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**SURVEY ARTICLE**

# Long time existence of Ricci-harmonic flow

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**Abstract** We give a survey about recent results on Ricci-harmonic flow.

**Keywords** Ricci-harmonic flow (RHF), curvature pinching estimates, bounded scalar curvature
**MSC** 53C44

## 1 Introduction

Consider the space-time $\mathbb{R}^{1,3}$ with a Lorentz metric

$$h = -V^2(x^1, x^2, x^3)\mathrm{d}t \otimes \mathrm{d}t + \sum_{1\leqslant i,j\leqslant 3} g_{ij}(x^1, x^2, x^3)\mathrm{d}x^i \otimes \mathrm{d}x^j,$$

where $V$ is a positive smooth function on $\mathbb{R}^3$ and $g$ is a Riemannian metric on $\mathbb{R}^3$. A solution to the *static Einstein vacuum equation*[1)] is a pair $(V, g)$, where $V$ is a positive smooth function on $\mathbb{R}^3$ and $g$ is a Riemannian metric on $\mathbb{R}^3$, satisfying

$$\mathrm{Ric}_g = V^{-1}\nabla_g^2 V, \quad V^{-1}\Delta_g V = 0. \tag{1.1}$$

In particular, when $V$ is a positive constant, we obtain the classical Einstein equation. Consider the conformal transformation

$$\widetilde{g} := \mathrm{e}^{2\alpha\psi} g, \quad \psi \in C^\infty(\Sigma),\ \dim\Sigma = n,\ \alpha \in \mathbb{R}.$$

Then

$$\begin{aligned} \widetilde{R}_{ij} &= R_{ij} - (n-2)\alpha\nabla_i\nabla_j\psi + (n-2)\alpha^2\nabla_i\psi\nabla_j\psi \\ &\quad - \alpha g_{ij}\Delta_g\psi - (n-2)\alpha^2|\nabla_g\psi|_g^2 g_{ij}, \\ \Delta_{\widetilde{g}} f &= \mathrm{e}^{-2\alpha\psi}[\Delta_g f + (n-2)\alpha\langle\nabla_g\psi, \nabla_g f\rangle_g], \quad f \in C^\infty(\Sigma). \end{aligned} \tag{1.2}$$

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1) For more physical background, see [27,40]

Let $u = \log V$, $\widetilde{g} = e^{2u} g$ in (1.1). We then obtain

$$\widetilde{R}_{ij} = 2\nabla_i u \nabla_j u, \quad \Delta_{\widetilde{g}} u = 0, \tag{1.3}$$

since

$$R_{ij} = \nabla_i \nabla_j u + \nabla_i u \nabla_j u.$$

To find a solution to the static Einstein vacuum equation, we shall try to solve (1.3). One method is to use the corresponding parabolic flow, that is, *Ricci-harmonic flow* (RHF).

Given a closed manifold $M$ of dimension $n$, the RHF is defined by

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)} + 2\alpha(t)\nabla_{g(t)}\phi(t) \otimes \nabla_{g(t)}\phi(t), \quad \partial_t \phi(t) = \Delta_{g(t)}\phi(t), \tag{1.4}$$

introduced in [27,28,32,34], where $g(t)$ is a family of Riemannian metric, $\phi(t)$ is a family of functions, and $t \in [0, T)$ (with $T \in (0, +\infty]$, and the existence of $T$ was proved in [27,32]). Here, $\alpha(t)$ is a time-dependent positive constant. In particular, we may choose $\alpha(t) \equiv \alpha$, a positive constant. If all functions $\phi(t) \equiv 0$, we obtain the Ricci flow (RF) introduced by Hamilton in his famous paper [22] and definitely used by Perelman [37–39] on his work about the Poincaré conjecture. The flow equations (1.1) come from mentioned static Einstein vacuum equations arising in the general relativity, and also arise as dimensional reductions of RF in higher dimensions [30].

Several analogous flows have been investigated in recent years. For example, connection Ricci flow [44], Ricci-Yang-Mills flow [43,46,56], and renormalization group flows [23,24,36,45].

Without loss of generality, we may assume $\alpha(t) \equiv 2$ in (1.4); thus, we consider the following RHF:

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)} + 4\nabla_{g(t)}\phi(t) \otimes \nabla_{g(t)}\phi(t), \quad \partial_t \phi(t) = \Delta_{g(t)}\phi(t), \tag{1.5}$$

For general $\alpha(t)$, the same results also hold (see [26–28,32,34]).

Throughout this paper, we fix a closed manifold $M$ of dimension $n$. For any Riemannian metric $g$ on $M$, let $\nabla_g$ denote the Levi-Civita connection induced by $g$. The Riemann curvature tensor $\mathrm{Rm}_g$, $\mathrm{Ric}_g$, and scalar curvature $R_g$ are, respectively, defined by

$$\mathrm{Rm}_g(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$R_{ijk}^{\ell} \frac{\partial}{\partial x^\ell} := \mathrm{Rm}_g\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^k}, \quad R_{ijk\ell} := g_{\ell p} R_{ijk}^p,$$

$$R_{ij} := \mathrm{Ric}_g\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) = \sum_{1 \leqslant k \leqslant m} R_{kijk}.$$

The volume element of $g$ is denoted by $\mathrm{d}V_g$.

## 2 Perelman-type functionals

As in [27], we define

$$\mathrm{Sic}_g := \mathrm{Ric}_g - 2\nabla_g\phi \otimes \nabla_g\phi, \quad S_g := R_g - \alpha|\nabla_g\phi|^2_g. \tag{2.1}$$

For any Riemannian metric $g$ and any smooth functions $\phi, f$ on $M$, we have a number of Perelman-type functionals:

$$\mathscr{F}(g,\phi,f) := \int_M (R_g + |\nabla_g f|^2_g - 2|\nabla_g\phi|^2_g)\mathrm{e}^{-f}\mathrm{d}V_g, \tag{2.2}$$

$$\mathscr{E}(g,\phi,f) := \int_M (R_g - 2|\nabla_g\phi|^2_g)\mathrm{e}^{-f}\mathrm{d}V_g, \tag{2.3}$$

$$\mathscr{F}_k(g,\phi,f) := \int_M (kR_g + |\nabla_g f|^2_g - 2k|\nabla_g\phi|^2_g)\mathrm{e}^{-f}\mathrm{d}V_g. \tag{2.4}$$

List [27] and Müller [34] showed that, under the system of evolution equations

$$\begin{aligned} \partial_t g(t) &= -2\mathrm{Ric}_{g(t)} + 4\nabla_{g(t)}\phi(t)\otimes\nabla_{g(t)}\phi(t), \\ \partial_t\phi(t) &= \Delta_{g(t)}\phi(t), \\ \partial_t f(t) &= -\Delta_{g(t)}f(t) - R_{g(t)} + |\nabla_{g(t)}f(t)|^2_{g(t)} + 2|\nabla_{g(t)}\phi(t)|^2_{g(t)}, \end{aligned} \tag{2.5}$$

the evolution equation for $\mathscr{F}$ is

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}(g(t),\phi(t),f(t)) \\ &\quad = 2\int_M |\mathrm{Sic}_{g(t)} + \nabla^2_{g(t)}f(t)|^2_{g(t)}\mathrm{e}^{-f(t)}\mathrm{d}V_{g(t)} \\ &\qquad + 4\int_M |\Delta_{g(t)}\phi(t) - \langle\nabla_{g(t)}\phi(t),\nabla_{g(t)}f(t)\rangle_{g(t)}|^2_{g(t)}\mathrm{e}^{-f(t)}\mathrm{d}V_{g(t)}, \end{aligned} \tag{2.6}$$

which is nonnegative. The evolution equations for other two functionals are derived in [25].

**Theorem 2.1** *Under* (2.5), *one has*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(g(t),\phi(t),f(t)) \\ &\quad = 2\int_M |\mathrm{Sic}_{g(t)}|^2_{g(t)}\mathrm{e}^{-f(t)}\mathrm{d}V_{g(t)} + 4\int_M \left|\Delta_{g(t)}\phi(t)\right|^2_{g(t)}\mathrm{e}^{-f(t)}\mathrm{d}V_{g(t)}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_k(g(t),\phi(t),f(t)) \\ &\quad = (k-1)\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(g(t),\phi(t),f(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}(g(t),\phi(t),f(t)). \end{aligned} \tag{2.8}$$

As a consequence, we give a new proof of the following result (see also [27,28,32,34]): there is no compact steady Ricci-harmonic breather unless the manifold $(M, g(t))$ is Ricci-flat and $\phi(t)$ is a constant. Recall that a solution $(g(t), \phi(t))$ of RHF is called a *Ricci-harmonic breather* if there exist $t_1 < t_2$, a diffeomorphism $\Psi\colon M \to M$, and a constant $\alpha > 0$ such that

$$g(t_2) = \alpha \Psi^* g(t_1), \quad \phi(t_2) = \Psi^* \phi(t_1).$$

The cases $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$, correspond to *shrinking*, *steady*, and *expanding Ricci-harmonic breathers*, respectively.

To deal with the expanding Ricci-harmonic breather, we need the following functionals (see [25,27,34]):

$$\mathscr{L}_+(g, \phi, \tau, f) := \tau^2 \int_M \Big(R_g + \frac{n}{2\tau} + \Delta_g f - 2|\nabla_g \phi|_g^2\Big) \mathrm{e}^{-f} \mathrm{d}V_g, \tag{2.9}$$

$$\mathscr{L}_{+,k}(g, \phi, \tau, f) := \tau^2 \int_M \Big[k\Big(R_g + \frac{n}{2\tau}\Big) + \Delta_g f - 2k|\nabla_g \phi|_g^2\Big] \mathrm{e}^{-f} \mathrm{d}V_g. \tag{2.10}$$

Here, $g, \phi, f$ are as above, and $\tau$ is a constant. Under the system of evolution equations

$$\begin{aligned} \partial_t g(t) &= -2\mathrm{Ric}_{g(t)} + 4\nabla_{g(t)}\phi(t) \otimes \nabla_{g(t)}\phi(t), \\ \partial_t \phi(t) &= \Delta_{g(t)}\phi(t), \\ \partial_t f(t) &= -\Delta_{g(t)} f(t) + |\nabla_{g(t)} f(t)|_{g(t)}^2 - R_{g(t)} + 2|\nabla_{g(t)}\phi(t)|_{g(t)}^2, \\ \frac{\mathrm{d}}{\mathrm{d}t}\tau(t) &= 1, \end{aligned} \tag{2.11}$$

we have the following result.

**Theorem 2.2** *Under* (2.11), *one has*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}_+(g(t), \phi(t), \tau(t), f(t)) \\ &\quad = 2\tau(t)^2 \int_M \Big|\mathrm{Sic}_{g(t)} + \nabla_{g(t)}^2 f(t) + \frac{1}{2\tau(t)} g(t)\Big|_{g(t)}^2 \mathrm{e}^{-f(t)} \mathrm{d}V_{g(t)} \\ &\qquad + 4\tau(t)^2 \int_M |\Delta_{g(t)}\phi(t) - \langle \nabla_{g(t)}\phi(t), \nabla_{g(t)} f(t)\rangle_{g(t)}|_{g(t)}^2 \mathrm{e}^{-f(t)} \mathrm{d}V_{g(t)}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}_{+,k}(g(t), \phi(t), \tau(t), f(t)) \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}_+(g(t), \phi(t), \tau(t), f(t)) \\ &\qquad + 2(k-1)\tau(t)^2 \int_M \Big|\mathrm{Sic}_{g(t)} + \frac{1}{2\tau(t)} g(t)\Big|_{g(t)}^2 \mathrm{e}^{-f(t)} \mathrm{d}V_{g(t)} \\ &\qquad + 4(k-1)\tau(t)^2 \int_M |\Delta_{g(t)}\phi(t)|_{g(t)}^2 \mathrm{e}^{-f(t)} \mathrm{d}V_{g(t)}. \end{aligned} \tag{2.13}$$

As a corollary, we obtain a new proof of the following result (see also [27,28, 32,34]): there is no expanding Ricci-harmonic breather on compact Riemannian manifolds unless the manifold is an Einstein manifold and $\phi(t)$ is a constant. For the proof, see [25,27].

Let us consider

$$\mu(g,\phi) := \inf\left\{\mathscr{F}(g,\phi,f)\colon f\in C^\infty(M), \int_M \mathrm{e}^{-f}\mathrm{d}V_g = 1\right\}, \tag{2.14}$$

which is the smallest eigenvalue of the operator

$$\Delta_{g,\phi} := -4\Delta_g + R_g - 2|\nabla_g\phi|_g^2.$$

For the related eigenvalue problems, see [19,25].

We finish this section by introducing Perelman-type $\mathscr{W}$ functional. For any Riemannian metric $g$, any smooth functions $\phi, f$, and any positive number $\tau$, we define

$$\mathscr{W}_\pm(g,\phi,f,\tau) := \int_M [\tau(S_g + |\nabla_g f|_g^2) \mp f \pm n]\frac{\mathrm{e}^{-f}}{(4\pi\tau)^{n/2}}\,\mathrm{d}V_g, \tag{2.15}$$

and set

$$\mu_\pm(g,\phi,\tau) := \inf\left\{\mathscr{W}_\pm(g,\phi,f,\tau)\colon f\in C^\infty(M), \int_M \frac{\mathrm{e}^{-f}\mathrm{d}V_g}{(4\pi\tau)^{n/2}} = 1\right\}, \tag{2.16}$$

$$\nu_+(g,\phi) := \sup\{\mu_+(g,\phi,\tau)\colon \tau > 0\}, \tag{2.17}$$

$$\nu_-(g,\phi) := \inf\{\mu_-(g,\phi,\tau)\colon \tau > 0\}. \tag{2.18}$$

In [24], we computed the first and second variations of $\nu_\pm(g,\phi)$. Consequently, if $\mathscr{W}_\pm(g,\phi,\cdot,\cdot)$ and $\nu_\pm(\cdot,\cdot)$ achieve their extremum, then $(M,g)$ is a gradient expanding and shrinker Ricci-harmonic soliton according to the sign; if $\mathscr{W}_\pm(g,\phi,\cdot,\cdot)$ achieve their minimum and $(g,\phi)$ is a critical point of $\nu_\pm(\cdot,\cdot)$, then $(M,g)$ is an Einstein manifold and $\phi$ is a constant function.

## 3 Long time existence of RHF

For the RF, Hamilton [22] showed that the short-time existence and

$$T < +\infty \implies \limsup_{t\to T}\big(\max_M |\mathrm{Rm}_{g(t)}|_{g(t)}^2\big) = +\infty. \tag{3.1}$$

Later, Sesum [41] improved Hamilton's result as

$$T < +\infty \implies \limsup_{t\to T}\big(\max_M |\mathrm{Ric}_{g(t)}|_{g(t)}^2\big) = +\infty \tag{3.2}$$

by blow-up argument. For the integral bounds, Ye [55] and Wang [50] independently proved that

$$T < +\infty \implies \left( \int_0^T \int_M |\mathrm{Rm}_{g(t)}|_{g(t)}^{(m+2)/2} \mathrm{d}V_{g(t)} \mathrm{d}t \right)^{2/(m+2)} = +\infty. \tag{3.3}$$

Moreover, Wang [50] proved another version for RF that

$$\mathrm{Ric}_{g(t)} \geqslant -C,\ T < +\infty \implies \left( \int_0^T \int_M |R_{g(t)}|_{g(t)}^{(m+2)/2} \mathrm{d}V_{g(t)} \mathrm{d}t \right)^{2/(m+2)} = +\infty. \tag{3.4}$$

Here, $C$ is a uniform constant. For other work on integral bounds, see, for example, [8,31,51,54].

A well-known conjecture (see [6]) about the extension of RF states that is it true for

$$\limsup_{t\to T} \big( \max_M R_{g(t)} \big) = +\infty? \tag{3.5}$$

Here, $T < +\infty$. This conjecture was settled for Kähler-Ricci flow by Zhang [57] and for type-I maximal solution of RF by Enders-Müller-Topping [10]. Cao [6] proved the following:

$$T < +\infty \implies \begin{array}{c} \text{either } \limsup\limits_{t\to T} \big( \max\limits_M R_{g(t)} \big) = +\infty, \text{ or} \\ \limsup\limits_{t\to T} \big( \max\limits_M R_{g(t)} \big) < +\infty \text{ but } \limsup\limits_{t\to T} \dfrac{|W_{g(t)}|_{g(t)}}{R_{g(t)}} = +\infty, \end{array} \tag{3.6}$$

where $W_{g(t)}$ denotes the Weyl tensor of $g(t)$.

For 4D RF, Simon [42] and Bamler-Zhang [5] independently proved

$$T < +\infty,\ |R_{g(t)}| \leqslant C \implies \int_M |\mathrm{Ric}_{g(t)}|_{g(t)}^2 \mathrm{d}V_{g(t)},\ \int_M |\mathrm{Rm}_{g(t)}|_{g(t)}^2 \mathrm{d}V_{g(t)} \leqslant C' \tag{3.7}$$

by different methods (for earlier work, see [49]).

On the other hand, for RHF, we have the following results. When $n = \dim M \geqslant 3$ and $T < +\infty$. Müller [32,34] showed that (3.1) is also true for RHF. Recently, Cheng and Zhu [9] extended Sesum's result [41] to RHF, that is, (3.2) is true for RHF. For more results about RHF, see [1–4,7,9,11–21,25,27–29, 32–35,47,48,52,53,58].

The first main result is an extension of Cao's result [6] to RHF.

**Theorem 3.1** *Let $(M, g(t), \phi(t))_{t\in[0,T)}$ be a solution to RHF (1.5) on a closed manifold $M$ with $n = \dim M \geqslant 3$ and $T < +\infty$. Either one has*

$$\limsup_{t\to T} \big( \max_M R_{g(t)} \big) = +\infty$$

*or*

$$\limsup_{t\to T} \big( \max_M R_{g(t)} \big) < +\infty, \quad \limsup_{t\to T} \max_M \frac{|W_{g(t)}|_{g(t)} + |\nabla^2_{g(t)} \phi(t)|^2_{g(t)}}{R_{g(t)}} = +\infty.$$

*Here, $W_{g(t)}$ is the Weyl part of* $\mathrm{Rm}_{g(t)}$.

The second main result focuses on the 4D RHF. According to Theorem 4.2 below, we can find a uniform constant $C$ such that $S_{g(t)}+C>0$ for all $t\in[0,T)$.

**Theorem 3.2** *Let $(M,g(t),\phi(t))_{t\in[0,T)}$ be a solution to RHF* (1.5) *on a closed manifold $M$ with $n=\dim M=4$ and $T\leqslant+\infty$. Choose a uniform constant $C$ in Theorem* 4.2 *such that $S_{g(t)}+C>0$. Then*

$$\int_M |\mathrm{Sic}_{g(s)}|_{g(s)}\mathrm{d}V_{g(s)} \leqslant 2c_0(M,g(0),\phi(0),s)+\frac{C}{2}\mathrm{Vol}(M,g(s)) + 1148\mathrm{e}^{36Cs}\int_0^s\int_M S_{g(t)}^2\mathrm{d}V_{g(t)}\mathrm{d}t, \tag{3.8}$$

$$\int_0^s\int_M |\mathrm{Sic}_{g(t)}|_{g(t)}^2\mathrm{d}V_{g(t)}\mathrm{d}t \leqslant 8c_0(M,g(0),\phi(0),s)+\frac{C^2}{4}\int_0^s\mathrm{Vol}(M,g(t))\mathrm{d}t + 4592\mathrm{e}^{36Cs}\int_0^s\int_M S_{g(t)}^2\mathrm{d}V_{g(t)}\mathrm{d}t, \tag{3.9}$$

*for all $s\in[0,T)$. Here,*

$$\begin{aligned} c_0&(M,g(0),\phi(0),s)\\ &=\frac{256\pi^2\chi(M)}{36C}(\mathrm{e}^{36Cs}-1)+\frac{416A_1^2\mathrm{Vol}(M,g(0))}{35C}(\mathrm{e}^{35Cs}-\mathrm{e}^{Cs})\\ &\quad+2\mathrm{e}^{37Cs}A_1\mathrm{Vol}(M,g(0))+\mathrm{e}^{36Cs}\int_M\frac{|\mathrm{Sic}_{g(0)}|_{g(0)}^2}{S_{g(0)}+C}\mathrm{d}V_{g(0)}, \end{aligned} \tag{3.10}$$

*where*

$$A_1=\max_M|\nabla_{g(0)}\phi(0)|_{g(0)}^2$$

*and $\chi(M)$ is the Euler characteristic of $M$.*

According to Theorem 4.2 below and following [42], we consider the basic assumption (BA) for a solution $(M,g(t),\phi(t))_{t\in[0,T)}$ to RHF:

(a) $M$ is a connected and closed 4-dimensional smooth manifold,

(b) $(M,g(t),\phi(t))_{t\in[0,T)}$ is a solution to RHF,

(c) $T<+\infty$,

(d) $\max_{M\times[0,T)}|S_{g(t)}|\leqslant 1$.

The upper bound 1 in condition (d) is not essential, since we can rescale the pair $(g(t),\phi(t))$ so that condition (d) is always satisfied. Furthermore, since

$$|\nabla_{g(t)}\phi(t)|_{g(t)}^2\leqslant A_1$$

(by (5.6)), it follows that condition (d) is equivalent to the uniform bound for $R_{g(t)}$.

**Theorem 3.3** *If $(M, g(t), \phi(t))_{t\in[0,T)}$ satisfies BA, then*

$$\int_M |\mathrm{Sic}_{g(s)}|^2_{g(s)} \mathrm{d}V_{g(s)} \leqslant b(M, g(0), \phi(0), s), \tag{3.11}$$

$$\int_0^s \int_M |\mathrm{Sic}_{g(t)}|^4_{g(t)} \mathrm{d}V_{g(t)} \mathrm{d}t \leqslant b(M, g(0), \phi(0), s), \tag{3.12}$$

*for any $s \in [0, T)$. Here,*

$$\begin{aligned} & b(M, g(0), \phi(0), s) \\ & := 9\mathrm{e}^{88s} \int_M |\mathrm{Sic}_{g(0)}|^2_{g(0)} \mathrm{d}V_{g(0)} + \frac{144}{11} \pi^2 \chi(M)(\mathrm{e}^{88s} - 1) \\ & \quad + \frac{936}{43} A_1^2 \mathrm{Vol}(M, g(0))(\mathrm{e}^{88s} - \mathrm{e}^{2s}) + 18A_1 \mathrm{Vol}(M, g(0))\mathrm{e}^{90s}. \end{aligned} \tag{3.13}$$

Define

$$\begin{aligned} c(M, g(0), \phi(0), T) := 9\mathrm{e}^{90T} \Big[ & \int_M |\mathrm{Sic}_{g(0)}|^2_{g(0)} \mathrm{d}V_{g(0)} + \pi^2 |\chi(M)| \\ & + (4A_1^2 + 2A_1)\mathrm{Vol}(M, g(0)) \Big]. \end{aligned} \tag{3.14}$$

Then

$$|b(M, g(0), \phi(0), T)| \leqslant c(M, g(0), \phi(0), T). \tag{3.15}$$

Theorem 3.3 now yields the following result.

**Theorem 3.4** *If $(M, g(t), \phi(t))_{t\in[0,T)}$ satisfies BA, then*

$$\sup_{t\in[0,T)} \int_M |\mathrm{Sic}_{g(t)}|^2_{g(t)} \mathrm{d}V_{g(t)} \leqslant c(M, g(0), \phi(0), T) < +\infty, \tag{3.16}$$

$$\begin{aligned} \sup_{t\in[0,T)} \int_M |\mathrm{Sm}_{g(t)}|^2_{g(t)} \mathrm{d}V_{g(t)} & \leqslant 32\pi^2 \chi(M) + 8c(M, g(0), \phi(0), T) \\ & \quad + 52A_1^2 \mathrm{Vol}(M, g(0))\mathrm{e}^{2T} \\ & < +\infty. \end{aligned} \tag{3.17}$$

The proof of Theorem 3.1 is based on a "curvature pinching estimate" for RHF (see Theorem 4.2). The new ingredient in the proof of Theorem 3.2 is an introduction of "Riemann curvature tensor" $\mathrm{Sm}_{g(t)}$ for RHF, so that we can express the Weyl tensor $W_{g(t)}$ in terms of $\mathrm{Sm}_{g(t)}$.

The proofs of Theorems 3.2–3.4 follow from the method of Simon [42]. As in [42], we define

$$Z_{ijk} := (\nabla_i S_{jk})(S_{g(t)} + C) - S_{jk}(\nabla_i S_{g(t)}), \quad Z_{g(t)} := (Z_{ijk}), \quad f := \frac{|\mathrm{Sic}_{g(t)}|^2_{g(t)}}{S_{g(t)} + C}.$$

Analogous to [42], we can show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M f\mathrm{d}V_{g(t)} = \int_M \bigg[-2\frac{|Z_{g(t)}|^2_{g(t)}}{(S_{g(t)}+C)^3} - 2f^2 + 4\frac{\mathrm{Sm}_{g(t)}(\mathrm{Sic}_{g(t)},\mathrm{Sic}_{g(t)})}{S_{g(t)}+C} - fS_{g(t)} - 4\bigg|\Delta_{g(t)}\phi(t)\frac{\mathrm{Sic}_{g(t)}}{S_{g(t)}+C} - \nabla^2_{g(t)}\phi(t)\bigg|^2_{g(t)} + 4|\nabla^2_{g(t)}\phi|^2_{g(t)}\bigg]\mathrm{d}V_{g(t)}.$$

The main difference is the last term on the right-hand side of the above equation. To control the integral of $|\nabla^2_{g(t)}\phi(t)|^2_{g(t)}$, we make use the evolution equation for $|\nabla_{g(t)}\phi(t)|^2_{g(t)}$ (see (5.6)) so that

$$2\int_0^t\int_M |\nabla^2_{g(s)}\phi(s)|^2_{g(s)}\mathrm{d}V_{g(s)}\mathrm{d}s + \int_M |\nabla_{g(t)}\phi(t)|^2_{g(t)}\mathrm{d}V_{g(t)} \leqslant \mathrm{e}^{Ct}A_1\mathrm{Vol}(M,g(0)).$$

The above estimate not only controls the space-time integral of $|\nabla^2_{g(t)}\phi(t)|^2_{g(t)}$, but also a uniform bound for the integral of $|\nabla_{g(t)}\phi(t)|^2_{g(t)}$. These two estimates play essential role in the following proof. In dimension four, the famous Gauss-Bonnet-Chern formula (5.10) should transform to (5.13), where the terms involving $|\nabla_{g(t)}\phi(t)|^2_{g(t)}$ can be bounded by the above discussion. A modification of [42] is now applied to the RHF.

## 4 Curvature pinching estimate for RHF

In this section, we give a proof of Theorem 3.1. Consider a solution $(M, g(t), \phi(t))_{t\in[0,T)}$ to RHF:

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)} + 4\nabla_{g(t)}\phi(t)\otimes\nabla_{g(t)}\phi(t), \quad \partial_t\phi(t) = \Delta_{g(t)}\phi(t). \tag{4.1}$$

Let

$$\Box_{g(t)} := \partial_t - \Delta_{g(t)}.$$

As in [27,28,32,34], we define the following notions:

$$\mathrm{Sic}_{g(t)} := \mathrm{Ric}_{g(t)} - 4\nabla_{g(t)}\phi(t)\otimes\nabla_{g(t)}\phi(t), \tag{4.2}$$

$$S_{g(t)} := \mathrm{tr}_{g(t)}\mathrm{Sic}_{\phi(t)} = R_{g(t)} - 2|\nabla_{g(t)}\phi(t)|^2_{g(t)}. \tag{4.3}$$

Motivated by RF, we introduce a "Riemann curvature" type for RHF:

$$S_{ijk\ell} := R_{ijk\ell} - (g_{j\ell}\nabla_i\phi\nabla_k\phi + g_{k\ell}\nabla_i\phi\nabla_j\phi). \tag{4.4}$$

Our notation for $S_{ijk\ell}$ implies that

$$S_{ij} := g^{k\ell}S_{ik\ell j} = R_{ij} - 2\nabla_i\phi\nabla_j\phi = g^{k\ell}S_{kij\ell},$$

which coincides with the components of $\mathrm{Sic}_{g(t)}$. The corresponding tensor field for $S_{ijk\ell}$ is denoted by $\mathrm{Sm}_{g(t)}$.

**Lemma 4.1** *For a solution* $(M, g(t), \phi(t))_{t\in[0,T)}$, *we have*

$$\Box_{g(t)}S_{g(t)} = 2|\mathrm{Sic}_{g(t)}|^2_{g(t)} + 4|\Delta_{g(t)}\phi(t)|^2_{g(t)}, \tag{4.5}$$

$$\Box_{g(t)}\mathrm{Sic}_{g(t)} = 2\mathrm{Sm}_{g(t)}(\mathrm{Sic}_{g(t)}, \cdot) - 2\mathrm{Sic}^2_{g(t)} + 4\Delta_{g(t)}\phi(t)\nabla^2_{g(t)}\phi(t), \tag{4.6}$$

*where*

$$\mathrm{Sic}^2_{g(t)} = (S_{ik}S_{j\ell}g^{k\ell})_{ij}, \quad \mathrm{Sm}_{g(t)}(\mathrm{Sic}_{g(t)}, \cdot) = (S_{kij\ell}S^{k\ell})_{ij}.$$

*Proof* The first equation can be found in [34, Corollary 4.5]. In the same corollary, we also have

$$\partial_t S_{ij} = \Delta_{g(t),L}S_{ij} + 4\Delta_{g(t)}\phi(t)\nabla_i\nabla_j\phi.$$

Here, $\Delta_{g(t),L}$ denotes the Lichnerowicz Laplacian with respect to $g(t)$ defined by

$$\Delta_{g(t),L}S_{ij} = \Delta_{g(t)}S_{ij} + 2R_{kij\ell}S^{k\ell} - R_{ik}S_j{}^k - R_{jk}S_i{}^k.$$

Then

$$\Box_{g(t)}S_{ij} = 2R_{kij\ell}S^{k\ell} - R_{ik}S_j{}^k - R_{jk}S_i{}^k + 4\Delta_{g(t)}\phi(t)\nabla_i\nabla_j\phi.$$

Plugging

$$S_{ij} = R_{ij} - 2\nabla_i\phi\nabla_j\phi$$

into the above equation yields the second desired equation. □

As a corollary of Lemma 4.1, we have (see [34, Corollary 5.2])

$$\min_M S_{g(t)} \geqslant \min_M S_{g(0)}. \tag{4.7}$$

**Theorem 4.2** (Curvature pinching estimate) *Let* $(M, g(t), \phi(t))_{t\in[0,T)}$ *be a solution to RHF on a closed manifold* $M$ *with* $n = \dim M \geqslant 3$ *and* $T \leqslant +\infty$. *Then there exist uniform constants* $C_1, C_2, C$, *depending only on* $n, g(0), \phi(0)$ *such that*

$$S_{g(t)} + C > 0, \quad \frac{|\mathrm{Sin}_{g(t)}|_{g(t)}}{S_{g(t)} + C} \leqslant C_1 + C_2 \max_{M\times[0,t]}\sqrt{\frac{|W_{g(s)}|_{g(s)} + |\nabla^2_{g(s)}\phi(s)|^2_{g(s)}}{S_{g(s)} + C}}, \tag{4.8}$$

*where*

$$\mathrm{Sin}_{g(t)} = \mathrm{Sic}_{g(t)} - \frac{S_{g(t)}}{n}g(t)$$

*is the trace-free part of* $\mathrm{Sic}_{g(t)}$ *and* $W_{g(t)}$ *is the Weyl tensor field of* $g(t)$.

*Proof* We give some critical steps of the proof and the entire detail can be found in [26]. The first inequality follows from (4.7). The quantity

$$f := \frac{|\mathrm{Sin}_{g(t)}|^2_{g(t)}}{(S_{g(t)} + C)^\gamma}, \quad \gamma > 0,$$

can be written as

$$f = \frac{|\mathrm{Sic}_{g(t)} + \frac{C}{n}\, g(t)|^2_{g(t)}}{(S_{g(t)} + C)^\gamma} - \frac{1}{n}\,(S_{g(t)} + C)^{2-\gamma}.$$

In the following, we always omit the subscripts $t$ and $g(t)$, and set

$$\mathrm{Sic}'_{g(t)} := \mathrm{Sic}_{g(t)} + \frac{C}{n}\, g(t), \quad S'_{g(t)} := S_{g(t)} + C = \mathrm{tr}_{g(t)}\mathrm{Sic}'_{g(t)}.$$

A direct computation shows that

$$\begin{aligned} \Box f = {} & \frac{2(\gamma-1)}{S'}\,\langle \nabla f, \nabla S'\rangle - \frac{2}{(S')^{\gamma+2}}\,|S'\nabla_i S'_{jk} - S'_{jk}\nabla_i S'|^2 \\ & - \frac{(2-\gamma)(\gamma-1)}{(S')^2}\,|\nabla S'|^2 f + \mathscr{Q}_1 + \mathscr{Q}_2 + \mathscr{Q}_3, \end{aligned} \tag{4.9}$$

where

$$\mathscr{Q}_1 := -\frac{2(2-\gamma)}{n}\,(S')^{1-\gamma}|\mathrm{Sic}'|^2 + \frac{4}{(S')^\gamma}\,\mathrm{Sm}(\mathrm{Sic}', \mathrm{Sic}') - \frac{2\gamma|\mathrm{Sic}'|^4}{(S')^{1+\gamma}}, \tag{4.10}$$

$$\mathscr{Q}_2 := \frac{4C}{n}\left[\frac{C(S'-2C)}{n(S')^2} - \frac{(1+\gamma)S' - 2\gamma C}{(S')^{\gamma+1}}\,|\mathrm{Sic}'|^2 - \frac{2-\gamma}{2n}\,\frac{C-2S'}{(S')^{\gamma-1}}\right], \tag{4.11}$$

$$\mathscr{Q}_3 := \frac{2}{(S')^\gamma}\,\langle \mathrm{Sic}', 4\Delta\phi\nabla^2\phi\rangle - \frac{2\alpha|\Delta\phi|^2}{(S')^{\gamma+1}}\left[\gamma|\mathrm{Sic}'|^2 + \frac{2-\gamma}{2}\,(S')^2\right]. \tag{4.12}$$

We can show that the above evolution equation can be written as

$$\begin{aligned} \Box f = {} & 2(\gamma-1)\langle \nabla f, \nabla \log S'\rangle - \frac{2}{(S')^\gamma}\,|\nabla_i S'_{jk} - S'_{jk}\nabla_i \log S'|^2 \\ & - (2-\gamma)(\gamma-1)|\nabla \log S'|^2 f + \mathscr{R}_1 + \mathscr{R}_2 + \mathscr{R}_3, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} \mathscr{R}_1 = {} & \frac{2}{(S')^{\gamma+1}}\Big[-\gamma(S')^{2\gamma}f^2 + \Big(\frac{2n-4}{n(n-1)} - \frac{\gamma}{n}\Big)(S')^{\gamma+2}f - \frac{4S'^4}{n-2}\,\frac{\mathrm{Sin}^3}{S'^3} \\ & + 2(S')^3 W\Big(\frac{\mathrm{Sin}}{S'}, \frac{\mathrm{Sin}}{S'}\Big) - \frac{2}{n-1}\Big(\frac{C}{n} + \frac{2|\nabla\phi|^2}{n-2}\Big)\Big(\frac{n-1}{n}\,S'^3 - (S')^{\gamma+1}f\Big) \\ & + \frac{4}{n-2}\Big\langle S'^2\mathrm{Sic}' - \frac{n}{2}\,S'\mathrm{Sic}'^2, \nabla\phi\otimes\nabla\phi\Big\rangle\Big], \end{aligned}$$

$$\mathscr{R}_2 = \frac{4C}{n^2}\left[\frac{C}{S'} - \frac{2C^2}{S'^2} + \frac{(3\gamma-2)C}{2(S')^{\gamma+1}} + \frac{1-\gamma}{(S')^{\gamma-2}} + \frac{f}{n}\Big(\frac{2\gamma C}{S'} - 1 - \gamma\Big)\right],$$

$$\mathscr{R}_3 = \frac{2}{(S')^\gamma}\,\langle \mathrm{Sic}', 4\Delta\phi\nabla^2\phi\rangle - \frac{4|\Delta\phi|^2}{(S')^{\gamma+1}}\left[\gamma(S')^\gamma f + \frac{2S'^2}{n}\right].$$

In particular, when $\gamma = 2$, we have

$$f = \frac{|\mathrm{Sin}|^2}{S'^2} = \frac{|\mathrm{Sic}'|^2}{S'^2}$$

and

$$\Box f = 2\langle \nabla f, \nabla \log S' \rangle - 2\Big|\nabla\Big(\frac{\mathrm{Sin}}{S'}\Big)\Big|^2 + \mathscr{S}_1 + \mathscr{S}_2 + \mathscr{S}_3, \tag{4.14}$$

where

$$\mathscr{S}_1 \leqslant 4S'\Big[ - f^2 - \frac{f}{n(n-1)} + \frac{2}{n-2} f^{3/2} + C_n \frac{|W|}{S'} f + C_0 f\Big],$$
$$\mathscr{S}_2 \leqslant 4S' C_0 (1+f), \quad \mathscr{S}_3 \leqslant 4S' C_0 \Big(1 + \frac{|\nabla^2 \phi|^2}{S'} f^{1/2}\Big),$$

where

$$C_n = C(n) > 0, \quad C_0 = C(n, g(0), \phi(0)) > 0.$$

Without loss of generality, we may assume that $f \geqslant 1$. In this case, we have

$$\Box f \leqslant 2\langle \nabla f, \nabla \log S' \rangle + 4S' f\Big[ - f - \frac{1}{n(n-1)} + \frac{2}{n-2} f^{1/2} + C_0 + C_0 \frac{|W| + |\nabla^2 \phi|^2}{S'}\Big]. \tag{4.15}$$

Applying the maximum principle to (4.15) yields (4.8). □

As an immediate consequently of Theorem 4.2, we can give a proof of Theorem 3.1.

*Proof of Theorem* 3.1 Suppose now that

$$\limsup_{t \to T} \big(\max_M R_{g(t)}\big) < +\infty, \quad \limsup_{t\to T} \max_M \frac{|W_{g(t)}|_{g(t)} + |\nabla^2_{g(t)} \phi(t)|^2_{g(t)}}{R_{g(t)}} < +\infty.$$

In this case, both $R_{g(t)}$ and $|W_{g(t)}|_{g(t)} + |\nabla^2_{g(t)} \phi(t)|^2_{g(t)}$ are uniformly bounded. Theorem 4.2 then implies that $|\mathrm{Sin}_{g(t)}|_{g(t)}$ is uniformly bounded. Since $\mathrm{Sin}_{g(t)} = \mathrm{Sic}_{g(t)} - \frac{S_{g(t)}}{n} g(t)$, it follows that $\mathrm{Sic}_{g(t)}$ is uniformly bounded. However, the assumption on $\alpha(t)$ tells us that $|\nabla_{g(t)} \phi(t)|^2_{g(t)}$ is uniformly bounded (e.g., [34, Proposition 5.5]). Thus, $\mathrm{Ric}_{g(t)}$ is uniformly bounded, contradicting with the fact (3.2). Therefore, we prove the theorem. □

## 5 4D Ricci-harmonic flow with bounded $S_{g(t)}$

Let the constant $C$ be given in Theorem 4.2 and we assume that $n = \dim M = 4$. As in [42], we define

$$Z_{ijk} := (\nabla_i S_{jk})(S_{g(t)} + C) - S_{jk}(\nabla_i S_{g(t)}), \quad Z_{g(t)} := (Z_{ijk}). \tag{5.1}$$

In the proof of Theorem 4.2, we actually have proved

$$\Box\frac{|\mathrm{Sic}|^2}{S+C} = -2\frac{|Z|^2}{(S+C)^3} - 2\frac{|\mathrm{Sic}|^4}{(S+C)^2} + 4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+C} - \frac{1}{(S+C)^2}\left[4|\Delta\phi|^2\,|\mathrm{Sic}|^2 - 2(S+C)\langle\mathrm{Sic}, 4\Delta\phi\nabla^2\phi\rangle\right]. \quad (5.2)$$

The bracket in the right-hand side of (5.2) can be expressed as

$$4[|\mathrm{Sic}\Delta\phi - (S+C)\nabla^2\phi|^2 - (S+C)^2|\nabla^2\phi|^2].$$

Therefore, identity (5.2) is equal to

$$\Box f = -2\frac{|Z|^2}{(S+C)^3} - 2f^2 + 4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+C} - 4\left|\Delta\phi\frac{\mathrm{Sic}}{S+C} - \nabla^2\phi\right|^2 + 4|\nabla^2\phi|^2, \quad (5.3)$$

where

$$f := \frac{|\mathrm{Sic}|^2}{S+C}, \quad (5.4)$$

which differs from the previous one in the proof of Theorem 4.2. Integrating (5.3) over $M$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M f\,\mathrm{d}V_{g(t)} = \int_M \Big[-2\frac{|Z|^2}{(S+C)^3} - 2f^2 + 4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+C} - fS - 4\left|\Delta\phi\frac{\mathrm{Sic}}{S+C} - \nabla^2\phi\right|^2 + 4|\nabla^2\phi|^2\Big]\mathrm{d}V_{g(t)}. \quad (5.5)$$

To control the integral of $|\nabla^2\phi|^2$, we recall the evolution equation for $|\nabla\phi|^2$ (see [34, Proposition 4.3]):

$$\Box|\nabla\phi|^2 = -4|\nabla\phi\otimes\nabla\phi|^2 - 2|\nabla^2\phi|^2. \quad (5.6)$$

In particular, we see that

$$|\nabla\phi|^2 \leqslant \max_M(|\nabla\phi|^2|_{t=0}) =: A_1. \quad (5.7)$$

Moreover, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla\phi|^2\mathrm{d}V_{g(t)} \leqslant \int_M [-(S+C)|\nabla\phi|^2 - 2|\nabla^2\phi|^2 + C|\nabla\phi|^2]\mathrm{d}V_{g(t)},$$

which shows that

$$2\int_0^t\int_M |\nabla^2\phi|^2\mathrm{d}V_{g(s)}\mathrm{d}s + \int_M |\nabla\phi|^2\mathrm{d}V_{g(t)} \leqslant \mathrm{e}^{Ct}A_1\mathrm{Vol}(M, g(0)). \quad (5.8)$$

Define

$$A_2(t) := \int_M |\nabla^2\phi|^2\mathrm{d}V_{g(t)}.$$

Plugging (5.8) into (5.5), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M f\mathrm{d}V_{g(t)} \leqslant \int_M \Big[-2f^2+4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+C}-fS\Big]\mathrm{d}V_{g(t)}+4A_2(t). \tag{5.9}$$

In the following, we restrict ourself in 4D RHF, i.e., $n=\dim M=4$. In the case, the famous Gauss-Bonnet-Chern formula says that

$$2^5\pi^2\chi(M)=\int_M [|\mathrm{Rm}|^2-4|\mathrm{Ric}|^2+R^2]\mathrm{d}V_{g(t)} \tag{5.10}$$

for any $t\in[0,T]$. In order to use formula (5.10), we should translate the integrand in (5.10) into a function in terms of $S_{ijk\ell}$.

**Lemma 5.1** *For any $m$-dimensional manifold $M$, one has*

$$\begin{aligned}|\mathrm{Rm}|^2-4|\mathrm{Ric}|^2+R^2=|\mathrm{Sm}|^2-4|\mathrm{Sic}|^2+S^2-2(n+9)|\nabla\phi|^4\\ -18\mathrm{Sic}(\nabla\phi,\nabla\phi)+4S|\nabla\phi|^2.\end{aligned} \tag{5.11}$$

*Proof* Using

$$S_{ijk\ell}=R_{ijk\ell}-(g_{j\ell}\nabla_i\phi\nabla_k\phi+g_{k\ell}\nabla_i\phi\nabla_j\phi),$$

we obtain

$$\begin{aligned}|\mathrm{Rm}|^2&=R_{ijk\ell}R^{ijk\ell}\\&=(S_{ijk\ell}+(g_{j\ell}\nabla_i\phi\nabla_k\phi+g_{k\ell}\nabla_i\phi\nabla_j\phi))\\&\quad\cdot(S^{ijk\ell}+(g^{j\ell}\nabla^i\phi\nabla^k\phi+g^{k\ell}\nabla^i\phi\nabla^j\phi))\\&=|\mathrm{Sm}|^2+2(n+1)|\nabla\phi|^4+2(S_{ijk\ell}g^{j\ell}\nabla^i\phi\nabla^k\phi+S_{ijk\ell}g^{k\ell}\nabla^i\phi\nabla^j\phi).\end{aligned}$$

Compute

$$\begin{aligned}S_{ijk\ell}g^{j\ell}&=-R_{ik}-(n+1)\nabla_i\phi\nabla_k\phi=-S_{ik}-(n+3)\nabla_i\phi\nabla_k\phi,\\ S_{ijk\ell}g^{k\ell}&=g^{k\ell}R_{ijk\ell}-(n+1)\nabla_i\phi\nabla_j\phi=-(n+1)\nabla_i\phi\nabla_j\phi,\end{aligned}$$

because of the first Bianchi identity

$$g^{k\ell}R_{ijk\ell}=-g^{k\ell}(R_{jki\ell}+R_{kij\ell})=-(-R_{ji}+R_{ij})=0.$$

Consequently,

$$\begin{aligned}|\mathrm{Rm}|^2&=|\mathrm{Sm}|^2+2(n+1)|\nabla\phi|^4\\&\quad+2[-\mathrm{Sic}(\nabla\phi,\nabla\phi)-(n+3)|\nabla\phi|^4-(n+1)|\nabla\phi|^4]\\&=|\mathrm{Sm}|^2-2\mathrm{Sic}(\nabla\phi,\nabla\phi)-2(n+3)|\nabla\phi|^4.\end{aligned}$$

Similarly, we can show that

$$|\mathrm{Ric}|^2=|\mathrm{Sic}|^2+4\mathrm{Sic}(\nabla\phi,\nabla\phi)+4|\nabla\phi|^4,\quad R^2=S^2+4|\nabla\phi|^4+4S|\nabla\phi|^2.$$

Combining these identities, we obtain (5.11). □

From (5.10) and (5.11), one has, in dimension $n = 4$,

$$\begin{aligned}
&\int_M [|\mathrm{Sm}|^2 - 4|\mathrm{Sic}|^2 + S^2] \mathrm{d}V_{g(t)} \\
&\quad = 2^5\pi^2\chi(M) + 26\int_M |\nabla\phi|^4 \mathrm{d}V_{g(t)} \\
&\qquad + 18\int_M \mathrm{Sic}(\nabla\phi, \nabla\phi)\mathrm{d}V_{g(t)} - 4\int_M S|\nabla\phi|^2 \mathrm{d}V_{g(t)}. \qquad (5.12)
\end{aligned}$$

Using the inequality

$$\mathrm{Sic}(\nabla\phi, \nabla\phi) \leqslant \varepsilon|\mathrm{Sic}|^2 + \frac{|\nabla\phi|^4}{4\varepsilon}, \quad \varepsilon := \frac{9}{52},$$

we obtain from (5.12) that, in dimension $n = 4$,

$$\begin{aligned}
&\int_M [|\mathrm{Sm}|^2 - 4|\mathrm{Sic}|^2 + S^2] \mathrm{d}V_{g(t)} \\
&\quad \leqslant 2^5\pi^2\chi(M) + 52\int_M |\nabla\phi|^4 \mathrm{d}V_{g(t)} \\
&\qquad + \frac{81}{26}\int_M |\mathrm{Sic}|^2 \mathrm{d}V_{g(t)} - 4\int_M S|\nabla\phi|^2 \mathrm{d}V_{g(t)}. \qquad (5.13)
\end{aligned}$$

For any $\varepsilon > 0$, we have

$$4\frac{\mathrm{Sm}(\mathrm{Sic}, \mathrm{Sic})}{S + C} \leqslant \varepsilon^2|\mathrm{Sm}|^2 + \frac{4|\mathrm{Sic}|^2}{\varepsilon^2(S + C)^2} = \frac{4}{\varepsilon^2}f^2 + \varepsilon^2|\mathrm{Sm}|^2$$

so that

$$\begin{aligned}
&-2f^2 + 4\frac{\mathrm{Sm}(\mathrm{Sic}, \mathrm{Sic})}{S + C} - fS \\
&\quad \leqslant -2f^2 + \frac{4}{\varepsilon^2}f^2 + \varepsilon^2|\mathrm{Sm}|^2 - fS \\
&\quad = -2f^2 + \varepsilon^2(|\mathrm{Sm}|^2 - 4|\mathrm{Sic}|^2 + S^2) + 4\varepsilon^2|\mathrm{Sic}|^2 - \varepsilon^2 S^2 + \frac{4}{\varepsilon^2}f^2 - fs \\
&\quad = \varepsilon^2(|\mathrm{Sm}|^2 - 4|\mathrm{Sic}|^2 + S^2) + (4\varepsilon^2 - 1)fS - \left(2 - \frac{4}{\varepsilon^2}\right)f^2 + 4C\varepsilon^2 f - \varepsilon^2 S^2.
\end{aligned}$$

Using estimate (5.13), we have

$$\begin{aligned}
&\int_M \left[-2f^2 + 4\frac{\mathrm{Sm}(\mathrm{Sic}, \mathrm{Sic})}{S + C} - fS\right] \mathrm{d}V_{g(t)} \\
&\quad \leqslant \varepsilon^2\left[32\pi^2\chi(M) + 52A_1^2\mathrm{Vol}(M, g(0))\mathrm{e}^{Ct} + \frac{81}{26}\int_M f(S + C)\mathrm{d}V_{g(t)}\right]
\end{aligned}$$

$$\begin{aligned}
&-\varepsilon^2\int_M f^2\mathrm{d}V_{g(t)}+(4\varepsilon^2-1)\int_M fS\mathrm{d}V_{g(t)}\\
&-\Big(2-\frac{4}{\varepsilon^2}\Big)\int_M f^2\mathrm{d}V_{g(t)}+4C\varepsilon^2\int_M f\mathrm{d}V_{g(t)}\\
=\ &\varepsilon^2\left[32\pi^2\chi(M)+52A_1^2\mathrm{Vol}(M,g(0))\mathrm{e}^{Ct}\right]\\
&+\int_M\Big[-\Big(2-\frac{4}{\varepsilon^2}\Big)f^2+\Big(\frac{55}{26}+4\varepsilon^2\Big)fS+\Big(\frac{81}{26}+4\varepsilon^2\Big)Cf-\varepsilon^2S^2\Big]\mathrm{d}V_{g(t)}.
\end{aligned}$$

For any $\eta>0$, we have

$$fS\leqslant\eta f^2+\frac{1}{4\eta}S^2,$$

and then

$$\begin{aligned}
&\int_M\Big[-2f^2+4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+C}-fS\Big]\mathrm{d}V_{g(t)}\\
\leqslant\ &\varepsilon^2[32\pi^2\chi(M)+52A_1^2\mathrm{Vol}(M,g(0))\mathrm{e}^{Ct}]+\int_M\Big[-\Big(2-\frac{4}{\varepsilon^2}-\Big(\frac{55}{26}+4\varepsilon^2\Big)\eta\Big)f^2\\
&+\Big(\frac{81}{26}+4\varepsilon^2\Big)Cf+\Big(\frac{\frac{55}{26}+4\varepsilon^2}{4\eta}-\varepsilon^2\Big)S^2\Big]\mathrm{d}V_{g(t)}.
\end{aligned}$$

If we choose

$$\eta=\frac{4/\varepsilon^2}{\frac{55}{26}+4\varepsilon^2},\quad \varepsilon>2, \tag{5.14}$$

then

$$\begin{aligned}
&\int_M\Big[-2f^2+4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+C}-fS\Big]\mathrm{d}V_{g(t)}\\
&\leqslant\varepsilon^2[32\pi^2\chi(M)+52A_1^2\mathrm{Vol}(M,g(0))\mathrm{e}^{Ct}]+\int_M\Big[-\Big(2-\frac{8}{\varepsilon^2}\Big)f^2\\
&\quad+\Big(\frac{81}{26}+4\varepsilon^2\Big)Cf+\Big(\frac{(\frac{55}{26}+4\varepsilon^2)^2}{16}-1\Big)\varepsilon^2S^2\Big]\mathrm{d}V_{g(t)}.
\end{aligned}$$

In particular, when $\varepsilon=2\sqrt{2}$ in (5.14), we arrive at

$$\begin{aligned}
&\int_M\Big[-2f^2+4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+C}-fS\Big]\mathrm{d}V_{g(t)}\\
&\leqslant\int_M(-f^2+36Cf+574S^2)\mathrm{d}V_{g(t)}\\
&\quad+8[32\pi^2\chi(M)+52A_1^2\mathrm{Vol}(M,g(0))\mathrm{e}^{Ct}].
\end{aligned}\tag{5.15}$$

Plugging (5.15) into (5.9) implies

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_M f\mathrm{d}V_{g(t)}\leqslant\ &\int_M(-f^2+36Cf+574S^2)\mathrm{d}V_{g(t)}\\
&+[256\pi^2\chi(M)+416A_1^2\mathrm{Vol}(M,g(0))\mathrm{e}^{Ct}+4A_2(t)].
\end{aligned}\tag{5.16}$$

As a consequence, we can prove (see [26]) Theorem 3.2 and the following result.

**Theorem 5.2** *Let $(M, g(t), \phi(t))_{t\in[0,T)}$ be a solution to RHF on a closed manifold $M$ with $n = \dim M = 4$ and $T \leqslant +\infty$. Suppose $\min_M S_{g(0)} > 0$. Then*

$$\int_M |\mathrm{Sic}_{g(s)}|_{g(s)} \mathrm{d}V_{g(s)} \leqslant 2a_0(M, g(0), \phi(0), s) + 1148 \int_0^s \int_M S^2_{g(t)} \mathrm{d}V_{g(t)} \mathrm{d}t, \tag{5.17}$$

$$\int_0^s \int_M |\mathrm{Sic}_{g(t)}|^2_{g(t)} \mathrm{d}V_{g(t)} \mathrm{d}t \leqslant 8a_0(M, g(0), \phi(0), s) + 4592 \int_0^s \int_M S^2_{g(t)} \mathrm{d}V_{g(t)} \mathrm{d}t, \tag{5.18}$$

$$\begin{aligned} &\int_0^s \int_M |\mathrm{Sm}_{g(t)}|^2_{g(t)} \mathrm{d}V_{g(t)} \mathrm{d}t \\ &\leqslant 32\pi^2 \chi(M) s + 13(\alpha A_1)^2 \mathrm{Vol}(M, g(0)) s \\ &\quad + \frac{881}{25} a_0(M, g(0), \phi(0), s) + \frac{1011469}{50} \int_0^s \int_M S^2_{g(t)} \mathrm{d}V_{g(t)} \mathrm{d}t, \end{aligned} \tag{5.19}$$

*for all $s \in [0, T)$. Here,*

$$\begin{aligned} a_0(M, g(0), \phi(0), s) &:= 256\pi^2 \chi(M) s + 416 A_1^2 \mathrm{Vol}(M, g(0)) s \\ &\quad + 2A_1 \mathrm{Vol}(M, g(0)) + \int_M \frac{|\mathrm{Sic}_{g(0)}|^2_{g(0)}}{S_{g(0)}} \mathrm{d}V_{g(0)}. \end{aligned} \tag{5.20}$$

Finally, we give proofs of Theorems 3.3 and 3.4. Recall the quantity

$$f := \frac{|\mathrm{Sic}_{g(t)}|^2_{g(t)}}{S_{g(t)} + 2}.$$

**Theorem 5.3** *If $(M, g(t), \phi(t))_{t\in[0,T)}$ satisfies BA, then*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_M f \mathrm{d}V_{g(t)} &\leqslant \int_M (-f^2 + 88f) \mathrm{d}V_{g(t)} + [128\pi^2 \chi(M) \\ &\quad + 208 A_1^2 \mathrm{Vol}(M, g(0)) \mathrm{e}^{2t} + 4A_2(t)]. \end{aligned} \tag{5.21}$$

*Proof* Recall the estimate (see [26])

$$4\frac{\mathrm{Sm}(\mathrm{Sic}, \mathrm{Sic})}{S+2} \leqslant \frac{4}{\varepsilon^2} f^2 + \varepsilon^2 (|\mathrm{Sm}|^2 - 4|\mathrm{Sic}|^2 + S^2) + 4\varepsilon^2 (S+2) f.$$

Since $-1 \leqslant S \leqslant 1$, it follows that

$$4\varepsilon^2 (S+2) f \leqslant 12\varepsilon^2 f.$$

Hence,

$$4\frac{\mathrm{Sm}(\mathrm{Sic}, \mathrm{Sic})}{S+2} \leqslant \frac{4}{\varepsilon^2} f^2 + \varepsilon^2 (|\mathrm{Sm}|^2 - 4|\mathrm{Sic}|^2 + S^2) + 12\varepsilon^2 f.$$

Using the inequality

$$-fS = -f(s+2) + 2f \leqslant 2f$$

and (5.13), we arrive at

$$\begin{aligned}
&\int_M \Big[-2f^2 + 4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+2} - fS\Big]\mathrm{d}V_{g(t)} \\
&\leqslant \int_M \Big[-2f^2 + \frac{4}{\varepsilon^2}\, f^2 + \varepsilon^2(|\mathrm{Sm}|^2 - 4|\mathrm{Sic}|^2 + S^2) + 12\varepsilon^2 f - fS\Big]\mathrm{d}V_{g(t)} \\
&= \int_M \Big[-\Big(2 - \frac{4}{\varepsilon^2}\Big)f^2 + (12\varepsilon^2+2)f + \varepsilon^2(|\mathrm{Sm}|^2 - 4|\mathrm{Sic}|^2 + S^2)\Big]\mathrm{d}V_{g(t)} \\
&\leqslant \int_M \Big[-\Big(2 - \frac{4}{\varepsilon^2}\Big)f^2 + (12\varepsilon^2+2)f\Big]\mathrm{d}V_{g(t)} \\
&\quad + \varepsilon^2[32\pi^2\chi(M) + 52\varepsilon^2 A_1^2\mathrm{Vol}(M,g(0))\mathrm{e}^{2t}] + \frac{243}{26}\,\varepsilon^2\int_M f\mathrm{d}V_{g(t)} \\
&= \int_M \Big[-\Big(2 - \frac{4}{\varepsilon^2}\Big)f^2 + \Big(\frac{555}{26}\,\varepsilon^2 + 2\Big)f\Big]\mathrm{d}V_{g(t)} \\
&\quad + \varepsilon^2[32\pi^2\chi(M) + 52\varepsilon^2 A_1^2\mathrm{Vol}(M,g(0))\mathrm{e}^{2t}].
\end{aligned}$$

From (5.9), we get (5.21) (compare with (5.16) when $C = 2$). □

**Theorem 5.4** *If $(M, g(t), \phi(t))_{t\in[0,T)}$ satisfies BA, then*

$$\int_M |\mathrm{Sic}_{g(s)}|^2_{g(s)}\mathrm{d}V_{g(s)} \leqslant b(M, g(0), \phi(0), s), \tag{5.22}$$

$$\begin{aligned}\int_M |\mathrm{Sm}_{g(s)}|^2_{g(s)}\mathrm{d}V_{g(s)} &\leqslant 32\pi^2\chi(M) + 676\alpha A_1^2\mathrm{Vol}(M,g(0))\mathrm{e}^{2s} \\ &\quad + \frac{185}{26}\, b(M, g(0), \phi(0), s),\end{aligned} \tag{5.23}$$

$$\int_0^s\int_M |\mathrm{Sic}_{g(t)}|^4_{g(t)}\mathrm{d}V_{g(t)}\mathrm{d}t \leqslant b(M, g(0), \phi(0), s), \tag{5.24}$$

$$\begin{aligned}\int_s^T\int_M |\mathrm{Sic}|^p_{g(t)}\mathrm{d}V_{g(t)}\mathrm{d}t &\leqslant |b(M, g(0), \phi(0), T)|^{p/4}\mathrm{e}^{T(4-p)/4} \\ &\quad \cdot [\mathrm{Vol}(M, g(0))]^{(4-p)/4}(T-s)^{(4-p)/4},\end{aligned} \tag{5.25}$$

*for any $s \in [0, T)$ and $0 < p < 4$. Here,*

$$\begin{aligned}
&b(M, g(0), \phi(0), s) \\
&\quad := 9\mathrm{e}^{88s}\int_M |\mathrm{Sic}_{g(0)}|^2_{g(0)}\mathrm{d}V_{g(0)} + \frac{144}{11}\,\pi^2\chi(M)(\mathrm{e}^{88s} - 1) \\
&\qquad + \frac{936}{43}\, A_1^2\mathrm{Vol}(M,g(0))(\mathrm{e}^{88s} - \mathrm{e}^{2s}) + 18A_1\mathrm{Vol}(M,g(0))\mathrm{e}^{90s}.
\end{aligned} \tag{5.26}$$

*Proof* Write

$$A_3(t) := 128\pi^2\chi(M) + 208A_1^2\mathrm{Vol}(M, g(0))\mathrm{e}^{2t} + 4A_2(t).$$

The inequality (5.21) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M f\mathrm{d}V_{g(t)} \leqslant A_3(t) + \int_M (-f^2 + 88f)\mathrm{d}V_{g(t)},$$

and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-88t}\int_M f\mathrm{d}V_{g(t)}\right) \leqslant -\mathrm{e}^{-88t}\int_M f^2\mathrm{d}V_{g(t)} + \mathrm{e}^{-88t}A_3(t).$$

Therefore,

$$\begin{aligned}
&\mathrm{e}^{-88s}\int_0^s\int_M f^2\mathrm{d}V_{g(t)}\mathrm{d}t + \mathrm{e}^{-88s}\int_M f\mathrm{d}V_{g(s)}\\
&\quad\leqslant \int_M f\mathrm{d}V_{g(0)} + \int_0^s \mathrm{e}^{-88t}[128\pi^2\chi(M) + 208A_1^2\mathrm{Vol}(M, g(0))\mathrm{e}^{2t} + 4A_2(t)]\mathrm{d}t\\
&\quad= \int_M f\mathrm{d}V_{g(0)} + \frac{16}{11}\,\pi^2\chi(M)(1-\mathrm{e}^{-88s}) + \frac{104}{43}\,A_1^2\mathrm{Vol}(M, g(0))(1-\mathrm{e}^{-86s})\\
&\qquad + 4\int_0^s\int_M |\nabla^2\phi|^2\mathrm{d}V_{g(t)}\mathrm{d}t\\
&\quad\leqslant \int_M f\mathrm{d}V_{g(0)} + \frac{16}{11}\,\pi^2\chi(M)(1-\mathrm{e}^{-88s}) + \frac{104}{43}\,A_1^2\mathrm{Vol}(M, g(0))(1-\mathrm{e}^{-86s})\\
&\qquad + 2A_1\mathrm{Vol}(M, g(0))\mathrm{e}^{2s}
\end{aligned}$$

by (5.8). Because $|S| \leqslant 1$, we have

$$\frac{1}{3}\,|\mathrm{Sic}|^2 \leqslant f \leqslant |\mathrm{Sic}|^2,$$

and hence,

$$\begin{aligned}
&\frac{\mathrm{e}^{-88s}}{9}\int_0^s\int_M |\mathrm{Sic}|^4\mathrm{d}V_{g(t)}\mathrm{d}t + \frac{\mathrm{e}^{-88s}}{3}\int_M |\mathrm{Sic}|^2\mathrm{d}V_{g(s)}\\
&\quad\leqslant \int_M |\mathrm{Sic}|^2\mathrm{d}V_{g(0)} + \frac{16}{11}\,\pi^2\chi(M)(1-\mathrm{e}^{-88s})\\
&\qquad + \frac{104}{43}\,A_1^2\mathrm{Vol}(M, g(0))(1-\mathrm{e}^{-86s}) + 2A_1\mathrm{Vol}(M, g(0))\mathrm{e}^{2s}.
\end{aligned}$$

This estimate yields (5.22) and (5.24).

For (5.23), we use (5.13) to get

$$\begin{aligned}
\int_M |\mathrm{Sm}|^2\mathrm{d}V_{g(t)} &\leqslant 32\pi^2\chi(M) + 52A_1^2\mathrm{Vol}(M, g(0))\mathrm{e}^{2t}\\
&\quad + \frac{81}{26}\int_M |\mathrm{Sic}|^2\mathrm{d}V_{g(t)} + 4\int_M |\mathrm{Sic}|^2\mathrm{d}V_{g(t)}\\
&= 32\pi^2\chi(M) + 52A_1^2\mathrm{Vol}(M, g(0))\mathrm{e}^{2t} + \frac{185}{26}\,b(M, g(0), \phi(0), t)
\end{aligned}$$

by (5.22).

For (5.25), we use

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}(M,g(t)) = -\int_M S\mathrm{d}V_{g(t)}$$

and $-1 \leqslant S \leqslant 1$ to deduce

$$\mathrm{e}^{-T}\mathrm{Vol}(M,g(0)) \leqslant \mathrm{Vol}(M,g(t)) \leqslant \mathrm{e}^{T}\mathrm{Vol}(M,g(0)).$$

Consequently, for any $0 < s < r < T$,

$$\begin{aligned}
&\int_s^r \int_M |\mathrm{Sic}|^p \mathrm{d}V_{g(t)}\mathrm{d}t \\
&\leqslant \left(\int_s^r \int_M |\mathrm{Sic}|^4 \mathrm{d}V_{g(t)}\mathrm{d}t\right)^{p/4} \left(\int_s^r \int_M \mathrm{d}V_{g(t)}\mathrm{d}t\right)^{(4-p)/4} \\
&\leqslant |b(M,g(0),\phi(0),T)|^{p/4}(r-s)^{(4-p)/4}\mathrm{e}^{T(4-p)/4}[\mathrm{Vol}(M,g(0))]^{(4-p)/4}.
\end{aligned}$$

Thus, we get (5.25). □

Using $c(M,g(0),\phi(0),T)$ and (3.15), Theorem 5.4 now yields

$$\sup_{t\in[0,T)} \int_M |\mathrm{Sic}_{g(t)}|^2_{g(t)}\mathrm{d}V_{g(t)} \leqslant c(M,g(0),\phi(0),T) < +\infty, \tag{5.27}$$

$$\begin{aligned}
\sup_{t\in[0,T)} \int_M |\mathrm{Sm}_{g(t)}|^2_{g(t)}\mathrm{d}V_{g(t)} &\leqslant 32\pi^2\chi(M) + 8c(M,g(0),\phi(0),T) \\
&\quad + 52A_1^2\mathrm{Vol}(M,g(0))\mathrm{e}^{2T} \\
&< +\infty.
\end{aligned} \tag{5.28}$$

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