From modular curves to Shimura curves: a $p$-adic approach

Laia Amorós Carafí

Université du Luxembourg
Universitat de Barcelona
1. Modular curves and elliptic curves
1. Modular curves and elliptic curves
2. Quaternion algebras and Shimura curves
1. Modular curves and elliptic curves
2. Quaternion algebras and Shimura curves
3. $p$-adic uniformisation of Shimura curves
1. Modular curves and elliptic curves
2. Quaternion algebras and Shimura curves
3. $p$-adic uniformisation of Shimura curves
4. Bad reduction of Shimura curves
1. MODULAR CURVES AND ELLIPTIC CURVES
Modular curves

\[ \mathcal{H}_\infty = \{ a + bi \in \mathbb{C} \mid b > 0 \} \subseteq \mathbb{C} \text{ complex upper half-plane.} \]
Modular curves

\[ \mathcal{H}_\infty = \{ a + bi \in \mathbb{C} \mid b > 0 \} \subseteq \mathbb{C} \text{ complex upper half-plane.} \]

\( SL_2(\mathbb{R}) \) acts on \( \mathcal{H}_\infty \):
Modular curves

\[ \mathcal{H}_\infty = \{ a + bi \in \mathbb{C} \mid b > 0 \} \subseteq \mathbb{C} \text{ complex upper half-plane.} \]

\( \text{SL}_2(\mathbb{R}) \) acts on \( \mathcal{H}_\infty \): \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \)
Modular curves

\[ \mathcal{H}_\infty = \{ a + bi \in \mathbb{C} \mid b > 0 \} \subseteq \mathbb{C} \text{ complex upper half-plane.} \]

\( \text{SL}_2(\mathbb{R}) \) acts on \( \mathcal{H}_\infty \):

\[ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad z \in \mathcal{H}_\infty \]
Modular curves

\[ \mathcal{H}_\infty = \{ a + bi \in \mathbb{C} \mid b > 0 \} \subseteq \mathbb{C} \] complex upper half-plane.

\( \text{SL}_2(\mathbb{R}) \) acts on \( \mathcal{H}_\infty \):

\[ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad z \in \mathcal{H}_\infty \]

\[ \alpha(z) := \frac{az + b}{cz + d} \in \mathcal{H}_\infty. \]
Modular curves

\[ \mathcal{H}_\infty = \{ a + bi \in \mathbb{C} \mid b > 0 \} \subseteq \mathbb{C} \text{ complex upper half-plane.} \]

\( \text{SL}_2(\mathbb{R}) \) acts on \( \mathcal{H}_\infty \):
\[ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad z \in \mathcal{H}_\infty \]

\[ \alpha(z) := \frac{az + b}{cz + d} \in \mathcal{H}_\infty. \]

\( \text{SL}_2(\mathbb{Z}) \) is a discrete subgroup that acts on \( \mathcal{H}_\infty \),
Modular curves

\[ H_\infty = \{ a + bi \in \mathbb{C} \mid b > 0 \} \subseteq \mathbb{C} \text{ complex upper half-plane.} \]

\( \text{SL}_2(\mathbb{R}) \) acts on \( H_\infty \):
\[ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad z \in H_\infty \]
\[ \alpha(z) := \frac{az + b}{cz + d} \in H_\infty. \]

\( \text{SL}_2(\mathbb{Z}) \) is a discrete subgroup that acts on \( H_\infty \), the \textbf{full modular group}. 

\[ Y(N) := \Gamma(N) / H_\infty \text{ Riemann surface, noncompact.} \]

Adding a finite number of points we can compactify it, we denote it by \( X(N) \).
This curve can be regarded, in a natural way, as an algebraic curve defined by some homogeneous polynomial(s) with coefficients in \( \mathbb{Q}(\zeta_N) \).
We call this the \textbf{modular curve} of level \( N \).
Modular curves

\[ \mathcal{H}_\infty = \{ a + bi \in \mathbb{C} \mid b > 0 \} \subseteq \mathbb{C} \] complex upper half-plane.

\( \text{SL}_2(\mathbb{R}) \) acts on \( \mathcal{H}_\infty \):
\[
\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad z \in \mathcal{H}_\infty
\]

\[
\alpha(z) := \frac{az + b}{cz + d} \in \mathcal{H}_\infty.
\]

\( \text{SL}_2(\mathbb{Z}) \) is a discrete subgroup that acts on \( \mathcal{H}_\infty \), the full modular group. For \( N \in \mathbb{Z} \), we define the principal congruence subgroup of level \( N \):

\[
\Gamma(N) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \} \subseteq \text{SL}_2(\mathbb{Z}).
\]
**Modular curves**

\[ \mathcal{H}_\infty = \{ a + bi \in \mathbb{C} \mid b > 0 \} \subseteq \mathbb{C} \] complex upper half-plane.

\[ \text{SL}_2(\mathbb{R}) \text{ acts on } \mathcal{H}_\infty : \quad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad z \in \mathcal{H}_\infty \]

\[ \alpha(z) := \frac{az + b}{cz + d} \in \mathcal{H}_\infty. \]

\[ \text{SL}_2(\mathbb{Z}) \text{ is a discrete subgroup that acts on } \mathcal{H}_\infty, \text{ the full modular group.} \]

For \( N \in \mathbb{Z} \), we define the **principal congruence subgroup** of level \( N \):

\[ \Gamma(N) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \} \subseteq \text{SL}_2(\mathbb{Z}). \]

\[ Y(N) := \Gamma(N)/\mathcal{H}_\infty \text{ Riemann surface, noncompact}. \]
Modular curves

$$\mathcal{H}_\infty = \{ a + bi \in \mathbb{C} \mid b > 0 \} \subseteq \mathbb{C} \text{ complex upper half-plane.}$$

$\text{SL}_2(\mathbb{R})$ acts on $\mathcal{H}_\infty$:

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad z \in \mathcal{H}_\infty$$

$$\alpha(z) := \frac{az + b}{cz + d} \in \mathcal{H}_\infty.$$ 

$\text{SL}_2(\mathbb{Z})$ is a discrete subgroup that acts on $\mathcal{H}_\infty$, the full modular group. For $N \in \mathbb{Z}$, we define the principal congruence subgroup of level $N$:

$$\Gamma(N) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ mod } N \} \subseteq \text{SL}_2(\mathbb{Z}).$$

$Y(N) := \Gamma(N)/\mathcal{H}_\infty$ Riemann surface, noncompact.

Adding a finite number of points we can compactify it, we denote it by $X(N)$. 
Modular curves

\[ \mathcal{H}_\infty = \{ a + bi \in \mathbb{C} \mid b > 0 \} \subseteq \mathbb{C} \text{ complex upper half-plane.} \]

\( \text{SL}_2(\mathbb{R}) \) acts on \( \mathcal{H}_\infty \): \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad z \in \mathcal{H}_\infty \)

\[ \alpha(z) := \frac{az + b}{cz + d} \in \mathcal{H}_\infty. \]

\( \text{SL}_2(\mathbb{Z}) \) is a discrete subgroup that acts on \( \mathcal{H}_\infty \), the full modular group. For \( N \in \mathbb{Z} \), we define the principal congruence subgroup of level \( N \):

\[ \Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\} \subseteq \text{SL}_2(\mathbb{Z}). \]

\( Y(N) := \Gamma(N)/\mathcal{H}_\infty \) Riemann surface, noncompact.

Adding a finite number of points we can compactify it, we denote it by \( X(N) \). This curve can be regarded, in a natural way, as an algebraic curve defined by some homogeneous polynomial(s) with coefficients in \( \mathbb{Q}(\zeta_N) \).
Modular curves

\[ \mathcal{H}_\infty = \{ a + bi \in \mathbb{C} \mid b > 0 \} \subseteq \mathbb{C} \text{ complex upper half-plane.} \]

\( \text{SL}_2(\mathbb{R}) \) acts on \( \mathcal{H}_\infty \):

\[ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad z \in \mathcal{H}_\infty \]

\[ \alpha(z) := \frac{az + b}{cz + d} \in \mathcal{H}_\infty. \]

\( \text{SL}_2(\mathbb{Z}) \) is a discrete subgroup that acts on \( \mathcal{H}_\infty \), the \textbf{full modular group}.

For \( N \in \mathbb{Z} \), we define the \textbf{principal congruence subgroup} of level \( N \):

\[ \Gamma(N) := \{ (a b) \equiv (1 0) \mod N \} \subseteq \text{SL}_2(\mathbb{Z}). \]

\( Y(N) := \Gamma(N)/\mathcal{H}_\infty \) Riemann surface, noncompact.

Adding a finite number of points we can compactify it, we denote it by \( X(N) \). This curve can be regarded, in a natural way, as an algebraic curve defined by some homogeneous polynomial(s) with coefficients in \( \mathbb{Q}(\zeta_N) \). We call this the \textbf{modular curve} of level \( N \).
Modular curves

Fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}_\infty$
Modular curves and elliptic curves

A **lattice** in $\mathbb{C}$ is a subset of the form $\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1, \omega_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$.
A **lattice** in $\mathbb{C}$ is a subset of the form $\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1, \omega_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$. We can always normalise our lattices: $\Lambda = \Lambda(\tau) := \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau$, for some $\tau \in \mathcal{H}_\infty$. 

We define $P(z) := \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda(\tau)} \frac{1}{(z - \lambda)^2 - 1}$.

This is a meromorphic function on $\mathbb{C}$, invariant under $\Lambda$, and the map $\mathbb{C}/\Lambda \rightarrow \mathbb{P}^2(\mathbb{C}) [z] \mapsto (P(z) : P'(z) : 1)$ defines an isomorphism of Riemann surfaces from $\mathbb{C}/\Lambda$ to $E(C)$, where $E$ is the elliptic curve $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$, with $g_2$ and $g_3$ are determined from the lattice $\Lambda$. 


Modular curves and elliptic curves

A lattice in $\mathbb{C}$ is a subset of the form $\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1, \omega_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$. We can always normalise our lattices:

$\Lambda = \Lambda(\tau) := \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau$, for some $\tau \in \mathcal{H}_\infty$.

$\Lambda(\tau) = \Lambda(\tau') \iff \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ such that $\tau' = \frac{a\tau + b}{c\tau + d}$.
Modular curves and elliptic curves

A lattice in \( \mathbb{C} \) is a subset of the form \( \Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \) with \( \omega_1, \omega_2 \in \mathbb{C} \) linearly independent over \( \mathbb{R} \). We can always normalise our lattices:

\[ \Lambda = \Lambda(\tau) := \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau, \text{ for some } \tau \in \mathcal{H}_\infty. \]

\( \Lambda(\tau) = \Lambda(\tau') \Leftrightarrow \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \) such that \( \tau' = \frac{a\tau + b}{c\tau + d} \).

The quotient \( \mathbb{C}/\Lambda \) is topologically a torus.
Modular curves and elliptic curves

A lattice in $\mathbb{C}$ is a subset of the form $\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1, \omega_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$. We can always normalise our lattices: $\Lambda = \Lambda(\tau) := \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau$, for some $\tau \in \mathcal{H}_\infty$.

$\Lambda(\tau) = \Lambda(\tau') \iff \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ such that $\tau' = \frac{a\tau + b}{c\tau + d}$.

The quotient $\mathbb{C}/\Lambda$ is topologically a torus. We define

$$\mathcal{P}(z) := \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right).$$
Modular curves and elliptic curves

A **lattice** in \( \mathbb{C} \) is a subset of the form \( \Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \) with \( \omega_1, \omega_2 \in \mathbb{C} \) linearly independent over \( \mathbb{R} \). We can always normalise our lattices:
\[
\Lambda = \Lambda(\tau) := \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau, \quad \text{for some } \tau \in \mathcal{H}_\infty.
\]

\( \Lambda(\tau) = \Lambda(\tau') \iff \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \text{ such that } \tau' = \frac{a\tau + b}{c\tau + d} \).

The quotient \( \mathbb{C}/\Lambda \) is topologically a torus. We define
\[
\mathcal{P}(z) := \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right).
\]

This is a meromorphic function on \( \mathbb{C} \), invariant under \( \Lambda \), and the map
\[
\mathbb{C}/\Lambda \quad \rightarrow \quad \mathbb{P}^2(\mathbb{C}) \\
[z] \quad \mapsto \quad (\mathcal{P}(z) : \mathcal{P}'(z) : 1)
\]
defines an isomorphism of Riemann surfaces from \( \mathbb{C}/\Lambda \) to \( E(\mathbb{C}) \), where \( E \) is the **elliptic curve**
\[
Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3,
\]
with \( g_2 \) and \( g_3 \) are determined from the lattice \( \Lambda \).
Modular curves and elliptic curves
Modular curves and elliptic curves

Real points of two elliptic curves

1. \( y^2 = x^3 - x \)
2. \( y^2 = x^3 - x + 1 \)
To **uniformise** a curve over \(\mathbb{C}\) means to express its complex points by means of coordinate functions. The algebraic relations that these coordinate functions satisfy will give some equations that define the curve and that allow us to compute its points.

**Examples:**
1. The **unit circle**.

Let \(S_1 : x^2 + y^2 = 1\) and consider \(\langle \tau \rangle \subseteq \text{Aut}(\mathbb{C})\), \(\tau : z \mapsto z + 2T\), for some \(T \in \mathbb{C}\).

Then there exist holomorphic functions \(F, G : U \subseteq \mathbb{C} \to \mathbb{C}\) invariant under the group \(\langle \tau \rangle\) such that there is a bijection \(\langle \tau \rangle \mathbb{C} \to S_1(\mathbb{C}) \subseteq \mathbb{P}^2(\mathbb{C})\) with:

- coordinate functions: \(F(t) = \cos(t)\) and \(G(t) = \sin(t)\)
- \(F\) and \(G\) are invariant under multiples of 2 \(T = 2\pi\)
- they satisfy the algebraic equation \(F(t)^2 + G(t)^2 = 1\)
To uniformise (intuitively)

To uniformise a curve over \( \mathbb{C} \) means to express its complex points by means of coordinate functions. The algebraic relations that these coordinate functions satisfy will give some equations that define the curve and that allow us to compute its points.

**Examples:** 1. The **unit circle**.

\[ S^1 : x^2 + y^2 = 1 \]
To uniformise (intuitively)

To **uniformise** a curve over \( \mathbb{C} \) means to express its complex points by means of coordinate functions. The algebraic relations that these coordinate functions satisfy will give some equations that define the curve and that allow us to compute its points.

**Examples:** 1. The **unit circle**. Let \( S^1 : x^2 + y^2 = 1 \) and consider \( \langle \tau \rangle \subseteq \text{Aut}(\mathbb{C}) \), \( \tau : z \mapsto z + 2T \), for some \( T \in \mathbb{C} \).
To uniformise (intuitively)

To **uniformise** a curve over $\mathbb{C}$ means to express its complex points by means of coordinate functions. The algebraic relations that these coordinate functions satisfy will give some equations that define the curve and that allow us to compute its points.

**Examples:** 1. The **unit circle**. Let $S^1 : x^2 + y^2 = 1$ and consider $\langle \tau \rangle \subseteq \text{Aut}(\mathbb{C})$, $\tau : z \mapsto z + 2T$, for some $T \in \mathbb{C}$.

Then there exist homomorphic functions $F, G : U \subseteq \mathbb{C} \to \mathbb{C}$ invariant under the group $\langle \tau \rangle$ such that there is a bijection

$$\langle \tau \rangle \setminus \mathbb{C} \to S^1(\mathbb{C}) \subseteq \mathbb{P}^2(\mathbb{C})$$

$$t \mapsto (F(t) : G(t) : 1)$$
To uniformise (intuitively)

To **uniformise** a curve over $\mathbb{C}$ means to express its complex points by means of coordinate functions. The algebraic relations that these coordinate functions satisfy will give some equations that define the curve and that allow us to compute its points.

**Examples:** 1. The **unit circle**. Let $S^1 : x^2 + y^2 = 1$ and consider $\langle \tau \rangle \subseteq \text{Aut}(\mathbb{C})$, $\tau : z \mapsto z + 2T$, for some $T \in \mathbb{C}$.

Then there exist homomorphic functions $F, G : U \subseteq \mathbb{C} \to \mathbb{C}$ invariant under the group $\langle \tau \rangle$ such that there is a bijection

$$\langle \tau \rangle \backslash \mathbb{C} \rightarrow S^1(\mathbb{C}) \subseteq \mathbb{P}^2(\mathbb{C})$$

$$t \mapsto (F(t) : G(t) : 1)$$

- coordinate functions: $F(t) = \cos(t)$ and $G(t) = \sin(t)$
To uniformise (intuitively)

To **uniformise** a curve over $\mathbb{C}$ means to express its complex points by means of coordinate functions. The algebraic relations that these coordinate functions satisfy will give some equations that define the curve and that allow us to compute its points.

**Examples:** 1. The **unit circle**. Let $S^1 : x^2 + y^2 = 1$ and consider $\langle \tau \rangle \subseteq \text{Aut}(\mathbb{C})$, $\tau : z \mapsto z + 2T$, for some $T \in \mathbb{C}$.

Then there exist homolorphic functions $F, G : U \subseteq \mathbb{C} \to \mathbb{C}$ invariant under the group $\langle \tau \rangle$ such that there is a bijection

$$
\langle \tau \rangle \setminus \mathbb{C} \to S^1(\mathbb{C}) \subseteq \mathbb{P}^2(\mathbb{C})
$$

$$
t \mapsto (F(t) : G(t) : 1)
$$

- coordinate functions: $F(t) = \cos(t)$ and $G(t) = \sin(t)$
- $F$ and $G$ are invariant under multiples of $2T = 2\pi$
To uniformise (intuitively)

To **uniformise** a curve over $\mathbb{C}$ means to express its complex points by means of coordinate functions. The algebraic relations that these coordinate functions satisfy will give some equations that define the curve and that allow us to compute its points.

**Examples:** 1. The **unit circle**. Let $S^1 : x^2 + y^2 = 1$ and consider $\langle \tau \rangle \subseteq \text{Aut}(\mathbb{C})$, $\tau : z \mapsto z + 2T$, for some $T \in \mathbb{C}$.

Then there exist homomorphic functions $F, G : U \subseteq \mathbb{C} \to \mathbb{C}$ invariant under the group $\langle \tau \rangle$ such that there is a bijection

$$\langle \tau \rangle \setminus \mathbb{C} \to S^1(\mathbb{C}) \subseteq \mathbb{P}^2(\mathbb{C})$$

$$t \mapsto (F(t) : G(t) : 1)$$

- coordinate functions: $F(t) = \cos(t)$ and $G(t) = \sin(t)$
- $F$ and $G$ are invariant under multiples of $2T = 2\pi$
- they satisfy the algebraic equation $F(t)^2 + G(t)^2 = 1$
To uniformise (intuitively)

2. **Elliptic curves:**

\[ E : y^2 = x^3 + ax^2 + bx + c \] elliptic curve over \( \mathbb{C} \), \( \Lambda = \langle T_1, T_2 \rangle \subseteq \mathbb{C} \) lattice.

There is a bijection \( \mathbb{C} / \Lambda \to E(\mathbb{C}) \)

\[ z \mapsto \left( P(z), P'(z), 1 \right) \]

• coordinate functions: \( P \) and \( P' \)

• \( P \) and \( P' \) are doubly periodic functions, invariant under translations on the lattice

• they satisfy the algebraic equation \( y^2 = x^3 + ax^2 + bx + c \)

3. **Modular curves:**

We have the following uniformisation:

\[ SL_2(\mathbb{Z}) \backslash \mathbb{H} \to Y(1)(\mathbb{C}) \]

\[ z \mapsto j(z) \]

where

\[ j(z) := q^{-1} + 744 + 196884q + 21493760q^2 + \ldots, \]

\( q = e^{2\pi iz} \) is the Klein \( j \)-function.
To uniformise (intuitively)

2. **Elliptic curves**: \( E : y^2 = x^3 + ax^2 + bx + c \) elliptic curve over \( \mathbb{C} \),
\( \Lambda = \langle T_1, T_2 \rangle_{\mathbb{Z}} \subseteq \mathbb{C} \) lattice.
To uniformise (intuitively)

2. **Elliptic curves**: \( E : y^2 = x^3 + ax^2 + bx + c \) elliptic curve over \( \mathbb{C} \), \( \Lambda = \langle T_1, T_2 \rangle \mathbb{Z} \subseteq \mathbb{C} \) lattice. There is a bijection

\[
\mathbb{C}/\Lambda \rightarrow E(\mathbb{C})
\]

\[
z \mapsto (P(z) : P'(z) : 1)
\]

• coordinate functions: \( P \) and \( P' \)
• \( P \) and \( P' \) are doubly periodic functions, invariant under translations on the lattice
• they satisfy the algebraic equation \( y^2 = x^3 + ax^2 + bx + c \)

3. Modular curves:

We have the following uniformisation:

\[
\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}_\infty \rightarrow Y(1)(\mathbb{C})
\]

\[
z \mapsto j(z)
\]

where \( j(z) := q^{-1} + 744 + 196884q + 21493760q^2 + \ldots \), \( q = e^{2\pi i z} \) is the Klein j-function.
To uniformise (intuitively)

2. **Elliptic curves:** $E : y^2 = x^3 + ax^2 + bx + c$ elliptic curve over $\mathbb{C}$, $\Lambda = \langle T_1, T_2 \rangle_{\mathbb{Z}} \subseteq \mathbb{C}$ lattice. There is a bijection

$$\mathbb{C}/\Lambda \rightarrow E(\mathbb{C})$$

$$z \mapsto (P(z) : P'(z) : 1)$$

- coordinate functions: $P$ and $P'$
To uniformise (intuitively)

2. **Elliptic curves:** $E : y^2 = x^3 + ax^2 + bx + c$ elliptic curve over $\mathbb{C}$, $\Lambda = \langle T_1, T_2 \rangle \subseteq \mathbb{C}$ lattice. There is a bijection

$$\mathbb{C}/\Lambda \rightarrow E(\mathbb{C})$$

$z \mapsto (P(z) : P'(z) : 1)$

- coordinate functions: $P$ and $P'$
- $P$ and $P'$ are doubly periodic functions, invariant under translations on the lattice
2. **Elliptic curves**: \( E : y^2 = x^3 + ax^2 + bx + c \) elliptic curve over \( \mathbb{C} \), \( \Lambda = \langle T_1, T_2 \rangle \mathbb{Z} \subseteq \mathbb{C} \) lattice. There is a bijection

\[
\mathbb{C}/\Lambda \rightarrow E(\mathbb{C}) \\
z \mapsto (\mathcal{P}(z) : \mathcal{P}'(z) : 1)
\]

- coordinate functions: \( \mathcal{P} \) and \( \mathcal{P}' \)
- \( \mathcal{P} \) and \( \mathcal{P}' \) are doubly periodic functions, invariant under translations on the lattice
- they satisfy the algebraic equation \( y^2 = x^3 + ax^2 + bx + c \)
To uniformise (intuitively)

2. **Elliptic curves:** \( E : y^2 = x^3 + ax^2 + bx + c \) elliptic curve over \( \mathbb{C} \), \( \Lambda = \langle T_1, T_2 \rangle_\mathbb{Z} \subseteq \mathbb{C} \) lattice. There is a bijection

\[
\mathbb{C}/\Lambda \rightarrow E(\mathbb{C}) \\
z \mapsto (P(z) : P'(z) : 1)
\]

- coordinate functions: \( P \) and \( P' \)
- \( P \) and \( P' \) are doubly periodic functions, invariant under translations on the lattice
- they satisfy the algebraic equation \( y^2 = x^3 + ax^2 + bx + c \)

3. **Modular curves:**
To uniformise (intuitively)

2. **Elliptic curves:** \( E : y^2 = x^3 + ax^2 + bx + c \) elliptic curve over \( \mathbb{C} \), \( \Lambda = \langle T_1, T_2 \rangle \mathbb{Z} \subseteq \mathbb{C} \) lattice. There is a bijection

\[
\mathbb{C}/\Lambda \rightarrow E(\mathbb{C}) \quad z \mapsto (P(z) : P'(z) : 1)
\]

- coordinate functions: \( P \) and \( P' \)
- \( P \) and \( P' \) are doubly periodic functions, invariant under translations on the lattice
- they satisfy the algebraic equation \( y^2 = x^3 + ax^2 + bx + c \)

3. **Modular curves:** We have the following uniformisation:
To uniformise (intuitively)

2. **Elliptic curves:** \( E : y^2 = x^3 + ax^2 + bx + c \) elliptic curve over \( \mathbb{C} \), \( \Lambda = \langle T_1, T_2 \rangle \mathbb{Z} \subseteq \mathbb{C} \) lattice. There is a bijection

\[
\mathbb{C}/\Lambda \rightarrow E(\mathbb{C})
\]

\[
z \mapsto (P(z) : P'(z) : 1)
\]

- coordinate functions: \( P \) and \( P' \)
- \( P \) and \( P' \) are doubly periodic functions, invariant under translations on the lattice
- they satisfy the algebraic equation \( y^2 = x^3 + ax^2 + bx + c \)

3. **Modular curves:** We have the following uniformisation:

\[
\text{SL}_2(\mathbb{Z}) \setminus \mathcal{H}_\infty \rightarrow Y(1)(\mathbb{C})
\]

\[
z \mapsto j(z)
\]
To uniformise (intuitively)

2. **Elliptic curves:** \( E : y^2 = x^3 + ax^2 + bx + c \) elliptic curve over \( \mathbb{C} \), \( \Lambda = \langle T_1, T_2 \rangle \mathbb{Z} \subseteq \mathbb{C} \) lattice. There is a bijection

\[
\begin{array}{ccc}
\mathbb{C}/\Lambda & \rightarrow & E(\mathbb{C}) \\
 z & \mapsto & (P(z) : P'(z) : 1)
\end{array}
\]

- coordinate functions: \( P \) and \( P' \)
- \( P \) and \( P' \) are doubly periodic functions, invariant under translations on the lattice
- they satisfy the algebraic equation \( y^2 = x^3 + ax^2 + bx + c \)

3. **Modular curves:** We have the following uniformisation:

\[
\begin{array}{ccc}
\text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}_\infty & \rightarrow & Y(1)(\mathbb{C}) \\
 z & \mapsto & j(z)
\end{array}
\]

where

\[
j(z) := q^{-1} + 744 + 196884q + 21493760q^2 + \ldots, \quad q = e^{2\pi iz}
\]

is the Klein \( j \)-function.
2. QUATERNION ALGEBRAS AND SHIMURA CURVES
SL_2(\mathbb{Q}) is a matrix algebra over \mathbb{Q};
Quaternions over $\mathbb{Q}$

$\text{SL}_2(\mathbb{Q})$ is a matrix algebra over $\mathbb{Q}$; it is a particular case of an \textit{indefinite quaternion algebra} over $\mathbb{Q}$.
**Quaternion algebras over \( \mathbb{Q} \)**

\( \text{SL}_2(\mathbb{Q}) \) is a matrix algebra over \( \mathbb{Q} \); it is a particular case **indefinite quaternion algebra** over \( \mathbb{Q} \).

**Definition.** Let \( a, b \in \mathbb{Q}^\times \). A **quaternion algebra** over \( \mathbb{Q} \) is a simple and central algebra over \( \mathbb{Q} \) of dimension 4

\[
B = \left( \frac{a, b}{\mathbb{Q}} \right) := \{ x + yi + zj + tk \mid x, y, z, t \in \mathbb{Q} \}
\]

such that \( i^2 = a, \ j^2 = b, \ k = ij = -ji \).
SL$_2$(\(\mathbb{Q}\)) is a matrix algebra over \(\mathbb{Q}\); it is a particular case **indefinite quaternion algebra** over \(\mathbb{Q}\).

**Definition.** Let \(a, b \in \mathbb{Q}^\times\). A **quaternion algebra** over \(\mathbb{Q}\) is a simple and central algebra over \(\mathbb{Q}\) of dimension 4

\[
B = \left( \frac{a, b}{\mathbb{Q}} \right) := \{x + yi + zj + tk \mid x, y, z, t \in \mathbb{Q}\}
\]

such that \(i^2 = a, j^2 = b, k = ij = -ji\).

- Hamilton quaternions: \(\mathbb{H} = \left( \frac{-1,-1}{\mathbb{Q}} \right)\) definite
Quat ternion algebras over \( \mathbb{Q} \)

\( SL_2(\mathbb{Q}) \) is a matrix algebra over \( \mathbb{Q} \); it is a particular case **indefinite quaternion algebra** over \( \mathbb{Q} \).

**Definition.** Let \( a, b \in \mathbb{Q}^\times \). A **quaternion algebra** over \( \mathbb{Q} \) is a simple and central algebra over \( \mathbb{Q} \) of dimension 4

\[
B = \left( \frac{a, b}{\mathbb{Q}} \right) := \{ x + yi + zj + tk \mid x, y, z, t \in \mathbb{Q} \}
\]

such that \( i^2 = a, \ j^2 = b, \ k = ij = -ji \).

- Hamilton quaternions: \( \mathbb{H} = \left( \frac{-1,-1}{\mathbb{Q}} \right) \) definite
- Matrices: \( M_2(\mathbb{Q}) = \left( \frac{1,-1}{\mathbb{Q}} \right) \) indefinite
Quat{e}rnion algebras over $\mathbb{Q}$

$\text{SL}_2(\mathbb{Q})$ is a matrix algebra over $\mathbb{Q}$; it is a particular case indefinite quaternion algebra over $\mathbb{Q}$.

**Definition.** Let $a, b \in \mathbb{Q}^\times$. A quaternion algebra over $\mathbb{Q}$ is a simple and central algebra over $\mathbb{Q}$ of dimension 4

$$B = \left( \frac{a, b}{\mathbb{Q}} \right) := \{ x + yi + zj + tk \mid x, y, z, t \in \mathbb{Q} \}$$

such that $i^2 = a, j^2 = b, k = ij = -ji$.

- Hamilton quaternions: $\mathbb{H} = (\frac{-1,-1}{\mathbb{Q}})$ definite
- Matrices: $M_2(\mathbb{Q}) = (\frac{1,-1}{\mathbb{Q}})$ indefinite

$B$ is indefinite when $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R})$ and definite when $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{H}$. 
B quaternion algebra

\[ B = \left( \frac{-1,-1}{\mathbb{Q}} \right) \]
Quaternion algebras over $\mathbb{Q}$

$B$ quaternion algebra

Discriminant of $B$

$B = \left( \frac{-1, -1}{\mathbb{Q}} \right)$

$D_B = 2$
Quaternion algebras over $\mathbb{Q}$

$B$ quaternion algebra

$B = \left( \frac{-1, -1}{\mathbb{Q}} \right)$

Discriminant of $B$

$D_B = 2$

$\mathcal{O}$ order, **non-commutative**

$\mathcal{O} = \mathbb{Z}[1, i, j, \frac{1+i+j+k}{2}]$
Quaternion algebras over $\mathbb{Q}$

- $B$ quaternion algebra
- Discriminant of $B$: $D_B = 2$
- $\mathcal{O}$ order, non-commutative
- Discriminant of $\mathcal{O}$: $D_\mathcal{O} = 2$, so $\mathcal{O}$ maximal order

$B = (\frac{-1, -1}{\mathbb{Q}})$

$\mathcal{O} = \mathbb{Z}[1, i, j, \frac{1+i+j+k}{2}]$
Quaternion algebras over $\mathbb{Q}$

$B$ quaternion algebra

discriminant of $B$

$\mathcal{O}$ order, non-commutative

discriminant of $\mathcal{O}$

integral quaternion

$B = \left( \frac{-1,-1}{\mathbb{Q}} \right)$

$D_B = 2$

$\mathcal{O} = \mathbb{Z}[1, i, j, \frac{1+i+j+k}{2}]$

$D_{\mathcal{O}} = 2$, so $\mathcal{O}$ maximal order

$h = 3/2 + 3/2i + 3/2j + 1/2k \in \mathcal{O}$
Quaternion algebras over $\mathbb{Q}$

$B$ quaternion algebra

discriminant of $B$

$\mathcal{O}$ order, **non-commutative**

discriminant of $\mathcal{O}$

integral quaternion

conjugate quaternion

\[ B = \left( \frac{-1, -1}{\mathbb{Q}} \right) \]

\[ D_B = 2 \]

\[ \mathcal{O} = \mathbb{Z}[1, i, j, \frac{1+i+j+k}{2}] \]

\[ D_{\mathcal{O}} = 2, \text{ so } \mathcal{O} \text{ maximal order} \]

\[ h = \frac{3}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{1}{2}k \in \mathcal{O} \]

\[ \bar{h} = \frac{3}{2} - \frac{3}{2}i - \frac{3}{2}j - \frac{1}{2}k \]
Quaternion algebras over $\mathbb{Q}$

$B$ quaternion algebra

discriminant of $B$

$\mathcal{O}$ order, non-commutative

discriminant of $\mathcal{O}$

integral quaternion

conjugate quaternion

trace, norm of a quaternion

\[ B = \left( \frac{-1}{Q}, \frac{-1}{Q} \right) \]

\[ D_B = 2 \]

\[ \mathcal{O} = \mathbb{Z}[1, i, j, \frac{1+i+j+k}{2}] \]

\[ D_{\mathcal{O}} = 2, \text{ so } \mathcal{O} \text{ maximal order} \]

\[ h = \frac{3}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{1}{2}k \in \mathcal{O} \]

\[ \overline{h} = \frac{3}{2} - \frac{3}{2}i - \frac{3}{2}j - \frac{1}{2}k \]

\[ \text{Tr}(h) = h + \overline{h} = 3, \quad \text{Nm}(h) = hh = 7 \]
Quaternion algebras over $\mathbb{Q}$

- $B$ quaternion algebra
  - $B = \left( \frac{-1}{\mathbb{Q}}, -1 \right)$

- Discriminant of $B$
  - $D_B = 2$

- $\mathcal{O}$ order, **non-commutative**
  - $\mathcal{O} = \mathbb{Z}[1, i, j, \frac{1+i+j+k}{2}]$
  - Discriminant of $\mathcal{O}$
    - $D_{\mathcal{O}} = 2$, so $\mathcal{O}$ maximal order

- Integral quaternion
  - $h = 3/2 + 3/2i + 3/2j + 1/2k \in \mathcal{O}$

- Conjugate quaternion
  - $\bar{h} = 3/2 - 3/2i - 3/2j - 1/2k$

- Trace, norm of a quaternion
  - $\text{Tr}(h) = h + \bar{h} = 3$, $\text{Nm}(h) = hh = 7$

- Normic form of $B$
  - $\text{Nm}_B(X, Y, Z, T) = X^2 + Y^2 + Z^2 + T^2$
We need some notation to define a Shimura curve:
We need some notation to define a Shimura curve:

- $B$ indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $D_B$
We need some notation to define a **Shimura curve**:

- \( B \) indefinite quaternion algebra over \( \mathbb{Q} \) of discriminant \( D_B \)
- \( \mathcal{O}_B \subseteq B \) order over \( \mathbb{Z} \) of level \( N \) coprime to \( D_B \)
We need some notation to define a **Shimura curve**:

- $B$ indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $D_B$
- $\mathcal{O}_B \subseteq B$ order over $\mathbb{Z}$ of level $N$ coprime to $D_B$
- $\Phi_{\infty} : B \rightarrow M_2(\mathbb{R})$
Canonical model of a Shimura curve

We need some notation to define a Shimura curve:

- $B$ indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $D_B$
- $\mathcal{O}_B \subseteq B$ order over $\mathbb{Z}$ of level $N$ coprime to $D_B$
- $\Phi_\infty : B \hookrightarrow M_2(\mathbb{R})$
- $\Gamma_\infty, + := \Phi_\infty(\{\alpha \in \mathcal{O}_B \mid \text{Nm}(\alpha) = 1\})/\mathbb{Z}^\times \subseteq \text{PSL}_2(\mathbb{R})$ discrete subgroup
Canonical model of a Shimura curve

We need some notation to define a **Shimura curve**:

- $B$ indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $D_B$
- $\mathcal{O}_B \subseteq B$ order over $\mathbb{Z}$ of level $N$ coprime to $D_B$
- $\Phi_\infty : B \hookrightarrow M_2(\mathbb{R})$
- $\Gamma_{\infty,+} := \Phi_\infty(\{\alpha \in \mathcal{O}_B \mid \text{Nm}(\alpha) = 1\})/\mathbb{Z}^\times \subseteq \text{PSL}_2(\mathbb{R})$ discrete subgroup
- $\Gamma_{\infty,+} \backslash \mathcal{H}_\infty$ compact Riemann surface ($\iff B \neq M_2(\mathbb{Q}) \iff D_B > 1$)
Shimura: there exists a model defined over $\mathbb{Q}$ for the Riemann surface $\Gamma_{\infty,+} \backslash \mathcal{H}_{\infty}$.
Canonical model of a Shimura curve

Shimura: there exists a model defined over $\mathbb{Q}$ for the Riemann surface $\Gamma_{\infty,+} \backslash \mathcal{H}_\infty$, i.e. there exist

- $X(D_B, N)$ proper algebraic curve defined over $\mathbb{Q}$
Shimura: there exists a model defined over $\mathbb{Q}$ for the Riemann surface $\Gamma_{\infty,+} \backslash \mathcal{H}_\infty$, i.e. there exist

- $X(D_B, N)$ proper algebraic curve defined over $\mathbb{Q}$
- $J : \Gamma_{\infty,+} \backslash \mathcal{H}_\infty \rightarrow X(D_B, N)(\mathbb{C})$ isomorphism

Moreover this model is characterised by certain arithmetic properties related to complex multiplication theory. It is called the canonical model of the Shimura curve $X(D_B, N)$ of discriminant $D_B$ and level $N$. The isomorphism $J$ is called the complex or $\infty$-adic uniformisation of the Shimura curve $X(D_B, N)$. The projective coordinates of $J$ are $\Gamma_{\infty,+} \backslash \mathcal{H}_\infty$, $\infty$-invariant functions $J_i : \mathcal{H}_\infty \rightarrow \mathbb{C}$ called uniformising functions. They satisfy some algebraic relations that give rise to the equations of $X(D_B, N)$. 
Shimura: there exists a model defined over $\mathbb{Q}$ for the Riemann surface $\Gamma_{\infty, +} \backslash \mathcal{H}_{\infty}$, i.e. there exist

- $X(D_B, N)$ proper algebraic curve defined over $\mathbb{Q}$
- $J : \Gamma_{\infty, +} \backslash \mathcal{H}_{\infty} \rightarrow X(D_B, N)(\mathbb{C})$ isomorphism

Moreover this model is characterised by certain arithmetic properties related to complex multiplication theory. It is called **canonical model of the Shimura curve** $X(D_B, N)$ of discriminant $D_B$ and level $N$. 
Shimura: there exists a model defined over \( \mathbb{Q} \) for the Riemann surface \( \Gamma_{\infty,+} \backslash \mathcal{H}_\infty \), i.e. there exist

- \( X(D_B, N) \) proper algebraic curve defined over \( \mathbb{Q} \)
- \( J : \Gamma_{\infty,+} \backslash \mathcal{H}_\infty \to X(D_B, N)(\mathbb{C}) \) isomorphism

Moreover this model is characterised by certain arithmetic properties related to complex multiplication theory. It is called **canonical model of the Shimura curve** \( X(D_B, N) \) of discriminant \( D_B \) and level \( N \).

The isomorphism \( J \) is called **complex** or **\( \infty \)-adic uniformisation of the Shimura curve** \( X(D_B, N) \).
Shimura: there exists a model defined over $\mathbb{Q}$ for the Riemann surface $\Gamma_{\infty,+} \setminus \mathcal{H}_\infty$, i.e. there exist

- $X(D_B, N)$ proper algebraic curve defined over $\mathbb{Q}$
- $J : \Gamma_{\infty,+} \setminus \mathcal{H}_\infty \to X(D_B, N)(\mathbb{C})$ isomorphism

Moreover this model is characterised by certain arithmetic properties related to complex multiplication theory. It is called **canonical model of the Shimura curve** $X(D_B, N)$ of discriminant $D_B$ and level $N$.

The isomorphism $J$ is called **complex** or **$\infty$-adic uniformisation of the Shimura curve** $X(D_B, N)$.

The projective coordinates of $J$ are $\Gamma_{\infty,+}$-invariant functions $J_i : \mathcal{H}_\infty \to \mathbb{C}$ called **uniformising functions**. They satisfy some algebraic relations that give rise to the equations of $X(D_B, N)$. 

Shimura: there exists a model defined over $\mathbb{Q}$ for the Riemann surface $\Gamma_{\infty,+} \backslash \mathcal{H}_{\infty}$, i.e. there exist

- $X(D_B, N)$ proper algebraic curve defined over $\mathbb{Q}$
- $J : \Gamma_{\infty,+} \backslash \mathcal{H}_{\infty} \rightarrow X(D_B, N)(\mathbb{C})$ isomorphism

Moreover this model is characterised by certain arithmetic properties related to complex multiplication theory. It is called **canonical model of the Shimura curve** $X(D_B, N)$ of discriminant $D_B$ and level $N$.

The isomorphism $J$ is called **complex** or $\infty$-adic uniformisation of the Shimura curve $X(D_B, N)$.

The projective coordinates of $J$ are $\Gamma_{\infty,+}$-invariant functions $J_i : \mathcal{H}_{\infty} \rightarrow \mathbb{C}$ called **uniformising functions**. They satisfy some algebraic relations that give rise to the equations of $X(D_B, N)$. 
Theorem. Let $p \nmid ND_B$ be a prime. Then $X(D_B, N)|_{\mathbb{F}_p}$ is smooth.
Reduction of Shimura curves

**Theorem.** Let $p \nmid ND_B$ be a prime. Then $X(D_B, N)|_{\mathbb{F}_p}$ is smooth.

If $p \mid ND_B$, $p$ is called a **bad reduction prime** of $X(D_B, N)$.
Reduction of Shimura curves

Theorem. Let $p \nmid ND_B$ be a prime. Then $X(D_B, N)|_{\mathbb{F}_p}$ is smooth.

If $p \mid ND_B$, $p$ is called a bad reduction prime of $X(D_B, N)$.

Theorem. Let $p \mid N$. Then the special fibre $X(D_B, N)|_{\mathbb{F}_p}$ has two irreducible components, each isomorphic to $X(D_B, N/p)|_{\mathbb{F}_p}$. 
Reduction of Shimura curves

**Theorem.** Let $p \nmid ND_B$ be a prime. Then $X(D_B, N)|_{\mathbb{F}_p}$ is smooth.

If $p | ND_B$, $p$ is called a **bad reduction prime** of $X(D_B, N)$.

**Theorem.** Let $p | N$. Then the special fibre $X(D_B, N)|_{\mathbb{F}_p}$ has two irreducible components, each isomorphic to $X(D_B, N/p)|_{\mathbb{F}_p}$.

**Theorem.** If $p | D_B$, the fibre $X(D_B, N)|_{\mathbb{F}_p}$ has totally degenerate (each component is isomorphic to $\mathbb{P}^1$) semistable (it is reduced, connected but possibly reducible, and its only singularities are ordinary double points) bad reduction.
Theorem. Let $p \nmid ND_B$ be a prime. Then $X(D_B, N)|_{\mathbb{F}_p}$ is smooth.

If $p \mid ND_B$, $p$ is called a bad reduction prime of $X(D_B, N)$.

Theorem. Let $p \mid N$. Then the special fibre $X(D_B, N)|_{\mathbb{F}_p}$ has two irreducible components, each isomorphic to $X(D_B, N/p)|_{\mathbb{F}_p}$.

Theorem. If $p \mid D_B$, the fibre $X(D_B, N)|_{\mathbb{F}_p}$ has totally degenerate (each component is isomorphic to $\mathbb{P}^1$) semistable (it is reduced, connected but possibly reducible, and its only singularities are ordinary double points) bad reduction.

We want to study the bad reduction of Shimura curves, i.e. its reduction modulo some prime $p \mid D_B$. 
3. $p$-ADIC UNIFORMISATION OF SHIMURA CURVES
Take $p \mid D_B$ bad reduction prime. Write $D_B = Dp$. 

$p$-adic uniformisation of Shimura curves
Take $p | D_B$ bad reduction prime. Write $D_B = Dp$.

$\mathbb{C} = \mathbb{C}_\infty$ is the algebraic closure of $\mathbb{Q}_\infty = \mathbb{R}$ and it is complete.
$p$-adic uniformisation of Shimura curves

Take $p \mid D_B$ bad reduction prime. Write $D_B = Dp$.

$\mathbb{C} = \mathbb{C}_\infty$ is the algebraic closure of $\mathbb{Q}_\infty = \mathbb{R}$ and it is complete. The algebraic closure of $\mathbb{Q}_p$ is not complete.
Take $p \mid D_B$ bad reduction prime. Write $D_B = Dp$.

$\mathbb{C} = \mathbb{C}_\infty$ is the algebraic closure of $\mathbb{Q}_\infty = \mathbb{R}$ and it is complete. The algebraic closure of $\mathbb{Q}_p$ is not complete. $\mathbb{C}_p = \text{completion of the algebraic closure of } \mathbb{Q}_p$. 
Take $p \mid D_B$ bad reduction prime. Write $D_B = Dp$.

$\mathbb{C} = \mathbb{C}_\infty$ is the algebraic closure of $\mathbb{Q}_\infty = \mathbb{R}$ and it is complete. The algebraic closure of $\mathbb{Q}_p$ is not complete. $\mathbb{C}_p = \text{completion of the algebraic closure of } \mathbb{Q}_p$.

We want to study the set of $\mathbb{C}_p$-points of $X(Dp, N)$ and its structure as rigid analytic variety ($p$-adic analog of Riemann surface). This knowledge will allow us to study the reductions mod $p$ of some integral models of $X(Dp, N)$.

$p$-adic uniformisation of Shimura curves
Take \( p \mid D_B \) \textbf{bad reduction prime}. Write \( D_B = D_p \).

\( \mathbb{C} = \mathbb{C}_\infty \) is the algebraic closure of \( \mathbb{Q}_\infty = \mathbb{R} \) and it is complete. The algebraic closure of \( \mathbb{Q}_p \) is not complete. \( \mathbb{C}_p = \) completion of the algebraic closure of \( \mathbb{Q}_p \).

We want to study the set of \( \mathbb{C}_p \)-points of \( X(D_p, N) \) and its structure as rigid analytic variety (\( p \)-adic analog of Riemann surface). This knowledge will allow us to study the reductions mod \( p \) of some integral models of \( X(D_p, N) \).

The theorem of Čerednik-Drinfel’d of \textbf{interchanging local invariants} tells us which is the group that uniformises the analytic variety \( X(D_p, N)(\mathbb{C}_p) \).
Take $p \mid D_B$ bad reduction prime. Write $D_B = Dp$.

$\mathbb{C} = \mathbb{C}_\infty$ is the algebraic closure of $\mathbb{Q}_\infty = \mathbb{R}$ and it is complete. The algebraic closure of $\mathbb{Q}_p$ is not complete. $\mathbb{C}_p = \text{completion of the algebraic closure of } \mathbb{Q}_p$.

We want to study the set of $\mathbb{C}_p$-points of $X(Dp, N)$ and its structure as rigid analytic variety ($p$-adic analog of Riemann surface). This knowledge will allow us to study the reductions mod $p$ of some integral models of $X(Dp, N)$.

The theorem of Čerednik-Drinfel’d of interchanging local invariants tells us which is the group that uniformises the analytic variety $X(Dp, N)(\mathbb{C}_p)$. It is an arithmetic group $\Gamma_{p,+} \subseteq \text{PGL}_2(\mathbb{Q}_p)$ such that

$$\Gamma_{p,+} \backslash \mathcal{H}_p \cong X(Dp, N)(\mathbb{C}_p),$$
Take $p \mid D_B$ bad reduction prime. Write $D_B = Dp$.

$\mathbb{C} = \mathbb{C}_\infty$ is the algebraic closure of $\mathbb{Q}_\infty = \mathbb{R}$ and it is complete.
The algebraic closure of $\mathbb{Q}_p$ is not complete. $\mathbb{C}_p =$ completion of the algebraic closure of $\mathbb{Q}_p$.

We want to study the set of $\mathbb{C}_p$-points of $X(Dp, N)$ and its structure as rigid analytic variety ($p$-adic analog of Riemann surface). This knowledge will allow us to study the reductions mod $p$ of some integral models of $X(Dp, N)$.

The theorem of Čerednik-Drinfel’d of interchanging local invariants tells us which is the group that uniformises the analytic variety $X(Dp, N)(\mathbb{C}_p)$. It is an arithmetic group $\Gamma_{p,+} \subseteq \text{PGL}_2(\mathbb{Q}_p)$ such that

$$\Gamma_{p,+} \backslash \mathcal{H}_p \cong X(Dp, N)(\mathbb{C}_p),$$

where $\mathcal{H}_p := \mathbb{P}^1(\mathbb{C}_p) \backslash \mathbb{P}^1(\mathbb{Q}_p)$ is the $p$-adic upper half-plane, the $p$-adic analog the the complex upper half-plane.
Take \( p \mid D_B \) bad reduction prime. Write \( D_B = Dp \).

\( \mathbb{C} = \mathbb{C}_\infty \) is the algebraic closure of \( \mathbb{Q}_\infty = \mathbb{R} \) and it is complete. The algebraic closure of \( \mathbb{Q}_p \) is not complete. \( \mathbb{C}_p = \) completion of the algebraic closure of \( \mathbb{Q}_p \).

We want to study the set of \( \mathbb{C}_p \)-points of \( X(Dp, N) \) and its structure as rigid analytic variety (\( p \)-adic analog of Riemann surface). This knowledge will allow us to study the reductions mod \( p \) of some integral models of \( X(Dp, N) \).

The theorem of Čerednik-Drinfel’d of interchanging local invariants tells us which is the group that uniformises the analytic variety \( X(Dp, N)(\mathbb{C}_p) \). It is an arithmetic group \( \Gamma_{p,+} \subseteq \text{PGL}_2(\mathbb{Q}_p) \) such that

\[
\Gamma_{p,+} \backslash \mathcal{H}_p \cong X(Dp, N)(\mathbb{C}_p),
\]

where \( \mathcal{H}_p := \mathbb{P}^1(\mathbb{C}_p) \backslash \mathbb{P}^1(\mathbb{Q}_p) \) is the \( p \)-adic upper half-plane, the \( p \)-adic analog the the complex upper half-plane.

This uniformisation is known as the \( p \)-adic uniformisation of the Shimura curve \( X(Dp, N) \).
The quaternion algebra that we need

Following Čerednik-Drinfel’d, we take the definite quaternion algebra $H$ of discriminant $D$ obtained from $B$ by interchanging the local invariants $p$ and $\infty$. 
The quaternion algebra that we need

Following Čerednik-Drinfel’d, we take the definite quaternion algebra $H$ of discriminant $D$ obtained from $B$ by interchanging the local invariants $p$ and $\infty$.

- $H$ definite quaternion algebra of discriminant $D$
The quaternion algebra that we need

Following Čerednik-Drinfel’d, we take the definite quaternion algebra $H$ of discriminant $D$ obtained from $B$ by interchanging the local invariants $p$ and $\infty$.

- $H$ definite quaternion algebra of discriminant $D$
- $\Phi_p : H \hookrightarrow M_2(\mathbb{Q}_p)$
The quaternion algebra that we need

Following Čerednik-Drinfel’d, we take the definite quaternion algebra $H$ of discriminant $D$ obtained from $B$ by interchanging the local invariants $p$ and $\infty$.

- $H$ definite quaternion algebra of discriminant $D$
- $\Phi_p : H \hookrightarrow M_2(\mathbb{Q}_p)$
- $\mathcal{O} \subset H$ maximal order over $\mathbb{Z}$
The quaternion algebra that we need

Following Čerednik-Drinfel’d, we take the definite quaternion algebra $H$ of discriminant $D$ obtained from $B$ by interchanging the local invariants $p$ and $\infty$.

- $H$ definite quaternion algebra of discriminant $D$
- $\Phi_p : H \hookrightarrow M_2(\mathbb{Q}_p)$
- $\mathcal{O} \subset H$ maximal order over $\mathbb{Z}$
- $\mathcal{O}[\frac{1}{p}] := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$ maximal order over $\mathbb{Z}[\frac{1}{p}]$
The quaternion algebra that we need

Following Čerednik-Drinfel’d, we take the definite quaternion algebra $H$ of discriminant $D$ obtained from $B$ by interchanging the local invariants $p$ and $\infty$.

- $H$ definite quaternion algebra of discriminant $D$
- $\Phi_p : H \hookrightarrow M_2(\mathbb{Q}_p)$
- $\mathcal{O} \subset H$ maximal order over $\mathbb{Z}$
- $\mathcal{O} \left[ \frac{1}{p} \right] := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z} \left[ \frac{1}{p} \right]$ maximal order over $\mathbb{Z} \left[ \frac{1}{p} \right]$
- $\Gamma_p := \Phi_p (\mathcal{O}[1/p]^\times)/\mathbb{Z}[1/p]^\times \subseteq \text{PGL}(\mathbb{Q}_p)$
The quaternion algebra that we need

Following Čerednik-Drinfel’d, we take the definite quaternion algebra $H$ of discriminant $D$ obtained from $B$ by interchanging the local invariants $p$ and $∞$.

- $H$ definite quaternion algebra of discriminant $D$
- $Φ_p : H \hookrightarrow M_2(\mathbb{Q}_p)$
- $\mathcal{O} \subset H$ maximal order over $\mathbb{Z}$
- $\mathcal{O}[\frac{1}{p}] := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$ maximal order over $\mathbb{Z}[\frac{1}{p}]$
- $Γ_p := Φ_p(\mathcal{O}[1/p]^\times)/\mathbb{Z}[1/p]^\times \subseteq \text{PGL}(\mathbb{Q}_p)$ unit group
The quaternion algebra that we need

Following Čerednik-Drinfel’d, we take the definite quaternion algebra $H$ of discriminant $D$ obtained from $B$ by interchanging the local invariants $p$ and $\infty$.

- $H$ definite quaternion algebra of discriminant $D$
- $\Phi_p : H \hookrightarrow M_2(\mathbb{Q}_p)$
- $\mathcal{O} \subset H$ maximal order over $\mathbb{Z}$
- $\mathcal{O}[\frac{1}{p}] := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$ maximal order over $\mathbb{Z}[\frac{1}{p}]$
- $\Gamma_p := \Phi_p(\mathcal{O}[1/p]^\times)/\mathbb{Z}[1/p]^\times \subseteq \text{PGL}(\mathbb{Q}_p)$ unit group
- $\Gamma_{p,+} := \{ \gamma \in \Gamma_p : v_p(\text{Nm}(\alpha)) \equiv 0 \mod 2 \}$
The quaternion algebra that we need

Following Čerednik-Drinfel’d, we take the definite quaternion algebra $H$ of discriminant $D$ obtained from $B$ by interchanging the local invariants $p$ and $\infty$.

- $H$ definite quaternion algebra of discriminant $D$
- $\Phi_p : H \hookrightarrow M_2(\mathbb{Q}_p)$
- $\mathcal{O} \subset H$ maximal order over $\mathbb{Z}$
- $\mathcal{O}[\frac{1}{p}] := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$ maximal order over $\mathbb{Z}[\frac{1}{p}]$
- $\Gamma_p := \Phi_p(\mathcal{O}[1/p]^\times)/\mathbb{Z}[1/p]^\times \subseteq \text{PGL}(\mathbb{Q}_p)$ unit group
- $\Gamma_{p,+} := \{ \gamma \in \Gamma_p : \nu_p(\text{Nm}(\alpha)) \equiv 0 \mod 2 \}$ “positive” units
The quaternion algebra that we need

Following Čerednik-Drinfel’d, we take the definite quaternion algebra $H$ of discriminant $D$ obtained from $B$ by interchanging the local invariants $p$ and $\infty$.

- $H$ definite quaternion algebra of discriminant $D$
- $\Phi_p : H \hookrightarrow M_2(\mathbb{Q}_p)$
- $\mathcal{O} \subset H$ maximal order over $\mathbb{Z}$
- $\mathcal{O}[\frac{1}{p}] := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$ maximal order over $\mathbb{Z}[\frac{1}{p}]$
- $\Gamma_p := \Phi_p(\mathcal{O}[1/p]^\times) / \mathbb{Z}[1/p]^\times \subseteq \text{PGL}(\mathbb{Q}_p)$ unit group
- $\Gamma_{p,+} := \{ \gamma \in \Gamma_p : \nu_p(\text{Nm}(\alpha)) \equiv 0 \mod 2 \}$ “positive” units
- $\Gamma_p(\xi) := \Phi_p(\{ \alpha \in \mathcal{O}[1/p]^\times \mid \alpha \equiv 1 \mod \xi\mathcal{O} \}) / \mathbb{Z}[1/p]^\times$, $\xi \in \mathcal{O}$
The quaternion algebra that we need

Following Čerednik-Drinfel’d, we take the definite quaternion algebra $H$ of discriminant $D$ obtained from $B$ by interchanging the local invariants $p$ and $\infty$.

- $H$ definite quaternion algebra of discriminant $D$
- $\Phi_p : H \hookrightarrow M_2(\mathbb{Q}_p)$
- $\mathcal{O} \subset H$ maximal order over $\mathbb{Z}$
- $\mathcal{O}[\frac{1}{p}] := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$ maximal order over $\mathbb{Z}[\frac{1}{p}]$
- $\Gamma_p := \Phi_p(\mathcal{O}[1/p]^\times)/\mathbb{Z}[1/p]^\times \subseteq \text{PGL}(\mathbb{Q}_p)$ unit group
- $\Gamma_{p,+} := \{ \gamma \in \Gamma_p : \nu_p(\text{Nm}(\alpha)) \equiv 0 \mod 2 \} \text{ “positive” units}$
- $\Gamma_p(\xi) := \Phi_p(\{\alpha \in \mathcal{O}[1/p]^\times \mid \alpha \equiv 1 \mod \xi \mathcal{O}\})/\mathbb{Z}[1/p]^\times$, $\xi \in \mathcal{O}$ elements congruent to 1 modulo $\xi$
Theorem. There is an isomorphism

$$\Gamma_p \backslash (\mathcal{H}_p \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^2) \simeq (X(Dp, N) \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\text{rig}}$$

of rigid analytic varieties over \( \mathbb{Q}_p \).
The Theorem of Čerednik and Drinfel’d

**Theorem.** There is an isomorphism

\[ \Gamma_p \backslash (\mathcal{H}_p \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^2) \simeq (X(Dp, N) \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\text{rig}} \]

of rigid analytic varieties over \( \mathbb{Q}_p \).

It follows from a theorem of Drinfel’d that the reduction-graph of the special fibre at \( p \) of (an adequate integral model) of \( X(Dp, N) \) is the quotient graph \( \Gamma_{p,+} \backslash \mathcal{T}_p \).
Theorem of Čerednik and Drinfel’d

**Theorem.** There is an isomorphism

\[ \Gamma_p \backslash (\mathcal{H}_p \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^2) \cong (X(Dp, N) \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\text{rig}} \]

of rigid analytic varieties over \( \mathbb{Q}_p \).

It follows from a theorem of Drinfel’d that the reduction-graph of the special fibre at \( p \) of (an adequate integral model) of \( X(Dp, N) \) is the quotient graph \( \Gamma_p, + \backslash \mathcal{T}_p \).

The group \( \Gamma_{p,+} \) is not always torsion-free but it admits a torsion-free normal and finite index subgroup.
Theorem. There is an isomorphism

\[ \Gamma_p \backslash (\mathcal{H}_p \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^2) \simeq (X(Dp, N) \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\text{rig}} \]

of rigid analytic varieties over \( \mathbb{Q}_p \).

It follows from a theorem of Drinfel’d that the reduction-graph of the special fibre at \( p \) of (an adequate integral model) of \( X(Dp, N) \) is the quotient graph \( \Gamma_{p,+} \backslash \mathcal{T}_p \).

The group \( \Gamma_{p,+} \) is not always torsion-free but it admits a torsion-free normal and finite index subgroup.
A lattice $M \subseteq \mathbb{Q}_p^2$ is a free $\mathbb{Z}_p$-module of rank 2.
The Bruhat-Tits tree

A lattice $M \subseteq \mathbb{Q}_p^2$ is a free $\mathbb{Z}_p$-module of rank 2. Two lattices $M, M' \subseteq \mathbb{Q}_p^2$ are homothetic if $\exists \lambda \in \mathbb{Q}_p^\times$ with $M' = \lambda M$. 
A lattice $M \subseteq \mathbb{Q}_p^2$ is a free $\mathbb{Z}_p$-module of rank 2. Two lattices $M, M' \subseteq \mathbb{Q}_p^2$ are homothetic if $\exists \lambda \in \mathbb{Q}_p^\times$ with $M' = \lambda M$.

We define the graph $\mathcal{T}_p$ as follows:
- $\text{Ver}(\mathcal{T}_p) = \text{homothety classes of lattices of } \mathbb{Q}_p^2$
- $\text{Ed}(\mathcal{T}_p) = \text{pairs of adjacent classes}$
The Bruhat-Tits tree

A lattice $M \subseteq \mathbb{Q}_p^2$ is a free $\mathbb{Z}_p$-module of rank 2.

Two lattices $M, M' \subseteq \mathbb{Q}_p^2$ are homothetic if $\exists \lambda \in \mathbb{Q}_p^\times$ with $M' = \lambda M$.

We define the graph $\mathcal{T}_p$ as follows:
- $\text{Ver}(\mathcal{T}_p) =$ homothety classes of lattices of $\mathbb{Q}_p^2$
- $\text{Ed}(\mathcal{T}_p) =$ pairs of adjacent classes.

It is a $(p + 1)$-regular tree known as the **Bruhat-Tits tree** associated to $\text{PGL}_2(\mathbb{Q}_p)$. 

The Bruhat-Tits tree

A lattice $M \subseteq \mathbb{Q}_p^2$ is a free $\mathbb{Z}_p$-module of rank 2. Two lattices $M, M' \subseteq \mathbb{Q}_p^2$ are homothetic if $\exists \lambda \in \mathbb{Q}_p^\times$ with $M' = \lambda M$.

We define the graph $T_p$ as follows:
- $\text{Ver}(T_p) =$ homothety classes of lattices of $\mathbb{Q}_p^2$
- $\text{Ed}(T_p) =$ pairs of adjacent classes.

It is a $(p + 1)$-regular tree known as the Bruhat-Tits tree associated to $\text{PGL}_2(\mathbb{Q}_p)$.

The group $\text{PGL}_2(\mathbb{Q}_p)$ acts transitively on $\text{Ver}(T_p)$: if $M = \langle u, v \rangle \subseteq \mathbb{Q}_p^2$ and $\gamma \in \text{GL}_2(\mathbb{Q}_p)$ then $\gamma \cdot M := \langle \gamma u, \gamma v \rangle$. 
The Bruhat-Tits tree

Bruhat-Tits tree $\mathcal{T}_p$ for $p = 2$

Picture taken from: *The Bruhat-Tits tree of SL(2), Bill Casselman*
4. BAD REDUCTION OF SHIMURA CURVES
The theory of $p$-adic uniformisation of curves developed by Mumford tells us that if $\Gamma \subseteq \text{PGL}_2(\mathbb{Q}_p)$ is a **Schottky group**, then there exists a curve $X_\Gamma$ over $\mathbb{Q}_p$ such that

- $X_\Gamma(\mathbb{C}_p) \cong \Gamma \backslash \mathbb{H}_p$,
- $\text{Red}_p(X_\Gamma) \cong \Gamma \backslash \mathcal{T}_p$,

where $\mathcal{T}_p$ is the Burhat-Tits tree attached to $\text{PGL}_2(\mathbb{Q}_p)$.

**Theorem.** Let $\Gamma \subseteq \text{PGL}_2(\mathbb{Q}_p)$ be a discontinuous and finitely generated group. Then there exists a normal subgroup $\Gamma_{\text{Sch}}$ of finite index which is torsion-free. In particular $\Gamma_{\text{Sch}}$ is a $p$-adic Schottky group.
The theory of $p$-adic uniformisation of curves developed by Mumford tells us that if $\Gamma \subseteq \text{PGL}_2(\mathbb{Q}_p)$ is a \textbf{Schottky group} (=discrete, finitely generated an torsion-free),
The theory of $p$-adic uniformisation of curves developed by Mumford tells us that if $\Gamma \subseteq \text{PGL}_2(\mathbb{Q}_p)$ is a **Schottky group** ($=$discrete, finitely generated an torsion-free), then there exists a curve $X_\Gamma$ over $\mathbb{Q}_p$ such that
The theory of $p$-adic uniformisation of curves developed by Mumford tells us that if $\Gamma \subseteq \text{PGL}_2(\mathbb{Q}_p)$ is a \textbf{Schottky group} (=discrete, finitely generated and torsion-free), then there exists a curve $X_\Gamma$ over $\mathbb{Q}_p$ such that

- $X_\Gamma(\mathbb{C}_p) \simeq \Gamma \setminus \mathcal{H}_p$, 

\textbf{Theorem.} Let $\Gamma \subseteq \text{PGL}_2(\mathbb{Q}_p)$ be a discontinuous and finitely generated group. Then there exists a normal subgroup $\Gamma_{\text{Sch}}$ of finite index which is torsion-free. In particular $\Gamma_{\text{Sch}}$ is a $p$-adic Schottky group.
The theory of $p$-adic uniformisation of curves developed by Mumford tells us that if $\Gamma \subseteq \text{PGL}_2(\mathbb{Q}_p)$ is a **Schottky group** (=discrete, finitely generated and torsion-free), then there exists a curve $X_\Gamma$ over $\mathbb{Q}_p$ such that

- $X_\Gamma(\mathbb{C}_p) \simeq \Gamma \setminus \mathcal{H}_p$,
- $\text{Red}_p(X_\Gamma) \simeq \Gamma \setminus \mathcal{T}_p$, 

**Theorem.** Let $\Gamma \subseteq \text{PGL}_2(\mathbb{Q}_p)$ be a discontinuous and finitely generated group. Then there exists a normal subgroup $\Gamma_{\text{Sch}}$ of finite index which is torsion-free. In particular $\Gamma_{\text{Sch}}$ is a $p$-adic Schottky group.
Mumford theory

The theory of \( p \)-adic uniformisation of curves developed by Mumford tells us that if \( \Gamma \subseteq \text{PGL}_2(\mathbb{Q}_p) \) is a Schottky group (=discrete, finitely generated and torsion-free), then there exists a curve \( X_\Gamma \) over \( \mathbb{Q}_p \) such that

- \( X_\Gamma(\mathbb{C}_p) \cong \Gamma \setminus \mathcal{H}_p \),
- \( \text{Red}_p(X_\Gamma) \cong \Gamma \setminus \mathcal{T}_p \),

where \( \mathcal{T}_p \) is the Burhat-Tits tree attached to \( \text{PGL}_2(\mathbb{Q}_p) \).

Theorem. Let \( \Gamma \subseteq \text{PGL}_2(\mathbb{Q}_p) \) be a discontinuous and finitely generated group. Then there exists a normal subgroup \( \Gamma_{\text{Sch}} \) of finite index which is torsion-free. In particular \( \Gamma_{\text{Sch}} \) is a \( p \)-adic Schottky group.
The theory of $p$-adic uniformisation of curves developed by Mumford tells us that if $\Gamma \subseteq \text{PGL}_2(\mathbb{Q}_p)$ is a **Schottky group** (=discrete, finitely generated and torsion-free), then there exists a curve $X_\Gamma$ over $\mathbb{Q}_p$ such that

- $X_\Gamma(\mathbb{C}_p) \simeq \Gamma \backslash \mathcal{H}_p$,
- $\text{Red}_p(X_\Gamma) \simeq \Gamma \backslash \mathcal{T}_p$,

where $\mathcal{T}_p$ is the Burhat-Tits tree attached to $\text{PGL}_2(\mathbb{Q}_p)$.

**Theorem.** Let $\Gamma \subseteq \text{PGL}_2(\mathbb{Q}_p)$ be a discontinuous and finitely generated group. Then there exists a normal subgroup $\Gamma^{Sch}$ of finite index which is torsion-free. In particular $\Gamma^{Sch}$ is a $p$-adic Schottky group.
Technical conditions

$H$ definite quaternion algebra,
Technical conditions

$H$ definite quaternion algebra, $\mathcal{O} \subseteq H$ order of level $N$ with $h(D, N) = 1$
Technical conditions

$H$ definite quaternion algebra, $\mathcal{O} \subseteq H$ order of level $N$ with $h(D, N) = 1$  
$\mathcal{O}^\times$ unit group, finite  

Take $\xi \in \mathcal{O}$
Technical conditions

$H$ definite quaternion algebra, $\mathcal{O} \subseteq H$ order of level $N$ with $h(D, N) = 1$

$\mathcal{O}^\times$ unit group, finite

Take $\xi \in \mathcal{O}$

**Definition 1.** $\alpha \in \mathcal{O}$ is $\xi$-primary with respect to $\{[1]\} \iff \alpha \equiv 1 \text{ mod } \xi\mathcal{O}$

$\iff \alpha - 1 \in \xi\mathcal{O}$. 
Technical conditions

$H$ definite quaternion algebra, $\mathcal{O} \subseteq H$ order of level $N$ with $h(D, N) = 1$

$\mathcal{O}^\times$ unit group, finite

Take $\xi \in \mathcal{O}$

**Definition 1.** $\alpha \in \mathcal{O}$ is $\xi$-primary with respect to $\{1\}$ $\Leftrightarrow \alpha \equiv 1 \mod \xi \mathcal{O}$

$\Leftrightarrow \alpha - 1 \in \xi \mathcal{O}$.

**Definition 2.** $\xi$ satisfies the right-unit property in $\mathcal{O}$ if

$$\#\mathcal{O}^\times = \#(\mathcal{O}/\xi \mathcal{O})^\times \text{ when } 2 \notin \xi \mathcal{O}$$

$$\#\mathcal{O}^\times/\mathbb{Z}^\times = \#(\mathcal{O}/\xi \mathcal{O})^\times \text{ when } 2 \in \xi \mathcal{O}.$$
Technical conditions

$H$ definite quaternion algebra, $\mathcal{O} \subseteq H$ order of level $N$ with $h(D, N) = 1$
$\mathcal{O}^\times$ unit group, finite

Take $\xi \in \mathcal{O}$

**Definition 1.** $\alpha \in \mathcal{O}$ is $\xi$-primary with respect to $\{[1]\}$ $\iff$ $\alpha \equiv 1 \mod \xi \mathcal{O}$
$\iff$ $\alpha - 1 \in \xi \mathcal{O}$.

**Definition 2.** $\xi$ satisfies the **right-unit property** in $\mathcal{O}$ if

$$\#\mathcal{O}^\times = \#(\mathcal{O}/\xi \mathcal{O})^\times$$ when $2 \notin \xi \mathcal{O}$

$$\#\mathcal{O}^\times /\mathbb{Z}^\times = \#(\mathcal{O}/\xi \mathcal{O})^\times$$ when $2 \in \xi \mathcal{O}$.

**Definition 3.** $p \nmid DN$ odd prime. Let

$$t_\xi(p) := \#\{\alpha \in \mathcal{O} \mid \text{Nm}(\alpha) = p, \alpha \equiv 1 \mod \xi \mathcal{O}, \text{Tr}(\alpha) = 0\}.$$
Technical conditions

$H$ definite quaternion algebra, $\mathcal{O} \subseteq H$ order of level $N$ with $h(D, N) = 1$
$\mathcal{O}^\times$ unit group, finite

Take $\xi \in \mathcal{O}$

**Definition 1.** $\alpha \in \mathcal{O}$ is $\xi$-primary with respect to $\{[1]\} \iff \alpha \equiv 1 \mod \xi \mathcal{O}$
$\iff \alpha - 1 \in \xi \mathcal{O}$.

**Definition 2.** $\xi$ satisfies the right-unit property in $\mathcal{O}$ if

$$\#\mathcal{O}^\times = \#(\mathcal{O}/\xi \mathcal{O})^\times \text{ when } 2 \notin \xi \mathcal{O}$$
$$\#\mathcal{O}^\times / \mathbb{Z}^\times = \#(\mathcal{O}/\xi \mathcal{O})^\times \text{ when } 2 \in \xi \mathcal{O}.$$  

**Definition 3.** $p \nmid DN$ odd prime. Let

$$t_\xi(p) := \#\{\alpha \in \mathcal{O} \mid \text{Nm}(\alpha) = p, \alpha \equiv 1 \mod \xi \mathcal{O}, \text{Tr}(\alpha) = 0\}.$$  

If $t_\xi(p) = 0$, we will say that $p$ satisfies the null-trace condition with respect to $\xi \mathcal{O}$. 
## Technical conditions

<table>
<thead>
<tr>
<th>$D$</th>
<th>$H$</th>
<th>$N$</th>
<th>$\mathcal{O}$</th>
<th>$#(\mathcal{O}^<em>/\mathbb{Z}^</em>)$</th>
<th>$\xi$</th>
<th>$\text{Nm}(\xi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\left( \frac{-1}{\mathbb{Q}} \right)$</td>
<td>1</td>
<td>$\mathbb{Z} \left[ 1, i, j, \frac{1}{2}(1 + i + j + k) \right]$</td>
<td>12</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>$\mathbb{Z} \left[ 1, 3i, -2i + j, \frac{1}{2}(1 - i + j + k) \right]$</td>
<td>3</td>
<td>$(-i + k)$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
<td>$\mathbb{Z} \left[ 1, 9i, -4i + j, \frac{1}{2}(1 - 3i + j + k) \right]$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11</td>
<td>$\mathbb{Z} \left[ 1, 11i, -10i + j, \frac{1}{2}(1 - 3i + j + k) \right]$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$\left( \frac{-1}{\mathbb{Q}} \right)$</td>
<td>1</td>
<td>$\mathbb{Z} \left[ 1, i, \frac{1}{2}(i + j), \frac{1}{2}(1 + k) \right]$</td>
<td>6</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>$\mathbb{Z} \left[ 1, 2i, \frac{1}{2}(-i + j), \frac{1}{2} - i + \frac{1}{2}k \right]$</td>
<td>2</td>
<td>$\frac{1}{2}(-1 - i - j + k)$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>$\mathbb{Z} \left[ 1, 4i, \frac{1}{2}(-5i + j), \frac{1}{2} - 3i + \frac{1}{2}k \right]$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>$\left( \frac{-2}{\mathbb{Q}} \right)$</td>
<td>1</td>
<td>$\mathbb{Z} \left[ 1, \frac{1}{2}(1 + i + j), j, \frac{1}{4}(2 + i + k) \right]$</td>
<td>3</td>
<td>$\frac{1}{2}(-1 + i - j)$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>$\mathbb{Z} \left[ 1, 1 + i + j, \frac{1}{2}(-1 - i + j), \frac{1}{4}(-i - 2j + k) \right]$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
<td>$\left( \frac{-2}{\mathbb{Q}} \right)$</td>
<td>1</td>
<td>$\mathbb{Z} \left[ 1, \frac{1}{2}(1 + i + j), j, \frac{1}{4}(2 + i + k) \right]$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

*Table:* Definite orders $\mathcal{O}$ with $\xi \in \mathcal{O}$ satisfying the right-unit property
About the groups $\Gamma_p$ and $\Gamma_{p,+}$

$\Gamma_p$ and $\Gamma_{p,+}$ are discrete and cocompact subgroups of $\text{PGL}_2(\mathbb{Q}_p)$. 
About the groups $\Gamma_p$ and $\Gamma_{p,+}$

$\Gamma_p$ and $\Gamma_{p,+}$ are discrete and cocompact subgroups of $\text{PGL}_2(\mathbb{Q}_p)$. BUT they may contain finite order elements, so they are not necessarily Schottky groups.
About the groups $\Gamma_p$ and $\Gamma_{p,+}$

$\Gamma_p$ and $\Gamma_{p,+}$ are discrete and cocompact subgroups of $\text{PGL}_2(\mathbb{Q}_p)$. BUT they may contain finite order elements, so they are not necessarily Schottky groups.

There exists a finite index normal subgroup $\Gamma_p(\xi) \subseteq \Gamma_p$ which IS torsion free, so it is Schottky.
About the groups $\Gamma_p$ and $\Gamma_{p,+}$

$\Gamma_p$ and $\Gamma_{p,+}$ are discrete and cocompact subgroups of $\text{PGL}_2(\mathbb{Q}_p)$. BUT they may contain finite order elements, so they are not necessarily Schottky groups.

There exists a finite index normal subgroup $\Gamma_p(\xi) \subseteq \Gamma_p$ which IS torsion free, so it is Schottky.

We want to describe the group

$$\Gamma_p = \phi_p(\mathcal{O}[1/p]^\times)/\mathbb{Z}[1/p]^\times$$
About the groups $\Gamma_p$ and $\Gamma_{p,+}$

$\Gamma_p$ and $\Gamma_{p,+}$ are discrete and cocompact subgroups of $\text{PGL}_2(\mathbb{Q}_p)$. BUT they may contain finite order elements, so they are not necessarily Schottky groups.

There exists a finite index normal subgroup $\Gamma_p(\xi) \subseteq \Gamma_p$ which IS torsion free, so it is Schottky.

We want to describe the group

$$\Gamma_p = \phi_p(\mathcal{O}[1/p]^\times)/\mathbb{Z}[1/p]^\times$$

and find a subgroup of $\Gamma_p$ which is a Schottky group.
**Theorem** (Amorós, Milione)

Let $O$ with $h(D,N) = 1$, $\xi \in O$. Then every $\alpha \in O[1/p]$ can be decomposed as a product $\alpha = n \cdot \prod_{i=1}^{r} \beta_i \cdot \varepsilon$, for $\varepsilon \in O \times$ unique (up to sign if $2 \in \xi O$), unique $n \in \mathbb{Z}$ and for unique $\beta_1, \ldots, \beta_r \in O$ primitive $\xi$-primary quaternions with $Nm(\beta_i) = p$, and such that no factor of the form $\beta_i \cdot \beta_i + 1 = p$ appears in the product.
Unique decomposition...

**Theorem (Amorós, Milione)**

Let $\mathcal{O}$ with $h(D, N) = 1$, $\xi \in \mathcal{O}$. Then every $\alpha \in \mathcal{O}[1/p]^\times$ can be decomposed as a product

$$\alpha = p^n \cdot \prod_{i=1}^{r} \beta_i \cdot \varepsilon,$$

where $\varepsilon \in \mathcal{O}^\times$ unique (up to sign if $2 \in \xi \mathcal{O}$), unique $n \in \mathbb{Z}$, and for unique $\beta_1, \ldots, \beta_r \in \mathcal{O}$ primitive $\xi$-primary quaternions with $N_{\mathcal{O}}(\beta_i) = p$, and such that no factor of the form $\beta_i \cdot \beta_{i+1} = p$ appears in the product.
**Theorem** (Amorós, Milione)

Let $\mathcal{O}$ with $h(D, N) = 1$, $\xi \in \mathcal{O}$. Then every $\alpha \in \mathcal{O}[1/p]^{\times}$ can be decomposed as a product

$$\alpha = p^n \cdot \prod_{i=1}^{r} \beta_i \cdot \varepsilon,$$

for $\varepsilon \in \mathcal{O}^{\times}$ unique (up to sign if $2 \in \xi \mathcal{O}$),
Theorem (Amorós, Milione)

Let $\mathcal{O}$ with $h(D, N) = 1$, $\xi \in \mathcal{O}$. Then every $\alpha \in \mathcal{O}[1/p]^{\times}$ can be decomposed as a product

$$\alpha = p^n \cdot \prod_{i=1}^{r} \beta_i \cdot \varepsilon,$$

for $\varepsilon \in \mathcal{O}^{\times}$ unique (up to sign if $2 \in \xi \mathcal{O}$), unique $n \in \mathbb{Z}$.
Theorem (Amorós, Milione)

Let $\mathcal{O}$ with $h(D, N) = 1$, $\xi \in \mathcal{O}$. Then every $\alpha \in \mathcal{O}[1/p]^\times$ can be decomposed as a product

$$\alpha = p^n \cdot \prod_{i=1}^{r} \beta_i \cdot \epsilon,$$

for $\epsilon \in \mathcal{O}^\times$ unique (up to sign if $2 \in \xi \mathcal{O}$), unique $n \in \mathbb{Z}$

and for unique $\beta_1, \ldots, \beta_r \in \mathcal{O}$ primitive $\xi$-primary quaternions with $Nm(\beta_i) = p$, 

Unique decomposition...
**Theorem (Amorós, Milione)**

Let $\mathcal{O}$ with $h(D, N) = 1$, $\xi \in \mathcal{O}$. Then every $\alpha \in \mathcal{O}[1/p]^\times$ can be decomposed as a product

$$\alpha = p^n \cdot \prod_{i=1}^{r} \beta_i \cdot \varepsilon,$$

for $\varepsilon \in \mathcal{O}^\times$ unique (up to sign if $2 \in \xi \mathcal{O}$), unique $n \in \mathbb{Z}$, and for unique $\beta_1, \ldots, \beta_r \in \mathcal{O}$ primitive $\xi$-primary quaternions with $Nm(\beta_i) = p$, and such that no factor of the form $\beta_i \cdot \beta_{i+1} = p$ appears in the product.
...as a consequence of the Zerlegungssatz

Theorem (Zerlegungssatz) (Amorós, Milione)
...as a consequence of the Zerlegungssatz

Theorem (Zerlegungssatz) (Amorós, Milione)

$H$ definite quaternion algebra of discriminant $D$, 
...as a consequence of the Zerlegungssatz

**Theorem (Zerlegungssatz) (Amorós, Milione)**

$H$ definite quaternion algebra of discriminant $D$, $\mathcal{O}$ order over $\mathbb{Z}$ of level $N$ with $h(D, N) = 1$.

Let $\xi \in \mathcal{O}$ such that $\mathcal{O}/\xi \mathcal{O}$ contains a $\xi$-primary class set $\mathcal{P}$. Take $\alpha \in \mathcal{O}$ primitive, $\xi$-primary such that its norm has a decomposition $N_m(\alpha) = p_1 \cdots p_s$. Then $\alpha$ admits a decomposition in primitive irreducible and $\xi$-primary quaternions: $\alpha = \pi_1 \cdots \pi_s$ with $N_m(\pi_i) = p_i$. Moreover, if $2 / 2 \in \xi \mathcal{O}$ this decomposition is unique, and if $2 \in \xi \mathcal{O}$, the decomposition is unique up to sign.
...as a consequence of the Zerlegungssatz

Theorem (Zerlegungssatz) (Amorós, Milione)

$H$ definite quaternion algebra of discriminant $D$, $\mathcal{O}$ order over $\mathbb{Z}$ of level $N$ with $h(D, N) = 1$.
Let $\xi \in \mathcal{O}$ such that $\mathcal{O}/\xi\mathcal{O}$ contains a $\xi$-primary class set $\mathcal{P}$. 

...as a consequence of the Zerlegungssatz

Theorem (Zerlegungssatz) (Amorós, Milione)

$H$ definite quaternion algebra of discriminant $D$, $\mathcal{O}$ order over $\mathbb{Z}$ of level $N$ with $h(D, N) = 1$.
Let $\xi \in \mathcal{O}$ such that $\mathcal{O}/\xi\mathcal{O}$ contains a $\xi$-primary class set $\mathcal{P}$.
Take $\alpha \in \mathcal{O}$ primitive, $\xi$-primary
Theorem (Zerlegungssatz) (Amorós, Milione)

$H$ definite quaternion algebra of discriminant $D$, $\mathcal{O}$ order over $\mathbb{Z}$ of level $N$ with $h(D, N) = 1$.

Let $\xi \in \mathcal{O}$ such that $\mathcal{O}/\xi \mathcal{O}$ contains a $\xi$-primary class set $\mathcal{P}$.

Take $\alpha \in \mathcal{O}$ primitive, $\xi$-primary such that its norm has a decomposition in prime factors

$$Nm(\alpha) = p_1 \cdot \ldots \cdot p_s.$$
Theorem (Zerlegungssatz) (Amorós, Milione)

$H$ definite quaternion algebra of discriminant $D$, $\mathcal{O}$ order over $\mathbb{Z}$ of level $N$ with $h(D, N) = 1$.
Let $\xi \in \mathcal{O}$ such that $\mathcal{O}/\xi\mathcal{O}$ contains a $\xi$-primary class set $\mathcal{P}$.
Take $\alpha \in \mathcal{O}$ primitive, $\xi$-primary such that its norm has a decomposition in prime factors

$$Nm(\alpha) = p_1 \cdot \ldots \cdot p_s.$$ 

Then $\alpha$ admits a decomposition in primitive irreducible and $\xi$-primary quaternions:

$$\alpha = \pi_1 \cdot \ldots \cdot \pi_s$$
Theorem (Zerlegungssatz) (Amorós, Milione)

$H$ definite quaternion algebra of discriminant $D$, $\mathcal{O}$ order over $\mathbb{Z}$ of level $N$ with $h(D, N) = 1$.
Let $\xi \in \mathcal{O}$ such that $\mathcal{O}/\xi \mathcal{O}$ contains a $\xi$-primary class set $\mathcal{P}$.

Take $\alpha \in \mathcal{O}$ primitive, $\xi$-primary such that its norm has a decomposition in prime factors

$$\text{Nm}(\alpha) = p_1 \cdot \ldots \cdot p_s.$$ 

Then $\alpha$ admits a decomposition in primitive irreducible and $\xi$-primary quaternions:

$$\alpha = \pi_1 \cdot \ldots \cdot \pi_s$$

with $\text{Nm}(\pi_i) = p_i$. 

...as a consequence of the Zerlegungssatz
Theorem (Zerlegungssatz) (Amorós, Milione)

$H$ definite quaternion algebra of discriminant $D$, $\mathcal{O}$ order over $\mathbb{Z}$ of level $N$ with $h(D, N) = 1$.

Let $\xi \in \mathcal{O}$ such that $\mathcal{O}/\xi\mathcal{O}$ contains a $\xi$-primary class set $\mathcal{P}$.

Take $\alpha \in \mathcal{O}$ primitive, $\xi$-primary such that its norm has a decomposition in prime factors

$$Nm(\alpha) = p_1 \cdot \ldots \cdot p_s.$$  

Then $\alpha$ admits a decomposition in primitive irreducible and $\xi$-primary quaternions:

$$\alpha = \pi_1 \cdot \ldots \cdot \pi_s$$

with $Nm(\pi_i) = p_i$.

Moreover, if $2 \not\in \xi\mathcal{O}$ this decomposition is unique, and if $2 \in \xi\mathcal{O}$, the decomposition is unique up to sign.
How to obtain a Schottky group in $\Gamma_p$

Take $\alpha_1, \ldots, \alpha_s \in \mathcal{O}$ all quaternions with norm $p$ and trace $= 0$ (up to sign and conjugation)
How to obtain a Schottky group in $\Gamma_p$

Take $\alpha_1, \ldots, \alpha_s \in \mathcal{O}$ all quaternions with norm $p$ and trace $= 0$ (up to sign and conjugation) and $\beta_1, \ldots, \beta_t$ all quaternions with norm $p$ and trace $\neq 0$ (up to sign).
How to obtain a Schottky group in $\Gamma_p$

Take $\alpha_1, \ldots, \alpha_s \in \mathcal{O}$ all quaternions with norm $p$ and trace $= 0$ (up to sign and conjugation) and $\beta_1, \ldots, \beta_t$ all quaternions with norm $p$ and trace $\neq 0$ (up to sign). Take $\xi \in \mathcal{O}$ such that it satisfies the right-unit property and $2 \in \xi \mathcal{O}$. Then

$$S = \{ \Phi_p(\alpha_1), \ldots, \Phi_p(\alpha_s), \Phi_p(\beta_1), \ldots, \Phi_p(\beta_t) \} \subseteq \text{PGL}_2(\mathbb{Q}_p)$$

is a system of generators of $\Gamma_p$. In particular, if $p$ satisfies the null-trace condition with respect to $\xi \mathcal{O}$, then $\Gamma_p(\xi \mathcal{O}) \subseteq \Gamma_p$ is a Schottky group of rank $s$. 
How to obtain a Schottky group in $\Gamma_p$

Take $\alpha_1, \ldots, \alpha_s \in \mathcal{O}$ all quaternions with norm $p$ and trace $= 0$ (up to sign and conjugation) and $\beta_1, \ldots, \beta_t$ all quaternions with norm $p$ and trace $\neq 0$ (up to sign). Take $\xi \in \mathcal{O}$ such that it satisfies the right-unit property and $2 \in \xi \mathcal{O}$. Then

$$S = \{\Phi_p(\alpha_1), \ldots, \Phi_p(\alpha_s), \Phi_p(\beta_1), \ldots, \Phi_p(\beta_t)\} \subseteq \text{PGL}_2(\mathbb{Q}_p)$$
How to obtain a Schottky group in $\Gamma_p$

Take $\alpha_1, \ldots, \alpha_s \in \mathcal{O}$ all quaternions with norm $p$ and trace $= 0$ (up to sign and conjugation) and $\beta_1, \ldots, \beta_t$ all quaternions with norm $p$ and trace $\neq 0$ (up to sign). Take $\xi \in \mathcal{O}$ such that it satisfies the right-unit property and $2 \in \xi \mathcal{O}$. Then

$$S = \{[\Phi_p(\alpha_1)], \ldots, [\Phi_p(\alpha_s)], [\Phi_p(\beta_1)], \ldots, [\Phi_p(\beta_t)]\} \subseteq \text{PGL}_2(\mathbb{Q}_p)$$

is a system of generators of $\Gamma_p(\xi)$. 

Take $\alpha_1, \ldots, \alpha_s \in \mathcal{O}$ all quaternions with norm $p$ and trace $= 0$ (up to sign and conjugation) and $\beta_1, \ldots, \beta_t$ all quaternions with norm $p$ and trace $\neq 0$ (up to sign). Take $\xi \in \mathcal{O}$ such that it satisfies the right-unit property and $2 \in \xi \mathcal{O}$. Then

$$S = \{[\Phi_p(\alpha_1)], \ldots, [\Phi_p(\alpha_s)], [\Phi_p(\beta_1)], \ldots, [\Phi_p(\beta_t)]\} \subseteq \text{PGL}_2(\mathbb{Q}_p)$$

is a system of generators of $\Gamma_p(\xi)$.

In particular, if $p$ satisfies the null-trace condition with respect to $\xi \mathcal{O}$, then $\Gamma_p(\xi) \subseteq \Gamma_p$ is a Schottky group of rank $s$. 
Theorem (Amorós, Milione)
Computation of the reduction-graph

**Theorem** (Amorós, Milione)

$H$ definite quaternion algebra of discriminant $D$

$p \nmid D$ odd prime

Conditions that we need:

1. $h(D, N) = 1$
2. There exists a quaternion $\xi \in O$ which satisfies the right-unit property in $O$ with $2 \in \xi$
3. The prime $p$ satisfies the null-trace condition with respect to $\xi$
Computation of the reduction-graph

**Theorem** (Amorós, Milione)

Let $H$ definite quaternion algebra of discriminant $D$ and

$\mathcal{O} = \mathcal{O}_H(N) \subseteq H$ order of level $N$. 

- The conditions that we need are:
  1. $h(D, N) = 1$
  2. There exists a quaternion $\xi \in \mathcal{O}$ which satisfies the right-unit property in $\mathcal{O}$ with $2 \in \xi \mathcal{O}$.
  3. The prime $p$ satisfies the null-trace condition with respect to $\xi \mathcal{O}$. 
Theorem (Amorós, Milione)

$H$ definite quaternion algebra of discriminant $D$
$\mathcal{O} = \mathcal{O}_H(N) \subseteq H$ order of level $N$
$p \nmid DN$ odd prime
Theorem (Amorós, Milione)

$H$ definite quaternion algebra of discriminant $D$

$\mathcal{O} = \mathcal{O}_H(N) \subseteq H$ order of level $N$

$p \nmid DN$ odd prime

Conditions that we need:
Theorem (Amorós, Milione)

$H$ definite quaternion algebra of discriminant $D$

$\mathcal{O} = \mathcal{O}_H(N) \subseteq H$ order of level $N$

$p \nmid DN$ odd prime

Conditions that we need:

(1) $h(D, N) = 1$
Computation of the reduction-graph

**Theorem** (Amorós, Milione)

\( H \) definite quaternion algebra of discriminant \( D \)
\( \mathcal{O} = \mathcal{O}_H(N) \subseteq H \) order of level \( N \)
\( p \nmid DN \) odd prime

Conditions that we need:

1. \( h(D, N) = 1 \)
2. there exists a quaternion \( \xi \in \mathcal{O} \) which satisfies the right-unit property in \( \mathcal{O} \) with \( 2 \in \xi \mathcal{O} \)
Computation of the reduction-graph

**Theorem** (Amorós, Milione)

$H$ definite quaternion algebra of discriminant $D$
$\mathcal{O} = \mathcal{O}_H(N) \subseteq H$ order of level $N$
$p \nmid DN$ odd prime

Conditions that we need:

1. $h(D, N) = 1$
2. There exists a quaternion $\xi \in \mathcal{O}$ which satisfies the right-unit property in $\mathcal{O}$ with $2 \in \xi\mathcal{O}$
3. The prime $p$ satisfies the null-trace condition with respect to $\xi\mathcal{O}$
First consequence

Then:

\[(a) \text{ the group } \Gamma_p(\xi) \text{ is a Schottky group of rank } \left(\frac{p + 1}{2}\right) \text{ generated by the transformations in } \text{PGL}_2(\mathbb{Q}_p) \text{ represented by the matrices in } \tilde{S} := \Phi_p(\{ \alpha \in \mathcal{O} | \text{Nm}(\alpha) = p, \alpha \equiv 1 \mod \xi \mathcal{O} \}).\]

Example. (computations done with an algorithm that we have implemented in Magma)

Let \(D = 3\), \(N = 2\), \(\xi = \frac{1}{2}(-1 - i - j + k)\).

Take \(p = 13\) and check that it satisfies the null-trace condition.

Then \(\Gamma_p(\xi)\) is a Schottky group with generators:

\[
\left( \begin{array}{cccc}
\frac{1}{2}(i - 6) & \frac{1}{2}(2i - 1) \\
\frac{1}{2}(6i + 3) & \frac{1}{2}(-i - 6) \\
\end{array} \right),
\]

\[
\left( \begin{array}{cccc}
\frac{1}{2}(-i - 6) & \frac{1}{2}(2i + 1) \\
\frac{1}{2}(6i - 3) & \frac{1}{2}(i - 6) \\
\end{array} \right),
\]

\[
\left( \begin{array}{cccc}
i - 3 & -1 \\
3 & -i - 3 \\
\end{array} \right),
\]

\[
\left( \begin{array}{cccc}
-1 & 2i \\
6i & -1 \\
\end{array} \right),
\]

\[
\left( \begin{array}{cccc}
\frac{1}{2}(3i - 2) & \frac{1}{2}(2i - 3) \\
\frac{1}{2}(6i + 9) & \frac{1}{2}(-3i - 2) \\
\end{array} \right),
\]

\[
\left( \begin{array}{cccc}
\frac{1}{2}(-3i - 2) & \frac{1}{2}(2i + 3) \\
\frac{1}{2}(6i - 9) & \frac{1}{2}(3i - 2) \\
\end{array} \right),
\]

\[
\left( \begin{array}{cccc}
-3i - 1 & -1 \\
3 & 3i - 1 \\
\end{array} \right).
\]
First consequence

Then:

(a) the group $\Gamma_p(\xi)$ is a Schottky group of rank $(p + 1)/2$
First consequence

Then:

(a) the group $\Gamma_p(\xi)$ is a Schottky group of rank $(p + 1)/2$ generated by the transformations in $\text{PGL}_2(\mathbb{Q}_p)$ represented by the matrices in

$$\tilde{S} := \Phi_p(\{\alpha \in \mathcal{O} \mid \text{Nm}(\alpha) = p, \alpha \equiv 1 \mod \xi \mathcal{O}\}).$$
First consequence

Then:

(a) the group $\Gamma_p(\xi)$ is a Schottky group of rank $(p + 1)/2$ generated by the transformations in $\text{PGL}_2(\mathbb{Q}_p)$ represented by the matrices in

$$\tilde{S} := \Phi_p(\{\alpha \in \mathcal{O} \mid \text{Nm}(\alpha) = p, \alpha \equiv 1 \mod \xi\mathcal{O}\}).$$

Example. (computations done with an algorithm that we have implemented in Magma)
Then:

(a) the group $\Gamma_p(\xi)$ is a Schottky group of rank $(p + 1)/2$ generated by the transformations in $\text{PGL}_2(\mathbb{Q}_p)$ represented by the matrices in

$$\tilde{S} := \Phi_p(\{\alpha \in \mathcal{O} \mid \text{Nm}(\alpha) = p, \alpha \equiv 1 \mod \xi \mathcal{O}\}).$$

Example. (computations done with an algorithm that we have implemented in Magma)

Let $D = 3$, $N = 2$, $\xi = \frac{1}{2}(-1 - i - j + k)$. 
First consequence

Then:

(a) the group $\Gamma_p(\xi)$ is a Schottky group of rank $(p + 1)/2$ generated by the transformations in $\text{PGL}_2(\mathbb{Q}_p)$ represented by the matrices in

$$\tilde{S} := \Phi_p(\{\alpha \in \mathcal{O} \mid \text{Nm}(\alpha) = p, \alpha \equiv 1 \mod \xi\mathcal{O}\}).$$

Example. (computations done with an algorithm that we have implemented in Magma)

Let $D = 3$, $N = 2$, $\xi = \frac{1}{2}(-1 - i - j + k)$. Take $p = 13$ and check that it satisfies the null-trace condition.
First consequence

Then:

(a) the group $\Gamma_p(\xi)$ is a Schottky group of rank $(p + 1)/2$ generated by the transformations in $\text{PGL}_2(\mathbb{Q}_p)$ represented by the matrices in

$$\tilde{S} := \Phi_p(\{\alpha \in \mathcal{O} \mid \text{Nm}(\alpha) = p, \alpha \equiv 1 \mod \xi\mathcal{O}\}).$$

Example. (computations done with an algorithm that we have implemented in Magma)

Let $D = 3$, $N = 2$, $\xi = \frac{1}{2}(-1 - i - j + k)$.
Take $p = 13$ and check that it satisfies the null-trace condition.
Then $\Gamma_p(\xi)$ is a Schottky group with generators:
First consequence

Then:

(a) the group $\Gamma_p(\xi)$ is a Schottky group of rank $(p + 1)/2$ generated by the transformations in $\text{PGL}_2(\mathbb{Q}_p)$ represented by the matrices in

$$\tilde{S} := \Phi_p(\{\alpha \in \mathcal{O} \mid \text{Nm}(\alpha) = p, \alpha \equiv 1 \mod \xi \mathcal{O}\}).$$

Example. (computations done with an algorithm that we have implemented in Magma)

Let $D = 3$, $N = 2$, $\xi = \frac{1}{2}(-1 - i - j + k)$.
Take $p = 13$ and check that it satisfies the null-trace condition.
Then $\Gamma_p(\xi)$ is a Schottky group with generators:

$$\begin{pmatrix} 1/2(i-6) & 1/2(2i-1) \\ 1/2(6i+3) & 1/2(-i-6) \end{pmatrix}, \begin{pmatrix} 1/2(-i-6) & 1/2(2i+1) \\ 1/2(6i-3) & 1/2(i-6) \end{pmatrix}, \begin{pmatrix} i-3 & -1 \\ 3 & -i-3 \end{pmatrix}, \begin{pmatrix} -1 & 2i \\ 6i & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1/2(3i-2) & 1/2(2i-3) \\ 1/2(6i+9) & 1/2(-3i-2) \end{pmatrix}, \begin{pmatrix} 1/2(-3i-2) & 1/2(2i+3) \\ 1/2(6i-9) & 1/2(3i-2) \end{pmatrix}, \begin{pmatrix} -3i-1 & -1 \\ 3 & 3i-1 \end{pmatrix}.$$
Second consequence

(b) A good fundamental domain for the action of $\Gamma_p(\xi)$ with respect to $\tilde{S}$ is
(b) A good fundamental domain for the action of $\Gamma_p(\xi)$ with respect to $\tilde{S}$ is

$$\mathcal{F}_p(\xi) := \mathbb{P}^{1,\text{rig}}(\mathbb{C}_p) \setminus \bigcup_{a \in \{0, \ldots, p-1, \infty\}} \mathbb{B}^-(a, 1/\sqrt{p}) \subseteq \mathcal{H}_p$$
(b) A good fundamental domain for the action of $\Gamma_p(\xi)$ with respect to $\tilde{S}$ is

$$\mathcal{F}_p(\xi) := \mathbb{P}^{1,\text{rig}}(\mathbb{C}_p) \setminus \bigcup_{a \in \{0, \ldots, p-1, \infty\}} \mathbb{B}^{-}(a, 1/\sqrt{p}) \subseteq \mathcal{H}_p$$

(c) Let $X_p(\xi)$ Mumford curve associated to $\Gamma_p(\xi)$. 
(b) A good fundamental domain for the action of $\Gamma_p(\xi)$ with respect to $\tilde{S}$ is

$$\mathcal{F}_p(\xi) := \mathbb{P}^{1,\text{rig}}(\mathbb{C}_p) \setminus \bigcup_{a \in \{0, \ldots, p-1, \infty\}} B^-(a, 1/\sqrt{p}) \subseteq \mathcal{H}_p$$

(c) Let $X_p(\xi)$ Mumford curve associated to $\Gamma_p(\xi)$.
Then the rigid analytic curve $X_p(\xi)^{\text{rig}}$ is obtained from the fundamental domain $\mathcal{F}_p(\xi)$ with the following pair-wise identifications:
(b) A good fundamental domain for the action of $\Gamma_p(\xi)$ with respect to $\tilde{S}$ is

$$\mathcal{F}_p(\xi) := \mathbb{P}^{1,\text{rig}}(\mathbb{C}_p) \setminus \bigcup_{a \in \{0, \ldots, p-1, \infty\}} \mathbb{B}^-(a, 1/\sqrt{p}) \subseteq \mathcal{H}_p$$

(c) Let $X_p(\xi)$ Mumford curve associated to $\Gamma_p(\xi)$. Then the rigid analytic curve $X_p(\xi)^{\text{rig}}$ is obtained from the fundamental domain $\mathcal{F}_p(\xi)$ with the following pair-wise identifications: for every $\gamma \in \tilde{S}$,
Second consequence

(b) A good fundamental domain for the action of $\Gamma_p(\xi)$ with respect to $\widetilde{S}$ is

$$\mathcal{F}_p(\xi) := \mathbb{P}^{1,\text{rig}}(\mathbb{C}_p) \setminus \bigcup_{a \in \{0, \ldots, p-1, \infty\}} \mathbb{B}^-(a, 1/\sqrt{p}) \subseteq \mathcal{H}_p$$

(c) Let $X_p(\xi)$ Mumford curve associated to $\Gamma_p(\xi)$. Then the rigid analytic curve $X_p(\xi)^{\text{rig}}$ is obtained from the fundamental domain $\mathcal{F}_p(\xi)$ with the following pair-wise identifications: for every $\gamma \in \widetilde{S}$,

$$\gamma \left( \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{B}^-(\tilde{\alpha}_\gamma, 1/\sqrt{p}) \right) = \mathbb{B}^+(\tilde{\alpha}_{\gamma^{-1}}, 1/\sqrt{p}),$$
Second consequence

(b) A good fundamental domain for the action of $\Gamma_p(\xi)$ with respect to $\tilde{S}$ is

$$
\mathcal{F}_p(\xi) := \mathbb{P}^1,\text{rig}(\mathbb{C}_p) \setminus \bigcup_{a \in \{0,\ldots, p-1, \infty\}} \mathbb{B}^-(a, 1/\sqrt{p}) \subseteq \mathcal{H}_p
$$

(c) Let $X_p(\xi)$ Mumford curve associated to $\Gamma_p(\xi)$. Then the rigid analytic curve $X_p(\xi)^{\text{rig}}$ is obtained from the fundamental domain $\mathcal{F}_p(\xi)$ with the following pair-wise identifications: for every $\gamma \in \tilde{S}$,

$$
\gamma (\mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{B}^-(\tilde{\alpha}_\gamma, 1/\sqrt{p})) = \mathbb{B}^+(\tilde{\alpha}_{\gamma^{-1}}, 1/\sqrt{p}),
$$

$$
\gamma (\mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{B}^+(\tilde{\alpha}_\gamma, 1/\sqrt{p})) = \mathbb{B}^-(\tilde{\alpha}_{\gamma^{-1}}, 1/\sqrt{p}),
$$
Second consequence

(b) A good fundamental domain for the action of \( \Gamma_p(\xi) \) with respect to \( \tilde{S} \) is

\[
\mathcal{F}_p(\xi) := \mathbb{P}^{1,\text{rig}}(\mathbb{C}_p) \setminus \bigcup_{a \in \{0, \ldots, p-1, \infty\}} \mathbb{B}^-(a, 1/\sqrt{p}) \subseteq \mathcal{H}_p
\]

(c) Let \( X_p(\xi) \) Mumford curve associated to \( \Gamma_p(\xi) \). Then the rigid analytic curve \( X_p(\xi)^{\text{rig}} \) is obtained from the fundamental domain \( \mathcal{F}_p(\xi) \) with the following pair-wise identifications: for every \( \gamma \in \tilde{S} \),

\[
\gamma \left( \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{B}^-(\tilde{a}_\gamma, 1/\sqrt{p}) \right) = \mathbb{B}^+(\tilde{a}_{\gamma^{-1}}, 1/\sqrt{p}),
\]

\[
\gamma \left( \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{B}^+(\tilde{a}_\gamma, 1/\sqrt{p}) \right) = \mathbb{B}^-(\tilde{a}_{\gamma^{-1}}, 1/\sqrt{p}),
\]

where \( \tilde{a}_\gamma \) and \( \tilde{a}_{\gamma^{-1}} \) are defined as the reduction in \( \mathbb{P}^1(\mathbb{F}_p) \) of the fixed points of the transformations \( \{\gamma, \gamma^{-1}\} \).
Our example

Fundamental domain for the action of $\Gamma_p(\xi)$ on $\mathcal{H}_p$
(d) The stable reduction-graph of $X_p(\xi)$ is the open subtree
$T_p^{(1)} \setminus \{v_0^{(1)}, \ldots, v_{p-1}^{(1)}, v_{\infty}^{(1)}\}$ of $T_p$. 
(d) The stable reduction-graph of $X_p(\xi)$ is the open subtree
\[ \mathcal{T}_p^{(1)} \setminus \{ v_0^{(1)}, \ldots, v_p^{(1)}, v_\infty^{(1)} \} \]
of $\mathcal{T}_p$ via the pair-wise identifications of the $p + 1$ oriented edges given by
\[ \gamma e_{\tilde{\alpha}_\gamma} = -e_{\tilde{\alpha}_{\gamma - 1}}, \]
for every $\gamma \in \tilde{S}$. 
Our Example

Reduction of the fundamental domain $\mathcal{F}_{13}(\xi)$
Our example

Stable reduction-graph of the Mumford curve associated to $\Gamma_{13}(\xi)$
Our example

Reduction-graphs with lengths $\Gamma_{13} \setminus \mathcal{T}_{13}$
Our example

Reduction-graphs with lengths $\Gamma_{13,+} \setminus T_{13}$ for the Shimura curve $X(3 \cdot 13, 2)$
Thank you!