HOW TO DISCRETIZE THE DIFFERENTIAL FORMS ON THE INTERVAL

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Abstract. We provide explicit quasi-isomorphisms between the following three algebraic structures associated to the unit interval: i) the commutative dg algebra of differential forms, ii) the non-commutative dg algebra of simplicial cochains and iii) the Whitney forms, equipped with a homotopy commutative and homotopy associative, i.e. $C_\infty$, algebra structure. Our main interest lies in a natural ‘discretization’ $C_\infty$ quasi-isomorphism $\varphi$ from differential forms to Whitney forms. We establish a uniqueness result that implies that $\varphi$ coincides with the morphism from homotopy transfer, and obtain several explicit formulas for $\varphi$, all of which are related to the Magnus expansion. In particular, we recover combinatorial formulas for the Magnus expansion due to Mielnik and Plebański.

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Introduction

The purpose of this paper is to construct several explicit quasi-isomorphisms between three algebraic structures associated to the unit interval $[0,1]$, and study some of their properties. The first algebraic structure we consider is the commutative dg algebra of differential forms $\Omega^*([0,1])$ – with the usual de Rham differential and wedge product. The other two structures are defined on the subcomplex $C^*([0,1]) \subset \Omega^*([0,1])$ of Whitney forms

$$C^*([0,1]) = \{a t + b(1-t) \mid a,b \in \mathbb{k}\} \oplus \{c \, dt \mid c \in \mathbb{k}\},$$

consisting of affine functions and constant one-forms on $[0,1]$. Notice that, as a complex, $C^*([0,1])$ is isomorphic to the complex of simplicial cochains on the one-dimensional simplex:
as such, it is equipped with a dg algebra structure via the usual cup product $\cup$ of cochains, and we denote this dg algebra by $C^\ast([0,1])$. The cup product $\cup$ is not graded commutative.

On the other hand, since the inclusion $\iota : C^\ast([0,1]) \to \Omega^\ast([0,1])$ is a quasi-isomorphism of complexes, the general transfer theorems of homotopical algebra guarantee the existence of a homotopy associative and commutative – i.e., a $C_\infty$ – algebra structure on $C^\ast([0,1])$. The latter was worked out explicitly in the papers [7, 9, 25], cf. Theorem 1.1 in Subsection 1.1 below, and its Taylor coefficients are given in terms of Bernoulli numbers. We denote $C^\ast([0,1])$, equipped with this $C_\infty$ algebra structure, by $C^\ast_\infty([0,1])$.

Furthermore, and again by homotopy transfer, one obtains a deformation of the inclusion $\iota : C^\ast([0,1]) \to \Omega^\ast([0,1])$ into a quasi-isomorphism $\mu : C^\ast_\infty([0,1]) \to \Omega^\ast([0,1])$ of $C_\infty$ algebras – see [7, 9, 25] for explicit formulas in terms of Bernoulli polynomials, and in particular [7] for the verification that one obtains indeed a morphism of $C_\infty$ algebras (a different proof can be found in Appendix B).

The $C_\infty$ algebra $C^\ast_\infty([0,1])$ contains – in a precise mathematical sense – the same information as the commutative dg algebra $\Omega^\ast([0,1])$. Since $C^\ast_\infty([0,1])$ is finite-dimensional, one can think of it as a natural discretization of $\Omega^\ast([0,1])$ – see also [25] for the corresponding interpretation of $C^\ast_\infty([0,1])$ in terms of a discretization of BF-theory on the interval. The map $\mu$ then provides a canonical way to related the discretization $C^\ast_\infty([0,1])$ to the original structure on $\Omega^\ast([0,1])$.

At this point, the following question naturally arises:

\textit{How to explicitly construct a homotopy inverse to $\mu : C^\ast_\infty([0,1]) \to \Omega^\ast([0,1])$?}

Or, to put it differently: how to provide a morphism from $\Omega^\ast([0,1])$ to its discretization? In principle, one can again invoke the general transfer theorems of homotopical algebra, such as those established in [16, 22]. However, this turns out to be a non-trivial task – in particular, we do not know how to obtain explicit formulas this way.

To circumvent this problem, we make use of the fact that, as a complex, $C^\ast([0,1])$ coincides with the simplicial cochains on $[0,1]$. The relation between the dg algebra of smooth, singular cochains $C^\ast(M)$ on a manifold $M$, and the dg algebra of differential forms $\Omega^\ast(M)$, is well-understood: First, recall that de Rham’s Theorem asserts that integration of forms over simplices provides a quasi-isomorphism of complexes

$$\int : \Omega^\ast(M) \to C^\ast(M).$$

Moreover, one can prove, cf. [12], that the chain map $\int$ admits a refinement to an $A_\infty$ quasi-isomorphism between $(\Omega^\ast(M), d, \wedge)$ and $(C^\ast(M), \delta, \cup)$, which implies that integration induces an isomorphism of algebras at the cohomology level. A rather explicit refinement in terms of Chen’s iterated integrals was provided by Gugenheim in [13]. In our setting, Gugenheim’s construction yields an explicit $A_\infty$ quasi-isomorphism

$$\lambda : \Omega^\ast([0,1]) \to C^\ast_\cup([0,1]).$$
We combine $\mu$ and $\lambda$ to produce the following diagram

\[
\begin{array}{ccc}
\Omega^*([0,1]) & \xrightarrow{\phi} & C^\infty_*([0,1]) \\
\downarrow{\lambda} & & \downarrow{\log} \\
C^*_*([0,1]) & \xrightarrow{\exp} & C^\infty_*([0,1])
\end{array}
\]

which yields, in particular, an explicit $C^\infty_*$ morphism $\phi$ from $\Omega^*([0,1])$ to its discretization $C^*_*([0,1])$. Let us briefly describe the constituencies of the diagram:

1. $\mu$ is the quasi-isomorphism from $C^\infty_*([0,1])$ to $\Omega^*([0,1])$ obtained by homotopy transfer.
2. $\lambda$ is a special case of Gugenheim’s $A_\infty$ morphism between differential forms and smooth, singular cochains.
3. $\exp$ is an isomorphism of $A_\infty$ algebras, defined as the composition

\[
\exp : C^\infty_*([0,1]) \xrightarrow{\mu} \Omega^*([0,1]) \xrightarrow{\lambda} C^*_*([0,1]),
\]

and $\log = (\exp)^{-1}$ is its inverse. Both have a simple description in Taylor coefficients: their linear part is the identity, and the higher Taylor coefficients vanish unless all their arguments are one-cochains, in which case we recover the Taylor coefficients of the functions $\exp(x) - 1$ and $\log(x + 1)$ respectively.
4. $\gamma$ is an $A_\infty$ morphism right inverse to $\lambda$, defined as the composition

\[
\gamma : C^*_0([0,1]) \xrightarrow{\log} C^\infty_*([0,1]) \xrightarrow{\mu} \Omega^*([0,1]).
\]

We derive explicit formulas for $\gamma$ in Proposition 1.7 in Subsection 1.2.
5. $\phi$ is a $C^\infty_*$ morphism left inverse to $\mu$, defined as the composition

\[
\phi : \Omega^*([0,1]) \xrightarrow{\lambda} C^*_0([0,1]) \xrightarrow{\log} C^\infty_*([0,1]).
\]

The morphism $\phi$ is the main object of this paper. We shall derive explicit, as well as recursive, formulas for $\phi$, and find interesting connections with Lie theory and the Magnus expansion.

Our main results concerning $\phi$ are:

1. $\phi$ is indeed a $C^\infty_*$ morphism, which is not evident from its definition as the composition of two $A_\infty$ morphism. We prove this directly in Corollary 2.7 in Subsection 2.1 and indirectly in Corollary 2.14 Subsection 2.3. For the direct argument, we show the identity $\phi_n = \lambda_n \circ E^*$, where $\phi_n, \lambda_n$ are the $n$th Taylor coefficients of $\phi$ and $\lambda$ respectively, and $E^*$ is a canonical projector vanishing on the image of the shuffle product. More precisely, $E^*$ is the adjoint to the first Eulerian idempotent $E$, which is a canonical projector from the tensor algebra onto the free Lie algebra, see [27].
2. $\phi$ (as well as $\lambda$, $\exp$, $\log$) is uniquely characterized by the property that its higher Taylor coefficients vanish whenever one of their arguments is a zero-form. As a consequence, we show that $\phi$ coincides with the morphism constructed via homotopy transfer formulas, as in [16, 22].
3. After scalar extension by a dg Lie algebra, our explicit formulas for $\phi$ recover known formulas for the Magnus expansion, see [20, 17, 23].
To add some perspective on the previous diagram, we remark that it continues to make sense after we replace the interval/one-simplex $[0,1]$ by any manifold/simplicial set $M$. We already observed this for Gugenheim’s $A\infty$ morphism $\lambda: \Omega^*(M) \to C^*_c(M)$. The $C\infty$ algebra $C\infty_c(M)$ and the $C\infty$ morphism $\mu: C\infty_c(M) \to \Omega^*(M)$ can be defined as before via homotopy transfer (along Dupont’s contraction, see [8]). Finally, the rest of the diagram can be defined as before: in particular, $\exp: C\infty_c(M) \xrightarrow{\sim} \Omega^*(M) \xrightarrow{\lambda} C\infty_c(M)$ continues to be an $A\infty$ isomorphism with linear part the identity. We remark that the previous diagram is natural in particular, $\exp$ from the previous diagram corresponds to an isomorphism of dg algebras $\Omega^*(M)$ from simplicial sets to (complete) dg Lie algebras, representing the underlying Quillen’s equivalence from rational homotopy theory, see [3]. In this context, the $A\infty$ isomorphism $\exp$ from the previous diagram corresponds to an isomorphism of dg algebras $\Omega^C_c(M) \xrightarrow{\sim} \mathcal{U}(L(M))$, where $\mathcal{U}(L(M))$ is the universal enveloping of $L(M)$ and $\Omega^C_c(M)$ is the natural simplicial analog of the Adams-Hilton model studied in [14, 21]. More concretely, $\Omega^C_c(M)$ is the cobar construction of the dg coalgebra $C_c(M)$ of normalized chains on $M$. It would be interesting to compare the cocommutative dg Hopf algebra structure induced on $\Omega^C_c(M)$ by the previous isomorphism and the one studied in the papers [14, 21] App. D], which is cocommutative only up to homotopy. This would open up the possibility to use the results of the latter reference to get explicit comparisons between $L(M)$ and other classical models for the rational homotopy type of $M$. We briefly address the particular case of $M = [0,1]$ in Remark 2.21 Subsection 2.3. In this case, the dg Lie algebra $L([0,1])$ recovers the well-studied Lawrence-Sullivan model of the interval [18] (as was proved in [7], thus answering a question posed by Sullivan).

- **Rational homotopy theory**: the composition of the functor $C\infty_c(-)$ and the Chevalley-Eilenberg functor from $C\infty$ algebras to (complete) dg Lie algebras yields a functor $L(-)$ from simplicial sets to (complete) dg Lie algebras, representing the underlying Quillen’s equivalence from rational homotopy theory, see [3]. In this context, the $A\infty$ isomorphism $\exp$ from the previous diagram corresponds to an isomorphism of dg algebras $\Omega^C_c(M) \xrightarrow{\sim} \mathcal{U}(L(M))$, where $\mathcal{U}(L(M))$ is the universal enveloping of $L(M)$ and $\Omega^C_c(M)$ is the natural simplicial analog of the Adams-Hilton model studied in [14, 21]. More concretely, $\Omega^C_c(M)$ is the cobar construction of the dg coalgebra $C_c(M)$ of normalized chains on $M$. It would be interesting to compare the cocommutative dg Hopf algebra structure induced on $\Omega^C_c(M)$ by the previous isomorphism and the one studied in the papers [14, 21] App. D], which is cocommutative only up to homotopy. This would open up the possibility to use the results of the latter reference to get explicit comparisons between $L(M)$ and other classical models for the rational homotopy type of $M$. We briefly address the particular case of $M = [0,1]$ in Remark 2.21 Subsection 2.3. In this case, the dg Lie algebra $L([0,1])$ recovers the well-studied Lawrence-Sullivan model of the interval [18] (as was proved in [7], thus answering a question posed by Sullivan).

- **Derived deformation theory**: the functor $L(-)$ from the previous paragraph is a left adjoint to Getzler’s higher generalization of the Deligne groupoid functor, see [11, 10] and the first author’s PhD Thesis. In this context, the previous diagram encodes the equivalences between three models of the derived deformation theory associated to a dg Lie algebra $\mathfrak{g}$: the one considered by Hinich in [15], the one considered by Getzler in [11] and the one considered by Behrend and Getzler in [2, Section 8] (the latter makes sense only for dg associative algebras, so either we assume that the Lie bracket on $\mathfrak{g}$ is the commutator of an associative product or we replace $\mathfrak{g}$ by its universal enveloping algebra). In the one-dimensional case, the three $L\infty$ algebras $\Omega^*([0,1];\mathfrak{g})$, $C\infty([0,1];\mathfrak{g})$ and $C\infty([0,1];\mathfrak{g})$ obtained via scalar extension by $\mathfrak{g}$ (again, the latter makes sense only in the associative setting) encode, via the respective Maurer-Cartan equations (cf. for instance [9, Section 7]), three different notions of gauge/homotopy equivalence between Maurer-Cartan elements in the dg Lie/associative algebra $\mathfrak{g}$. As is well-known, these three equivalence relations coincide, and our diagram established this fact by providing direct comparisons.

- **Mathematical physics**: Let $M$ be an oriented manifold and $\mathfrak{g}$ a Lie algebra. From these data one obtains a topological field theory on $M$, known as BF-theory. Its classical action functional reads

$$S_{BF}: \Omega^1(M;\mathfrak{g}) \oplus \Omega^{p-2}_c(M;\mathfrak{g}^*) \to \mathbb{R},$$

$$(A, B) \mapsto \int_M < B, dA + \frac{1}{2}[A, A]> = \int_M < B, F_A >,$$
where $\langle \cdot, \cdot \rangle : g^* \times g \to \mathbb{R}$ is the natural pairing. In this theory, the induced $L_\infty$ algebra structure on Whitney forms with values in $g$ corresponds to the tree-level effective action functional $S^{\text{tree}}_{\text{eff}}$ on the space of infrared fields, obtained by integrating out ultraviolet fields, see [25]. Moreover, the Wilson loop observable $W_\gamma$, given by

$$W_\gamma(A, B) = \text{tr}(\text{hol}_\gamma(A)),$$

where $\gamma : S^1 \to M$ is a loop and $\text{hol}_\gamma(A)$ is the holonomy of the connection $A$ around $\gamma$, can be expressed in terms of Chen’s iterated integrals. For the case of $[0, 1]$, one is therefore naturally led to consider $\lambda$ and $\log \circ \lambda$. We remark that several higher dimensional generalizations of the Wilson loop observables were constructed and studied in the mathematical physics literature, see for instance [5, 26].

Let us conclude the introduction of this paper with a brief outline of its structure.

In Section 1, we recall the $C_\infty$ algebra structure on the space of Whitney forms $C^*([0, 1])$, along with the $C_\infty$ morphism $\mu$ from $C^*(0, 1)$ to $\Omega^*([0, 1])$, and Gugenheim’s morphism $\lambda$ from differential forms to simplicial cochains. In Subsection 1.3, we compute $\exp := \lambda \circ \mu : C_\infty([0, 1]) \to C_\infty^*([0, 1])$, as well as its inverse log. Moreover, we work out the morphism

$$\gamma : C^*_\infty([0, 1]) \xrightarrow{\log} C_\infty^*([0, 1]) \xrightarrow{\mu} \Omega^*([0, 1])$$

in Subsection 1.4.

In Section 2, we introduce and study the morphism

$$\varphi : \Omega^*([0, 1]) \xrightarrow{\lambda} C_\infty^*([0, 1]) \xrightarrow{\log} C^*_\infty([0, 1]).$$

We start in 2.1 by establishing explicit formulas for $\varphi$. The first formula expresses the $n$th Taylor coefficient $\varphi_n$ of $\varphi$ in terms of an integral over the geometric $n$-simplex, see Theorem 2.2. In Proposition 2.6 we express the Taylor coefficients of $\varphi$ in terms of the adjoint $E^*$ to the first Eulerian idempotent. Together with a symmetry property of $E^*$, Proposition 2.6 implies that $\varphi$ is a morphism of $C_\infty$ algebras, see Corollary 2.7. In Theorem 2.10 we establish a recursive description of $\varphi$, which is inspired by [17]. In Subsection 2.3, we establish a uniqueness result for morphisms between (very) special $A_\infty$ algebras, and $C_\infty$ algebras, respectively. This result applies to $\varphi$, and as consequence we deduce that 1) $\varphi$ coincides with the morphism obtained from homotopy transfer and 2) we obtain a second proof that it is a morphism of $C_\infty$ algebras.

In Section 3 we study the pushforward along $\varphi$, after extension of scalars to a dg algebra $A$, or a dg Lie algebra $g$, respectively. In the latter case, we recover known formulas for the Magnus expansion.

The two appendices provide background material on $A_\infty$, $L_\infty$ and $C_\infty$ algebras.

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1. Differential forms on the interval

We briefly review three algebraic structures associated to the interval $[0, 1]$, as well as the known morphisms between them. We refer the reader to the Appendix for an explanation of our notation and terminology concerning $A_\infty$, $L_\infty$ and $C_\infty$ algebras.
1.1. Differential forms and Whitney forms. Throughout the article, \( \Omega^*(\mathbb{[0,1]} \) denotes the graded vector space of differential forms on the closed interval \([0,1]\). To be more precise, there are two variants of \( \Omega^*(\mathbb{[0,1]} \) which we will consider:

- the space of real-valued, smooth differential forms on \([0,1]\), denoted by \( \Omega^*_\text{dr}(\mathbb{[0,1]} \), equipped with the structure of a commutative dg algebra via the de Rham differential \( d \) and the wedge product \( \wedge \).
- the space of \( k \)-polynomial forms on \([0,1]\), where \( k \) is a field of characteristic zero, denoted by \( \Omega^*_k(\mathbb{[0,1]} \): formally, \( \Omega^*_k(\mathbb{[0,1]} = \Omega^0_k(\mathbb{[0,1]} \oplus \Omega^1_k(\mathbb{[0,1]} = \mathbb{k}[t] \oplus \mathbb{k}[t]dt \), where \( \mathbb{k}[t] \) is the polynomial algebra over \( \mathbb{k} \). Again, this is a commutative dg algebra via the wedge product \( p(t) \wedge q(t) = p(t)q(t) \), \( p(t) \wedge q(t)dt = p(t)q(t)dt \), \( p(t)dt \wedge q(t)dt = 0 \) and the differential \( d : p(t) \mapsto p'(t)dt \) and \( d : p(t)dt \mapsto 0 \).

Since most of our constructions work in both contexts, we will usually just use \( \Omega^*(\mathbb{[0,1]} \) to refer to either variant.

The subcomplex of Whitney forms is the graded vector subspace of \( \Omega^*(\mathbb{[0,1]} \) given by the affine functions and constant one-forms, i.e.

\[
\mathcal{C}^*(\mathbb{[0,1]} = \mathcal{C}^0(\mathbb{[0,1]} \oplus \mathcal{C}^1(\mathbb{[0,1]} = \{ at + b(1 - t) \mid a, b \in \mathbb{k} \} \oplus \{ c dt \mid c \in \mathbb{k} \}.
\]

Notice that this space is closed under the differential \( d \), but not under multiplication. However, \( \mathcal{C}^*(\mathbb{[0,1]} \) can be identified with the complex of simplicial \( k \)-valued cochains on the standard 1-dimensional simplex. As such, we might equip \( \mathcal{C}^*(\mathbb{[0,1]} \) with the cup product \( \cup \), which is determined by the fact that the constant function 1 is a unit and that the relations

\[
t \cup t = t, \quad t \cup dt = 0 \quad \text{and} \quad dt \cup t = dt
\]

hold. The cup product is associative and compatible with \( d \), hence it makes \( \mathcal{C}^*(\mathbb{[0,1]} \) into a dg algebra. We denote this dg algebra by \( \mathcal{C}_*^*(\mathbb{[0,1]} \), and emphasize that the cup product is not graded commutative.

In order to retain some form of commutativity on \( \mathcal{C}^*(\mathbb{[0,1]} \), one can use homological perturbation theory, as done in the references \([16, 22]\), to transfer the wedge product on \( \Omega^*(\mathbb{[0,1]} \) down to a homotopy associative and homotopy commutative algebra structure, i.e., a \( C_\infty \)-algebra structure, on \( \mathcal{C}^*(\mathbb{[0,1]} \). We refer to the Appendix for a short reminder on these algebraic structures. To carry out the transfer of the wedge product from \( \Omega^*(\mathbb{[0,1]} \) to \( \mathcal{C}^*(\mathbb{[0,1]} \), we first need to fix suitable contraction data from \( \Omega^*(\mathbb{[0,1]} \) to \( \mathcal{C}^*(\mathbb{[0,1]} \). Following \([7, 9, 25]\) we consider Dupont’s contraction (cf. \([8]\), which is given by the inclusion \( \iota : \mathcal{C}^*(\mathbb{[0,1]} \hookrightarrow \Omega^*(\mathbb{[0,1]} \) ), the chain map

\[
\pi : \Omega^0(\mathbb{[0,1]} \rightarrow \mathcal{C}^0(\mathbb{[0,1]} \), \quad f \mapsto f(1)t + f(0)(1 - t)
\]

\[
\pi : \Omega^1(\mathbb{[0,1]} \rightarrow \mathcal{C}^1(\mathbb{[0,1]} \), \quad a(t) dt \mapsto \left( \int_0^1 a(\tau) d\tau \right) dt
\]

and the chain homotopy

\[
h : \Omega^1(\mathbb{[0,1]} \rightarrow \Omega^0(\mathbb{[0,1]} \), \quad a(t) dt \mapsto t \int_0^1 a(\tau) d\tau - \int_0^t a(\tau) d\tau.
\]

We notice that the side-conditions

\[
h \circ h = 0, \quad h \circ \iota = 0 \quad \text{and} \quad \pi \circ h = 0
\]

are satisfied.

The resulting homotopy algebra structure on \( \mathcal{C}^*(\mathbb{[0,1]} \) was explicitly worked out in \([7, 9, 25]\). Below we denote by \( s \) the suspension endofunctor on the category of graded vector spaces, see Appendix \([A]\) for our conventions related to \( A_\infty \) algebras.
There is a unital \( C^\infty \)-algebra. Here \( B_n \) is the \( n \)’th Bernoulli number, defined in terms of the generating function

\[
\frac{z}{e^z - 1} = \sum_{n \geq 0} \frac{z^n}{n!} B_n.
\]

We denote \( C^*([0,1]) \), equipped with the \( C^\infty \) algebra structure given by the maps \( (d,m_2,m_3,\ldots) \), by \( C^\infty_\ast([0,1]) \).

The \( C^\infty \) algebra structure on \( C^*([0,1]) \) comes with a quasi-isomorphism of \( C^\infty \) algebras

\[
\mu : C^\infty_\ast([0,1]) \rightarrow \Omega^\ast([0,1]),
\]

whose linear part is the inclusion \( C^\ast([0,1]) \hookrightarrow \Omega^\ast([0,1]) \), see \cite{7}. Explicit formulas for \( \mu \) were worked out in \cite{9,25}.

**Proposition 1.2** (\cite{7,9,25}). There is a \( C^\infty \) morphism

\[
\mu : C^\infty_\ast([0,1]) \rightarrow \Omega^\ast([0,1])
\]

whose Taylor coefficients are determined as follows:

- \( \mu \) is unital.
- The linear part \( \mu_1 \) is the inclusion.
- For \( n \geq 1 \), \( \mu_{n+1} \) vanishes unless precisely one of the inputs is a function and one has

\[
\mu_{n+1}((sdt)^{\otimes i} \otimes st \otimes (sdt)^{\otimes n-i}) = s(-1)^i \binom{n}{i} \frac{B_{n+1}(t) - B_{n+1}}{(n+1)!}.
\]

Here \( B_n(t) \) is the \( n \)’th Bernoulli polynomial, defined in terms of the generating function

\[
\frac{z e^t z}{e^z - 1} = \sum_{n \geq 1} \frac{z^n}{n!} B_n(t).
\]

1.2. Gugenheim’s \( A^\infty \)-morphism \( \lambda \). Let \( X \) be a smooth manifold. In \cite{13} Gugenheim constructed an \( A^\infty \) quasi-isomorphism \( \lambda_X \) from the de Rham dg algebra \( \Omega_{dR}^\ast(X) \) of smooth, real-valued differential forms on \( X \) to the dg algebra of singular, smooth \( \mathbb{R} \)-valued cochains on \( X \).

The construction relies on Chen’s theory of iterated integrals \cite{6}, see also the exposition in \cite{14}.

We obtain the following result when we specialize Gugenheim’s construction to \( X = [0,1] \):

**Theorem 1.3**. There is a unital \( A^\infty \)-morphism \( \lambda : \Omega^\ast([0,1]) \rightarrow C^\ast_\ast([0,1]) \) whose Taylor coefficients are determined as follows:

- The linear part \( \lambda_1 \) is the chain map \( \pi \) from Subsection 1.1.
- For \( n > 1 \), \( \lambda_n \) vanishes on tensor products that contain a factor which is a zero-form.
- For \( n \geq 1 \) we have

\[
\lambda_n(sa_1(t)dt \otimes \cdots \otimes sa_n(t)dt) = \left( \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} a_1(t_1) \cdots a_n(t_n)dt_1 \cdots dt_n \right) sdt.
\]
We provide a direct proof of this fact below.

**Remark 1.4.** We remark that the previous theorem remains true when $\Omega^*([0,1]) = \Omega^*([0,1])$ is the dg algebra of $k$-polynomial forms: in this case, given $p(t_1, \ldots, t_n) \in k[t_1, \ldots, t_n]$ and $s \in k$, the integral $\int_{0 \leq t_1 \leq \cdots \leq t_n \leq s} p(t_1, \ldots, t_n) dt_1 \cdots dt_n$ can be evaluated formally by setting

$$\int_{0 \leq t_1 \leq \cdots \leq t_n \leq s} t_1^{l_1-1} \cdots t_n^{l_n-1} dt_1 \cdots dt_n = \frac{s^{l_1+\cdots+l_n}}{l_1(l_1 + l_2) \cdots (l_1 + \cdots + l_n)}$$

for all positive integers $l_1, \ldots, l_n$.

**Proof.** Let us evaluate the defining relations for $\lambda$ to be an $A_\infty$ morphism on a tensor product of elements in $s\Omega^*([0,1])$. We do this by considering three separate cases, which cover all possibilities.

First, suppose all factors are one-forms. Then by degree reasons, the defining relation takes values in the component of degree two of $C^*([0,1])$, which is zero.

The second case to consider is that two or more of the factors are zero-forms. Since $\lambda_n$ vanishes for $n > 1$ if one of the inputs is a zero-form, the defining relation is trivially satisfied in this case as well, unless we consider precisely $sf_1(t) \otimes sf_2(t)$. Then the defining relation for $\lambda$ to be an $A_\infty$ morphism reads

$$\pi(f_1(t)f_2(t)) = \pi(f_1(t)) \cup \pi(f_2(t)),$$

which follows immediately from the definitions of $\pi$ and $\cup$.

Finally, we consider an element of the form

$$sa_1(t)dt \otimes \cdots \otimes sa_i(t)dt \otimes sf(t) \otimes sa_{i+1}(t)dt \otimes \cdots \otimes sa_n(t)dt$$

with $n > 0$ and work out the defining relations of $\lambda$ being an $A_\infty$ morphism, evaluated on such an element.

If $0 < i < n$ we obtain

$$\int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} a_1(t_1) \cdots a_i(t_i) \left( \frac{df}{dt}(t_{i+1}) \right) a_{i+1}(t_{i+2}) \cdots a_n(t_{n+1}) dt_1 \cdots dt_{n+1}$$

$$= \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} a_1(t_1) \cdots a_i(t_i)\left(f(t_{i+1})a_{i+1}(t_{i+1})\right) a_{i+2}(t_{i+2}) \cdots a_n(t_n) dt_1 \cdots dt_n$$

$$- \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} a_1(t_1) \cdots a_{i-1}(t_{i-1})\left(a_i(t_i)f(t_i)\right)a_{i+1}(t_{i+1}) \cdots a_n(t_n) dt_1 \cdots dt_n$$

which is a consequence of Stokes theorem. For the extremal case $i = 0$, we obtain,

$$\int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} \left( \frac{df}{dt}(t_1) \right) a_1(t_2) \cdots a_n(t_{n+1}) dt_1 \cdots dt_{n+1}$$

$$= \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} \left(f(t_1)a_1(t_1)\right) a_2(t_2) \cdots a_n(t_n) dt_1 \cdots dt_n$$

$$- f(0) \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} a_1(t_1) \cdots a_n(t_n) dt_1 \cdots dt_n,$$
while for \(i = n\), we obtain
\[
\int_{0 \leq t_1 \leq \cdots \leq t_{n+1} \leq 1} a_1(t_1) \cdots a_n(t_n) \left( \frac{df}{dt}(t_{n+1}) \right) dt_1 \cdots dt_{n+1} =
\]
\[
\left( \int_{0 \leq t_1 \leq \cdots \leq t_{n} \leq 1} a_1(t_1) \cdots a_n(t_n) dt_1 \cdots dt_k \right) f(1)
\]
\[
- \int_{0 \leq t_1 \leq \cdots \leq t_{n} \leq 1} a_1(t_1) \cdots a_{k-1}(t_{k-1}) \left( a_n(t_n) f(t_n) \right) dt_1 \cdots dt_n.
\]

Also the latter two equations are immediate consequences of Stokes theorem. \(\square\)

It is well known that iterated integrals behave well with respect to the shuffle product \([6]\): in the case of the interval, we have the following proposition, which we will use in the next section.

**Proposition 1.5.** We denote by \(p\lambda : \mathcal{T}(s\Omega^1([0,1])) \rightarrow sC^1([0,1]) = k\) the corestriction of the degree zero part of Gugenheim’s morphism: then \(p\lambda\) is a morphism of commutative algebras, where we equip \(\mathcal{T}(s\Omega^1([0,1]))\) with the shuffle product \(\circ\) (cf. Appendix B).

**Proof.** Denoting by \(\Delta_i = \{(t_1, \ldots, t_i) \in \mathbb{R}^i \text{ s.t. } 0 \leq t_1 \leq \cdots \leq t_i \leq 1\}\) the \(i\)-dimensional simplex, we have
\[
\lambda_j(sa_1(t)dt \otimes \cdots \otimes sa_j(t)dt) \cdot \lambda_k(sa_{j+1}(t)dt \otimes \cdots \otimes sa_{j+k}(t)dt) = \int_{\Delta_j \times \Delta_k} a_1(t_1) \cdots a_{j+k}(t_{j+k}) dt_1 \cdots dt_{j+k},
\]

Using the natural triangulation of \(\Delta_j \times \Delta_k\)
\[
\prod_{\sigma \in S(j,k)} \Delta_n \xrightarrow{\sigma} \Delta_j \times \Delta_k, \quad (t_1, \ldots, t_n) \xrightarrow{\sigma} (t_{\sigma(1)}, \ldots, t_{\sigma(n)}),
\]

where \(S(j,k)\) is the set of \((j,k)\)-unshuffles, we can rewrite the right hand side of the previous equation as
\[
\int_{\Delta_j \times \Delta_k} a_1(t_1) \cdots a_{j+k}(t_{j+k}) dt_1 \cdots dt_{j+k} =
\]
\[
= \sum_{\sigma \in S(j,k)} \int_{\Delta_n} a_1(t_{\sigma(1)}) \cdots a_{j+k}(t_{\sigma(j+k)}) dt_1 \cdots dt_{j+k}
\]
\[
= \sum_{\sigma \in S(j,k)} \int_{\Delta_n} a_{\sigma^{-1}(1)}(t_1) \cdots a_{\sigma^{-1}(j+k)}(t_{j+k}) dt_1 \cdots dt_{j+k}
\]
\[
= \lambda_k(sa_1(t)dt \otimes \cdots \otimes sa_j(t)dt) \otimes (sa_{j+1}(t)dt \otimes \cdots \otimes sa_{j+k}(t)dt),
\]

by definition of the shuffle product \(\otimes\). \(\square\)

### 1.3. Comparing two structures on Whitney forms. We can now combine the \(C_\infty\) morphism \(\mu : C_\infty([0,1]) \rightarrow \Omega^*([0,1])\) with the \(A_\infty\) morphism \(\lambda : \Omega^*([0,1]) \rightarrow C_\infty([0,1])\). Since the linear part of the composition \(\lambda \circ \mu\) is the identity, we obtain an \(A_\infty\) isomorphism between \(C_\infty([0,1])\) and \(C_\infty([0,1])\).

**Proposition 1.6.** The Taylor coefficients of the unital \(A_\infty\) isomorphism
\[
\exp := \lambda \circ \mu : C_\infty([0,1]) \rightarrow C_\infty^*([0,1])
\]
are determined as follows:
• The linear part $\exp_1$ is the identity.
• For $n > 1$, $\exp_n$ vanishes on tensor products that contain a factor of degree 0.
• For $n \geq 1$, we have
  $$\exp_n(sdt \otimes \cdots \otimes sdt) = \frac{1}{n!} sdt.$$  

The inverse $\log : C^*_\cap([0,1]) \to C^*_\cap([0,1])$ to $\exp$ is the unital $A_\infty$ isomorphism whose Taylor coefficients are determined as follows:

• The linear part of $\log_1$ is the identity.
• For $n > 1$, $\log_n$ vanishes on tensor products that contain a factor of degree 0.
• For $n \geq 1$, we have
  $$\log_n(sdt \otimes \cdots \otimes sdt) = (-1)^{n+1} \frac{n}{n!} sdt.$$  

Proof. Let us first consider the map $\exp$: we already observed the assertion about the linear part.

If we evaluate $\exp_{n+1}$, $n \geq 1$, on a tensor product of the form
  $$(sdt)^{\otimes i} \otimes st \otimes (sdt)^{\otimes n-i},$$
only the contribution from $\pi\mu_{n+1}((sdt)^{\otimes i} \otimes st \otimes (sdt)^{\otimes n-i})$ can be non-zero, since all higher order terms of $\lambda$ map tensor products which contain a factor that is a zero-form to zero. Hence we obtain
  $$\exp_{n+1}((sdt)^{\otimes i} \otimes st \otimes (sdt)^{\otimes n-i}) = (-1)^i \binom{n}{i} \left( \frac{B_{n+1}(1) - B_{n+1}}{(n+1)!} \right) st = 0,$$
since $B_{n+1}(1) = B_{n+1}$ for $n \geq 1$.

On the other hand, only $\lambda \mu^{\otimes n}_1$ contributes to the evaluation of $\exp_n$ on the tensor product $(sdt)^{\otimes n}$, since the higher order terms of $\mu$ vanish unless precisely one argument is a function, and we find
  $$\exp_n(sdt \otimes \cdots \otimes sdt) = \left( \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} dt_1 \cdots dt_n \right) sdt = \frac{1}{n!} sdt,$$
as desired. Finally, it is clear by degree reasons that $\exp_n$ vanishes if two or more arguments are functions.

It is straightforward to check that $\log$ as defined in the proposition is indeed the inverse to $\exp$.  

1.4. A one-sided inverse to $\lambda$. We define an $A_\infty$ morphism $\gamma$ as the composition
  $$\gamma : C^*_\cap([0,1]) \xrightarrow{\log} C^*_\cap([0,1]) \xrightarrow{\mu} \Omega^*([0,1]).$$
By construction, we have $\lambda \circ \gamma = \lambda \circ \mu \circ \log = \exp \circ \log = \text{id}$.

Proposition 1.7. The Taylor coefficients of the unital $A_\infty$ morphism
  $$\gamma = \mu \circ \log : C^*_\cap([0,1]) \to \Omega^*([0,1])$$
are determined as follows:

• The linear part $\gamma_1$ is the inclusion $C^*([0,1]) \hookrightarrow \Omega^*([0,1])$.  


• For $i \geq 0$, $j \geq 0$, we have
\[
\gamma_{i+j+1}(\text{sd}t^{\otimes i} \otimes \text{st} \otimes \text{sd}t^{\otimes j}) = s \sum_{l=0}^{i} \binom{1-t}{l} \binom{t}{i+j+1-l},
\]
where $\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}$.

• For $n \geq 1$, we have
\[
\gamma_n(\text{sd}t \otimes \cdots \otimes \text{sd}t) = \frac{(-1)^{n+1}}{n} \text{sd}t.
\]

Proof. We introduce the following generating function
\[
F(z, w) := \sum_{i,j \geq 0} \gamma_{i+j+1}(\text{sd}t^{\otimes i} \otimes \text{st} \otimes \text{sd}t^{\otimes j}) z^i w^j
\]
and compute
\[
F(z, w) = \sum_{i,j \geq 0} \sum_{i_1+\cdots+i_p=i, j_1+\cdots+j_q=j} \frac{(-1)^{i+p} (-1)^{j+q}}{i_1! \cdots i_p! j_1! \cdots j_q!} \mu_{p+q+1} (\text{sd}t^{\otimes p} \otimes \text{st} \otimes \text{sd}t^{\otimes q})
\]
\[
= \sum_{p,q \geq 0} \log(1 + z)^p \log(1 + w)^q \mu_{p+q+1} (\text{sd}t^{\otimes p} \otimes \text{st} \otimes \text{sd}t^{\otimes q})
\]
\[
= \sum_{n \geq 0} \left( \sum_{p+q=n} \binom{n}{p} (-1)^p \log(1 + z)^p \log(1 + w)^q \right) \frac{B_{n+1}(t) - B_{n+1}}{(n+1)!}
\]
\[
= \sum_{n \geq 0} \frac{B_{n+1}(t) - B_{n+1}}{(n+1)!} (\log(1 + w) - \log(1 + z))^n
\]
\[
= G \left( \log \left( \frac{1 + w}{1 + z} \right) \right),
\]
where $G(u)$ is the formal power series
\[
G(u) = \sum_{r \geq 0} \frac{B_{r+1}(t) - B_{r+1}}{(r+1)!} u^r = e^u - 1.
\]

Hence we find
\[
F(z, w) = \frac{(1+w)^t}{1+w} - 1 = (1+z)^{1-t} \frac{1}{w-z} \left( (1+w)^t - (1+z)^t \right).
\]
Since $z$ and $w$ are formal variables, we can apply Newton’s generalized binomial Theorem to obtain
\[
\frac{1}{w-z} \left( (1+w)^t - (1+z)^t \right) = \frac{1}{w-z} \sum_{k \geq 0} \binom{t}{k+1} (w^{k+1} - z^{k+1}) = \sum_{r,s \geq 0} \binom{t}{r+s+1} w^r z^s
\]
where, by definition,
\[
\binom{t}{k} := \frac{t(t-1)\cdots(t-k+1)}{k!}.
\]
We therefore have
\[
F(z, w) = \left( \sum_{l \geq 0} \binom{1-t}{l} z^l \right) \left( \sum_{r,s \geq 0} \binom{t}{r+s+1} w^r z^s \right).
\]
Consequently, the coefficient for $z^i w^j$ of $F(z, w)$ is
\[
\sum_{l=0}^{i} \binom{1-t}{l} \left( i + j + 1 - l \right).
\]

We conclude that
\[
\gamma_{i+j+1}( (sdt)^{\otimes i} \otimes st \otimes (sdt)^{\otimes j} ) = s \sum_{l=0}^{i} \binom{1-t}{l} \left( i + j + 1 - l \right).
\]

Since $\mu_n$ vanishes for $n > 1$ if we evaluate it on a tensor product that contains only elements of degree one, the only relevant contribution to $\gamma_n(sdt \otimes \cdots \otimes sdt)$ is $\frac{(\mu_1 \circ \log_n)(sdt \otimes \cdots \otimes sdt)}{n}$. \hfill \Box

2. The $C_\infty$ morphism $\varphi$ from $\Omega^*([0, 1])$ to $C_\infty^*([0, 1])$

In this section we study the composition

\[
\varphi : \Omega^*([0, 1]) \xrightarrow{\lambda} C_\infty^*([0, 1]) \xrightarrow{\log} C_\infty^*([0, 1]).
\]

Our main results concerning $\varphi$ are as follows:

1. We provide several formulas for the Taylor coefficients of $\varphi$.
2. We show that $\varphi$ is a $C_\infty$ morphism of $C_\infty$ algebras.
3. We prove that $\varphi$ is unique within a certain class of $A_\infty$ morphisms, and, as a consequence, that it coincides with the morphism obtained via homotopy transfer along Dupont’s contraction, cf. Subsection 1.2.

2.1. An explicit formula. The aim of this subsection is to make the $A_\infty$ morphism

\[
\varphi : \Omega^*([0, 1]) \to C_\infty^*([0, 1]),
\]

defined as the composition of $\lambda : \Omega^*([0, 1]) \to C_\infty^*([0, 1])$ from Subsection 1.2 and $\log : C_\infty^*([0, 1]) \to C_\infty^*([0, 1])$ from Subsection 1.3 explicit.

Definition 2.1. The descent number $d_\sigma$ of a permutation $\sigma \in S_n$ is the non-negative integer $d_\sigma := |\{i \in \{1, \ldots, n-1\} \text{ such that } \sigma(i) > \sigma(i+1)\}|$.

Theorem 2.2. The higher Taylor coefficients $\varphi_n$, $n \geq 2$, of $\varphi$ vanish unless all of the inputs are one-forms, in which case one has

\[
\varphi_n(sa_1(t) dt \otimes \cdots \otimes sa_n(t) dt) = \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} \frac{1}{n} \sum_{\sigma \in S_n} \left( \frac{(-1)^{d_\sigma}}{(n-1)^{d_\sigma}} \right) a_1(t_{\sigma(1)}) \cdots a_n(t_{\sigma(n)}) dt_1 \cdots dt_n s dt.
\]

Proof. Since both the higher Taylor coefficients of $\lambda$ and $\log$ vanish unless all of the inputs are one-forms, the first assertion is clear. When all the inputs are one-forms, by definition of $\log$ and $\lambda$ we have

\[
\varphi_n(sa_1(t) dt \otimes \cdots \otimes sa_n(t) dt) = \sum_{m=1}^{n} \frac{(-1)^{m+1}}{m} \sum_{i_1+\cdots+i_m=n} \left( \int_{\Delta_{i_1}} a_1(t_1) \cdots a_{i_1}(t_{i_1}) dt_1 \cdots dt_{i_1} \cdots \int_{\Delta_{i_m}} a_{n-i_m+1}(t_1) \cdots a_n(t_{i_m}) dt_1 \cdots dt_{i_m} \right) s dt.
\]
According to (the proof of) Proposition 1.5, the right hand side of the previous equation equals $sdt$ multiplied by the scalar

$$\sum_{\sigma \in S_n} \left( \sum_{i_1 + \cdots + i_m = n \atop \sigma \in S(i_1, \ldots, i_m)} \frac{(-1)^{m+1}}{m} \right) \int_{\Delta_n} a_1(t_{\sigma(1)}) \cdots a_n(t_{\sigma(n)}) dt_1 \cdots dt_n.$$ 

Hence the proof is completed by the following lemma. \hfill \Box

**Lemma 2.3.** Given a positive integer $n$ and a permutation $\sigma \in S_n$, we have

$$\sum_{i_1 + \cdots + i_m = n \atop \sigma \in S(i_1, \ldots, i_m)} \frac{(-1)^{m+1}}{m} = \frac{(-1)^{d_\sigma}}{n^{(n-1)}},$$

where the sum runs over all ordered partitions $i_1 + \cdots + i_m = n$ of $n$ such that $\sigma$ is an $(i_1, \ldots, i_m)$-unshuffle.

**Proof.** Let us consider partitions $i_1 + \cdots + i_m = n$ with a fixed $m$ such that $\sigma \in S(i_1, \ldots, i_m)$. One sees that there are

$$\binom{n-d_\sigma-1}{m-d_\sigma-1}$$

of those. Hence we find

$$\sum_{i_1 + \cdots + i_m = n \atop \sigma \in S(i_1, \ldots, i_m)} \frac{(-1)^{m+1}}{m} = \sum_{m=d_\sigma+1}^{n} \frac{(-1)^{m+1}}{m} \frac{n-d_\sigma-1}{m-d_\sigma-1} = \sum_{j=0}^{n-d_\sigma-1} \frac{(-1)^{d_\sigma+j}}{d_\sigma+j+1} \frac{n-d_\sigma-1}{j}.$$ 

The latter sum can be identified with the $n$'th Taylor coefficient of $\log (1 + z)(1 + z)^{n-d_\sigma-1}$ at $z = 0$ and is given by

$$(-1)^{d_\sigma} \frac{(n-d_\sigma-1)!d_\sigma!}{n!} = \frac{(-1)^{d_\sigma}}{n^{(n-1)}}.$$ 

\hfill \Box

**Definition 2.4.** We denote by $C_{n,d}$ the numbers $C_{n,d} := \frac{(-1)^d}{n^{(n-1)}}$, $n \geq 1$, $0 \leq d < n$: these satisfy the identities

$$C_{n,d} = C_{n-1,d} + C_{n,d+1}, \quad C_{n,d} = (-1)^{n+1}C_{n,n-d-1},$$

as it follows by straightforward computations. For $n \leq 6$, they are given by

$$\begin{array}{cccccccc}
1 & & & & & & & \\
0 & 1 & & & & & & \\
0 & 0 & 1 & & & & & \\
0 & 0 & 0 & 1 & & & & \\
0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 0 & 1 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{array}$$

The element $e_{\frac{1}{n}}^{[1]} := \sum_{\sigma \in S_n} C_{n,d_\sigma} \sigma \in k[S_n]$ of the group algebra of the symmetric group is called the (first) Eulerian idempotent, see [19, 27].

**Remark 2.5.** There is a natural action of $S_n$ on the functions on the $n$-cube, by permuting the variables, and a projector corresponding to $e_{\frac{1}{n}}^{[1]}$: then the integrand of Theorem 2.2 is precisely the image of $a_1(t_1) \cdots a_n(t_n)$ under this projector.
It is well known that \( e_n^{[1]} \) is an idempotent of the group algebra, and in fact a \textit{Lie idempotent}. The latter means the following: let \( V \) be a vector space and \( \mathcal{T}(V) \) the reduced tensor algebra on \( V \), then the mapping
\[
E : \mathcal{T}(V) \to \mathcal{T}(V), \quad v_1 \otimes \cdots \otimes v_n \mapsto \sum_{\sigma \in S_n} C_{n,d_\sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}
\]
is a projector from \( \mathcal{T}(V) \) onto the subspace \( \text{Lie}(V) \) spanned by Lie words, i.e., onto the free Lie algebra on \( V \), see \cite{19} \cite{27}. Notice that the restriction of \( E \) to \( n \)\textsuperscript{th} fold tensor products is precisely the projector corresponding to \( e_n^{[1]} \) under the natural action of \( S_n \) on \( T^n(V) \). It is not immediately clear how to express \( E(v_1 \otimes \cdots \otimes v_n) \) as a linear combination of Lie words: one way to do it is to compose \( E \) with a second Lie idempotent, for instance the Dynkin idempotent
\[
\gamma : \mathcal{T}(V) \to \mathcal{T}(V), \quad v_1 \otimes \cdots \otimes v_n \mapsto \frac{1}{n!}[v_1, \ldots, [v_{n-1}, v_n] \ldots].
\]
Since both \( E \) and \( \gamma \) are projectors with image \( \text{Lie}(V) \), we see that \( E = \gamma \circ E \), that is
\[
E(v_1 \otimes \cdots \otimes v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{d_\sigma \cdot \text{sign} \sigma} (v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}).
\]
By definition of the shuffle product, a straightforward application of Lemma \cite{23} yields the more explicit formula
\[
E^*(v_1 \otimes \cdots \otimes v_n) = \sum_{\sigma \in S_n} C_{n,d_\sigma} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.
\]

**Proposition 2.6.** The corestriction \( \varphi : \mathcal{T}(\mathfrak{g}(\lambda([0,1]), C, \sigma_n)) \to C^*(\mathfrak{g}(\lambda([0,1]), C, \sigma_n)) \) of the \( A_\infty \) morphism \( \varphi : \Omega^*(\mathfrak{g}(\lambda([0,1]), C, \sigma_n)) \to C^*(\mathfrak{g}(\lambda([0,1]), C, \sigma_n)) \) equals
\[
\varphi = (\lambda \circ E^*),
\]
where \( \lambda : \mathcal{T}(\mathfrak{g}(\lambda([0,1]), C, \sigma_n)) \to C^*(\mathfrak{g}(\lambda([0,1]), C, \sigma_n)) \) is the corestriction of Gugenheim’s \( A_\infty \) morphism \( \lambda \) and \( E^* : \mathcal{T}(\mathfrak{g}(\lambda([0,1]), C, \sigma_n)) \to \mathcal{T}(\mathfrak{g}(\lambda([0,1]), C, \sigma_n)) \) is defined as above.

**Proof.** This is a straightforward consequence of the previous formula for \( E^* \): if one of the arguments is a zero-form, both the left- and the right-hand side of the claimed identity vanish, otherwise, we see that
\[
(\lambda \circ E_\sigma)(\mathcal{A}(t) dt \otimes \cdots \otimes \mathcal{A}(t) dt) = \sum_{\sigma \in S_n} C_{n,d_\sigma} \int_{\Delta_n} a_{\sigma^{-1}(1)}(t_1) \cdots a_{\sigma^{-1}(n)}(t_n) dt_1 \cdots dt_n = \sum_{\sigma \in S_n} C_{n,d_\sigma} \int_{\Delta_n} a_1(t_{\sigma(1)}) \cdots a_n(t_{\sigma(n)}) dt_1 \cdots dt_n = \varphi_n(\mathcal{A}(t) dt \otimes \cdots \otimes \mathcal{A}(t) dt).
\]
\[\square\]
Corollary 2.7. The map \( \varphi : \Omega^*([0,1]) \to C_\infty^*([0,1]) \) is a \( C_\infty \) morphism.

Proof. Recall, cf. Appendix [3] that since \( \varphi \) is a morphism of \( A_\infty \) algebras, we only have to check that the Taylor coefficients \( \varphi_n, n \geq 2 \), vanish on the image of the shuffle product
\[
\otimes : T(s\Omega^*([0,1])) \otimes T(s\Omega^*([0,1])) \to T(s\Omega^*([0,1])).
\]
This follows from the previous proposition, since \( E^* \) vanishes on the image of the shuffle product, compare with the proof of [27, Theorem 6.3]. \( \square \)

We shall give another proof of the previous corollary below, in Subsection 2.3.

2.2. A recursive formula. In this subsection we derive an alternative presentation of the \( C_\infty \) morphism
\[
\varphi : \Omega^*([0,1]) \to C_\infty^*([0,1]),
\]
closely related to Magnus expansion (see [20, 17], cf. Section 3 below.

Definition 2.8. For all \( n \geq 1 \) we define maps
\[
\mathcal{M}_n : (\Omega^0([0,1]))^\otimes n \to \Omega^0([0,1])
\]
as follows:

- for \( n = 1 \), we set \( \mathcal{M}_1(a_1(t))(s) = \int_0^s a_1(t_1)dt_1 \),
- for \( n \geq 2 \), we apply the recursive formula (where the suspension points inside parentheses are to be filled by the arguments in the order \( a_1, \ldots, a_n \), and we denote by \( p_j \) the partial sum \( p_j = \sum_{h \leq j} i_h \))
\[
\mathcal{M}_n(a_1(t) \otimes \cdots \otimes a_n(t))(s) = \sum_{k=1}^{n-1} \frac{B_k}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j}
\]
\[
\sum_{i_1 + \cdots + i_k = n-1} \int_0^s \mathcal{M}_{i_1}(\cdots)(t_1) \cdots \mathcal{M}_{i_k}(\cdots)(t_1) a_{p_j+1}(t_1) \mathcal{M}_{i_{j+1}}(\cdots)(t_1) \cdots \mathcal{M}_{i_k}(\cdots)(t_1) dt_1.
\]

Definition 2.9. We denote by \( (\beta_s)_{s \in \mathbb{K}} \) (\( s \in [0,1] \) in the smooth case) the one-parameter family of maps given by \( \beta_s(t) = s \cdot t \). We refer to the corresponding endomorphisms \( \beta_s^* \) of the dg algebra \( \Omega^*([0,1]) \) as the scaling morphisms and define a one-parameter family of \( C_\infty \)-morphisms from \( \Omega^*([0,1]) \) to \( C_\infty^*([0,1]) \) by setting \( \varphi_s := \varphi \circ \beta_s^* \).

Theorem 2.10. The \( n \)th Taylor coefficient \( \varphi_{s,n} \) of the \( C_\infty \) morphism \( \varphi_s \) is given by
\[
\varphi_{s,n}(sa_1(t)dt \otimes \cdots \otimes sa_n(t)dt) = \mathcal{M}_n(a_1(t) \otimes \cdots \otimes a_n(t))(s)sdt.
\]

Proof. We proceed by showing that the family of maps
\[
\nu_n : (\Omega^0([0,1]))^\otimes n \to \Omega^0([0,1])
\]
defined by \( \nu_n(a_1(t) \otimes \cdots \otimes a_n(t))(s)sdt = \varphi_{s,n}(sa_1(t)dt \otimes \cdots \otimes sa_n(t)dt) \) obeys the same recursion as the family of maps \( (\mathcal{M}_n)_{n \geq 1} \) from above.

Recall that by definition we have \( \varphi_s = \varphi \circ \beta_s^* \). Let \( X \) be an arbitrary element of \( T(s\Omega^1([0,1])) \) and consider the curve
\[
s \mapsto \varphi_s(X) \in T(sC^1([0,1])) \cong T(\mathbb{K}).
\]
If we differentiate it with respect to \( s \), we find
\[
\frac{d}{ds} \varphi_s(X) = \varphi \left( \frac{d}{ds} \beta_s^*(X) \right).
\]
Now suppose we find a one-parameter family of elements \( Y_s \in T(s\Omega^*([0,1])) \) such that \( \frac{d}{ds}\beta_s^*X = Q(Y_s) \), where \( Q \) denotes the codifferential which encodes the dg algebra structure on \( \Omega^*([0,1]) \). We would then conclude that

\[
\frac{d}{ds}(\varphi_s(X)) = \varphi(\frac{d}{ds}\beta_s^*(X)) = \varphi(Q(Y_s)) = M(\varphi(Y_s))
\]

holds, where \( M \) is the codifferential on \( T(sC^*([0,1])) \) which encodes the \( C_\infty \) algebra structure on \( C^*([0,1]) \) from Theorem 1.1.

We now consider \( X = sa_1(t)dt \otimes \cdots \otimes sa_n(t)dt \) and claim that an appropriate \( Y_s \) is given by

\[
Y_s = -\sum_{j=0}^{n-1} s\beta_s^*(a_1(t)dt) \otimes \cdots \otimes s\beta_s^*(a_j(t)dt) \otimes s(ta_{j+1}(st)) \otimes s\beta_s^*(a_{j+2}(t)dt) \otimes \cdots \otimes s\beta_s^*(a_n(t)dt).
\]

It is straightforward to show that applying the linear part \( Q_1 \) of the codifferential \( Q \), which encodes the de Rham differential, yields \( Q_1(Y_s) = X \). One checks that the contribution from the quadratic part \( Q_2 \) of the codifferential, which encodes the wedge product, vanishes. This is due to the equality

\[
\beta_s^*(a_j(t)dt)(ta_{j+1}(st)) = sta_j(st)a_{j+1}(st)dt = (ta_j(st))\beta_s^*(a_{j+1}(t)dt).
\]

From this we infer that

\[
\frac{d}{ds}(\varphi_s(sa_1(t)dt \otimes \cdots \otimes sa_n(t)dt)) =
\]

\[
- M\varphi \left( \sum_{j=0}^{n-1} s\beta_s^*(a_1(t)dt) \otimes \cdots \otimes s\beta_s^*(a_j(t)dt) \otimes s(ta_{j+1}(st)) \otimes s\beta_s^*(a_{j+2}(t)dt) \otimes \cdots \otimes s\beta_s^*(a_n(t)dt) \right).
\]

Recall from Subsection 1.1 that the Taylor coefficients of \( M \) are only non-zero on tensor products which contain exactly one factor in \( sC^0([0,1]) \), while the Taylor coefficients \( (\varphi_n) \) all vanish for \( n \geq 2 \) whenever one of the factors is a function. Furthermore, we notice that \( \varphi_1 = \pi \) evaluates on \( ta_{j+1}(st) \) to \( a_{j+1}(s)t \). We thus obtain that the projection of

\[
- M\varphi \left( \sum_{j=0}^{n-1} s\beta_s^*(a_1(t)dt) \otimes \cdots \otimes s\beta_s^*(a_j(t)dt) \otimes s(ta_{j+1}(st)) \otimes s\beta_s^*(a_{j+2}(t)dt) \otimes \cdots \otimes s\beta_s^*(a_n(t)dt) \right)
\]

to \( sC^*([0,1]) \) equals (by definition of the functions \( (\nu_n)_{n \geq 1} \))

\[
- \sum_{\ell \geq 1} \sum_{p=0}^{\ell} (-1)^{\ell+1} \binom{\ell}{p} \frac{B_{\ell}}{\ell!} \sum_{j=0}^{n-1} \nu_{n_1} (\cdots) (s) \cdots \nu_{n_p} (\cdots) (s) a_{j+1}(s) \nu_{n_{p+1}} (\cdots) (s) \cdots \nu_{n_\ell} (\cdots) (s)
\]

times \( sdt \).

On the other hand, the projection of

\[
\frac{d}{ds}(\varphi_s(sa_1(t)dt \otimes \cdots \otimes sa_n(t)dt))
\]

to \( sC^*([0,1]) \) equals

\[
\frac{d}{ds}(\varphi_{s,n}(sa_1(t)dt \otimes \cdots \otimes sa_n(t)dt)) = \frac{d}{ds}(\nu_n(a_1(t) \otimes \cdots \otimes a_n(t))(s) sdt)
\]
and hence we finally arrive at the recursion
\[
\frac{d}{ds} (\nu_n(a_1(t) \otimes \cdots \otimes a_n(t))(s)) =
\]
\[
= \sum_{\ell \geq 1} \sum_{p=0}^{\ell} (-1)^p \frac{\ell!}{p!} B_{\ell} \sum_{j=0}^{n-1} \sum_{n_1+\cdots+n_\ell=n-1} \nu_{n_1}(\cdots)(s) \cdots \nu_{n_\ell}(\cdots)(s) a_j(s) \nu_{n_\ell+1}(\cdots)(s) \cdots \nu_{n_\ell}(\cdots)(s).
\]
This is precisely the recursion which the family of maps \((M_n)_{n \geq 1}\) obeys.

\[\square\]

### 2.3. Some uniqueness results

The aim of this section is to show that the \(C_\infty\) morphism \(\varphi : \Omega^*([0,1]) \to C_\infty^*([0,1])\) coincides with the \(A_\infty\) morphism induced via homotopy transfer along Dupont’s contraction, cf. Section 1.1. We do so by showing a uniqueness result for \(A_\infty\) morphisms satisfying some properties in the following lemma. We shall rely heavily on the notations and results from the Appendix.

**Lemma 2.11.** Let \((V,Q_1,\ldots,Q_n,\ldots)\) be an \(A_\infty\) algebra, together with a decomposition of \(V\) in the direct sum of graded subspaces \(V = X \oplus Y\) such that \(sY \subset Q_1(sX)\). Let \((W,R_1,\ldots,R_n,\ldots)\) be a second \(A_\infty\) algebra and \(G,G' : V \to W\) two \(A_\infty\) morphisms such that

1. The linear parts of \(G\) and \(G'\) are equal and
2. Whenever there exists \(1 \leq i \leq n\) such that \(v_i \in X\), then
   \[G_n(sv_1 \otimes \cdots \otimes sv_n) = G'_n(sv_1 \otimes \cdots \otimes sv_n).\]

Under these conditions, the morphisms \(G\) and \(G'\) coincide.

In particular, if there exists an \(A_\infty\) morphism \(F : V \to W\) with a given linear part and the property that its higher Taylor coefficients \(F_n, n \geq 2,\) vanish whenever at least one of their arguments is in \(X\), it is unique.

**Proof.** We have to prove that in the given hypotheses for all \(n \geq 1\) and \(y_1,\ldots,y_n \in Y\) we have \(G_n(sy_1 \otimes \cdots \otimes sy_n) = G'_n(sy_1 \otimes \cdots \otimes sy_n)\). We use induction, knowing by hypothesis that \(G_1 = G'_1\). We denote by \(Q_i^n\) the composition \(sV \otimes \cdots \otimes T(sV) \overset{Q_i}{\to} T(sV) \to sV \otimes \cdots \otimes T(sV)\) and similarly for \(G_i^n, G'_i^n : sV \otimes \cdots \otimes T(sV) \to \cdots\) notice that for \(i \geq 2\) \(G_i^n = G'_i^n\) by the inductive hypothesis, since they only depend on \(G_1 = G'_1,\ldots,G_{n-1} = G'_{n-1}\). Since \(G\) is an \(A_\infty\) morphism we have the identity
\[
\sum_{i=1}^n R_i G_i^n = \sum_{j=1}^n G_j Q_j^n,
\]
and similarly for \(G'\). Finally, we choose \(x_1 \in X\) such that \(sy_1 = Q_1(sx_1)\), then by the hypotheses of the lemma, \(G_n(sx_1 \otimes sy_2 \otimes \cdots \otimes sy_n) = G'_n(sx_1 \otimes sy_2 \otimes \cdots \otimes sy_n)\), and together with the inductive hypothesis this shows that (notice that \(Q_1(sY) = 0)\)
\[
G_n(sy_1 \otimes \cdots \otimes sy_n) = G_n Q_1(sx_1 \otimes sy_2 \otimes \cdots \otimes sy_n) =
\]
\[
= \left( \sum_{i=1}^n R_i G_i^n \right) - \left( \sum_{j=1}^{n-1} G_j Q_j^n \right) (sx_1 \otimes sy_2 \otimes \cdots \otimes sy_n)
\]
\[
= G'_n Q_1(sx_1 \otimes sy_2 \otimes \cdots \otimes sy_n) = G'_n(sy_1 \otimes \cdots \otimes sy_n).
\]

\[\square\]
Corollary 2.12. The $A_\infty$ morphisms
\[ \lambda : \Omega^*([0,1]) \to C^*_\mathcal{C}([0,1]), \quad \exp : C^*_\mathcal{C}([0,1]) \to C^*_\mathcal{C}([0,1]) \quad \text{and} \quad \varphi : \Omega^*([0,1]) \to C^*_\mathcal{C}([0,1]) \]
are the only $A_\infty$ morphisms with linear parts equal to $\pi$, id and $\pi$ respectively, and the property that their higher order Taylor coefficients vanish whenever one of the arguments is a zero-form.

Proof. Apply the last assertion of the preceding lemma to $V = \Omega^*([0,1])$, $X = \Omega^0([0,1])$ and $Y = \Omega^1([0,1])$ (respectively, $V = C^*([0,1])$, $X = C^0([0,1])$ and $Y = C^1([0,1])$).

Lemma 2.13. Under the same hypotheses as in the previous lemma, suppose moreover that $V$ and $W$ are $C_\infty$ algebras. If there exists $F : V \to W$ as in the final claim, then $F$ is a $C_\infty$ morphism.

Proof. We denote the Taylor coefficients of the $C_\infty$ algebra structure on $V$ and $W$ by $(Q_i)_{i \geq 1}$ and $(R_i)_{i \geq 1}$, respectively. We have to show $F_n((sy_1 \otimes \cdots \otimes sy_i) \oplus (sy_{i+1} \otimes \cdots \otimes sy_n)) = 0$ for all $n \geq 1$, $1 \leq i < n$ and $y_1, \ldots, y_n \in Y$. The case $n = 1$ being empty, we use induction: in particular, we can consider the morphism of graded coalgebras $F_{<n} : \overline{T}(sV) \to \overline{T}(sW)$, whose Taylor coefficients are $(F_{<n})_i = F_i$ for $i < n$ and $(F_{<n})_i = 0$ for $i \geq n$, and according to the inductive hypothesis and Lemma B.2, this is a morphism of graded bialgebras. We define coderivations $Q_{\geq 2}$ on $\overline{T}(sV)$ and $R_{\geq 2}$ on $\overline{T}(sW)$ by declaring their Taylor coefficients to be $(Q_{\geq 2})_i = Q_i$, $(R_{\geq 2})_i = R_i$ if $i \geq 2$ and $(Q_{\geq 2})_1 = (R_{\geq 2})_1 = 0$: according to Lemma B.3, these are biderivations. Thus $H := R_{>2}F_{<n} - F_{<n}Q_{>2} : \overline{T}(sV) \to \overline{T}(sW)$ is an $F_{<n}$-biderivation by Lemma B.2, and in particular it sends the image of the shuffle product in $\overline{T}(sV)$ into the image of the shuffle product in $\overline{T}(sW)$. As in the proof of the previous lemma we choose $x_1$ such that $Q_1(sx_1) = sy_1$: we finally compute, denoting by $p : \overline{T}(sW) \to sW$ the natural projection, that
\[
F_n((sy_1 \otimes \cdots \otimes sy_i) \oplus (sy_{i+1} \otimes \cdots \otimes sy_n)) = (F_nQ_1 - R_nF_n)((sx_1 \otimes sy_2 \otimes \cdots \otimes sy_i) \oplus (sy_{i+1} \otimes \cdots \otimes sy_n))
\]
\[
= \sum_{i=2}^{n} \sum_{j=1}^{n-1} R_if_i^n - F_j^n((sx_1 \otimes sy_2 \otimes \cdots \otimes sy_i) \oplus (sy_{i+1} \otimes \cdots \otimes sy_n))
\]
\[
= pH((sx_1 \otimes sy_2 \otimes \cdots \otimes sy_i) \oplus (sy_{i+1} \otimes \cdots \otimes sy_n)) = 0,
\]
since $p$ vanishes on the image of the shuffle product. \qed

Corollary 2.14. $\varphi : \Omega^*([0,1]) \to C^*_\mathcal{C}([0,1])$ is a morphism of $C_\infty$ algebras.

Remark 2.15. Since $\lambda$, exp, log, $\varphi$ are all compatible with the simplicial structure on $[0,1]$, we can extend them to morphisms over any 1-dimensional simplicial complex $T$, and Corollary 2.14 still holds. If, moreover, $H_1(T) = 0$, we can apply the previous lemmas to $\Omega^*(T) = \Omega^0(T) \oplus \Omega^1(T)$ and $C^*(T) = C^0(T) \oplus C^1(T)$, respectively, to obtain uniqueness results parallel to Corollary 2.12.

Lemma 2.16. For $n > 1$ the Taylor coefficients of the $A_\infty$ morphism $\Omega^*([0,1]) \to C^*_\mathcal{C}([0,1])$ obtained via homological perturbation theory vanish on $n$-fold tensor products which contain a factor that is a zero-form.

Proof. Given the contraction data $\iota, \pi, h$ from $\Omega^*([0,1])$ to $C^*([0,1])$ as in Section 1.1, we denote by
\[
H^n = \sum_{i=0}^{n-1} \text{id}^{\otimes i} \otimes (-h) \otimes (\iota \pi)^{\otimes n-i-1} : T^n(s\Omega^*([0,1])) \to T^n(s\Omega^*([0,1])),
\]
where \( Q_2 : T^2(\Omega^*([0,1])) \to \mathfrak{s}\Omega^*([0,1]) \) is the quadratic part of the codifferential, encoding the wedge product on \( \Omega^*([0,1]) \), and finally by \( \pi_n : T^n(\mathfrak{s}\Omega^*([0,1])) \to sC^*([0,1]) \) the Taylor coefficients of the \( A_\infty \) morphism \( \pi_\infty : \Omega^*([0,1]) \to C^*([0,1]) \) induced via homotopy transfer. According to the usual perturbation formulas, cf. [16, 22], the maps \( \pi_n \) are determined recursively by \( \pi_1 = \pi \),

\[
\pi_n = \pi_{n-1}Q_n^{-1}H_n \quad \text{for } n \geq 2,
\]

and we want to prove that for \( n \geq 2 \) they vanish on tensor products of total degree less than \( 0 \). For \( n = 2 \) we have \( \pi_2 = \pi Q_2 H_2^2 \); if both arguments are functions, this vanishes by degree reasons, while if exactly one argument is a function, it vanishes since \( \pi \) vanishes on functions of the form \( f(t)h(a(t)dt) \), as \( \pi : \Omega^0([0,1]) \to C^0([0,1]) \) is strictly multiplicative and the image of \( h \) is contained in the kernel of \( \pi \). For \( n \geq 3 \) the thesis follows by a straightforward induction, since \( Q_n^{-1}H_n : T^n(\mathfrak{s}\Omega([0,1])) \to T^{n-1}(\mathfrak{s}\Omega([0,1])) \) preserves the total degree. \( \square \)

We obtain the following result as an immediate consequence of Lemma 2.11.

**Corollary 2.17.** The \( A_\infty \) morphism \( \Omega^*([0,1]) \to C^*_\infty([0,1]) \) obtained from homological perturbation theory – see [16, 22] – coincides with the \( C_\infty \) morphism \( \varphi : \Omega^*([0,1]) \to C^*_\infty([0,1]) \).

We next show that similar uniqueness results hold for the \( C_\infty \) algebra structure on \( C^*_\infty([0,1]) \) and the \( C_\infty \) morphism \( \mu \).

**Proposition 2.18.** The unital \( C_\infty \) algebra structure on \( C^*_\infty([0,1]) \), as in Theorem 1.1, is the only one with linear part \( m_1(st) = -std \), quadratic part satisfying \( m_2(st \otimes st) = st \) and higher Taylor coefficients vanishing unless precisely one argument is a zero-form. The morphism \( \mu \) is the only unital \( C_\infty \) morphism from \( C^*_\infty([0,1]) \) to \( \Omega^*([0,1]) \) with linear part the inclusion.

**Proof.** We shall denote by \( M \) the codifferential on \( T(sC^*([0,1])) \) encoding the \( C_\infty \) algebra structure, by \( m_n \) its Taylor coefficients as in Theorem 1.1 and by \( M_n \) the composition

\[
M_n : T^n(sC^*([0,1])) \looparrowright T(sC^*([0,1])) \overset{\pi}{\longrightarrow} T(sC^*([0,1])) \to T^n(sC^*([0,1])).
\]

Notice that the higher coefficients \( m_{n+1}, n \geq 1 \), vanish by degree reasons unless precisely one or two of the arguments are zero-forms. To illustrate the result we check the first claim directly for \( m_2 \): by the \( C_\infty \) property \( 0 = m_2(st \otimes sdt) = m_2(st \otimes sdt) + m_2(sdt \otimes st) \), where \( \otimes \) is the shuffle product (cf. Appendix [3]), hence

\[
2m_2(st \otimes sdt) = m_2(st \otimes sdt) - m_2(sdt \otimes st) = m_2 M_2^2(st \otimes st) = -m_2 M_2 M_2(st \otimes st) = sdt,
\]

from which we get \( m_2(st \otimes sdt) = \frac{1}{2}sdt = -m_2(sdt \otimes st) \). Next we assume inductively to have shown the thesis up to a certain \( n \). First of all, the \( C_\infty \) property implies

\[
m_{n+1}(sdt)^{\otimes i} \otimes st \otimes (sdt)^{\otimes n-i} = (-1)^i \binom{n}{i} m_{n+1}(st \otimes (sdt)^{\otimes n}).
\]

In fact, since this is clear for \( i = 0 \), it follows in general by induction on \( i \) and

\[
0 = m_{n+1}(sdt \otimes ((sdt)^{\otimes i-1} \otimes st \otimes (sdt)^{\otimes n-i})) = \]

\[
= im_{n+1}((sdt)^{\otimes i} \otimes st \otimes (sdt)^{\otimes n-i}) + (n - i + 1)m_{n+1}((sdt)^{\otimes i-1} \otimes st \otimes (sdt)^{\otimes n-i+1}).
\]

Combined with the fact that \( M \) is a codifferential, this shows

\[
(n + 1)m_{n+1}(st \otimes (sdt)^{\otimes n}) = m_{n+1} M_{n+1}^{n+1}(st^{\otimes 2} \otimes (sdt)^{\otimes n-1}) = -\sum_{i=1}^{n} m_i M_{n+1}^{n+1}(st^{\otimes 2} \otimes (sdt)^{\otimes n-i}).
\]
Since by hypothesis $m_{n+1}(st^\otimes 2 \otimes (sdt)^{\otimes n-1}) = 0$ for $n \geq 2$,

$$(n+1)m_{n+1}(st \otimes (sdt)^{\otimes n}) = -\sum_{i=2}^{n} m_{i}M_{n+1}^{i}(st^{\otimes 2} \otimes (sdt)^{\otimes n-1}),$$

which proves the thesis inductively, as the right hand side only depends on $m_{2}, \ldots, m_{n}$.

The claim about $\mu$ is proven similarly. For $n \geq 2$, by degree reasons $\mu_{n}$ vanishes unless precisely one or none of the arguments are zero-forms. In the latter case, by the $C_{\infty}$ property $\mu_{n}((sdt)^{\otimes n}) = \frac{1}{n!}\mu_{n}((sdt)^{\otimes n}) = 0$. In the former case, the $C_{\infty}$ property implies $\mu_{n+1}((sdt)^{\otimes i} \otimes st \otimes (sdt)^{\otimes n-1}) = (-1)^{i}m_{n+1}(st \otimes (sdt)^{\otimes n})$ as before. In particular, we see that $(n+1)\mu_{n+1}(st \otimes (sdt)^{\otimes n}) = \mu_{n+1}M_{n+1}^{1}(st^{\otimes 2} \otimes (sdt)^{\otimes n-1})$, and using the facts that $\mu_{n+1}(st^{\otimes 2} \otimes (sdt)^{\otimes n-1}) = 0$ by degree reasons and $\mu$ commutes with the codifferentials, we conclude as before that the right hand side only depends on $\mu_{1}, \ldots, \mu_{n}$.

**Remark 2.19.** In contrast with the final claim of the previous proposition, there can be several $A_{\infty}$ morphisms $C^{*}_{\infty}([0,1]) \to \Omega^{*}([0,1])$ whose linear part is the inclusion. For instance, a direct verification shows that $F : C^{*}_{\infty}([0,1]) \to \Omega^{*}([0,1])$, defined in Taylor coefficients $F_{1}, \ldots, F_{n}, \ldots$ by

- $F$ is unital and $F_{1}$ is the inclusion;
- $F_{n}((sdt)^{\otimes n}) = (-1)^{n-1}s(t^{n-1}dt)$ for $n \geq 1$;
- $F_{n}((sdt)^{\otimes (n-1)} \otimes st) = (-1)^{n}s(t^{n-1}(1-t))$ for $n \geq 2$;
- For $n \geq 2$, $F_{n}$ vanishes if an argument different from the rightmost one is a zero-form;
- is a unital $A_{\infty}$ morphism (which is right inverse to $\lambda$, by a straightforward application of Lemma 2.11). Therefore $F \circ \exp : C^{*}_{\infty}([0,1]) \to \Omega^{*}([0,1])$ is an $A_{\infty}$ morphism different from $\mu$, whose linear part is the inclusion.

The proof of Proposition 2.18 leads to the following result, which is of independent interest:

**Proposition 2.20.** Let Aut$_{\infty}(C^{*}_{\infty}([0,1]))$ be the group of unital $C_{\infty}$ automorphisms of $C^{*}_{\infty}([0,1])$, and $GL(C^{*}([0,1])) \cong \mathbb{k}^{*} \ltimes \mathbb{k} \cong \text{Aff}(\mathbb{k})$ the group of those automorphisms of the complex $C^{*}([0,1])$ which map 1 to itself. The correspondence

$$r : \text{Aut}_{\infty}(C^{*}_{\infty}([0,1])) \to \text{GL}(C^{*}([0,1])) \cong \text{Aff}(\mathbb{k})$$

$$\psi = (\psi_{1}, \psi_{2}, \ldots) \mapsto \psi_{1}$$

is an isomorphism of groups.

**Proof.** That the map $r$ is a morphism of groups is clear. The same argument as in the proof of Proposition 2.18 shows that a $C_{\infty}$ morphism with domain $C^{*}_{\infty}([0,1])$ is uniquely determined by its linear part, hence $r$ is injective.

To conclude the proof, we have to show that $r$ is surjective. Let us fix an automorphism $\xi$ of the complex $C^{*}([0,1])$ mapping 1 to itself. Evidently, $\xi$ is determined by its value on $t$, given by

$$\xi(t) = \alpha t + \beta,$$

for $\alpha \in \mathbb{k}^{*}$ and $\beta \in \mathbb{k}$ two constants. Our aim is to show that $\xi$ lies in the image of $r$. We define an automorphism $\rho$ of the unital dg algebra $\Omega^{*}_{\mathbb{k}}([0,1]) = \mathbb{k}[t] \oplus \mathbb{k}[t]dt$ by declaring its action on the generator $t$ to be $\rho(t) = \alpha t + \beta$. The composition

$$\tilde{\xi} : C^{*}_{\infty}([0,1]) \xrightarrow{\mu} \Omega^{*}_{\mathbb{k}}([0,1]) \xrightarrow{\rho} \Omega^{*}_{\mathbb{k}}([0,1]) \xrightarrow{\lambda} C^{*}_{\infty}([0,1]),$$

is a unital $C_{\infty}$ automorphism of $C^{*}_{\infty}([0,1])$ such that $r(\tilde{\xi}) = \xi$.

We close this section by sketching a relation with the papers [14, 21], which was also briefly outlined in the introduction.
Remark 2.21. We denote by $L([0,1]) = \tilde{L}(x,y,a)$ the Lawrence-Sullivan model of the interval: this is the free complete graded Lie algebra on generators $x$ and $y$ in degree one and $a$ in degree zero, and the unique differential such that $x, y$ are Maurer-Cartan elements and $a$ is a gauge equivalence between them, see [18], namely,

$$d(x) = -\frac{1}{2}[x,x], \quad d(y) = -\frac{1}{2}[y,y], \quad d(a) = \text{ad}_a(y) + \sum_{n \geq 0} \frac{B_n}{n!}(\text{ad}_a)^n(y-x),$$

where $\text{ad}_a(-) = [a,-]$ is the adjoint. As observed in the paper [7], this is also the Chevalley-Eilenberg dg Lie algebra associated to the $C_\infty$ algebra $C^*_\infty([0,1])$. We shall denote by $U(L([0,1]))$ its universal enveloping algebra. Following the notations from the introduction, we shall denote by $\Omega C_*([0,1])$ the cobar construction of the dg coalgebra of normalized chains on $[0,1]$, i.e., the complete tensor algebra $\hat{T}(x,y,a)$ over generators $x,y,a$ as before, equipped with the differential

$$d(x) = -x^2, \quad d(y) = -y^2, \quad d(a) = (1+a)y - x(1+a).$$

Notice that both $\Omega C_*([0,1])$ and $U(L([0,1]))$ have the same underlying graded algebra $\hat{T}(x,y,a)$, and only the differentials differ. The $A_\infty$ isomorphism $\exp : C^*_\infty([0,1]) \to C^*_\infty([0,1])$ from Subsection 1.3 yields an isomorphism of dg algebras

$$\Omega C_*([0,1]) \xrightarrow{\approx} U(L([0,1])), \quad x \mapsto x, \quad y \mapsto y, \quad a \mapsto e^a - 1.$$

In particular, there is an induced cocommutative dg Hopf algebra structure on $\Omega C_*([0,1])$, and it is easy to check that the induced diagonal $\Delta : \Omega C_*([0,1]) \to \Omega C_*([0,1]) \otimes \Omega C_*([0,1])$ is

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \Delta(y) = y \otimes 1 + 1 \otimes y, \quad \Delta(a) = a \otimes 1 + 1 \otimes a + a \otimes a.$$

Finally, it can be proved, in the spirit of this subsection, that the above $\Delta$ may be characterized as the unique morphism of unital augmented dg algebras satisfying $\Delta(x) = x \otimes 1 + 1 \otimes x, \Delta(y) = y \otimes 1 + 1 \otimes y$; details are left to the interested reader [7]. From this, one can deduce that the diagonal $\Delta$ coincides with the Alexander-Whitney cobar diagonal on $\Omega C_*([0,1])$, constructed as in the paper [14].

3. Pushforward and the Magnus expansion

In this subsection we present implications of our previous results for differential forms on $[0,1]$ with values in a dg algebra $A$ or a dg Lie algebra $\mathfrak{g}$, respectively.

Remark 3.1. We will extend the scalars for $\Omega^*([0,1])$ and $C^*([0,1])$ from $k$ to either a dg algebra $A$ or a dg Lie algebra $\mathfrak{g}$. In order for our previous discussion to remain meaningful, we have to guarantee existence and convergence of certain constructions. Two instances where this works are:

1. **Pro-case**: Assume that $A$ is unital and augmented and that the augmentation ideal $\bar{A}$ is pro-nilpotent. Correspondingly, assume that $\mathfrak{g}$ is pro-nilpotent. Then consider polynomial differential forms on $[0,1]$ with values in $A$ or $\mathfrak{g}$.

2. **Finite-dimensional case**: Assume that $A$ and $\mathfrak{g}$ are finite-dimensional and consider smooth differential forms on $[0,1]$ with values in $A$ or $\mathfrak{g}$.

---

1We have $\Delta(a) = \sum_{i,j \geq 0} r_{i,j} a^i \otimes a^j$ for certain constants $r_{i,j} \in k$: then $r_{0,0} = 0$, since $\Delta$ is a morphism of augmented dg algebras, and one checks, using the fact that $\hat{T}(x,y,a)$ is a free algebra, that the remaining $r_{i,j}$ are uniquely determined by $\Delta(x) = x \otimes 1 + 1 \otimes x, \Delta(y) = y \otimes 1 + 1 \otimes y$ and the requirement that $\Delta \circ d(a) = \Delta(y-x) + \Delta(a)\Delta(y) - \Delta(x)\Delta(a) = (d \otimes \text{id} + \text{id} \otimes d) \circ \Delta(a)$. 

---
In both cases we obtain dg algebras $\Omega^*[0,1; A)$ and $C_*^*[0,1; A)$, an $A_{\infty}$ algebra $C_*^*[0,1; A)$, as well as a dg Lie algebra $\Omega^*[0,1; g)$ and an $L_\infty$ algebra $C_*^*[0,1; g)$. The latter two were described in [25] and [9]. Observe that, since $C_*^*[0,1; ]$ is not commutative, extension of scalars to $g$ is not meaningful in this case (within the world of algebras).

3.1. Forms with values in a dg algebra. We first consider extension by a unital dg algebra $A$. The family of $C_{\infty}$ quasi-isomorphisms $\varphi_s : \Omega^*[0,1] \rightarrow C_*^*[0,1]$ extends to a one-parameter family of $A_{\infty}$ quasi-isomorphism

$$\varphi_s : \Omega^*[0,1; A) \rightarrow C_*^*[0,1; A),$$

see Definition [2,9]. The explicit formulas from Theorem [2,2] and [2,10] remain valid in this setting, i.e. they are compatible with scalar extension by $A$ (essentially, because they keep the arguments in order). Notice however that Proposition [2,6] fails in the non-commutative case.

**Definition 3.2.** The pushforward along $\varphi_s$ is the mapping

$$(\varphi_s)_* : s(\Omega^0([0,1]; A^1) \oplus \Omega^1([0,1]; A^0)) \rightarrow s(C_*^0([0,1]; A^1) \oplus C_*^1([0,1]; A^0) \cong sA^1 \oplus sA^1 \oplus A^0$$

$$f(t) + a(t)dt \mapsto \sum_{n \geq 1} \varphi_{s,n}(s(f(t) + a(t)dt) \times \cdots \times s(f(t) + a(t)dt)).$$

**Remark 3.3.** Since $\varphi_{s,n}$ vanishes for $n > 1$ whenever one of the inputs is a zero-form, we find

$$\sum_{n \geq 1} \varphi_{s,n}(s(f(t) + a(t)dt) \times \cdots \times s(f(t) + a(t)dt)) =$$

$$s(f(s)t + f(0)(1-t)) + \sum_{n \geq 1} \varphi_{s,n}(sa(t)dt \times \cdots \times sa(t)dt).$$

Therefore, we see that the essential information is the restriction of $(\varphi_s)_*$ to $s\Omega^1([0,1]; A^0)$.

We remark that in the pro-case, we have to restrict the domain of definition of $(\varphi_s)_*$ to

$$s(\Omega^0([0,1]; A^1) \oplus \Omega^1([0,1]; A^0)), $$

i.e. we have to require the one-forms to take values in the augmentation ideal $A^0$ of $A^0$. The reason is that this guarantees that the potentially infinite series in the definition of $(\varphi_s)_*$ is well-defined. Whenever we consider the pro-case, we will from now on apply this restriction.

The following result was established independently by Burghart-Mnëv-Steinebrunner in [4].

**Proposition 3.4.** For a given $a(t)dt \in \Omega^1([0,1]; A^0)$, consider the curve

$$[0,1] \rightarrow A^0, \quad s \mapsto A(s)sdA := (\varphi_s)_*(sa(t)dt).$$

Its exponential

$$e^{A(s)} := 1_A + \sum_{k \geq 0} \frac{1}{k!}(A(s))^k.$$

satisfies the differential equation

$$\frac{d}{ds}e^{A(s)} = e^{A(s)}a(s), \quad e^{A(0)} = 1_A.$$

**Proof.** Since pushforward is compatible with composition of morphisms, we find that

$$\exp_s \circ (\varphi_s)_* = \exp_s \circ \log_s \circ (\lambda_s)_* = (\lambda_s)_*, $$

where $\lambda_s = \lambda \circ \beta_s^*$, with $\beta_s^*$ being the scaling morphism from Definition [2,9]. We therefore have

$$e^{A(s)} = 1_A + (\lambda_s)_*(sa(t)dt).$$
The pushforward along \( \lambda_s \) is given by
\[
(\lambda_s)_*(sa(t)dt) = \sum_{n \geq 1} \int_{0 \leq t_1 \leq \cdots \leq t_n \leq s} a(t_1) \cdots a(t_n)dt_1 \cdots dt_n.
\]
Differentiation with respect to \( s \) yields
\[
d \frac{d}{ds} (1_A + (\lambda_s)_*(a(t)dt)) = (1_A + (\lambda_s)_*(a(t)dt))a(s)
\]
and for \( s = 0 \), we have \((1_A + (\lambda_0)_*(a(t)dt)) = 1_A\). This concludes the proof. □

**Remark 3.5.**

1. By Theorem 2.2, we can write the pushforward along \( \varphi \) as
\[
(\varphi)_*(sa(t)dt) = \left( \sum_{n \geq 1} \frac{1}{n!} \int_{0 \leq t_1 \leq \cdots \leq t_n \leq s} \sum_{\sigma \in S_n} \left( \frac{(-1)^{d_\sigma}}{n!^{d_\sigma}} a(t_{\sigma(1)}) \cdots a(t_{\sigma(n)}) \right) dt_1 \cdots dt_n \right) s dt.
\]
2. Alternatively, we may use Theorem 2.10 to describe the pushforward along \( \varphi_s \) as follows. We define maps \( M_n : \Omega^0([0, 1]; A^0)^{\otimes n} \to \Omega^0([0, 1], A^0) \) recursively as in Definition 2.8. Given \( a(t)dt \in \Omega^1([0, 1]; A^0) \), we simplify the notations and put \( M_k(s) := M_k(a(t)^{\otimes k})(s), M_\infty(s) := \sum_{k \geq 1} M_k(s) \). Then, according to Theorem 2.10
\[
M_\infty(s)sdT = \sum_{k \geq 1} (M_k(s)sdT) := \sum_{n \geq 1} \varphi_{s,n}(sa(t)dt \otimes \cdots \otimes sa(t)dt) = (\varphi)_*(sa(t)dt).
\]

Differentiating the defining recursion for the maps \( M_k \), we find
\[
\frac{d}{ds} M_\infty(s) = a(s) + \sum_{k \geq 1} \frac{B_k}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} \sum_{i_1, \ldots, i_k \geq 1} M_{i_1}(s) \cdots M_{i_j}(s) a(s) M_{i_{j+1}}(s) \cdots M_{i_k}(s)
\]
\[
= \sum_{k \geq 0} \frac{B_k}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} M_\infty(s)^j a(s) M_\infty(s)^{k-j}
\]
\[
= \sum_{k \geq 0} \frac{B_k}{k!} \left[ \cdots [a(s), M_\infty(s)] \cdots, M_\infty(s) \right],
\]
which is equivalent to
\[
\sum_{k \geq 0} \frac{1}{(k+1)!} \left[ \cdots \left\{ \frac{d}{ds} M_\infty(s), M_\infty(s) \right\} \cdots, M_\infty(s) \right] = a(s).
\]

According to a classical result by Hausdorff, compare with [17, Theorem 2.1], this shows that \( e^{M_\infty(s)} \) is the solution to the differential equation \( \frac{d}{ds} e^{M_\infty(s)} = e^{M_\infty(s)}a(s) \) with initial condition \( e^{M_\infty(0)} = 1_A \), and provides another proof of Proposition 3.4.

**3.2. Forms with values in a dg Lie algebra.** For \( g \) a dg Lie algebra, we obtain a one-parameter family of \( L_\infty \) quasi-isomorphisms
\[
\varphi_s : \Omega^*[0, 1]; g \to C_\infty([0, 1]; g)
\]
from \( \varphi : \Omega^*[0, 1] \to C_\infty^*[0, 1] \) by extension of scalars (cf. 24 for the definiton of scalar extension of a \( C_\infty \) algebra by a dg Lie algebra). By compatibility between scalar extension and homotopy transfer, together with Corollary 2.17 this is the same as the composition of the scaling morphism \( \beta^*_s \) and the \( L_\infty \) morphism induced via homotopy transfer along the obvious extension of Dupont’s contraction (cf. 24).
We denote the universal enveloping dg algebra of $g$ by $U(g)$. By compatibility with the symmetrization functor from $A_\infty$ algebra to $L_\infty$ algebras, $\varphi_s$ may also be characterized by the commutative diagram of $L_\infty$ algebras and $L_\infty$ morphisms,

\[
\begin{array}{ccc}
\Omega^*([0,1];g) & \xrightarrow{\varphi_s} & \Omega^*([0,1];U(g)) \\
\downarrow^{\text{sym}(\varphi_s)} & & \downarrow^{\text{sym}(\varphi_s)} \\
C^*_L([0,1];g) & \xrightarrow{\text{sym}(\varphi_s)} & C^*_L([0,1];U(g))
\end{array}
\]

where the horizontal arrows are the strict inclusions and the right vertical arrow is the symmetrization of the $A_\infty$ morphism studied in the previous subsection. For convenience, let us define maps

\[M_n : \bigodot^n (\Omega^0([0,1];g)) \to \Omega^0([0,1];g)\]

by setting $M_n(l_1(t) \odot \cdots \odot l_n(t))(s) dt := \varphi_{s,n}(sl_1(t) dt \odot \cdots \odot sl_n(t) dt)$.

**Theorem 3.6.** (1) The maps $(M_n)_{n \geq 1}$ are given by

\[
M_n(l_1(t) \odot \cdots \odot l_n(t))(s) = \int_{0 \leq t_1 \leq \cdots \leq t_n \leq s} \left( \frac{1}{n^2} \sum_{\sigma, \tau \in S_n} \varepsilon(\tau)(\frac{(-1)^{d_\sigma}}{(n-1)!}) [l_{\tau(1)}(t_{\tau(1)}), \ldots, [l_{\tau(n-1)}(t_{\tau(n-1)}), l_{\tau(n)}(t_{\tau(n)\ldots})] \ldots ] \right) dt_1 \cdots dt_n,
\]

where $\varepsilon(\tau)$ is the Koszul sign associated to $\tau$, i.e. the sign given by $l_1(t) \odot \cdots \odot l_n(t) = \varepsilon(\tau)l_{\tau(1)}(t) \odot \cdots \odot l_{\tau(n)}(t)$.

(2) Equivalently, we may define the maps $(M_n)$ recursively by putting $M_1(l_1(t))(s) = \int_0^s l_1(t_1) dt_1$ for $n = 1$, and for $n > 1$

\[
M_n(l_1(t) \odot \cdots \odot l_n(t))(s) = \sum_{k=1}^{n-1} (-1)^k \frac{B_k}{k!} \sum_{i_1 + \cdots + i_k = n - 1} \sum_{\sigma \in S_n} \varepsilon(\sigma) \int_0^s [M_{i_1} \cdots M_{i_k} (\cdots)(t_n), \ldots [M_{i_1} \cdots M_{i_k} (\cdots)(t_n), l_{\sigma(n)}(t_n)] \cdots ] dt_n,
\]

where the suspension points inside $M_{i_1} \cdots M_{i_k} (\cdots)$ have to be filled by the arguments in the order $l_{\sigma(1)}(t), \ldots, l_{\sigma(n-1)}(t)$.

**Proof.** The first explicit presentation follows by symmetrizing the formulas for the $A_\infty$-morphism $\varphi_s : \Omega^*([0,1];U(g)) \to C^*_L([0,1];U(g))$ coming from Theorem 2.2 where now the arguments $l_i(t) dt$ are elements in $\Omega^1([0,1];g^0) \subset \Omega^1([0,1];U^0(g))$: we see that $\varphi_{s,n}(l_1(t) dt \odot \cdots \odot l_n(t) dt)$ is the integral over the $n$th simplex of the image of

\[
\sum_{\tau \in S_n} \varepsilon(\tau) l_{\tau(1)}(t_1) \cdots l_{\tau(n)}(t_n) dt_1 \cdots dt_n \in \Omega^n([0,1]^n;U^0(g))
\]

under the Eulerian projector $E : U^0(g) \to g^0$. We recall, compare [27], that the latter may also be understood as the composition $E = p \circ \text{PBW}^{-1}$ of the inverse of the Poincaré-Birkhoff-Witt isomorphism $\text{PBW} : S(g) \to U(g)$ and the natural projection $p : S(g) \to g$. Finally, we get the desired formula for $M_n$ by composing $E$ with the Dynkin idempotent, as we did in formula (2) on page 13.

The claimed recursive presentation for the maps $M_n$ is precisely the one we get, after symmetrization, from the corresponding one in the $A_\infty$ case coming from Definition 2.8 as it follows.
by straightforward computations, keeping in mind the formula

\[ \sum_{\sigma \in S_k} \varepsilon(\sigma) [x_{\sigma(1)}, \cdots \cdots [x_{\sigma(k)}, y] \cdots ] = \]

\[ = \sum_{\sigma \in S_k} \varepsilon(\sigma) \sum_{j=0}^{k} (-1)^{k-j} + \sum_{h>j} |y||x_h| \binom{n}{j} x_{\sigma(1)} \cdots x_{\sigma(j)} y x_{\sigma(j+1)} \cdots x_{\sigma(k)}, \]

valid in any associative graded algebra. Thus, the second claim follows by Theorem 2.10. \[ \square \]

**Remark 3.7.** The first few instances of the previous recursion are

\[ M_1(l_1(t))(s) = \int_0^s l_1(t_1) dt_1, \]

\[ M_2(l_1(t) \circ l_2(t))(s) = \sum_{\sigma \in S_2} \varepsilon(\sigma) \frac{1}{2} \int_0^s \int_0^{t_2} \int_0^{t_1} l_{\sigma(1)}(t_1) dt_1 l_{\sigma(2)}(t_2) dt_2 \]

\[ M_3(l_1(t) \circ l_2(t) \circ l_3(t))(s) = \sum_{\sigma \in S_3} \varepsilon(\sigma) \frac{1}{12} \int_0^s \int_0^{t_3} \int_0^{t_2} \int_0^{t_1} l_{\sigma(1)}(t_1) dt_1 l_{\sigma(2)}(t_2) dt_2 \]

\[ + \sum_{\sigma \in S_3} \varepsilon(\sigma) \frac{1}{12} \int_0^s \int_0^{t_3} \int_0^{t_2} \int_0^{t_1} l_{\sigma(1)}(t_1) dt_1 \int_0^{t_3} l_{\sigma(2)}(t_2) dt_2 l_{\sigma(3)}(t_3) dt_3. \]

For the pushforward, we find the following general recursion, where for simplicity we put

\[ M_n(s) = M_n(l(t)^{\otimes n})(s), M_\infty(s) = \sum_{n \geq 1} \frac{1}{n!} M_n(s), \]

\[ (\varphi_s)_*(l(t) dt) = M_\infty(s) dt = \sum_{n \geq 1} \frac{1}{n!} M_n(s) dt = \]

\[ = \left( \int_0^s l(t_1) dt_1 + \sum_{n \geq 2} \frac{(-1)^{n-1} B_k}{k!} \sum_{i_1 + \cdots + i_k = n-1} \int_0^s [M_{i_1}(t_1), \cdots, [M_{i_k}(t_n), l(t_n)] \cdots ] dt_1 \cdots dt_n \right) dt. \]

Modulo the switch from \( [\cdot, \cdot] \) to the opposite bracket \( [x, y]^{op} := [y, x] \), this is precisely the recursive expansion given by Magnus, see [20, 17], for the solution of the differential equation \( \frac{d}{ds} e^{M_\infty(s)} = e^{M_\infty(s)} l(s) \) in the enveloping algebra \( U(g^0) \), compare with Proposition 3.4. By the previous theorem, we also find

\[ M_\infty(s) = \sum_{n \geq 0} \int_{t_1 \leq \cdots \leq t_n \leq s} \left( \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\mu_{\sigma}} \left[ l(t_{\sigma(1)}), \cdots, [l(t_{\sigma(n-1)}), l(t_{\sigma(n)})] \cdots \right] dt_1 \cdots dt_n. \]

This formula for the Magnus expansion was found by Miernik and Plabański [23].

**APPENDIX A. REVIEW OF \( A_\infty \) AND \( L_\infty \) ALGEBRAS**

We briefly describe our terminology and notations concerning \( A_\infty \) and \( L_\infty \) algebras. In the next section we shall review in more detail some results concerning \( C_\infty \) algebras.

- The suspension endofunctor \( s \) maps a graded vector space \( V \) to its suspension \( sV \), whose component \( (sV)^i \) in degree \( i \in \mathbb{Z} \) is \( V^{i+1} \).
\[ \mathcal{T}(V) = \bigoplus_{n \geq 1} T^n(V) \] denotes the reduced tensor coalgebra on a graded vector space, with the deconcatenation coproduct \( \Delta : \mathcal{T}(V) \to \mathcal{T}(V) \otimes \mathcal{T}(V) \),

\[ \Delta(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n-1} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n). \]

It is the cofree object over \( V \) in the category of coassociative, locally conilpotent (i.e., the union of the kernels of the iterated coproducts is exhaustive) graded coalgebras.

- We denote by \( S_n \) the \( n \)’th symmetric group. Given an integer \( n \geq 1 \) and an ordered partition \( i_1 + \cdots + i_k = n \), we denote by \( S(i_1, \ldots, i_k) \subset S_n \) the set of \((i_1, \ldots, i_k)\)-unshuffles, i.e., permutations \( \sigma \in S_n \) such that \( \sigma(i) < \sigma(i+1) \) for \( i \neq i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_{k-1} \).

- The symmetric group \( S_n \) acts on \( T^n(V) \) by \( \sigma(x_1 \otimes \cdots \otimes x_n) = \varepsilon(\sigma)x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \), where \( \varepsilon(\sigma) = \varepsilon(\sigma; x_1, \ldots, x_n) \) is the usual Koszul sign. We denote the space of coinvariants either by \( S^n(V) \) or by \( \bigodot^n(V) \), and by \( x_1 \otimes \cdots \otimes x_n \) the image of \( x_1 \otimes \cdots \otimes x_n \) under the natural projection \( T^n(V) \to S^n(V) \). The reduced symmetric coalgebra over \( V \) is the space \( S(V) = \bigoplus_{n \geq 1} S^n(V) \), with the unshuffle coproduct

\[ \Delta(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n-1} \sum_{\sigma \in S(i,n-i)} \varepsilon(\sigma)(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(i)}) \otimes (x_{\sigma(i+1)} \otimes \cdots \otimes x_{\sigma(n)}). \]

This is the cofree, coassociative, cocommutative and locally conilpotent graded coalgebra over \( V \).

- Let \((C, \Delta)\) be a graded coalgebra. A map \( Q : (C, \Delta) \to (C, \Delta) \) of degree 1 is a codifferential if \( Q \circ Q = 0 \) and \( \Delta \circ Q = (Q \otimes \text{id} + \text{id} \otimes Q) \circ \Delta \) hold true.

- An \( A_\infty \) algebra structure on a graded vector space \( V \) is a codifferential \( Q \) of the graded coalgebra \( (\overline{T}(sV), \Delta) \). Similarly, an \( L_\infty \) algebra structure on \( V \) is a codifferential \( Q \) of the graded coalgebra \( (\overline{S}(sV), \overline{\Delta}) \).

- A morphism of \( A_\infty \) algebras from \( A_\infty \) algebra \( V \) to \( A_\infty \) algebra \( W \) is a morphism of the corresponding dg coalgebras \( F : (\overline{T}(sV), \Delta, Q_V) \to (\overline{T}(sW), \Delta, Q_W) \). In the same manner one defines morphisms of \( L_\infty \) algebras.

- An \( A_\infty \) algebra structure \( Q \) on \( V \) is determined by its Taylor coefficients \((Q_n)_{n \geq 1}\), which are the maps given by

\[ T^n(sV) \longrightarrow \overline{T}(sV) \overset{Q}{\longrightarrow} \overline{T}(sV) \overset{p}{\longrightarrow} T^1(sV) \cong sV. \]

Moreover, a morphism \( F \) of \( A_\infty \) algebras from \( V \) to \( W \) is determined by its Taylor coefficients \( F_n : T^n(sV) \to sW \), which are defined in the same manner as the Taylor coefficients of an \( A_\infty \) algebra structure.

- Similarly, an \( L_\infty \) algebra structure \( Q \) on \( V \) is determined by its Taylor coefficients \( Q_n : \bigodot^n(sV) \to sV \), for \( n \geq 1 \), and a \( L_\infty \) algebra morphism \( F \) from \( V \) to \( W \) is determined by its Taylor coefficients \( F_n : \bigodot^n(sV) \to sW \).

- A morphism of \( A_\infty \) algebras, respectively \( L_\infty \) algebras, is called a quasi-isomorphism if its first Taylor coefficient induces an isomorphism on cohomology.

- The category of dg algebras embeds into the category of \( A_\infty \) algebra via the embedding

\[ (A, \cdot, d) \mapsto (\overline{T}(sA), Q), \]

where \( Q \) is the coderivation whose non-trivial Taylor coefficients are \( Q_1(sa) = -s(da) \) and \( Q_2(sa \otimes sb) = (-1)^{|a|} s(a \cdot b) \). Similar formulas define an embedding of the category of dg Lie algebras into the category of \( L_\infty \) algebras.
• The forgetful functor from dg associative algebras to dg Lie algebras admits the following higher generalization. Given a graded vector space \( V \), we denote by \( \text{sym}_n \), \( n \geq 1 \), the maps

\[
sym_n : S^n(sV) \to T^n(sV), \quad sx_1 \odot \cdots \odot sx_n \mapsto \sum_{\sigma \in S_n} \varepsilon(\sigma) sx_{\sigma(1)} \odot \cdots \odot sx_{\sigma(n)}.
\]

If \( Q_n : T^n(sV) \to sV \), \( n \geq 1 \), are the Taylor coefficients of an \( A_{\infty} \) algebra structure on \( V \), then the \( Q_n \circ \text{sym}_n : S^n(sV) \to sV \) are the Taylor coefficients of an \( L_{\infty} \) algebra structure \( \text{sym}(Q) \) on \( V \). Similarly, if \( F_n : T^n(sV) \to sW \) are the Taylor coefficients of an \( A_{\infty} \) morphism \( F : (V, Q_V) \to (W, Q_W) \), then \( F_n \circ \text{sym}_n : S^n(sV) \to sW \) are the Taylor coefficients of an \( L_{\infty} \) morphism \( \text{sym}(F) : (V, \text{sym}(Q_V)) \to (W, \text{sym}(Q_W)) \). This defines the \textit{symmetrization functor} from the category of \( A_{\infty} \) algebras to the one of \( L_{\infty} \) algebras.

**Appendix B. Review of \( C_{\infty} \) algebras**

\( C_{\infty} \) algebra structures are \( A_{\infty} \) algebra structures which are compatible with the shuffle product on the reduced tensor coalgebra. To be precise, the reduced tensor coalgebra \( (\mathcal{T}(V), \mathcal{T}) \) can be equipped with the structure of a graded bialgebra by introducing the shuffle product

\[
(v_1 \odot \cdots \odot v_p) \odot (v_{p+1} \odot \cdots \odot v_n) = \sum_{\sigma \in S(p,q)} \varepsilon(\sigma)v_{\sigma^{-1}(1)} \odot \cdots \odot v_{\sigma^{-1}(n)},
\]

where \( S(p,q) \) is the set of \((p,q)\)-unshuffles, i.e. a permutation \( \sigma \) of \( \{1, \ldots, n\} \) such that \( \sigma(i) < \sigma(i+1) \) for all \( i \neq p \).

**Definition B.1.** A \( C_{\infty} \) algebra structure on a graded space \( V \) is a dg bialgebra structure \( Q : \mathcal{T}(sV) \to \mathcal{T}(sV) \) on the graded bialgebra \( (\mathcal{T}(sV), \mathcal{T}, \mathcal{T}) \). A \( C_{\infty} \) morphism \( F : V \to W \) between \( C_{\infty} \) algebras \( V \) and \( W \) is a morphism of dg bialgebras \( F : \mathcal{T}(sV) \to \mathcal{T}(sW) \).

Let \((C, \Delta_C, m_C)\) and \((D, \Delta_D, m_D)\) be graded bialgebras with coproducts \( \Delta_C, \Delta_D \) and products \( m_C, m_D \) respectively. Recall that given a morphism \( F : (C, \Delta_C) \to (D, \Delta_D) \) of graded coalgebras, a linear map \( R : C \to D \) is an \( F \)-coderivation if it satisfies the identity \( \Delta_D R = (R \otimes F + F \otimes R) \Delta_C \). Similarly, given a morphism of graded algebras \( F : (C, m_C) \to (D, m_D) \), a linear map \( R : C \to D \) is an \( F \)-derivation if it satisfies the identity \( R m_C = m_D (R \otimes F + F \otimes R) \). Finally, given a morphism of graded bialgebras \( F : (C, \Delta_C, m_C) \to (D, \Delta_D, m_D) \), a linear map \( R : C \to D \) is an \( F \)-biderivation if it is both an \( F \)-coderivation and an \( F \)-derivation. When \( F = \text{id}_C \) we recover the usual definition of a (resp.: \( co, bi \))derivation on \( C \). The proof of the following lemma is a straightforward verification.

**Lemma B.2.** Given a morphism of (resp.: \( co, bi \))algebras \( F : C \to D \) and (resp.: \( co, bi \))derivations \( Q : C \to C, Q' : D \to D \), then the maps \( FQ, Q'F : C \to D \) are \( F \)-(resp.: \( co, bi \))derivations.

We say that a graded coalgebra \((C, \Delta_C)\) is locally conilpotent if \( C = \bigcup_{n \geq 1} \ker(\Delta^n_C) \), where \( \Delta^n_C : C \to C^{\otimes n+1} \) is the iterated coproduct. Recall that \((\mathcal{T}(V), \mathcal{T})\) is the cofree locally conilpotent graded coalgebra over \( V \); in particular, if \( C \) is locally conilpotent every morphism of graded coalgebras \( F : C \to \mathcal{T}(V) \) (resp.: every \( F \)-coderivation \( R : C \to \mathcal{T}(V) \)) is determined by its corestriction \( pF : C \to V \) (resp.: \( pR : C \to V \)), where we denote by \( p : \mathcal{T}(V) \to V \) the natural projection. This applies to \( C = \mathcal{T}(V) \otimes \mathcal{T}(V) \), equipped with the induced (locally conilpotent) coalgebra structure: in particular, the shuffle product \( \mathcal{T}(V) \otimes \mathcal{T}(V) \to \mathcal{T}(V) \) is the only morphism of graded coalgebras with vanishing corestriction \( 0 = p \circ : \mathcal{T}(V) \otimes \mathcal{T}(V) \to \mathcal{T}(V) \) ′.
Lemma B.3. A coderivation \( Q : \mathcal{T}(V) \to \mathcal{T}(V) \) of a reduced tensor coalgebra is also a derivation with respect to the shuffle product \( \circ \) if and only if its Taylor coefficients \( Q_n : V^\otimes n \to V \) vanish on the image of \( \circ \).

Proof. We have to show \( Q \circ = \circ(Q \otimes \mathrm{id} + \mathrm{id} \otimes Q) \). Since both the left and the right hand side are \( \circ \)-coderivations by Lemma B.2 it suffices to show that they have the same corestriction: as \( p \circ = 0 \), this happens if and only if the composition \( \mathcal{T}(V) \otimes \mathcal{T}(V) \xrightarrow{\circ} \mathcal{T}(V) \xrightarrow{Q} \mathcal{T}(V) \xrightarrow{\circ} V \) also vanishes.

Lemma B.4. A morphism of graded coalgebras \( F : (\mathcal{T}(V), \Delta) \to (\mathcal{T}(W), \Delta) \) is also a morphism of graded bialgebras if and only if its Taylor coefficients \( F_n : V^\otimes n \to W \) vanish on the image of the shuffle product.

Proof. As for the previous lemma, the two morphisms of graded locally conilpotent coalgebras \( \circ(F \otimes F), F \circ : \mathcal{T}(V) \otimes \mathcal{T}(V) \to \mathcal{T}(W) \) coincide if and only if they have the same corestriction if and only if the composition \( \mathcal{T}(V) \otimes \mathcal{T}(V) \xrightarrow{\circ} \mathcal{T}(V) \xrightarrow{F} \mathcal{T}(W) \xrightarrow{\circ} W \) vanishes.

Given an \( A_\infty \) algebra \( V \), whose Taylor coefficients are \( (Q_i)_{i \geq 1} \), and contraction data

\[
\begin{array}{ccc}
\text{sW} & \xrightarrow{F_1} & \text{sV}, \\
G_1 & \xrightarrow{F_n} & K \end{array}
\]

The usual \( A_\infty \) homotopy transfer theorem – see [16, 22] – tells us that the maps \( R_n : \text{sW}^\otimes n \to \text{sW} \) (where \( R_1 \) is the differential on \( \text{sW} \)) and \( F_n : \text{sW}^\otimes n \to \text{sV} \), defined recursively by

\[
R_n = \sum_{i=2}^n G_1 Q_i F_n^i, \quad F_n = \sum_{i=2}^n K Q_i F_n^i,
\]

where \( F_n^i = \sum_{j_1 + \cdots + j_i = n} F_{j_1} \otimes \cdots \otimes F_{j_i} : \text{sW}^\otimes n \to \text{sV}^\otimes i \), are respectively the Taylor coefficients of an \( A_\infty \) algebra structure on \( W \) and an \( A_\infty \) quasi-isomorphism \( F : W \to V \).

Theorem B.5. In the above hypotheses, if \( V \) is a \( C_\infty \) algebra then \( R_n, F_n \) as in formula (3) are the Taylor coefficients of a \( C_\infty \) algebra structure on \( \text{sW} \) and a \( C_\infty \) quasi-isomorphism respectively.

This result was established by Cheng and Getzler [7] with a different proof.

Proof. Suppose inductively we have shown that \( R_i, F_i \) vanish on the image of the shuffle product for all \( i < n \), the induction starting at \( n = 2 \) where it is trivial: then the morphism of graded coalgebras \( F_{<n} : \mathcal{T}(\text{sW}) \to \mathcal{T}(\text{sV}) \) with Taylor coefficients \( (F_{<n})_i = F_i \) if \( i < n \) and \( (F_{<n})_i = 0 \) if \( i \geq n \), is also a morphism of graded bialgebras. Moreover, since \( V \) is a \( C_\infty \) algebra the coderivation \( Q_{\geq 2} : \mathcal{T}(\text{sV}) \to \mathcal{T}(\text{sV}) \) with vanishing linear part \( (Q_{\geq 2})_1 = 0 \) and the same higher Taylor coefficients as \( Q \) if \( i \geq 2 \), is also a biderivation. Both statements follow by the previous two lemmas. Finally, \( R_n \) and \( F_n \) are respectively the composition with \( G_1 : \text{sV} \to \text{sW} \) and \( K : \text{sV} \to \text{sV} \) of the map

\[
\text{sW}^\otimes n \xrightarrow{F_{<n}} \mathcal{T}(\text{sW}) \xrightarrow{Q_{\geq 2}} \mathcal{T}(\text{sV}) \xrightarrow{p} \text{sV},
\]

and the latter vanishes on the image of the shuffle product: in fact, so does the corestriction map \( p : \mathcal{T}(\text{sV}) \to \text{sV} \), and by Lemma B.2 the composition \( Q_{\geq 2} F_{<n} : \mathcal{T}(\text{sW}) \to \mathcal{T}(\text{sV}) \) is an \( F_{<n} \)-biderivation, hence it sends the image of the shuffle product in \( \mathcal{T}(\text{sW}) \) into the image of the shuffle product in \( \mathcal{T}(\text{sV}) \).
References


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