta(1-\tau)+\rho)^2}+\frac{\gamma(1-\mu)(1-\tau)}{\rho(1-\phi)}\right)}{\left(1-\frac{\gamma\delta\mu(1-\tau)}{\delta(1-\tau)+\rho}\right)^2}(g_L-0) < 0,\end{aligned}$$

$$
\begin{aligned}
\frac{d\beta^*}{d\phi} &\approx \left.\frac{d\beta^*}{d\phi}\right|_{g_L=0} + \left.\frac{\partial^2\beta^*}{\partial\phi\partial g_L}\right|_{g_L=0} (g_L - 0) \\
&= 0 + \frac{-\frac{\gamma\eta\theta\lambda\mu(1-\tau)}{(1-\phi)^2(\delta(1-\tau)+\rho)^2} - \frac{\gamma\lambda(1-\mu)(1-\tau)(\eta\theta-\eta+1)}{\rho^2(1-\phi)^2}}{1-\frac{\gamma\delta\mu(1-\tau)}{\delta(1-\tau)+\rho}} (g_L - 0) \\
&- \frac{\left(\frac{\gamma\mu(1-\tau)}{\delta(1-\tau)+\rho} + \frac{\gamma(1-\mu)(1-\tau)}{\rho}\right)\left(\frac{\gamma\delta\eta\theta\lambda\mu(1-\tau)}{(1-\phi)^2(\delta(1-\tau)+\rho)^2} + \frac{\gamma\lambda(1-\mu)(1-\tau)}{\rho(1-\phi)^2}\right)}{\left(1-\frac{\gamma\delta\mu(1-\tau)}{\delta(1-\tau)+\rho}\right)^2} (g_L - 0) < 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\beta^*}{d\theta} &\approx \left.\frac{\partial\beta^*}{\partial\theta}\right|_{g_L=0} + \left.\frac{\partial^2\beta^*}{\partial\theta\partial g_L}\right|_{g_L=0} (g_L - 0) \\
&= 0 - \frac{\gamma\eta\lambda(1-\tau)\left(\delta^2(1-\mu)(1-\tau)^2(1-\gamma\mu) + 2\delta(1-\mu)\rho(1-\tau) + \rho^2\right)}{\rho^2(1-\phi)(\delta(1-\tau)(1-\gamma\mu)+\rho)^2} (g_L - 0) < 0.
\end{aligned}
$$

To verify that indeed $d\beta^*/dg_L \gtreqless 0$ consider a calibration exercise with the following parameter values:

$$\gamma = 0.36, \quad \rho = 0.06, \quad \delta = 0.15, \quad \theta = 2,$$

$$\tau = 0.25, \quad \lambda = 0.4, \quad \phi = 0.45.$$

Then, it holds that

$$\left.\frac{d\beta^*}{dg_L}\right|_{g_L=0} < 0 \quad \text{if} \quad \mu \in [\gamma, 0.914144),$$

$$\left.\frac{d\beta^*}{dg_L}\right|_{g_L=0} > 0 \quad \text{if} \quad \mu \in (0.914144, 1),$$

$$\left.\frac{d\beta^*}{dg_L}\right|_{g_L=0} = 0 \quad \text{if} \quad \mu = 0.914144.$$

■

6.15 Proof of Proposition 15

Recall that $\alpha_J^* = \tilde{r}_J^* \times \beta_J^*$. Then, it holds that

$$
\begin{aligned}
\left.\frac{d\alpha_J^*}{d\delta}\right|_{g_L=0} &= \frac{d\bar{\alpha}^*}{d\delta} < 0, \quad \left.\frac{d\alpha_J^*}{d\gamma}\right|_{g_L=0} = \frac{d\bar{\alpha}^*}{d\gamma} > 0, \\
\left.\frac{d\alpha_J^*}{d\rho}\right|_{g_L=0} &= \frac{d\bar{\alpha}^*}{d\rho} > 0, \quad \left.\frac{d\alpha_J^*}{d\tau}\right|_{g_L=0} = \frac{d\bar{\alpha}^*}{d\tau} > 0, \\
\left.\frac{d\alpha_J^*}{d\mu}\right|_{g_L=0} &= \frac{d\bar{\alpha}^*}{d\mu} < 0.
\end{aligned}
$$

One readily verifies that α_J^* does not depend on L, a, and σ. The following first-order Taylor approximations deliver the signs of the remaining comparative statics:

$$\left.\frac{d\alpha^*}{d\theta}\right|_{g_L=0} \approx \left.\frac{\partial\alpha^*}{\partial\theta}\right|_{g_L=0} + \left.\frac{\partial^2\alpha^*}{\partial\theta\partial g_L}\right|_{g_L=0}(g_L-0) = 0+0=0,$$

$$\left.\frac{d\alpha^*}{dg_L}\right|_{g_L=0} = -\frac{\gamma^2\lambda(1-\mu)(1-\tau)\left(\delta^2(1-\mu)(1-\tau)^2-\delta\rho(\theta-(2-\mu)(1-\tau))+\rho^2\right)}{\rho(1-\phi)(\delta(1-\tau)(1-\gamma\mu)+\rho)^2} \gtreqless 0,$$

$$\left.\frac{d\alpha^*}{d\lambda}\right|_{g_L=0} \approx \left.\frac{\partial\alpha^*}{\partial\lambda}\right|_{g_L=0} + \left.\frac{\partial^2\alpha^*}{\partial\lambda\partial g_L}\right|_{g_L=0}(g_L-0)$$

$$= 0-\frac{\gamma^2(1-\mu)(1-\tau)\left(\delta^2(1-\mu)(1-\tau)^2-\delta\rho(\theta-(2-\mu)(1-\tau))+\rho^2\right)}{\rho(1-\phi)(\delta(1-\tau)(1-\gamma\mu)+\rho)^2}(g_L-0) \gtreqless 0,$$

$$\left.\frac{d\alpha^*}{d\phi}\right|_{g_L=0} \approx \left.\frac{\partial\alpha^*}{\partial\phi}\right|_{g_L=0} + \left.\frac{\partial^2\alpha^*}{\partial\phi\partial g_L}\right|_{g_L=0}(g_L-0)$$

$$= 0+\frac{\gamma^2\lambda(1-\mu)(1-\tau)\left(-\delta^2(1-\mu)(1-\tau)^2+\delta\rho(\theta-(2-\mu)(1-\tau))-\rho^2\right)}{\rho(1-\phi)^2(\delta(1-\tau)(1-\gamma\mu)+\rho)^2}(g_L-0) \gtreqless 0.$$

To verify that indeed the comparative statics with respect to g_L, λ and ϕ are not unequivocal consider a calibration exercise with the following parameter values:

$$\gamma=0.36, \quad \rho=0.06, \quad \delta=0.15, \quad \theta=2,$$

$$\tau=0.25, \quad \lambda=0.7, \quad \phi=0.9.$$

Then, it holds that

$$\left.\frac{d\alpha^*}{dg_L}\right|_{g_L=0} < 0 \quad \text{if} \quad \mu\in[\gamma, 0.605797),$$

$$\left.\frac{d\alpha^*}{dg_L}\right|_{g_L=0} > 0 \quad \text{if} \quad \mu\in(0.605797, 1),$$

$$\left.\frac{d\alpha^*}{dg_L}\right|_{g_L=0} = 0 \quad \text{if} \quad \mu=0.605797.$$

Moreover, we have

$$\left.\frac{d\alpha^*}{d\phi}\right|_{g_L=0} \approx \left.\frac{\partial\alpha^*}{\partial\phi}\right|_{g_L=0} + \left.\frac{\partial^2\alpha^*}{\partial\phi\partial g_L}\right|_{g_L=0}(g_L - 0) < 0 \quad \text{if} \quad \mu \in [\gamma, 0.605797),$$

$$\left.\frac{d\alpha^*}{d\phi}\right|_{g_L=0} \approx \left.\frac{\partial\alpha^*}{\partial\phi}\right|_{g_L=0} + \left.\frac{\partial^2\alpha^*}{\partial\phi\partial g_L}\right|_{g_L=0}(g_L - 0) > 0 \quad \text{if} \quad \mu \in (0.605797, 1),$$

$$\left.\frac{d\alpha^*}{d\phi}\right|_{g_L=0} \approx \left.\frac{\partial\alpha^*}{\partial\phi}\right|_{g_L=0} + \left.\frac{\partial^2\alpha^*}{\partial\phi\partial g_L}\right|_{g_L=0}(g_L - 0) = 0 \quad \text{if} \quad \mu = 0.605797.$$

The same qualitative results obtain for a Taylor approximation of $d\alpha^*/d\lambda$ around $g_L = 0$. ■

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